QIS-oriented primer on Hamiltonian lattice gauge theories

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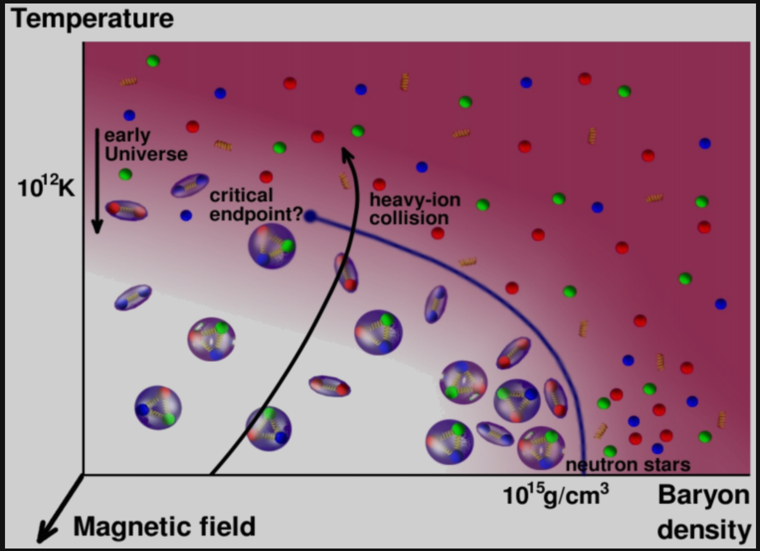
Big picture

Physics targets:

- Simulation of quantum chromodynamics
  - Hadronization
  - Microscopic understanding of nuclear interactions
- Complete phase diagram of QCD
- Equation of state for nuclear matter



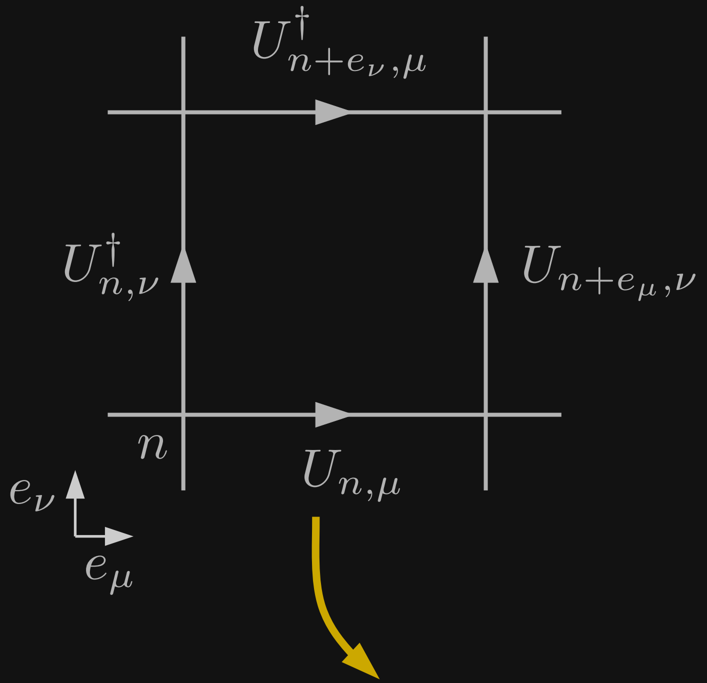
How to make these predictions?

- Nonperturbative problems
  - → **Numerically simulate** QCD degrees of freedom

Conjectured phase diagram credit: G. Endrődi J.Phys.Conf.Ser. 503 (2014) 012009

Traditional lattice field theory

$$x^\mu \to a n^\mu$$

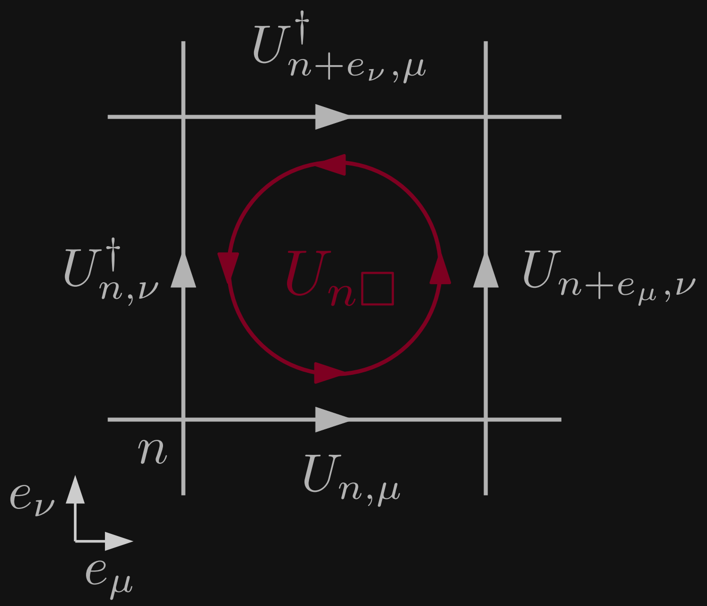


$$\begin{pmatrix} -0.7485 & -0.2744 - 0.6037i \\ 0.2744 - 0.6037i & -0.7485 \end{pmatrix}$$

- Defines a field theory nonperturbatively
- Spacetime discretized with a lattice (e.g. square, cubic, hypercubic)
- Matter particles such as quarks are described "live" on the sites
- Gauge bosons live on oriented links joining sites
- Gauge fields belonging to some Lie group–the "gauge group" G

Traditional lattice field theory

$x^\mu \to a n^\mu$



Wilson's gauge action, S_W

"link operators" $U_{n,\mu}$ in gauge group G

$$S_W = -\beta \sum_{n,\mu} \mathrm{tr}\big(\underbrace{U_{n,\mu} U_{n+e_\mu,\nu} U^\dagger_{n+e_\nu,\mu} U^\dagger_{n,\nu}}_{U_\square} + U^\dagger_\square\big)$$

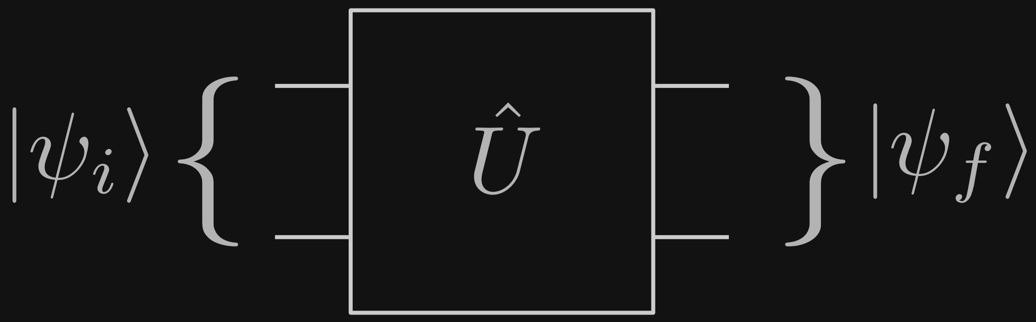
"plaquette" operator

for non-Abelian

- In classical simulations, exp(-S_W) acts like a probably weight for the configuration
- Real-time dynamics and nonzero baryon density both suffer from 'sign problems' in classical simulations

Classical problems.. quantum solutions?

Digital quantum computers:



- Unitary gates: $e^{-it\hat{H}}$ with Hamiltonian of interest
- Want to simulate nonperturbative gauge theory
  - ➔ Gauge theory on the lattice
  - ➔ Hamiltonian lattice gauge theory
- Has no apparent sign problems

General problem:

How to map a Hilbert space $\mathcal{H}$, and $\hat{H}$, on to qubits & quantum gates?

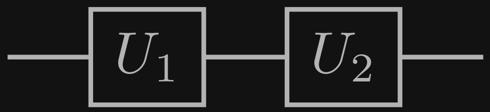
Gate-based model

Two state, qubit system – computational basis

$$|\uparrow\rangle \leftrightarrow |0\rangle$$
$$|\downarrow\rangle \leftrightarrow |1\rangle$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

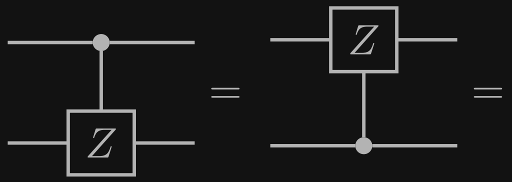
$|\psi\rangle$



$U_2 U_1 |\psi\rangle$

Two qubit basis: |00>, |01>, |10>, |11>

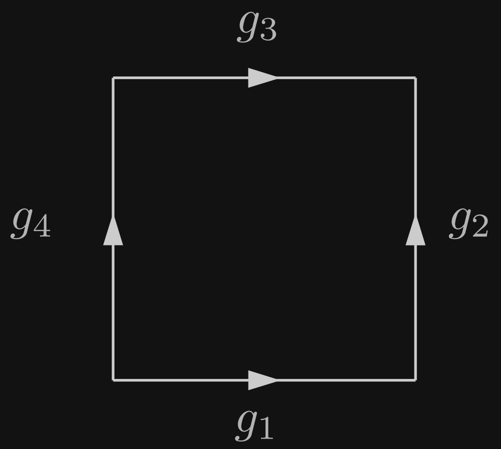
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Nielsen & Chuang (2001)

Warm-up: Z(2) plaquette



Local link bases

$$g_i \in \{+1, -1\}$$

Qubit mapping

$$g_i = +1 \rightarrow |0\rangle$$
$$g_i = -1 \rightarrow |1\rangle$$

$$Z\,|b\rangle = (-)^b\,|b\rangle \qquad b = 0, 1$$

Global/lattice basis

$$|b_1, b_2, b_3, b_4\rangle \equiv$$
$$|b_1\rangle \otimes |b_2\rangle \otimes |b_3\rangle \otimes |b_4\rangle$$

Electric fields $\quad Q_\ell = X_\ell$

Link operators $\quad U_\ell = Z_\ell$
"A$_\mu$ basis"

On-link algebra $\quad QUQ = -U$

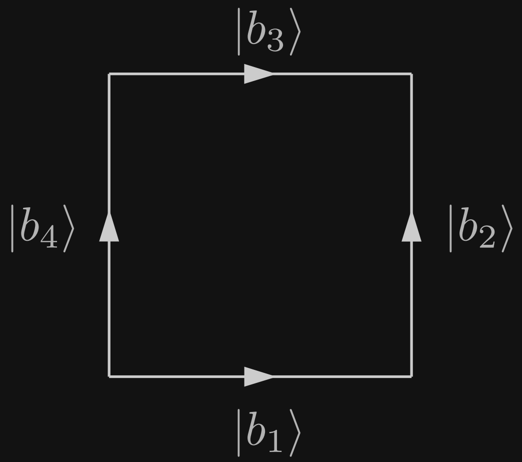
Gauge symmetry

$$G_{12} = Q_1 Q_2$$
$$G_{12} U_{1/2} G_{12} = -U_{1/2}$$
$$G_{12} U_{3/4} G_{12} = U_{3/4}$$

Gauge invariant Hamiltonian

$$H_E = \sum_{\ell=1}^{4} Q_\ell$$
$$H_B = -\lambda\, P = -\lambda\, U_1 U_2 U_3 U_4$$

Z(2) plaquette



Real-time evolution

Qubit Hamiltonian

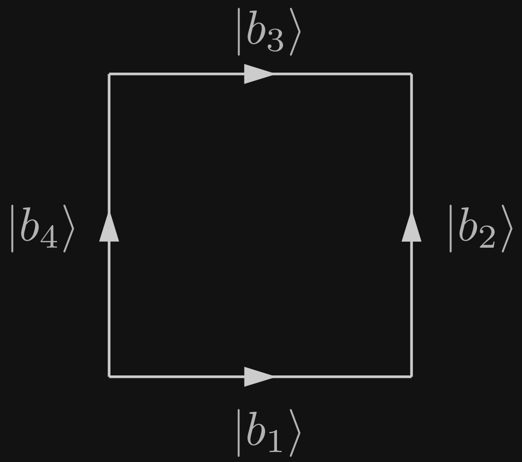
$$H = \sum_{\ell=1}^{4} X_\ell - \lambda\, Z_1 Z_2 Z_3 Z_4$$

Trotter-Suzuki approximation

$$e^{-iHt} = \left(e^{-iHt/s}\right)^s$$

$$e^{-iH\,\delta t} = e^{-iH_E\,\delta t} e^{-iH_B\,\delta t} + O(\,\delta t^2)$$

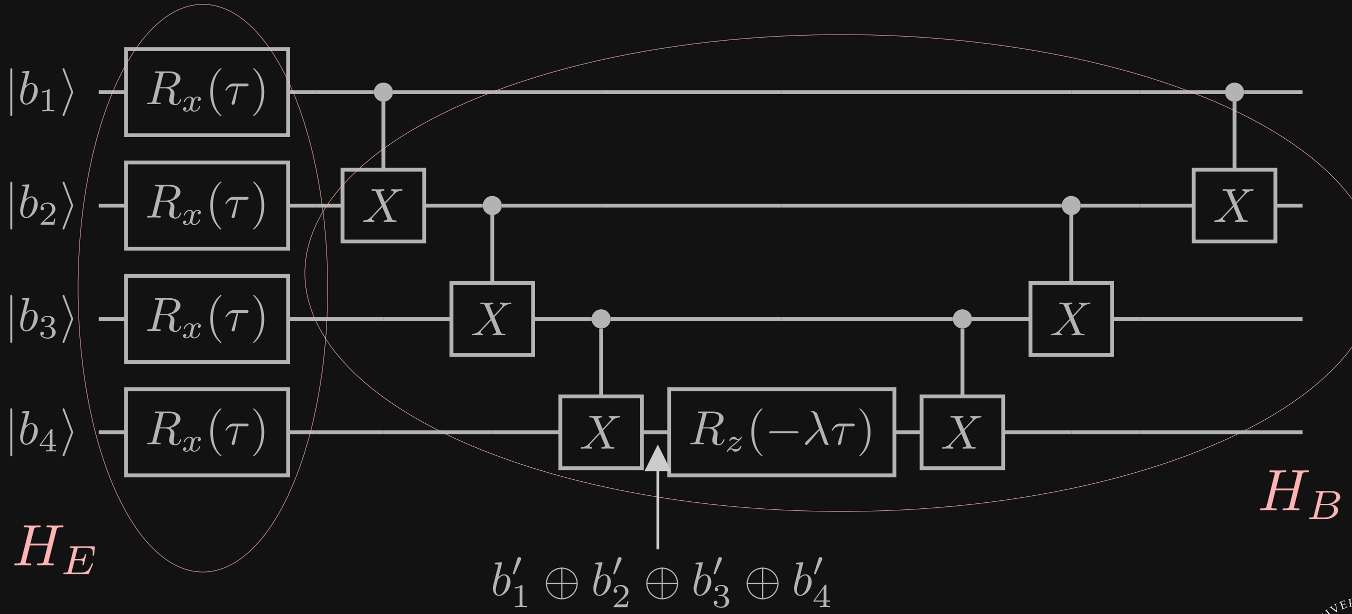
Z(2) plaquette



Real-time evolution

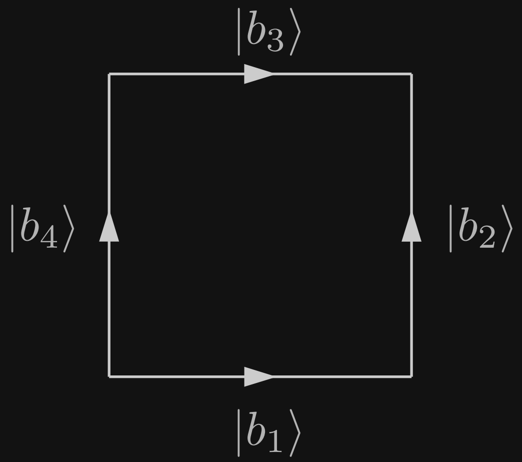
$$e^{iH_E\tau} = e^{iX_1\tau}e^{iX_2\tau}e^{iX_3\tau}e^{iX_4\tau}$$

$$e^{iH_B\tau} = e^{-i\lambda Z_1 Z_2 Z_3 Z_4 \tau}$$



$$R_z(\theta)\,|b_1 \oplus b_2 \oplus b_3 \oplus b_4\rangle = e^{i\theta(-)^{b_1\oplus b_2\oplus b_3\oplus b_4}}\,|b_1 \oplus b_2 \oplus b_3 \oplus b_4\rangle$$

Z(2) plaquette: Electric basis



Qubit Hamiltonian, electric basis

$$H = \sum_\ell Z_\ell - \lambda X_1 X_2 X_3 X_4$$

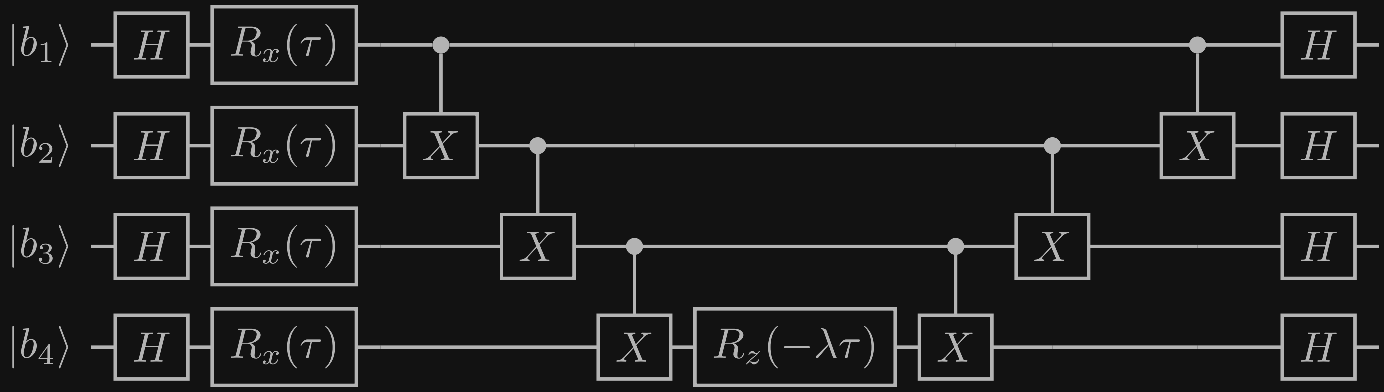
$$G_{12} = Z_1 Z_2$$

$$e^{iH_E\tau} = e^{iZ_1\tau} e^{iZ_2\tau} e^{iZ_3\tau} e^{iZ_4\tau}$$

$$e^{iH_B\tau} = e^{-i\lambda X_1 X_2 X_3 X_4 \tau}$$

$$HXH = Z$$

$$HZH = X$$



Hamiltonian lattice gauge theory

Lattice gauge theory Hilbert space structure

- An Abelian group, U(1)

$$\langle \phi | q \rangle = \frac{1}{\sqrt{2\pi}} e^{i\phi q}$$

group element or "coordinate" basis for link

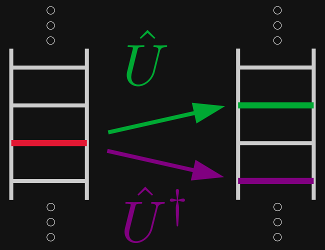
electric representation or "momentum" basis

Quantized with canonical, same-link commutation relations.

$$[E, U] = U$$

$$U |q\rangle = |q+1\rangle$$

"U raises E"



Gauge transformations:

$$\hat{U}_{n,i} \rightarrow e^{i(\theta_n - \theta_{n+e_i})} \hat{U}_{n,i}$$

$$\hat{H}_E = \frac{g^2}{2} \sum_{\vec{n},i} \hat{E}^2_{\vec{n},i}$$

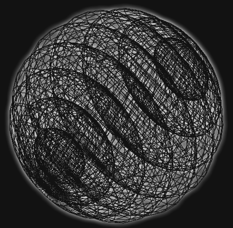
$$\hat{H}_B = -\sum_{\vec{n}} \frac{1}{2g^2} \mathrm{Re}(\hat{U}_{\vec{n},\Box})$$

Kogut & Susskind (1975); Creutz (1983), Smit (2002)

Hamiltonian lattice gauge theory

Lattice gauge theory Hilbert space structure

- Non-Abelian group, e.g. SU(2)



$$\langle g | j, m, n \rangle = \sqrt{\frac{d_j}{|G|}} D^{(j)}_{m,n}(g)$$

group element basis

representation basis

canonical, same-link commutation relations

$$[E^a_{L/R}, E^b_{L/R}] = i f^{abc} E^c_{L/R}$$

$$[E^a_R, U] = U T^a$$

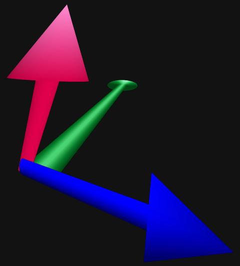
$$[E^a_L, U] = -T^a U$$

Gauge transformations:

$$\hat{U}_{n,i} \rightarrow \Omega_n \hat{U}_{n,i} \Omega^\dagger_{n+e_i}$$

"Left" and "right" electric fields to generate the independent left/right rotations.

Left and right electric fields each have 'colored' components in addition to spatial components



True gluons would have 8 such components

Hamiltonian lattice gauge theory

Lattice gauge theory Hilbert space structure

- Non-Abelian group, e.g. SU(2)

"*U* adds representations"

$$U_{m,m'} |j, M, M'\rangle = C_+(j, m, m', M, M')\times \times |j + 1/2, M + m, M' + m'\rangle + C_-(j, m, m', M, M')\times \times |j - 1/2, M + m, M' + m'\rangle$$

SU(2) example for the 2x2 link operator

Zohar & Burrello (2015)

Non-Abelian Hamiltonian

$$\hat{H}_E = \frac{g^2}{2} \sum_{n,i} \hat{E}^{\alpha}_{n,i} \hat{E}^{\alpha}_{n,i}$$

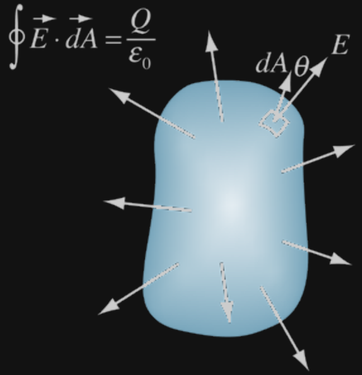
$$\hat{H}_B = -\sum_{n} \frac{1}{2g^2} \mathrm{tr}(\hat{U}_{n,\square} + \hat{U}^{\dagger}_{n,\square})$$

Hamiltonian lattice gauge theory

Plus Gauss law constraints

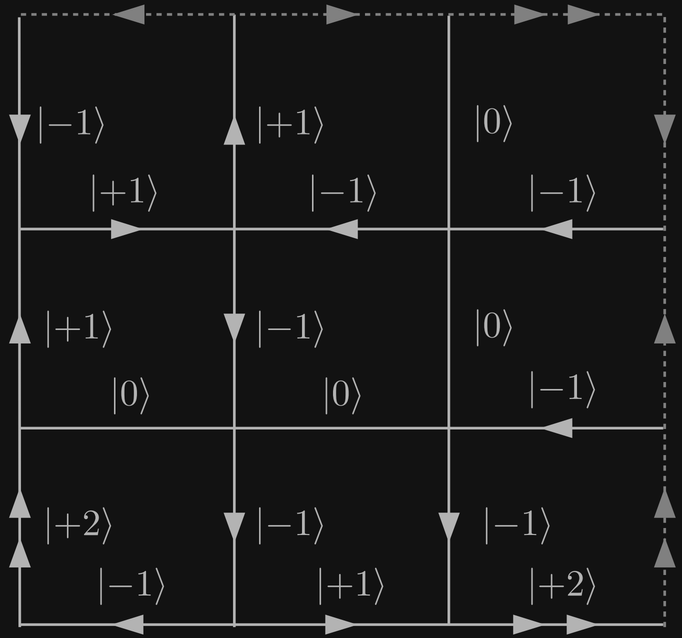
U(1) $\nabla \cdot \mathbf{E} - \rho = 0$

$\hat{\mathcal{G}}_n$ ↗ $\rho = \psi^\dagger \psi$



SU(N) $\mathbf{D} \cdot \mathbf{E}^a - \rho^a = 0$

$\hat{\mathcal{G}}_n^a$ ↗ $\rho^a = \psi^\dagger T^a \psi$



compact U(1)
electric eigenbasis

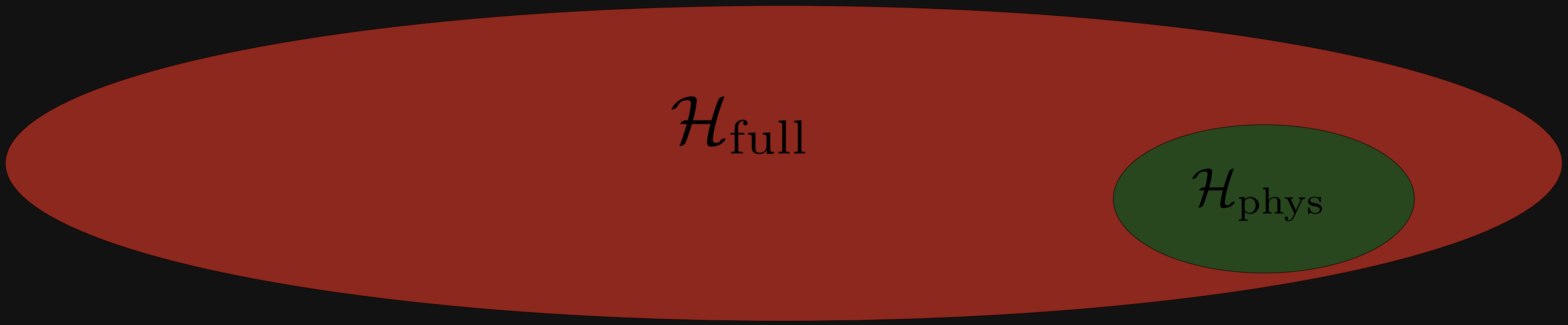
Gauss's law ⟷ gauge invariance

gauge invariance ⟷ charge conservation

Gauss's law ⟷ charge conservation

Potential issues simulating KS formulation

- Qubits wasted on physical states



- Non-Abelian constraints mean individual basis states are virtually never allowed by themselves
- Quantum noise will create components along unphysical directions
- Gauge invariance not necessarily respected by algorithms, even for noiseless simulation

Schwinger model [U(1)]

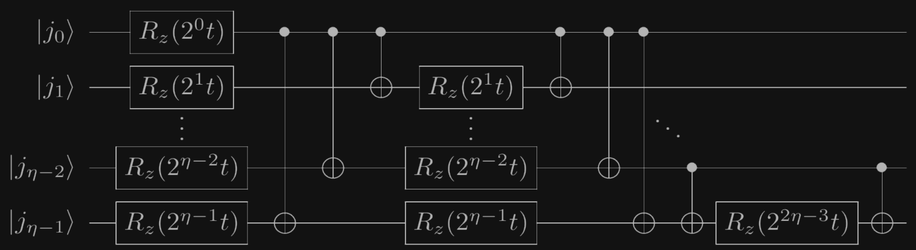
- Less trivial example:
  Schwinger model

Martinez et al (2016); Klco et al (2018)
Shaw, Lougovski, JRS, Wiebe (2020)

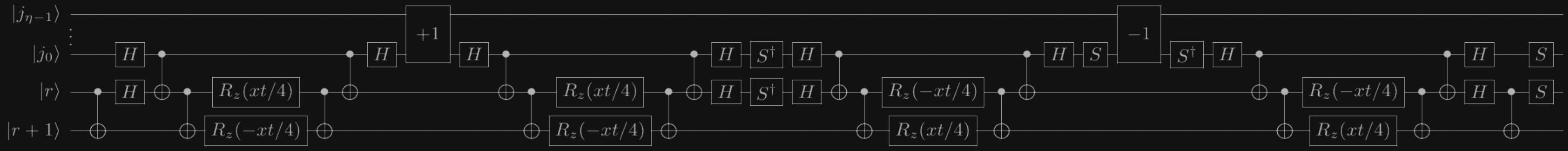
$$|j\rangle = \left|\sum_{n=0}^{\eta-1} j_n 2^n\right\rangle = \bigotimes_{n=0}^{\eta-1} |j_n\rangle$$

$$j = E - E_{\min}$$

H_E



H_I



SU(2) plaquettes example: Klco, Savage, JRS (2020)

Further reading

- Approaches
  - Quantum Link Models (Wiese, Chandrasekharan, et al)
  - Schwinger bosons (Raychowdhury et al)
  - Loop-string-hadron (Raychowdhury, JRS)
  - group element digitization (Zohar, Burrello, Lamm, Lawrence, Yamauchi)
  - tensor methods (Meurice, Unmuth-Yockey, et al)
  - dual variables (Kaplan, JRS, et al)
  - particle bases (Mueller*)
  - 'Qubit models' (Chandrasekharan, Singh, et al)
  - Tensor Networks (Cirac, Dalmonte, Tagliacozzo, Zoller, et al)
- Also **analog** approaches (Zohar et al, Davoudi et al, …) – older

For more complete list see these recent reviews:
1910.00257
1911.00003