Theory of strong QED field at RHIC

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Online workshop: RHIC Science Programs Informative Toward EIC in the Coming Years
May 26, 2021

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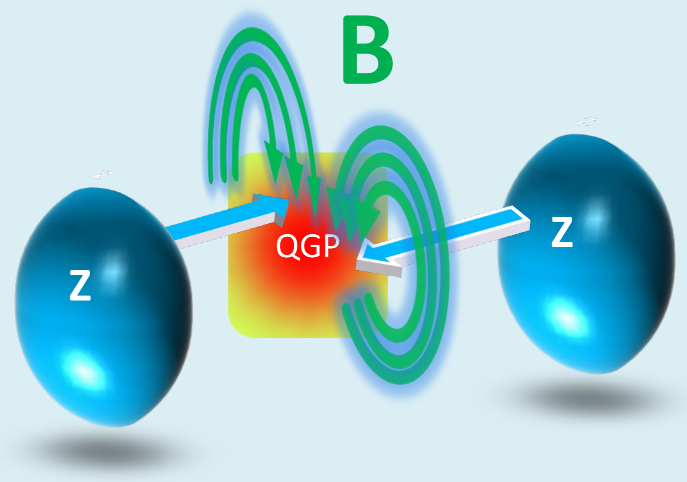
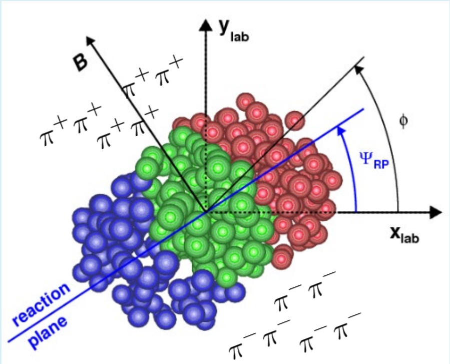
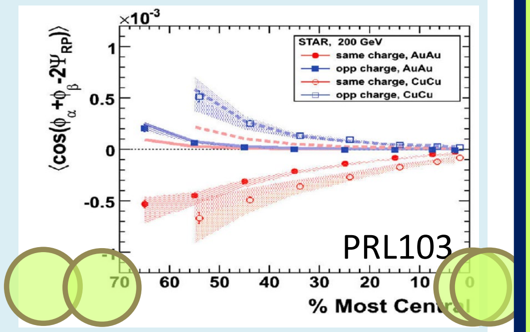
Peripheral collisions



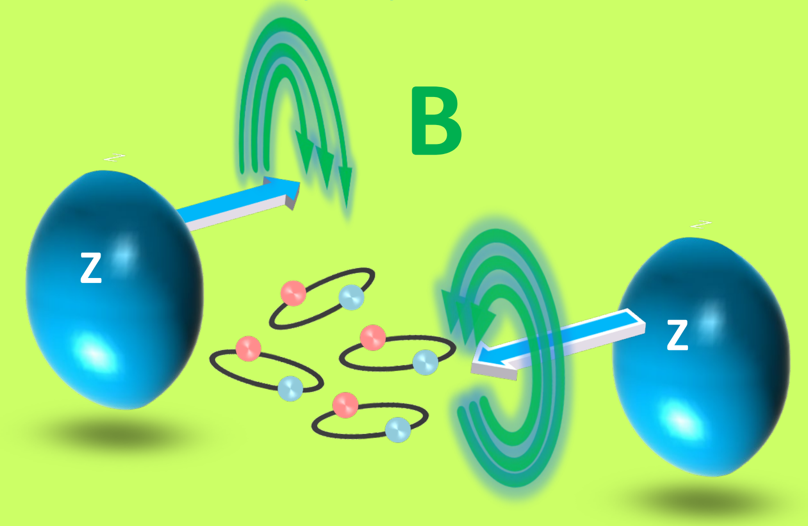
Novel transport phenomena in QGP

- Chiral magnetic effect
- magnetohydrodynamics
- thermal radiations in B
- etc.





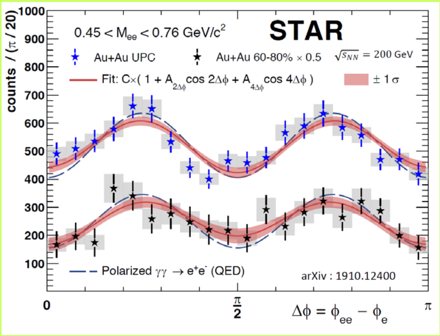
Ultraperipheral collisions



"Strong-field QED" without QGP

--- Vacuum physics in strong fields

- Photon-photon interactions
- Vacuum fluctuations in B
- etc.

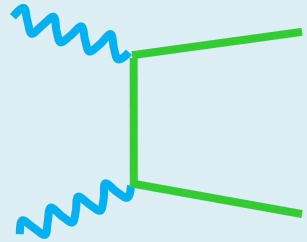


Photon-photon interactions

DECEMBER 15, 1934 PHYSICAL REVIEW VOLUME 46

Collision of Two Light Quanta

G. Breit* and John A. Wheeler,** *Department of Physics, New York University*
(Received October 23, 1934)



Vacuum fluctuations in external strong fields

Consequences of Dirac's Theory of the Positron
W. Heisenberg and H. Euler (1935)

Abstract

According to Dirac's theory of the positron, an electromagnetic field tends to create pairs of particles which leads to a change of Maxwell's equations in the vacuum. These changes are calculated in the special case that no real electrons or positrons are present and the field varies little over a Compton wavelength. The resulting effective Lagrangian of the field reads:

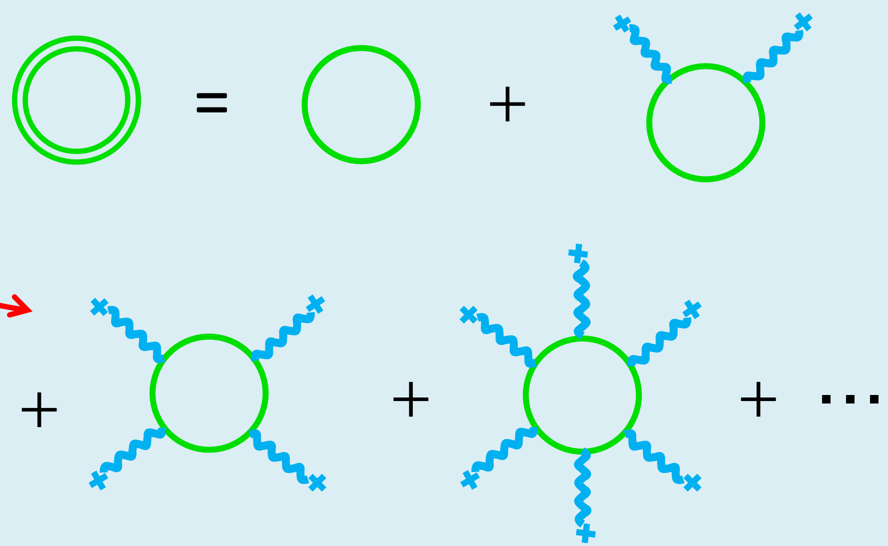
$$\mathfrak{L}=\frac{1}{2}(\mathfrak{E}^2-\mathfrak{B}^2)+\frac{e^2}{\hbar c}\int_0^\infty e^{-\eta}\frac{d\eta}{\eta^3}\left\{i\eta^2(\mathfrak{E}\mathfrak{B})\cdot\frac{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2-\mathfrak{B}^2+2i(\mathfrak{E}\mathfrak{B})}\right)+\mathrm{conj.}}{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2-\mathfrak{B}^2+2i(\mathfrak{E}\mathfrak{B})}\right)-\mathrm{conj.}}+|\mathfrak{E}_k|^2+\frac{\eta^2}{3}(\mathfrak{B}^2-\mathfrak{E}^2)\right\}$$

$\mathfrak{E},\mathfrak{B}$ field strengths

$|\mathfrak{E}_k|=\frac{m^2c^3}{e\hbar}=\frac{1}{137}\frac{e}{(e^2/mc^2)^2}=$ critical field strengths

The expansion terms in small fields (compared to $\mathfrak{E}$) describe light-light scattering. The simplest term is already known from perturbation theory. For large fields, the equations derived here differ strongly from Maxwell's equations. Our equations will be compared to those proposed by Born.

Resummation wrt the number of external legs
-- Furry's theorem: Only even-order diagrams contribute in C-even systems.



What happens when photons go through strong fields?

THE DISPERSION RELATION FOR LIGHT

AND

ITS APPLICATION TO PROBLEMS INVOLVING ELECTRON PAIRS

A Dissertation

Presented to the

Faculty of Princeton University

in Candidacy for the Degree

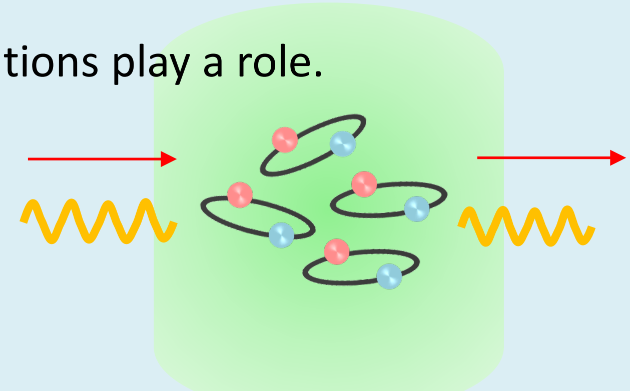
of Doctor of Philosophy.

by

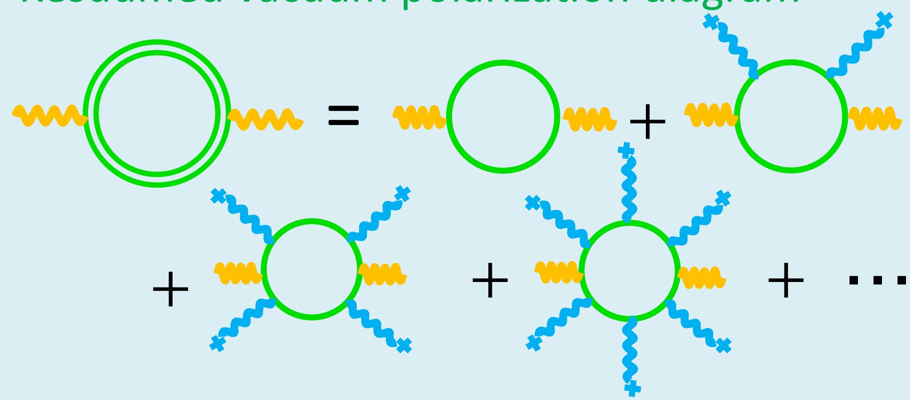
John Sampson Toll

Princeton University 1952.

Vacuum fluctuations play a role.



Resuumed vacuum polarization diagram



PHOTON SPLITTING IN A STRONG MAGNETIC FIELD

S. L. Adler, J. N. Bahcall,* C. G. Callan, and M. N. Rosenbluth
The Institute for Advanced Study, Princeton, New Jersey 08540

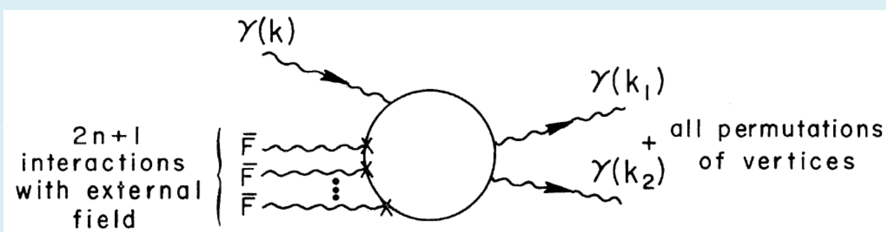
(Received 6 August 1970)

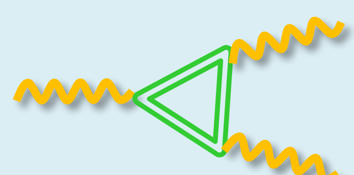
ANNALS OF PHYSICS: 67, 599–647 (1971)

Photon Splitting and Photon Dispersion in a Strong Magnetic Field

STEPHEN L. ADLER
Institute for Advanced Study, Princeton, New Jersey 08540

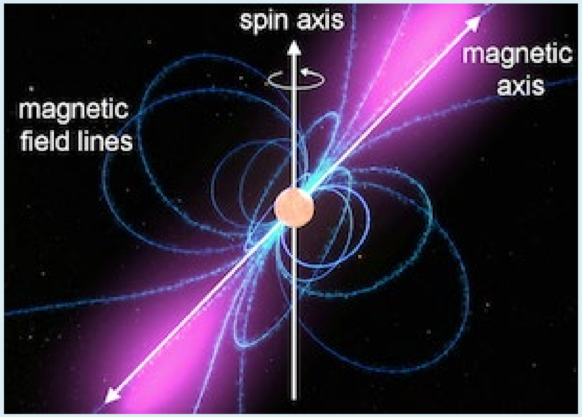
Received January 27, 1971





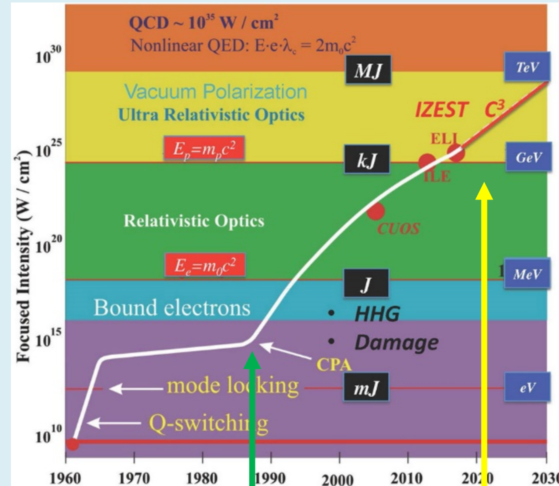
Strong fields in laboratories and nature (other than HIC)

Magnetospheres of neutron stars/magnetars



Dipole strength estimated from the intervals of the pulsation.

High-intensity laser field



2021

Mourou&Strickland (2018 Nobel laureates)

https://www.nobelprize.org/prizes/physics/2018/mourou/lecture/

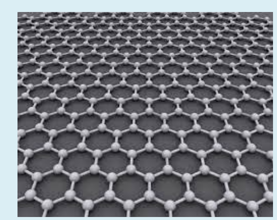
Supercritical Coulomb fields

$Z > 137$ [Coulomb potential: $\frac{Ze^2}{4\pi r}$]

Transient high-Z atoms (once studied at GSI)

See an anecdotal review "Probing QED Vacuum with Heavy Ions" Rafelski, Kirsch, Mueller, Reinhardt, and W. Greiner [1604.08690]

Renewed interests realized with graphene; "Atomic collapse"



v << c → Larger effective α.

→ Kinetic energy << Coulomb potential

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0. Introduction

- Strong magnetic fields in laboratories and nature

- Brief historical overview

1-1. Photon propagation in a magnetic field (at zero T and μ)

- Vacuum birefringence and (real) photon decay

- (Diagrammatic technique by the proper-time method)

1-2. Differential dilepton spectrum in a magnetic field

- (Ritus basis formalism)

- "Helicity suppression" in the ratio of di-electron yield to di-muon yield

2. Summary

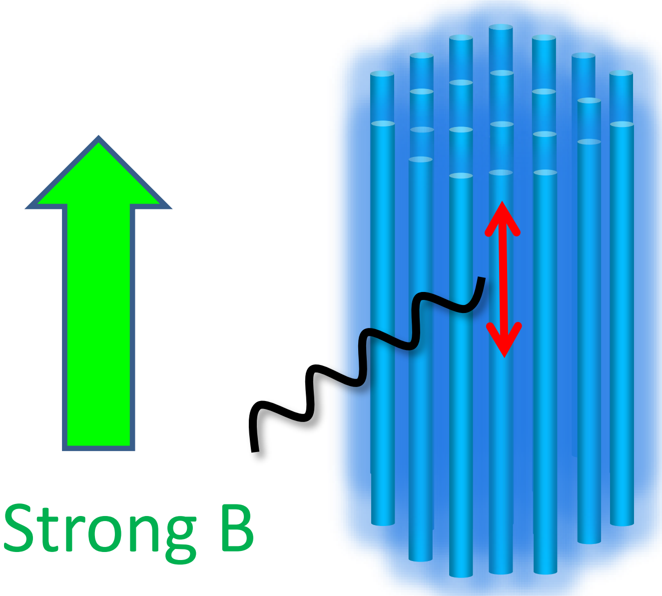
(1) Refractive index of photon in strong B-fields - Old but unsolved problem

KH and K.Itakura, “Vacuum birefringence in strong magnetic fields”:
(I) Photon polarization tensor with all the Landau levels,” (2013);
(II) Complex refractive index from the lowest Landau level,” (2013).

Photon propagation in magnetic fields
(in four dimensions)

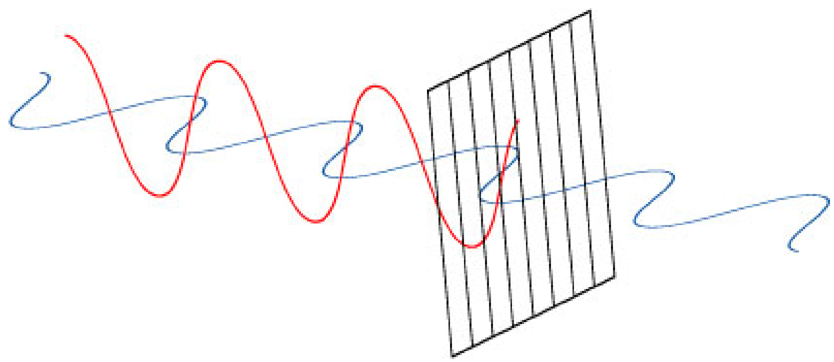
~~Lorentz~~ & gauge symmetries → $n \neq 1$ in general

Preferred orientation provided by an external B
→ ``Vacuum birefringence”



Strong B

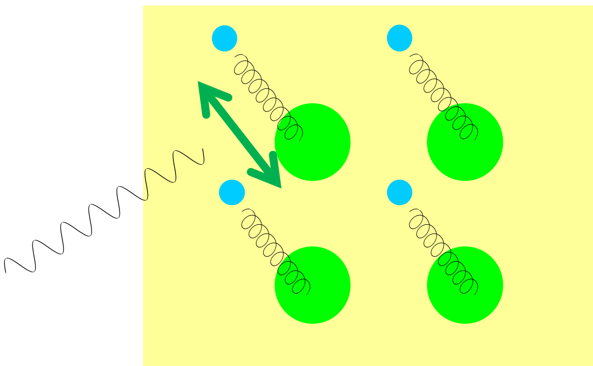
Cyclotron motion in B field



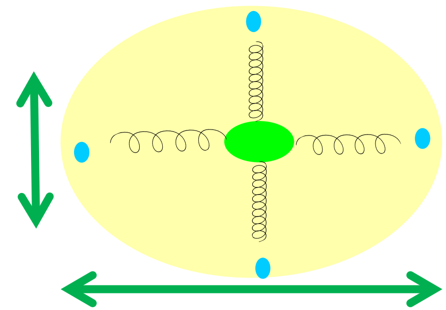
Polarizer

What is birefringence？

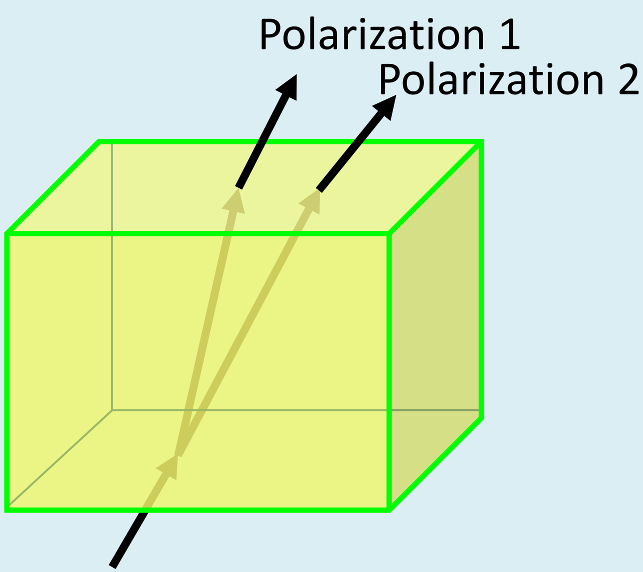
Response of electrons to incident lights

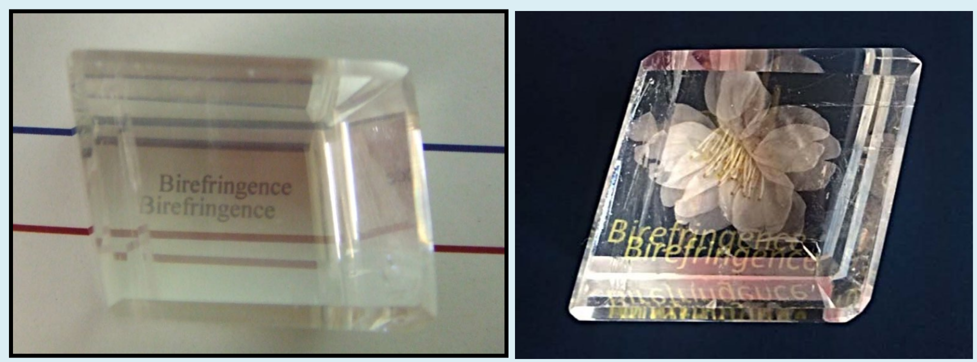


Structured ions
→ Anisotropic spring constants



Birefringence = Polarization-dependent refractive indices



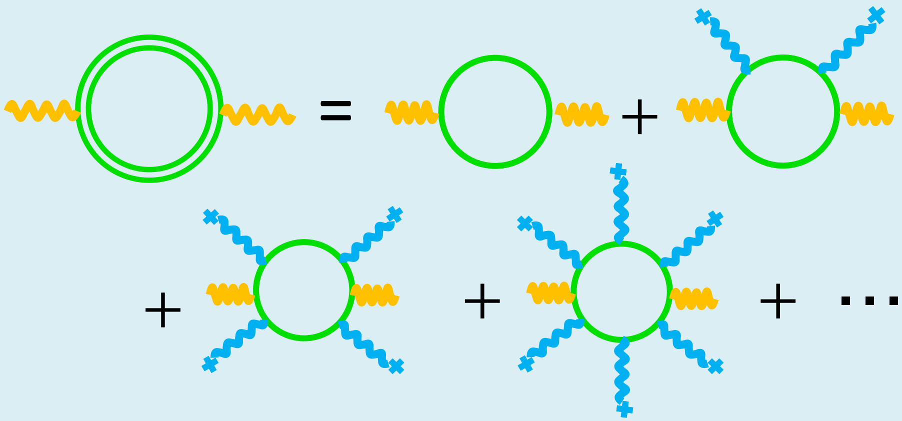


Birefringent substance
“Calcite” (方解石)

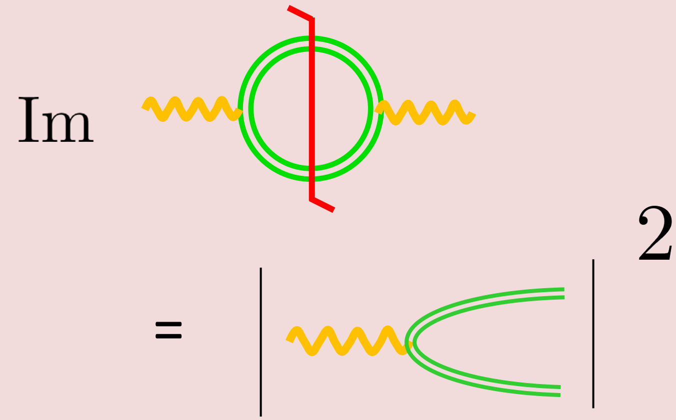
Lesson: The fermion spectrum is important for the photon spectrum.

Complex refractive indices

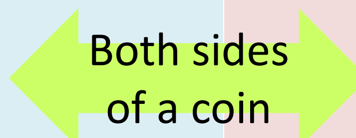
Resummed vacuum polarization diagram



Optical theorem for the imaginary part



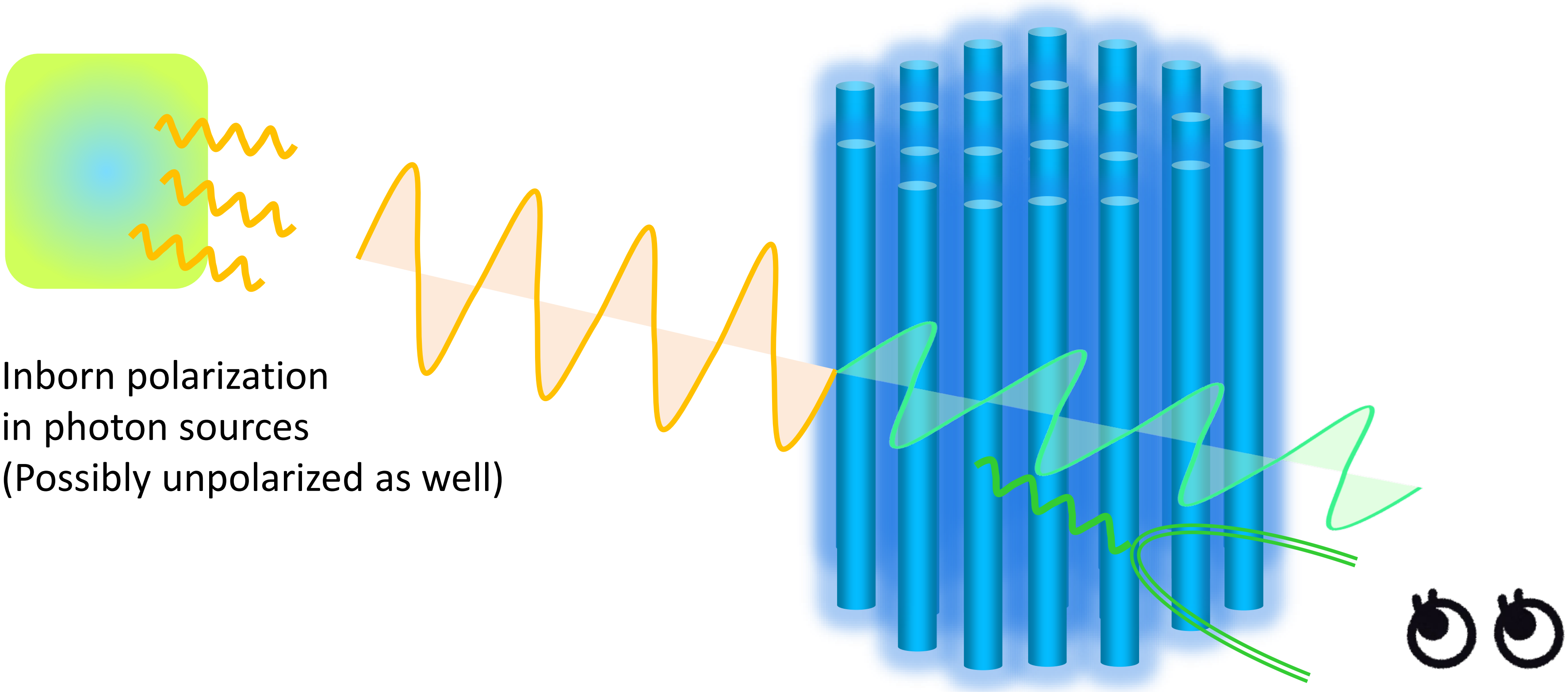
Photon refraction



Dispersion integral

Di-fermion production

Rotating photon polarization



Inborn polarization
in photon sources
(Possibly unpolarized as well)

- Acquired polarization due to the birefringence
- Some of photons decay into fermion pairs

Cf. Cotton-Mouton effect, Faraday effect, etc in optics

Vacuum birefringence
from the resummed vacuum polarization tensor

Maxwell eq. w/ quantum corrections: $[q^2 g^{\mu\nu} - q^\mu q^\nu - \Pi^{\mu\nu}]A_\nu(q) = 0$

U(1) gauge symmetry constrains possible tensor structures $\quad q_\mu \Pi^{\mu\nu} = 0$

$$\Pi^{\mu\nu} = -[\chi_0 P_0^{\mu\nu} + \chi_1 P_1^{\mu\nu} + \chi_2 P_2^{\mu\nu}]$$

$$P_0^{\mu\nu} = q^2 \eta^{\mu\nu} - q^\mu q^\nu$$

$$P_1^{\mu\nu} = q_\parallel^2 \eta_\parallel^{\mu\nu} - q_\parallel^\mu q_\parallel^\nu$$

$$P_2^{\mu\nu} = q_\perp^2 \eta_\perp^{\mu\nu} - q_\perp^\mu q_\perp^\nu$$

B-induced structures

Preferred orientation in B

$$\boldsymbol{B} = (0, 0, B)$$

$$\eta_\parallel^{\mu\nu} = \mathrm{diag}(1, 0, 0, -1)$$

$$\eta_\perp^{\mu\nu} = \mathrm{diag}(0, -1, -1, 0)$$

$$q_\parallel^\mu = (q^0, 0, 0, q^3)$$

$$q_\perp^\mu = (0, q^1, q^2, 0)$$

(Boost invariance along B)

Refractive indices from the Maxwell eq.

$$n = \frac{|\boldsymbol{q}|}{\omega}$$

Two (physical) polarization modes in || and ⊥ to B.

$$n_\parallel^2 = \frac{1 + \chi_0 + \chi_1}{1 + \chi_0 + \chi_1 \cos^2\theta} \rightarrow 1$$

$$n_\perp^2 = \frac{1 + \chi_0}{1 + \chi_0 + \chi_2 \sin^2\theta} \rightarrow 1$$

Direct consequence of the gauge symmetry and the breaking of one spatial rotational symmetry.

Vanishing B limit: $\chi_0 \rightarrow \Pi_{\mathrm{vac}}\ ,\ \ \chi_{1,2} \rightarrow 0$

Proper-time method for external strong fields

Schwinger (1951)

$$G(p|A) = \frac{i\left(\not{p} - e\not{A} + m\right)}{(\not{p} - e\not{A})^2 - m^2 + i\epsilon}$$

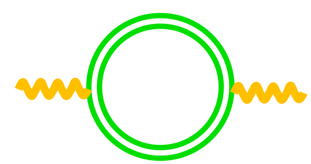
A^μ for external (constant) fields

$$= i\left(\not{p} - e\not{A} + m\right) \times \frac{1}{i}\int_0^\infty d\tau\; e^{i\tau\left\{(\not{p}-e\not{A})^2-(m^2-i\varepsilon)\right\}}$$

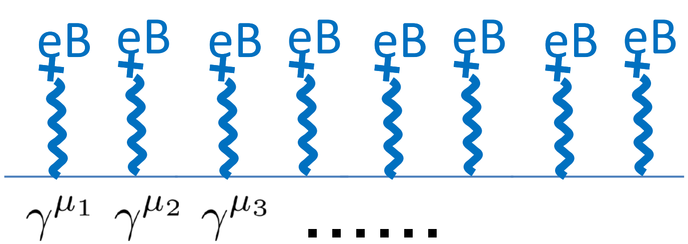
τ : proper-time

No p and A in the denominator → Gaussian form

$$i\Pi^{\mu\nu}_{\rm ex}(q) = e^2 \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\, \gamma^\mu G(p|A)\gamma^\nu G(p+q|A)\,\right]$$



Nonlinear wrt the external fields



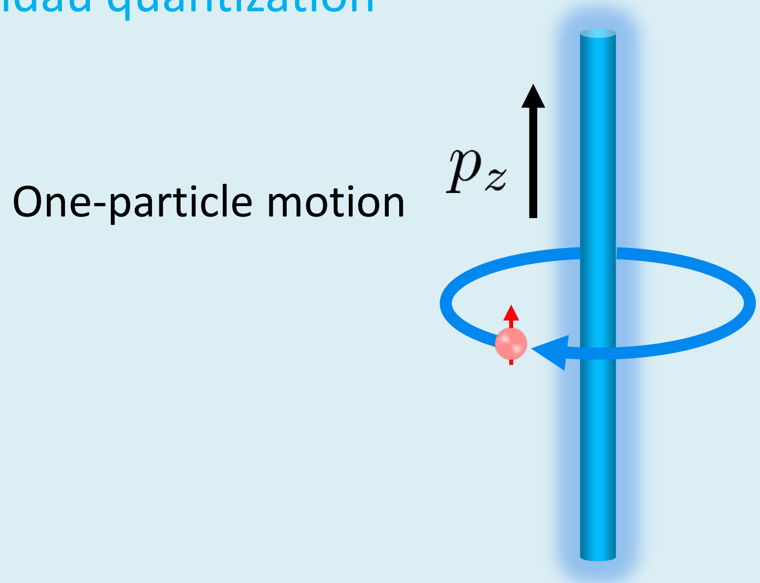
Technically demanding.
For example, one needs to perform the Dirac trace with an "infinite" number of gamma matrices.

Peskin & Schoeder tell us only tr[γ γ γ γ] or a little more…

Fermion spectrum in a magnetic field

Remember the lesson: Photon spectrum depends on the fermion spectrum.

Landau quantization

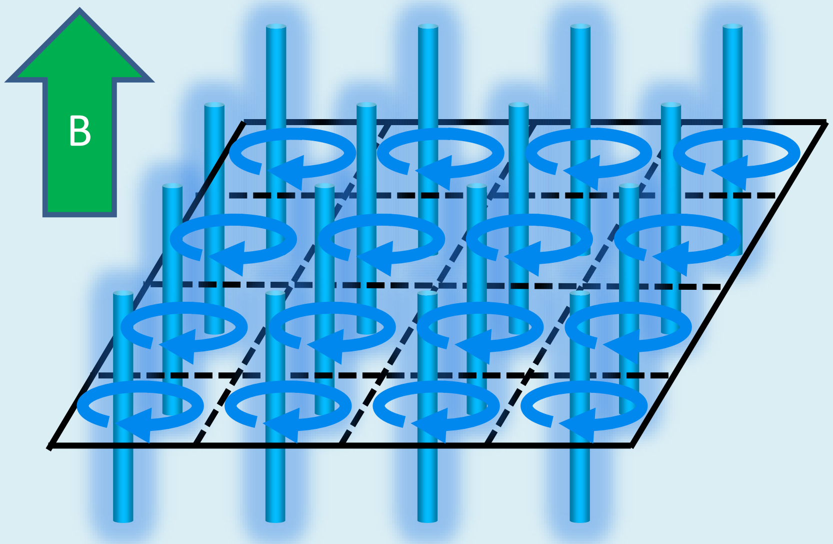


1. No effects on the longitudinal motion
2. Cyclotron motion ~ Harmonic oscillation
3. Spin polarization (Zeeman effect)

Energy spectrum

$$\epsilon_n^2 = p_z^2 + m^2 + (2n+1)|eB| \pm \frac{g}{2}|eB|$$

Landau degeneracy



No energetically favored position.
→ Degeneracy

$$\text{Landau degeneracy } \rho_B = \frac{eB}{2\pi}$$

Fermion pair spectrum in the imaginary part
--- Thresholds at the Landau levels

Only one possible source of the imaginary part:

$$\frac{1}{\sqrt{4ac-b^2}}\left[\arctan\left(\frac{b+2a}{\sqrt{4ac-b^2}}\right)-\arctan\left(\frac{b-2a}{\sqrt{4ac-b^2}}\right)\right]$$

$$a=q_{\parallel}^2/(4m^2),\ \ b=-(n-\ell)eB/m^2, c=(1-a)+(\ell+n)eB/m^2$$

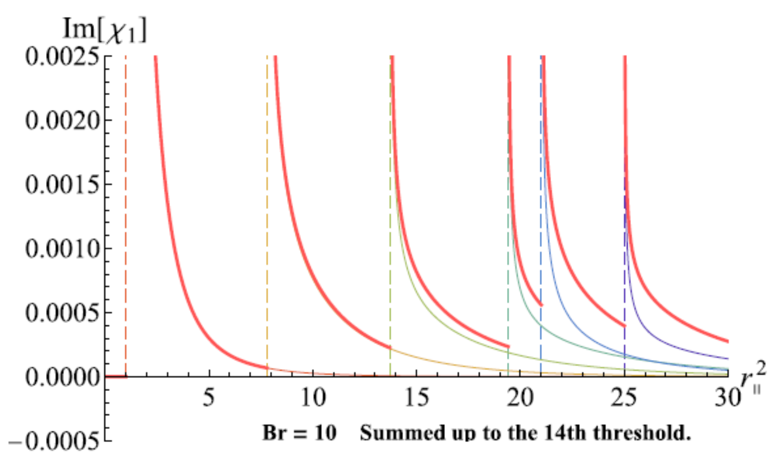
Polarization tensor acquires an imaginary part when $4ac-b^2\leq 0$

Threshold condition

$$q_{\parallel}^2 \geq (\epsilon_\ell+\epsilon_n)^2 \qquad \epsilon_n=\sqrt{p_z^2+2neB+m^2}$$

The integers are identified with the Landau levels.

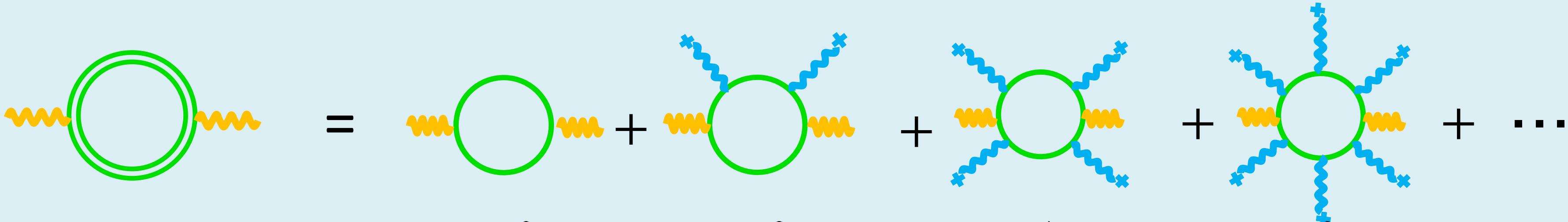
- $q_{\parallel}^2=\omega^2-q_z^2$: Photon energy in the frame where $q_z=0$.
  (Boost invariance along a constant B-field)
- Compatible with the on-shell condition $q^2=0$.
- Real photon can decay in B.
  (Cyclotron radiation is also a 1-to-2 process in B.)



Vacuum polarization tensor in two different series representations

1. Naïve perturbative series when eB is small.

Naïve perturbation breaks down when eB is large!



$\mathcal{O}\left((eB)^0\right)$ $\mathcal{O}\left((eB)^2\right)$ $\mathcal{O}\left((eB)^4\right)$ $\mathcal{O}\left((eB)^6\right)$ $\mathcal{O}\left((eB)^{2n}\right)$

Weak-field approximation
Adler, etc.

2. Landau level representation

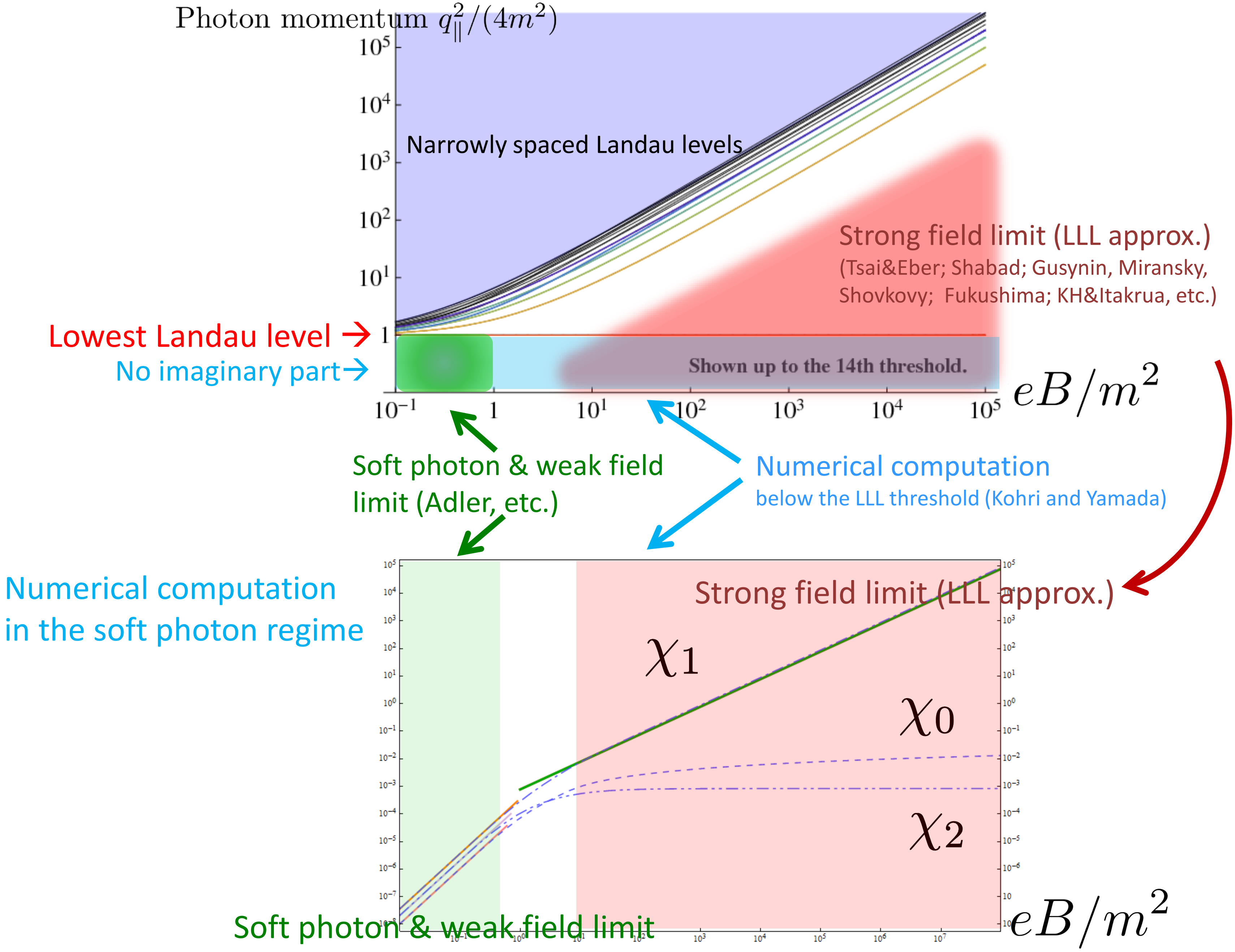
$$= \frac{eB}{2\pi} \sum_{\ell=0}^{\infty} \sum_{n=0}^{\infty} F_{\ell,n}(q_\parallel^2, q_\perp^2)$$

Lowest Landau approximation
(ℓ = n = 0) when eB is large.

KH&Itakrua [1209.2663, 1212.1897]

Cf. For the Landau level representation of the HE effective action, see KH, Itakura, Ozaki [2001.06131].

Summary of relevant scales and preceding calculations



Refractive indices and decay rate

$$\Pi^{\mu\nu} \to n_{\parallel,\perp}$$

Refractive indices with the LLL fluctuations

Refractive indices at the LLL(ℓ=n=0)

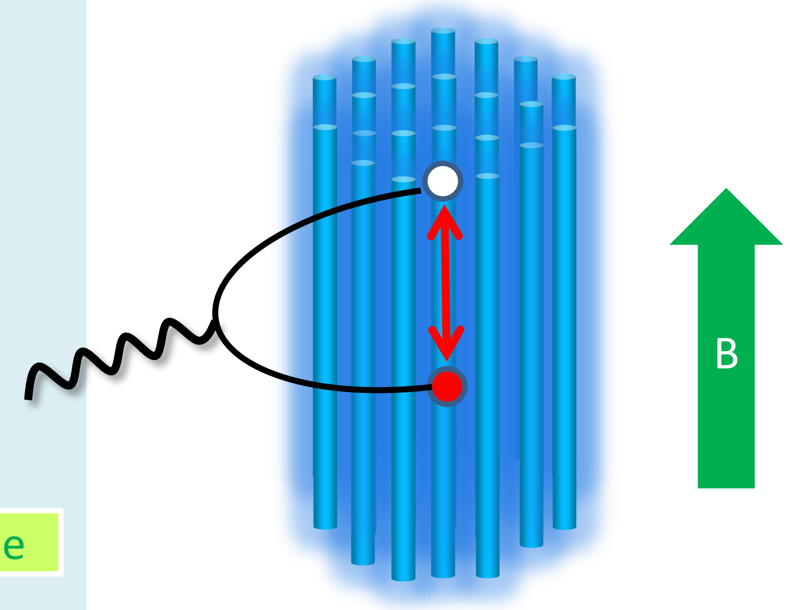
Polarization excites only along the magnetic field
``Vacuum birefringence''

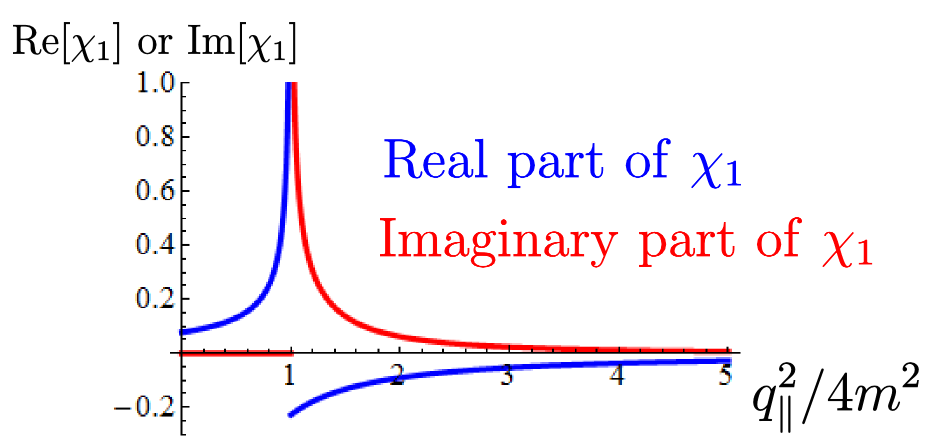
$$\chi_1 \neq 0, \quad \chi_0 = \chi_2 = 0$$

$$\begin{cases} n_\parallel^2 = \dfrac{1+\chi_1}{1+\chi_1 \cos^2\theta} \\ n_\perp^2 = 1 \end{cases}$$

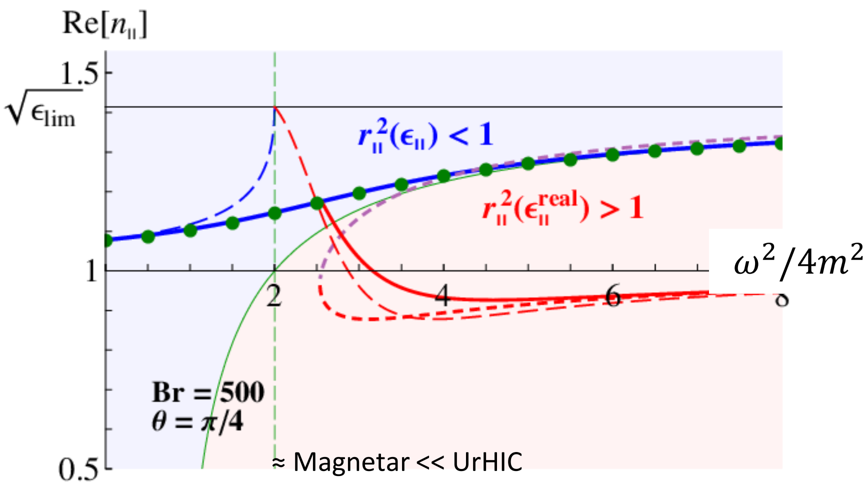
← No modification in the ⊥ mode

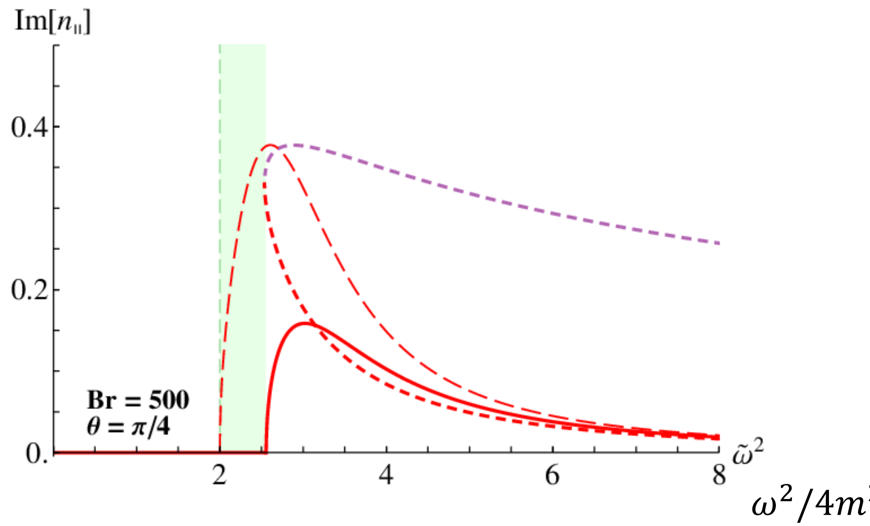
(1+1)-dimensional fluctuations





Complex refractive indices





Final results shown by solid lines

cf. air n = 1.0003, water n = 1.3, prism n = 1.5

Refraction

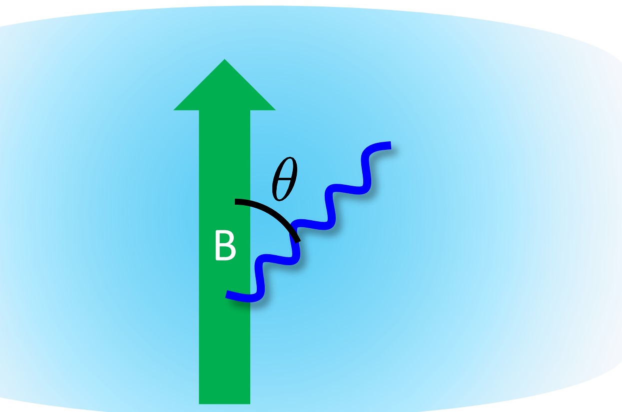
Difermion production

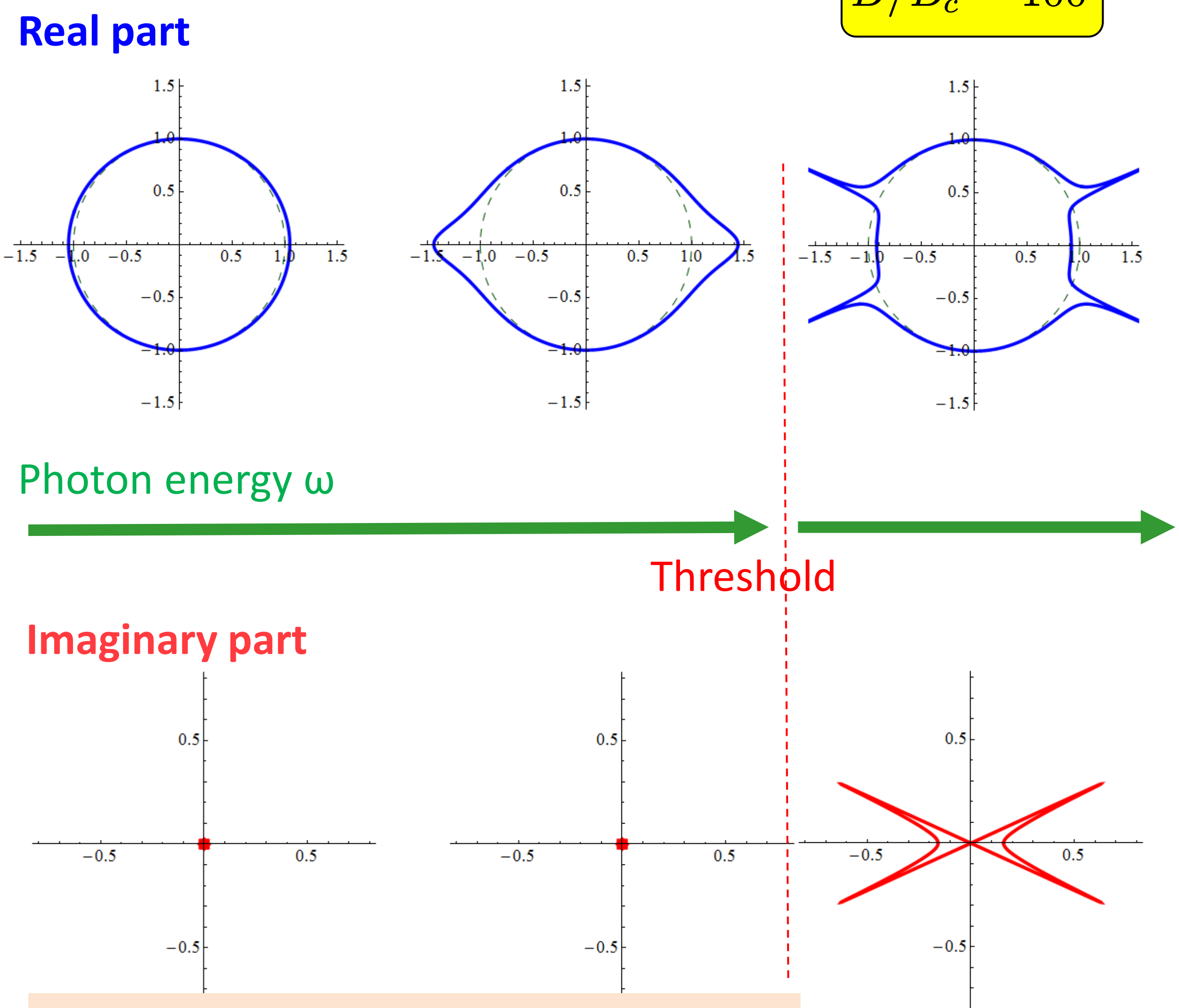
Anisotropy of the refraction index

Angle : Direction of the photon propagation

Radius : Magnitude of the refraction index

$B/B_c = 100$





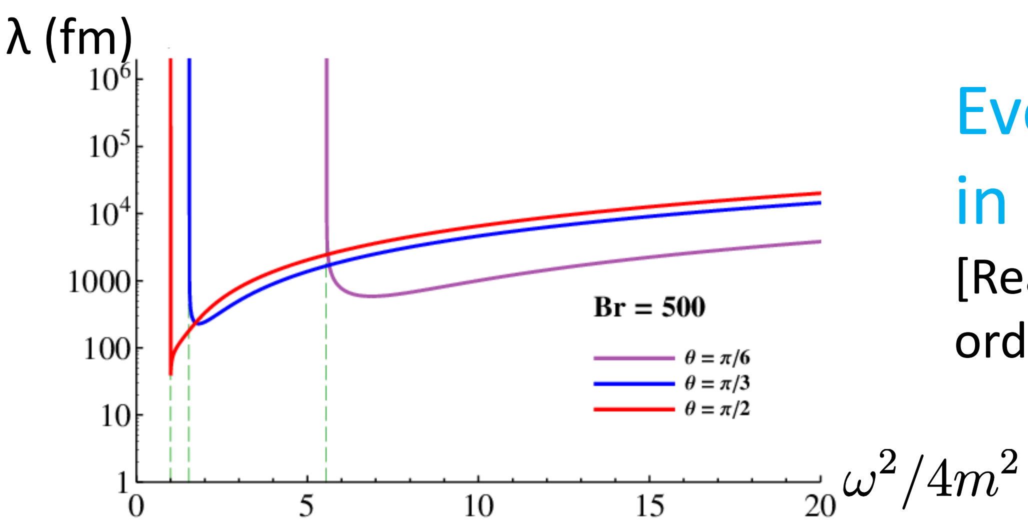
No imaginary part below the threshold

"Mean-free-path" of photons in B-fields

When the refractive index has an imaginary part,

$$\text{Photon flux}: \; I \propto \exp\{-\lambda^{-1}\hat{\boldsymbol{q}}\cdot\boldsymbol{x}\}$$

"Mean-free-path" $$\lambda = \frac{1}{2\omega n_{\mathrm{imag}}}$$



Even real photons decay in a microscopic scale!

[Real photons never decay in ordinary vacuum without B-field.]

$\omega \sim 1\ \mathrm{GeV}$
$\to \lambda \sim 1 - 10\ \mathrm{fm}.$

Smaller mfp for a larger energy ~ 1/ω

Differential dilepton spectrum

--- Better accessibility than the photon polarization

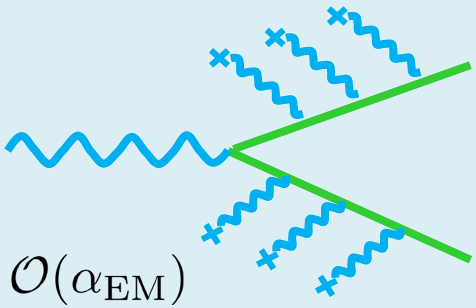
KH, Hidetoshi Taya, Shinsuke Yoshida, “Di-lepton production from a single photon in strong magnetic fields: vacuum dichroism”, [2010.13492]

Pair production in a magnetic field

LO w/ B-field

$\gamma \to (f\bar{f})_B$

$\gamma^* \to (f\bar{f})_B$



$|\mathcal{M}|^2 \sim \mathcal{O}(\alpha_{\rm EM})$

Nonperturbatively dressed fermions

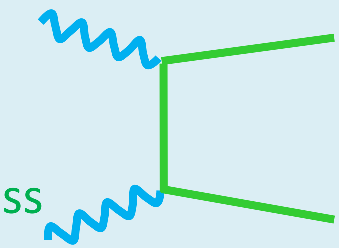
Both <u>on-shell and off-shell photons</u> can decay in B.

LO w/o B-field

No kinematical window for
<u>a single on-shell photon when B = 0.</u>

→ Starts only from 2 photons

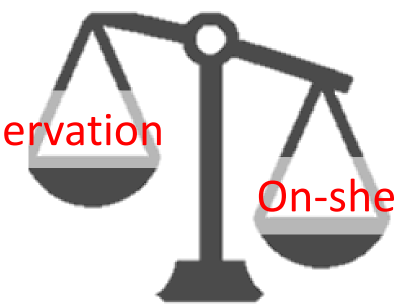
$|\mathcal{M}|^2 \sim \mathcal{O}(\alpha_{\rm EM}^2)$



E.g., Breit-Wheeler process

Not compatible with each other without B.

$$q_\gamma^2 = (\epsilon_f + \epsilon_{\bar{f}})^2_{\rm CoM} \neq 0$$



E-m conservation

On-shell conditions

Finite B opens a kinematical window for an on-shell γ → f fbar.

In B-field, $q_{\parallel}^2 \geq (\epsilon_\ell + \epsilon_n)^2$

Compatible with $q^2 = 0$

Ritus basis formalism

See a review part in [2010.13492]

= Mode expansion with the exact fermion wave functions in B-field.

$$(i\not{D}_{\rm ext} - m)\psi = 0$$

$$D^\mu_{\rm ext} = \partial^\mu + i|e|A^\mu_{\rm ext}$$

$$\mathrm{rot}\,\boldsymbol{A}_{\rm ext} = \boldsymbol{B}$$

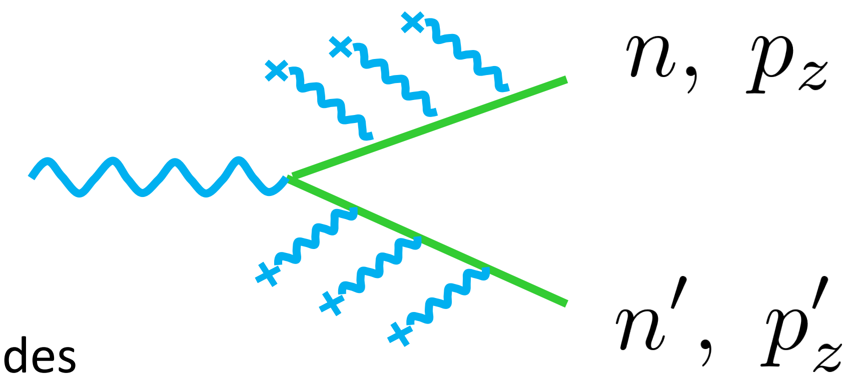
- Mode expansion with the eigenspinor basis.
- Fermion propagator gets simplified.
- Price: Convolution of the wave functions at the vertex is no longer a delta function.

Pair production rate for general kinematics

Fermion pair in the Landau levels

One-shell/off-shell photon

$$q^\mu,\ \ \epsilon^\mu$$



Differential information includes

- Photon polarization (ε^μ)
- Photon momentum (q^μ) with a general direction and invariant mass
- Landau levels (n, n') and continuous momentum (p_z, p_z') along B

Consistency checks done: Ward identity, etc.

Muon-pair excess over electron pairs

KH, Taya, Yoshida [2010.13492]

"Helicity suppression" in pion decay

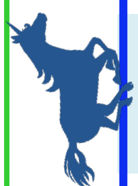
$\pi^- \to \mu^- \bar{\nu}_\mu \quad 99.9877\ \%$

$\pi^- \to e^- \bar{\nu}_e \quad 10.23 \times 10^{-4}\ \%$

PDG

Opposite momentum

Opposite spin | Same helicity



$\ell^- \quad S_\ell = \pm\frac{1}{2} \quad h_\ell = \pm\frac{1}{2}$

$\pi^- \quad S = 0$

Opposite chirality
→ Needs chirality mixing by finite mass.

$\bar{\nu} \quad S_{\bar{\nu}} = \mp\frac{1}{2} \quad h_{\bar{\nu}} = \pm\frac{1}{2}$

The LLL (= soft photon)

$$\frac{N_{\mu^+\mu^-}}{N_{e^+e^-}} \propto \frac{m_\mu^2}{m_e^2} \sim 4.4 \times 10^4$$

Opposite spin along B-field | Same helicity

$S_{\ell^-} = -\frac{1}{2} \quad h_{\ell^-} = -\frac{1}{2}$

$S_\gamma = 0$

Superposition of $(\pm, L)$

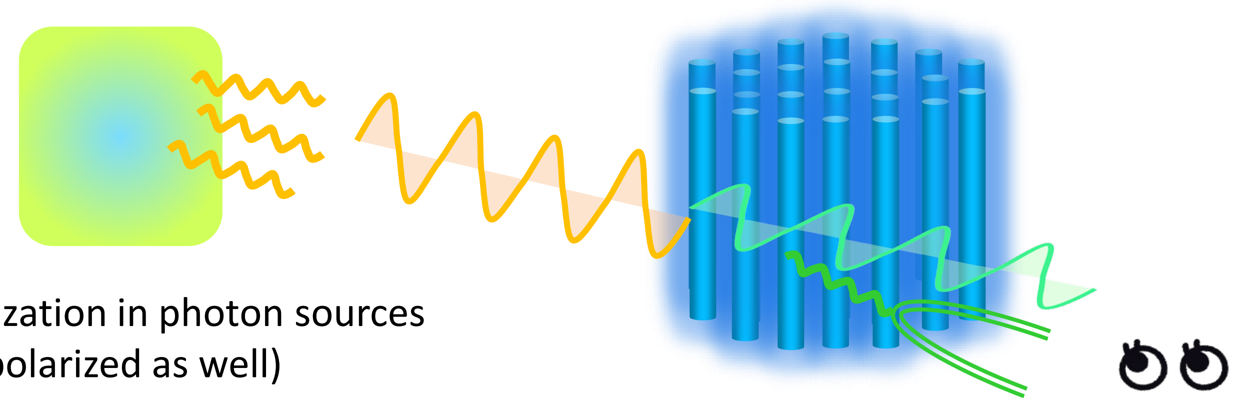
Opposite chirality

$S_{\ell^+} = +\frac{1}{2} \quad h_{\ell^+} = -\frac{1}{2}$

Especially, R neutrino (L antineutrino) does not exist at the QCD scale.
(However, this is not an essential reason for the helicity suppression.)

Feasibility with HIC?

Ideal set-up

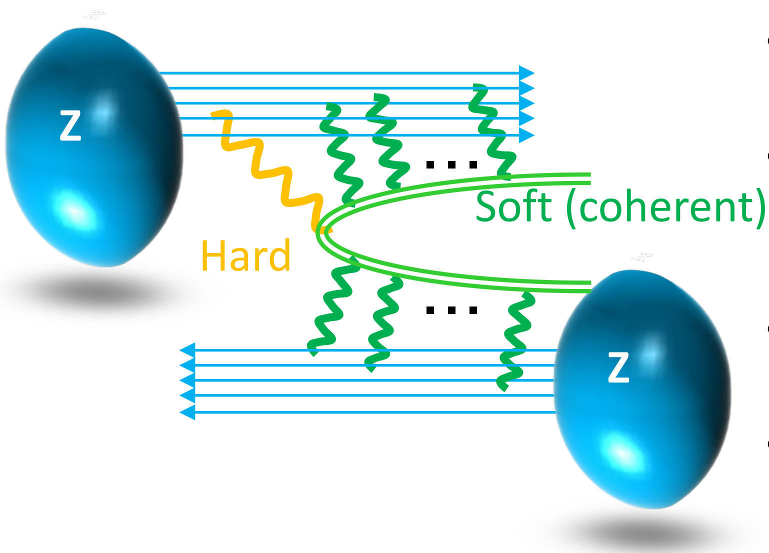


Inborn polarization in photon sources
(Possibly unpolarized as well)

- Acquired polarization due to the birefringence
- Some of photons decay into fermion pairs

UPC events

What is a photon source and what is a strong B field in HIC?



- There is no a priori difference: Both are Coulomb electric fields in the nucleus rest frame.
- The photon source has a momentum distribution: Fourier transform of the charge distribution, e.g., the Woods-Saxon profile.
- The hard and soft components of the distribution may be regarded as "photons" and a "magnetic field".
- Needs quantitative estimates with a separation scale.

Summary

- Vacuum birefringence (polarization-dependent refractive indices)
- $\gamma \rightarrow$ ee for both on-shell and off-shell photons
- Differential di-lepton spectrum
  → "Helicity suppression" in the electron/muon

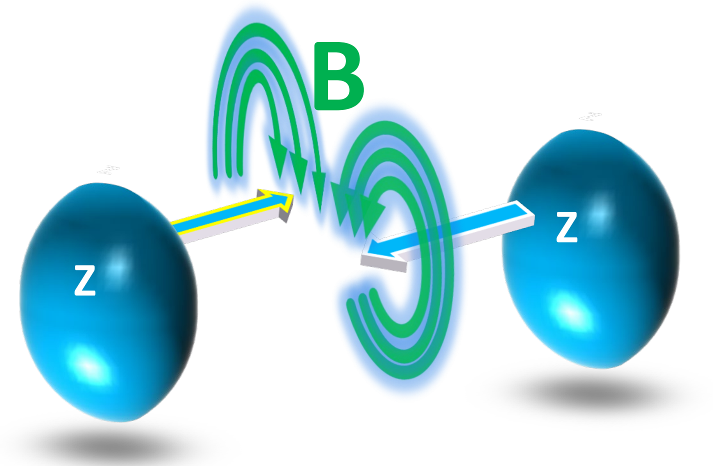
Prospects for the UPC

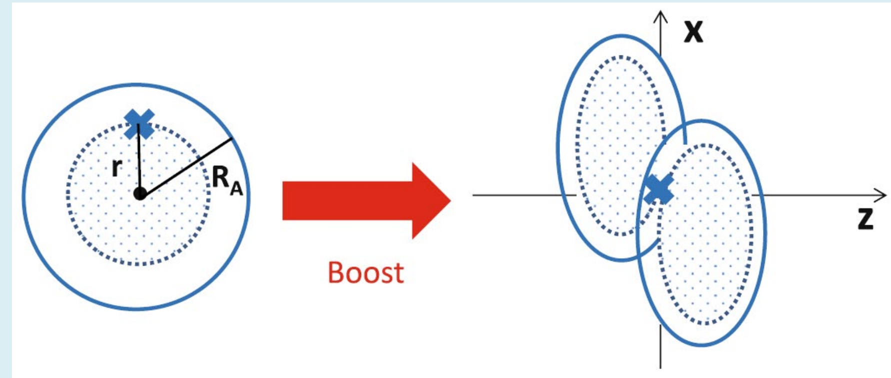
- Needs convolution with the photon distribution function
- Time dependence of a magnetic field

KH, Xu-Guang Huang, Hidetoshi Taya, Shinsuke Yoshida, In progress.

Back-up slides

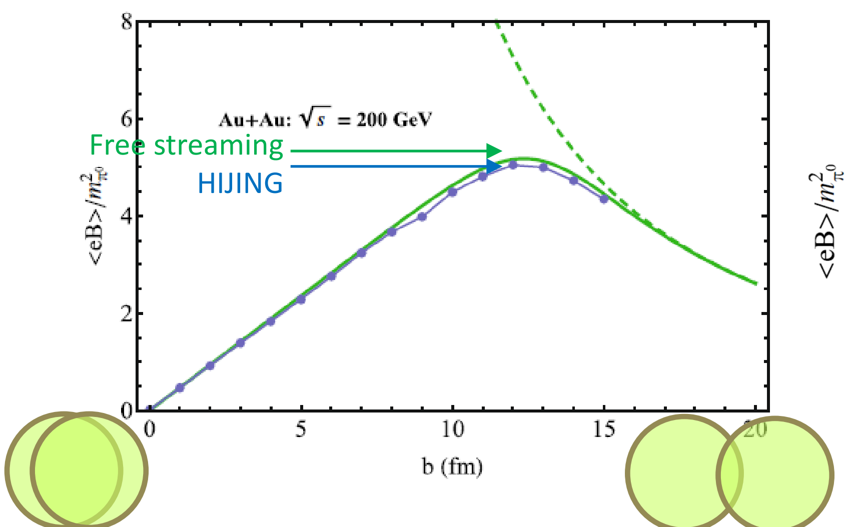
Strong magnetic fields induced by relativistic heavy-ion collisions

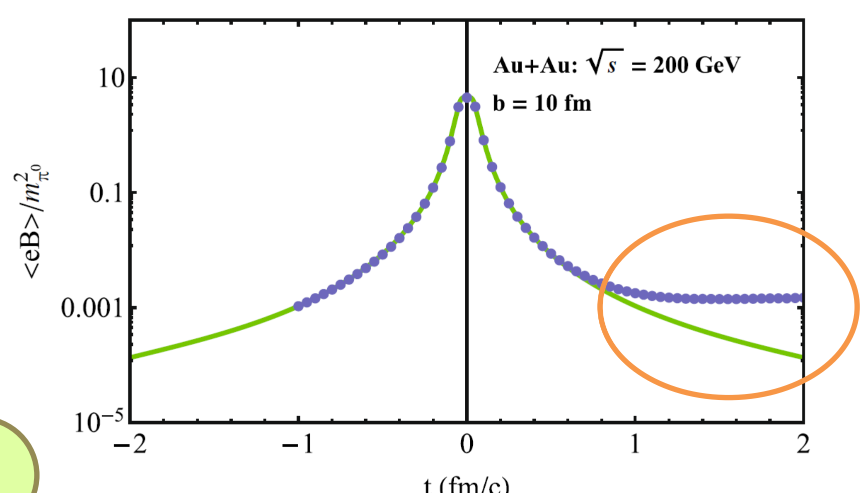




Static Coulomb field

Boost → Lienard-Wiechert potential





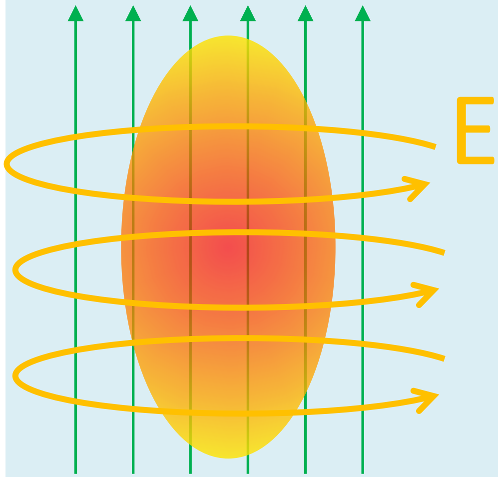
Central coll. Peripheral coll.

Deng & Huang; KH & Huang [1609.00747]

Lifetime of the B-field after the collisions

B(t)

A longer lifetime due to the Lenz's law?

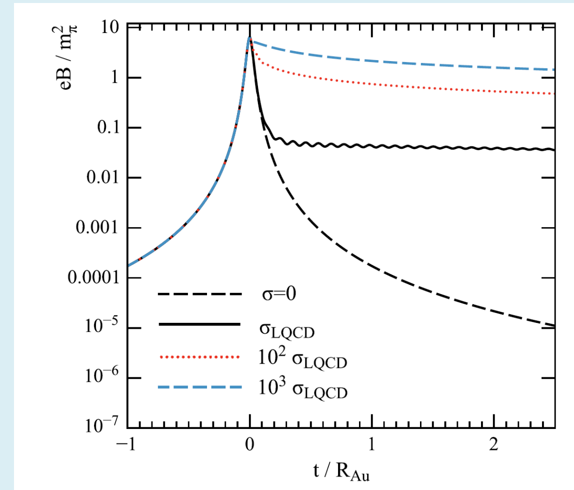


Tuchin

Time dependent B induces E.

E induces J if QGP is conducting.

→ Induced J sustains B.



McLerran & Skokov

Important to know the conductivity of QGP in magnetic fields. KH & Satow; KH, Li, Satow, Yee; Fukushima & Hidaka

Isobaric collisions will help us to understand the backgrounds and to extract the magnetic-field effects.

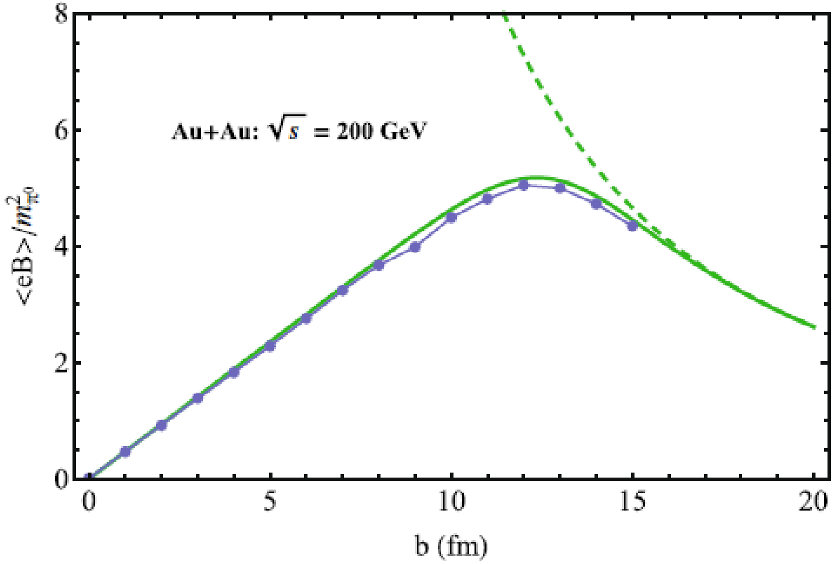
$$\Delta Z \sim 10\ \% \ \ \rightarrow \Delta B \sim 10\ \%,$$
but little difference in the flow effects

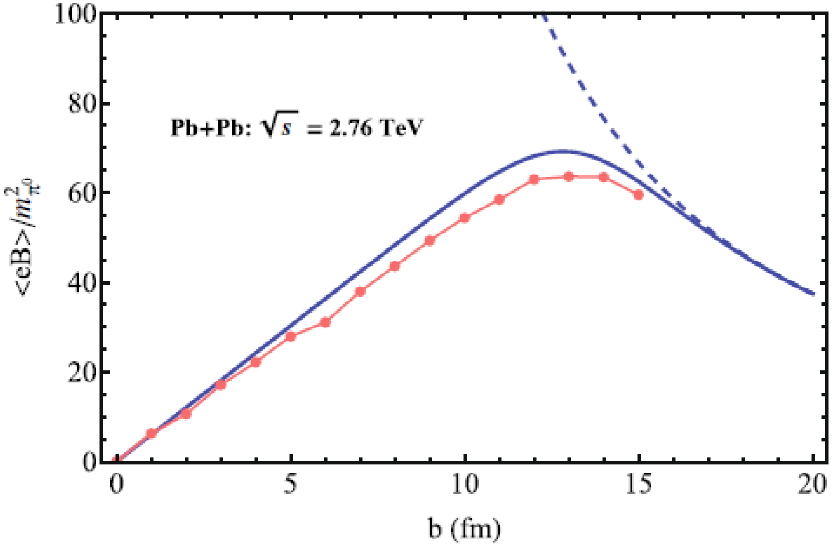
$^{96}_{40}$Zirconium

$^{96}_{44}$Ruthenium

Cf. Deng, Huang, Ma, Wang for recent estimates

Analytic and numerical estimates of the strong B
-- Impact parameter dependences

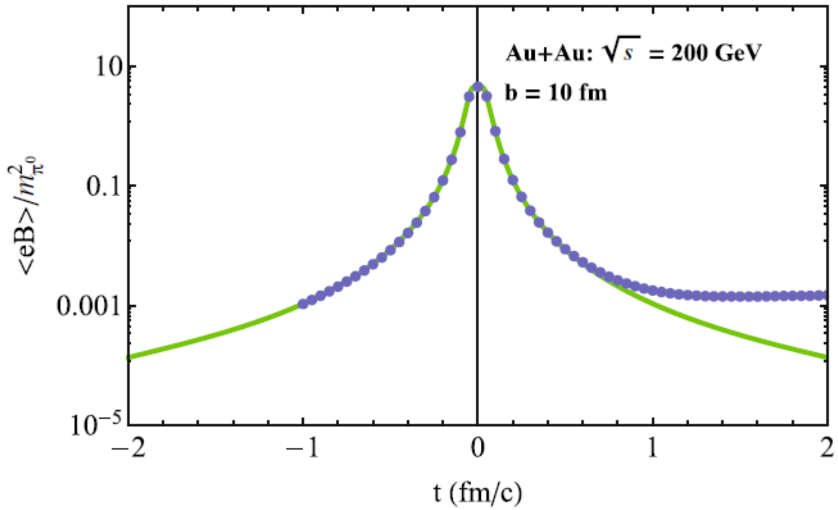


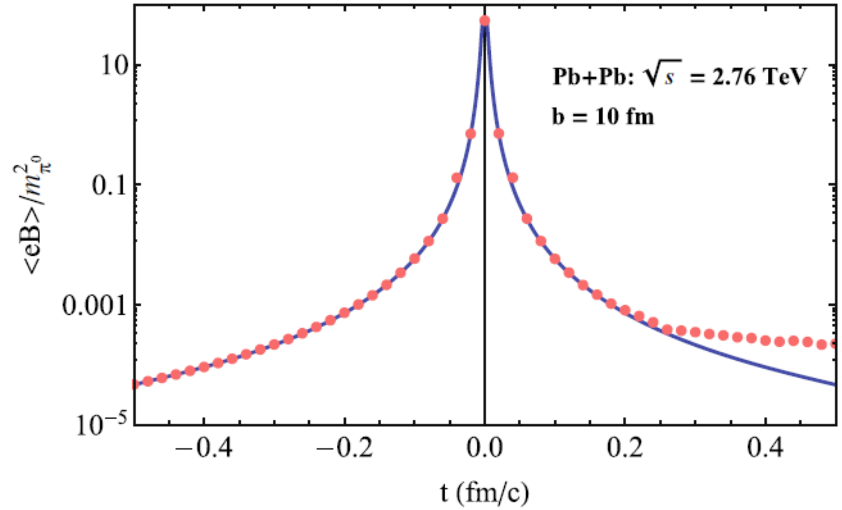


t = 0 (at the collision)

W.-T. Deng & X.-G. Huang, KH and X.-G. Huang

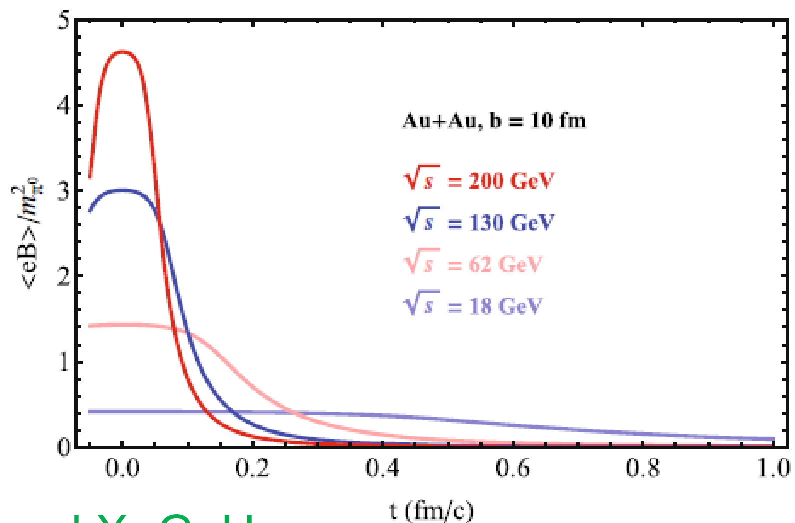
Time dependences



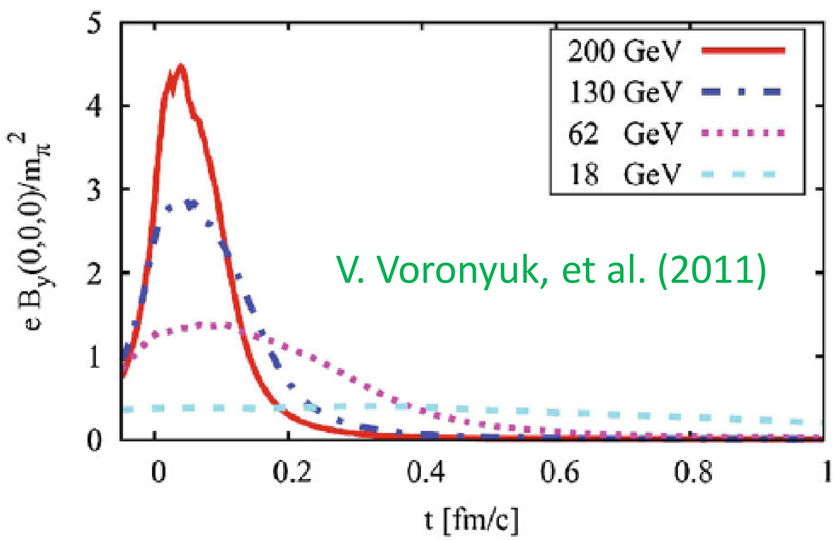


KH and X.-G. Huang

Collision-energy dependences



KH and X.-G. Huang



V. Voronyuk, et al. (2011)

Canonical quantization in B-field

See a review part in [2010.13492]

$$(i\not{D}_{\rm ext} - m)\psi = 0$$

$$D^\mu_{\rm ext} = \partial^\mu + i|e|A^\mu_{\rm ext}$$

$$\mathrm{rot}\,\boldsymbol{A}_{\rm ext} = \boldsymbol{B}$$

Ritus basis: Eigenspinor in B-field

$$\mathcal{R}_n(x_\perp) = \phi_n \mathcal{P}_+ + \phi_{n-1}\mathcal{P}_-$$

$\phi_n(x_\perp)$: Wave function at the Landau level n

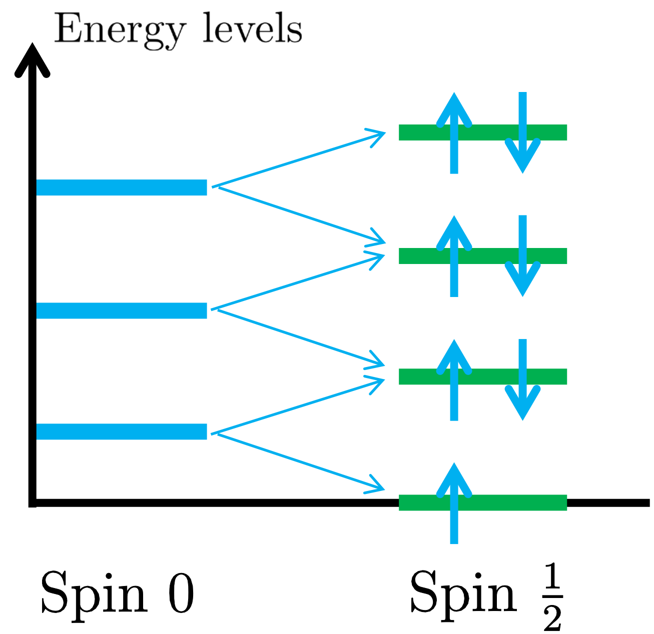
$\mathcal{P}_\pm$: Spin projection operator along B-field

Mode expansion can be performed in this basis.

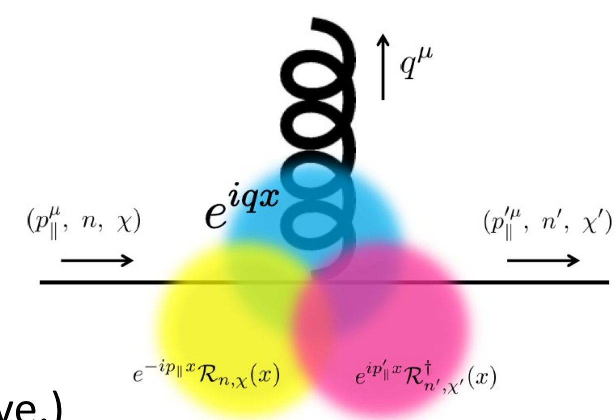
Fermion propagator gets simplified.

$$\mathcal{R}_n \frac{i}{\not{D}_{\rm ext} - m}\mathcal{R}^\dagger_{n'} = \frac{i}{\not{p}_n - m}\delta_{nn'}$$

$$p^\mu_n = (p^0, \sqrt{2n|eB|}, 0, p^3)$$



Spin up and down states are degenerated except for n = 0.

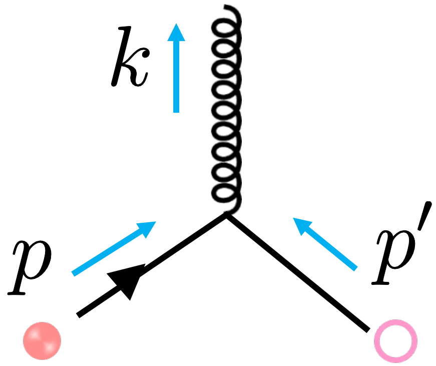


Price: Convolution of the wave functions at the vertex gets complicated. Fermion wave functions are not orthogonal to photon wave function. (Photon wave function is a plane wave.)

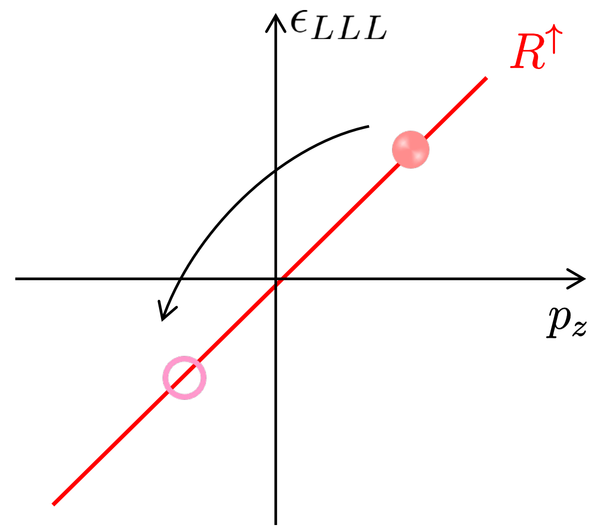
Possible 1-2 processes in the massless LLL

Perturbative vertex does not mix the R and L.

Possible pair creation/annihilation



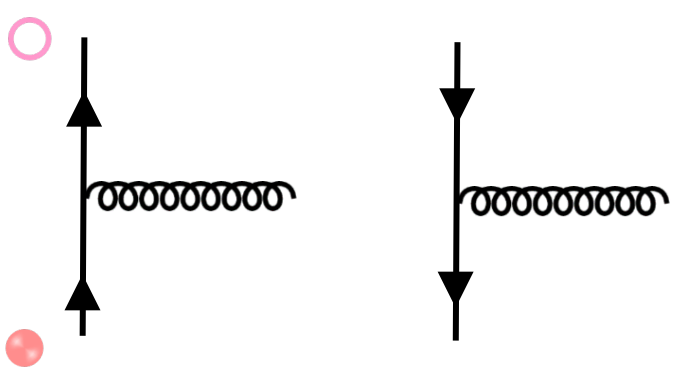
$$\epsilon_{\rm quark}^2 = p_z^2$$
$$\epsilon_{\rm gluon}^2 = k_z^2 + |\boldsymbol{k}_\perp|^2$$



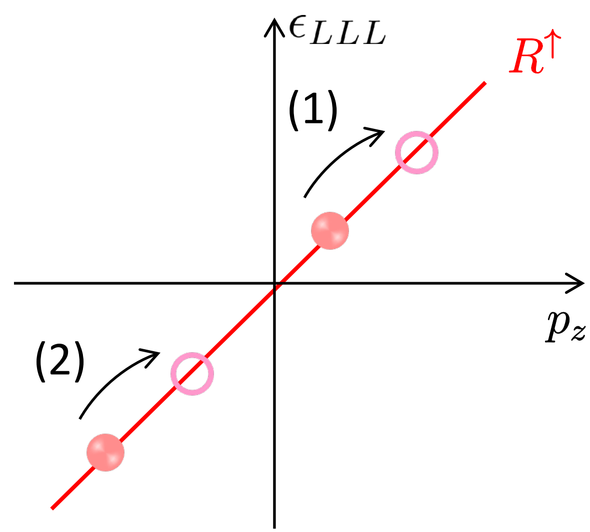
$|\boldsymbol{k}_\perp|$ works as a gluon mass for 2D kinematics.

Analogue of a massive weak boson production from $q\bar{q}$ annihilation in 4D.

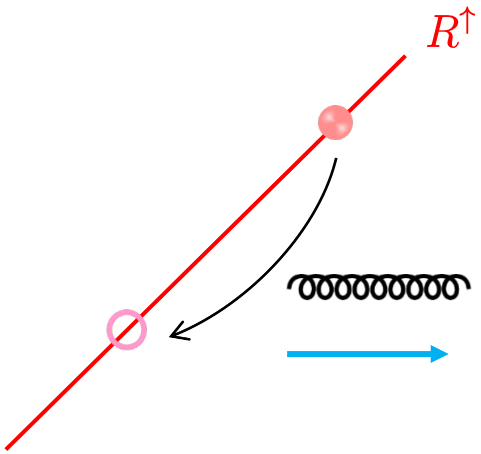
Possible scatterings



(1) particles (2) antiparticles



Possible quantum numbers carried by gluons



When the LLL fermion moves from one LLL to another,

No changes of quantum numbers but ε and p_z.

No spin! (The LLL is completely polarized.)

$$\delta\epsilon,\ \delta p_z \neq 0$$

$$\delta q_{\mathrm{electric,flavor}} = 0$$

$$\delta s_z = 0$$

Possible quantum numbers carried by gluons are ε = p_z (and color), but no spin!

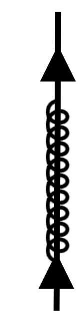
⇒ No coupling to the transverse gluons.

Possibly coupled to scalar fields such as phonons (?)

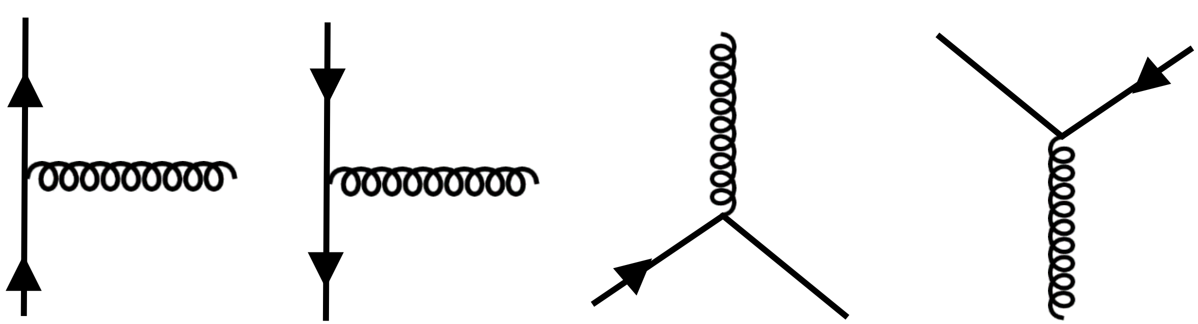
Kinematics in the massless limit is satisfied only in the "collinear limit."

The scattering in the (1+1) D is *prohibited* by the chiral symmetry in the massless limit (cf., CME).

$$j_\mu A^\mu = 0$$



Scatterings allowed only in a massive case



$$\sim g m_f$$

m_f: Fermion mass

Resummed polarization tensor

Integrands with strong oscillations

$$\begin{cases} \Gamma_0(\tau,\beta) = \cos(\beta\tau) - \beta\sin(\beta\tau)\cot(\tau) \\ \Gamma_1(\tau,\beta) = (1-\beta^2)\cos(\tau) - \Gamma_0(\tau,\beta) \\ \Gamma_2(\tau,\beta) = 2\frac{\cos(\beta\tau)-\cos(\tau)}{\sin^2(\tau)} - \Gamma_0(\tau,\beta) \quad . \end{cases}$$

$$\phi_\parallel(r_\parallel^2, B_\mathrm{r}) = \frac{1}{B_\mathrm{r}}\left\{1-(1-\beta^2)\, r_\parallel^2\right\}$$

$$\phi_\perp(r_\parallel^2, B_\mathrm{r}) = -\frac{2r_\perp^2}{B_\mathrm{r}} \cdot \frac{\cos(\beta\tau)-\cos(\tau)}{\sin(\tau)}$$

$$r_\parallel^2 = q_\parallel^2/4m^2$$

Exponentiated trig-functions generate strongly oscillating behavior with arbitrarily high frequency.

$$\chi_i(r_\parallel^2, r_\perp^2, B_\mathrm{r}) = \frac{\alpha B_\mathrm{r}}{4\pi}\int_{-1}^{1} d\beta \int_0^\infty d\tau\, \frac{\Gamma_i(\tau,\beta)}{\sin(\tau)}\, e^{-i(\phi_\parallel+\phi_\perp)\tau}$$

Proper time integrals on the two fermion lines

Vanishing B limit: $\chi_0 \to \Pi_{\mathrm{vac}}\,,\ \ \chi_{1,2} \to 0$

Schwinger, Adler, Shabad, Urrutia, Tsai and Eber, Dittrich and Gies

Decomposing exponential factors

$$\chi_i = \frac{\alpha}{4\pi}\int_{-1}^{1} d\beta \int_0^\infty d\tau\, \frac{\Gamma_i(\tau,\beta)}{\sin\tau}\, \mathrm{e}^{-iu\cos(\beta\tau)}\, \mathrm{e}^{i\eta\cot\tau}\, \mathrm{e}^{-i\phi_\parallel \tau}$$

Contains arbitrarily higher harmonics

Linear w.r.t. τ in exp.

$$= 1 + c_1\cos(\beta\tau) + c_2\cos^2(\beta\tau) + \cdots c_n\cos^n(\beta\tau) + \cdots$$
$$= 1 + d_1\cos(\beta\tau) + d_2\cos(2\beta\tau) + \cdots + d_n\cos(n\beta\tau) + \cdots$$

1st step: "Partial wave decomposition"

$$e^{-iu\cos(\beta\tau)} = \sum_{n=0}^{\infty}(2-\delta_{n0}) I_n(-iu)\, e^{in\beta\tau}$$

Linear w.r.t. τ in exp.

2nd step: Getting Laguerre polynomials

Put $z = \exp(-2i\tau)$ and $x = y = \eta$

Associated Laguerre polynomial

$$\exp\left(-\frac{(x+y)z}{1-z}\right) I_n\left(\frac{2\sqrt{xyz}}{1-z}\right) = (1-z)(xyz)^{\frac{n}{2}}\sum_{\ell=0}^{\infty}\frac{\ell!}{\Gamma(\ell+n+1)} L_\ell^n(x) L_\ell^n(y)\, e^{-2i\ell\tau}$$

Linear w.r.t. τ in exp.

All terms fall in one of three elementary integrals.

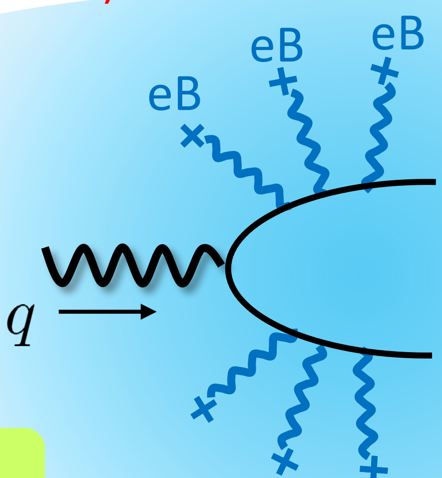
$$
\begin{aligned}
F^n_\ell(r^2_\parallel, B_\mathrm{r}) &= \frac{i}{B_\mathrm{r}} \int_{-1}^{1} d\beta \int_0^\infty d\tau \; e^{-i\left(\phi_\parallel + 2\ell - n\beta + n\right)\tau} \\
G^n_\ell(r^2_\parallel, B_\mathrm{r}) &= \frac{i}{B_\mathrm{r}} \int_{-1}^{1} d\beta \int_0^\infty d\tau \; \beta \, e^{-i\left(\phi_\parallel + 2\ell - n\beta + n\right)\tau} \\
H^n_\ell(r^2_\parallel, B_\mathrm{r}) &= \frac{i}{B_\mathrm{r}} \int_{-1}^{1} d\beta \int_0^\infty d\tau \; \beta^2 \, e^{-i\left(\phi_\parallel + 2\ell - n\beta + n\right)\tau}
\end{aligned}
$$

$$
\phi_\parallel(q^2_\parallel, B) = \frac{m^2}{eB}\left\{1 - (1-\beta^2)\,\frac{q^2_\parallel}{4m^2}\right\}
$$

What are the integers ℓ and n intorudced in the mathematical formulas?

Integers specify the fermion spectrum encoded in the photon spectrum.
(Remember the lesson in introduction)

$$
\mathrm{Im}\,\Pi^{\mu\nu} =
$$



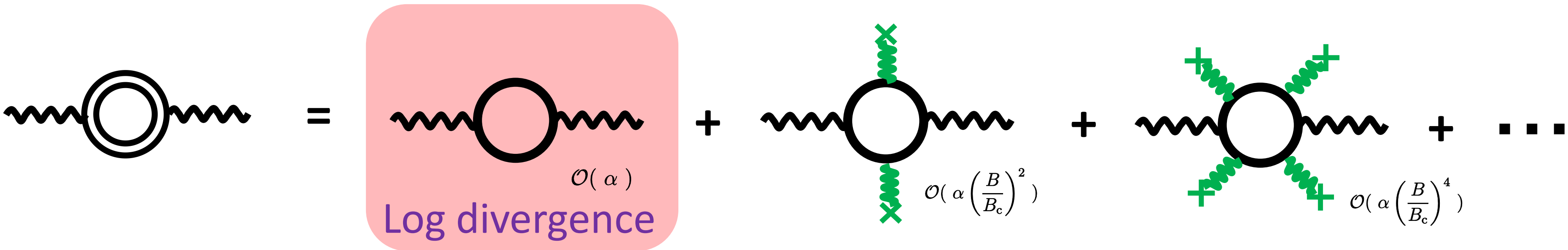
Invariant mass:

$$
q^2 \geq (\epsilon_f + \epsilon_{\bar{f}})^2
$$

Fermion-antifermion spectrum

Square of the decay amplitude
(Optical theorem)

Renormalization

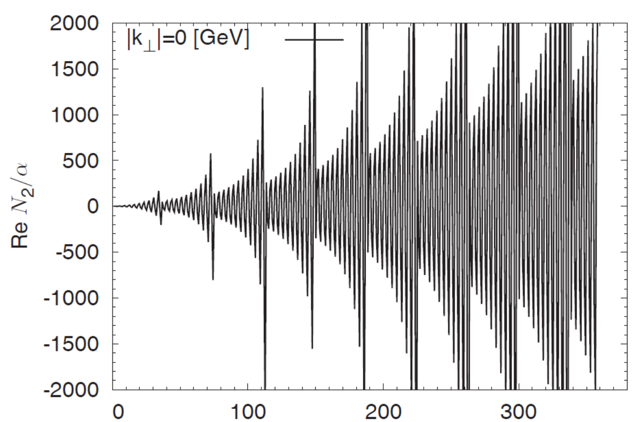


$$\begin{aligned}\Pi_{\rm ren}(q^2) &= \Pi(q_{\|}^2, q_{\perp}^2) - \Pi_{\rm vac}(q^2=0) \\ &= \underbrace{\Pi(q_{\|}^2, q_{\perp}^2) - \Pi(0, q_{\perp}^2)}_{\text{Term-by-term subtraction}} + \{ \underbrace{\Pi(0, q_{\perp}^2) - \Pi_{\rm vac}(q^2=0)}_{\text{Finite}} \}\end{aligned}$$

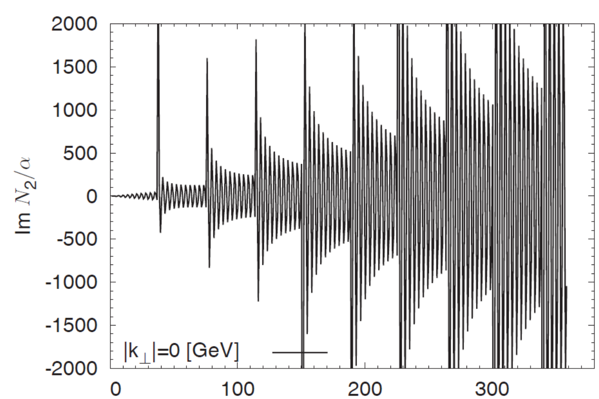
$\Pi(0, q_{\perp}^2)$ can be evaluated both by directly integrating the proper-time integrals and decomposing into the seires of Landau levels.

Ishikawa, Kimura, Shigaki, Tsuji (2013)

Re



Im



Taken from Ishikawa, et al. (2013)