Variations on the Maiani-Testa approach and the inverse problem

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based on 2012.11488

BNL HET Seminar, (Virtual)
February 25$^{\mathrm{th}}$, 2021

Motivations

Standard Model of particle physics extremely successful

we know is incomplete: neutrino masses, dark matter, etc..

Search for new physics at energy and intensity frontiers

muon anomaly: hadronic vacuum polarization and light-by-light

e.g. $\gamma \to \pi^+\pi^-$, $\pi^0 \to \gamma\gamma$

flavor physics, e.g. CP violation in strange and charm decays

e.g. study ε'/ε from $K \to \pi\pi$

Study of hadronic amplitudes very important

Reliable non-perturbative predictions from QCD (and SM)

$\to$ Lattice QCD

Textbook amplitudes

Infinite volume, Minkowski, $0 \to 2$ process

$$G_{\rm c}(q_1, q_2) = \int d^4x_1\, d^4x_2\, e^{-iq_1\cdot x_1 - iq_2\cdot x_2}\, \langle 0|T\big\{\phi(x_1)\phi(x_2)J(0)\big\}|0\rangle_{\rm c}$$

$G_{\rm c}$ has a pole $\dfrac{1}{q_1^2 - m^2 - i\varepsilon}\dfrac{1}{q_2^2 - m^2 - i\varepsilon}$

residue ($= q_1, q_2$ on-shell) is connected amplitude

form factor $\mathcal{F} = \langle \boldsymbol{q}_1, \boldsymbol{q}_2, \mathrm{out}|J(0)|0\rangle$

this is the standard LSZ theorem

Revisiting LSZ - I

$$F(\boldsymbol{q}_1, q_2) = \int d^3\boldsymbol{x}_2\, e^{-i\boldsymbol{q}_2\cdot\boldsymbol{x}_2} \int dx_2^0\, e^{-iq_2^0 x_2^0}\, \theta(x_2^0)\langle \boldsymbol{q}_1|\phi(x_2)J(0)|0\rangle_c$$

Using $\phi_{\boldsymbol{p}}(x^0) = \int d^3\boldsymbol{x} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\phi(x)$ and $\phi_{\boldsymbol{p}}(x^0) = e^{i\hat{H}x^0}\phi_{\boldsymbol{p}}(0)e^{-i\hat{H}x^0}$

Complete set of (out) states $\sum_n \int d\Phi_n |\boldsymbol{p}_1, \ldots \boldsymbol{p}_n, \mathrm{out}\rangle\langle \boldsymbol{p}_1, \ldots \boldsymbol{p}_n, \mathrm{out}|$

$$F(\boldsymbol{q}_1, q_2) = \sum_n \int d\Phi_n \int dx_2^0\, e^{-i(q_2^0 + E(\boldsymbol{q}_1) - E_n + i\varepsilon)x_2^0}\, \theta(x_2^0)$$
$$\times \langle \boldsymbol{q}_1|\phi_{\boldsymbol{q}_2}(0)|n\rangle\langle n|J(0)|0\rangle_c$$

Using the usual trick $\int d\mathcal{E}\delta(E_n - \mathcal{E})$

$$F(\boldsymbol{q}_1, q_2) = \int d\mathcal{E} \int dx_2^0\, e^{-i(q_2^0 + E(\boldsymbol{q}_1) - \mathcal{E})x_2^0}\, \theta(x_2^0)\, \rho(\mathcal{E})$$

$\rho(\mathcal{E})$ is the underlying spectral function

Revisiting LSZ - II

$$
\begin{aligned}
F(\boldsymbol{q}_1, q_2) &= \int d^4x_2 e^{-iq_2x_2}\theta(x_2^0)\,\langle \boldsymbol{q}_1|\phi_{\boldsymbol{q}_2}(0)J(0)|0\rangle \\
&= \int \frac{d\mathcal{E}}{\pi}\frac{i}{q_2^0 + E(\boldsymbol{q}_1) - \mathcal{E} + i\varepsilon}\rho(\mathcal{E}) \\
&= \frac{\sqrt{Z}}{2E(\boldsymbol{q}_2)}\frac{\mathcal{F}(\boldsymbol{q}_1, \boldsymbol{q}_2)}{q_2^0 - E(\boldsymbol{q}_2) + i\varepsilon} + \cdots
\end{aligned}
$$

what are the ...? off-shell unphysical terms (regular)

the limit $q_2^2 \to m^2$ kills them, leaving physical $\mathcal{F}$

Take home message:

integral spectral function at on-shell kinematics $\leftrightarrow$ physical amplitude

Lattice field theories

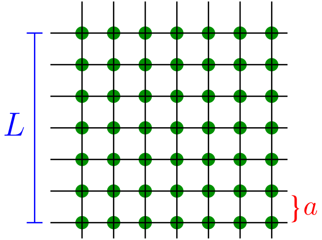
Due to confinement → non-perturbative formulation is necessary

lattice spacing a → regulate UV divergences

finite size L → infrared regulator

Continuum theory $a \to 0$, $L \to \infty$

Euclidean metric → Boltzman interpretation of path integral



$$\langle O \rangle = \mathcal{Z}^{-1} \int [DU] e^{-S[U]} O(U) \approx \frac{1}{N} \sum_{i=1}^{N} O[U_i]$$

Very high dimensional integral → Monte-Carlo methods

Markov Chain of gauge field configs $U_0 \to U_1 \to \cdots \to U_N$

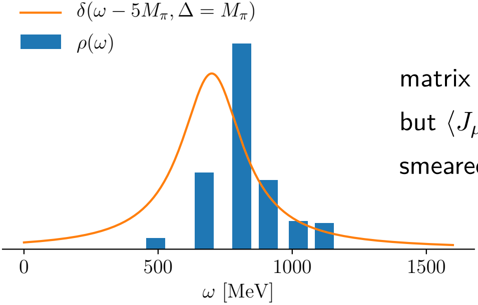
Spectral decomposition

EM current J_μ, projected to momentum $\boldsymbol{p}$

two-point function $\to$ hadronic vacuum polarization and R-ratio

$$\langle J_\mu(t,\boldsymbol{p}) J_\mu(0,\boldsymbol{p})\rangle = \sum_n |\langle 0|J_\mu(0)|n,\boldsymbol{p}\rangle|^2 e^{-E_n t}$$

$$\int d\omega\, e^{-\omega t} \underbrace{\sum_n \delta(\omega - E_n) |\langle 0|J_\mu(0)|n,\boldsymbol{p}\rangle|^2}_{=\rho(\omega)}$$



matrix elements $1/L^k$ finite vol. effects

but $\langle J_\mu J_\mu\rangle$ errors of $O(e^{-M_\pi L})$

smeared $\delta \to$ smeared ρ FV $O(e^{-\Delta L})$

Inverse problem - I

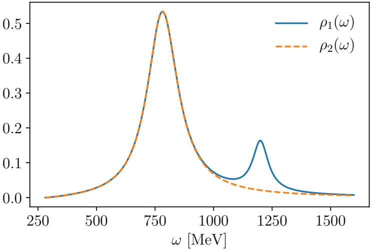
Understanding the role of Euclidean metric

we assume large (infinite) spatial volume

goal is predicting physical time-like amplitudes

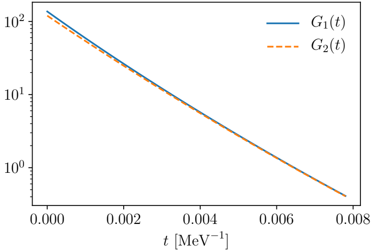
Minkowski

$\rho_i(\omega)$ time-like cross sections



Euclidean

$G_i(t) = \int d\omega \, e^{-\omega|t|} \rho_i(\omega)$



Euclidean $\rightarrow$ Minkowski requires solution inverse Laplace

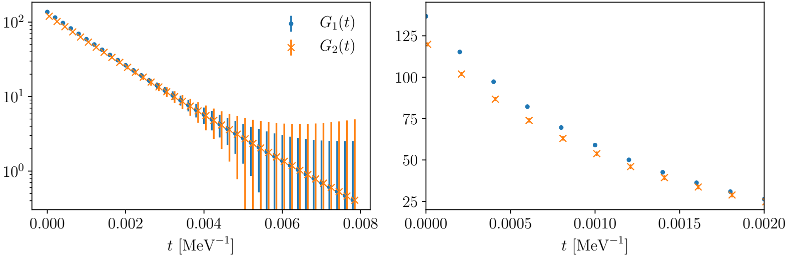
Inverse problem - II

$$\int dt\, e^{\omega_0 t}\, G(t) = \int d\omega \overbrace{\int dt e^{(\omega_0-\omega)t}}^{=\delta(\omega_0-\omega)} \rho(\omega) = \rho(\omega_0)$$

Lattice QCD

1. finite (discrete) subset of points
2. statistical (and syst.) errors

= inverse Laplace numerically ill-defined



extremely hard problem:

long-distance growing statistical noise (exponential)

short-distance large discretization errors

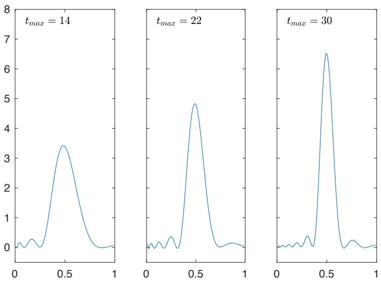
Inverse problem - III

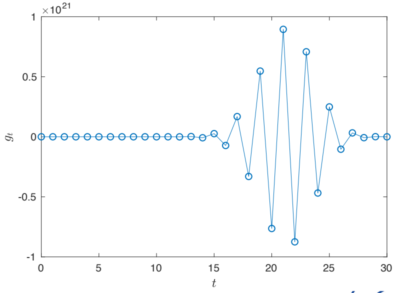
Approx. numerical solutions [Backus, Gilbert '68][Hansen, Lupo, Tantalo '19]

$$\sum_t c_t \langle J_\mu(t) J_\mu(0) \rangle \approx \sum_n |\langle 0|J_\mu|n\rangle|^2 \delta(E_n - \omega, \Delta)$$

coefficients c_t numerically obtained for each ω, Δ

highly oscillatory $\rightarrow$ regulate with cov. matrix or large Δ

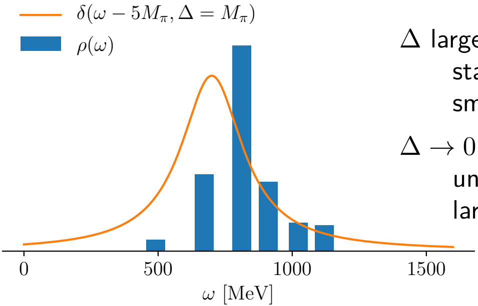




Large oscillations $\rightarrow$ large statistical errors

Inverse problem - Recap

Numerical methods to approximate smeared $\rho(\omega)$ from $G(t)$



Δ large for finite vol. errors
- stable reconstruction
- smaller stat. errors

$\Delta \to 0$ for physics
- unstable reconstruction
- large stat. errors

Current available lattices
- $\Delta \approx m_\pi$, preventing extraction $\rho(770)$ resonance
- ordered $L \to \infty, \Delta \to 0$ still very challenging

The problem

"It is of considerable interest to identify the physical quantities, if any, which can be extracted directly from euclidean correlation functions, avoiding analytic continuation" [Maiani, Testa '90]

Maiani-Testa - I

Goal: the form factor $\langle \pi(-\boldsymbol{q})\pi(\boldsymbol{q})\mathrm{out}|\hat{J}|0\rangle$ for $\boldsymbol{q} \neq 0$

J scalar current

Starting point: euclidean correlator $\langle \widetilde{\pi}_{\boldsymbol{q}_1}(t_1)\widetilde{\pi}_{\boldsymbol{q}_2}(t_2)J(0)\rangle$

$\widetilde{\pi}_{\boldsymbol{q}}$ pion interpolating operator projected to $\boldsymbol{q}$

$\omega_{\boldsymbol{q}_i} = \sqrt{M_\pi^2 + \boldsymbol{q}_i^2}$ pion energy

0. Limit of large t_1

$$\langle \widetilde{\pi}_{\boldsymbol{q}_1}(t_1)\widetilde{\pi}_{\boldsymbol{q}_2}(t_2)J(0)\rangle \overset{t_1\to\infty}{\simeq} [2\omega_{\boldsymbol{q}_1}]^{-1}\sqrt{Z_\pi}e^{-\omega_{\boldsymbol{q}_1}t_1}\underbrace{\langle \pi, \boldsymbol{q}_1|\widetilde{\pi}_{\boldsymbol{q}_2}(t_2)\, J(0)|0\rangle}$$

1. Use physical Hamiltonian $\hat{H}$ to remove t_2 dependence

$$e^{\omega_{\boldsymbol{q}_2}t_2} \times \langle \pi, \boldsymbol{q}_1|\widetilde{\pi}_{\boldsymbol{q}_2}(t_2)\, J(0)|0\rangle = \langle \pi, \boldsymbol{q}_1|e^{+\hat{H}t_2}\widetilde{\pi}_{\boldsymbol{q}_2}(0)\, e^{-\hat{H}t_2} \times e^{\omega_{\boldsymbol{q}_2}t_2}\, J(0)|0\rangle$$

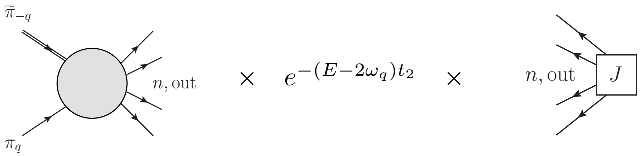
2. set $\boldsymbol{q}_1 = -\boldsymbol{q}_2 = \boldsymbol{q} \to \langle \pi, \boldsymbol{q}|\widetilde{\pi}_{-\boldsymbol{q}}(0)\, e^{-(\hat{H}-2\omega_{\boldsymbol{q}})t_2}\, J(0)|0\rangle$

$$\langle \pi, \boldsymbol{q} | \widetilde{\pi}_{-\boldsymbol{q}}(0)\, e^{-(\hat{H}-2\omega_q)t_2} \qquad\qquad\qquad\qquad J(0)|0\rangle$$

1. insert complete set of (out) states

$$\langle \pi, \boldsymbol{q} | \widetilde{\pi}_{-\boldsymbol{q}}(0)\; e^{-(E-2\omega_q)t_2} \sum_n \int d\Phi_n |n, \mathrm{out}\rangle\langle n, \mathrm{out}| J(0)|0\rangle$$

$d\Phi_n$: n-particle phase space

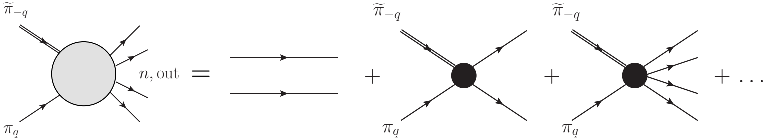


$\mathcal{F}_n = \langle n, \mathrm{out}|J(0)|0\rangle$: time-like $1 \to n$-particles form factor

Maiani-Testa - II

$\langle \pi, \boldsymbol{q} | \widetilde{\pi}_{-\boldsymbol{q}}(0) | n, \mathrm{out} \rangle =$ not the $2 \to n$ scattering amplitude!

2. identify off-shell contributions in $\langle \pi, \boldsymbol{q} | \widetilde{\pi}_{-\boldsymbol{q}}(0) | n, \mathrm{out} \rangle$



$$= \mathrm{discon} \times \delta_{2n} + \frac{\sqrt{Z_\pi}}{\eta + i\epsilon} \boldsymbol{\mathcal{M}}^*_{22}(\eta) + \frac{\sqrt{Z_\pi}}{\eta + i\epsilon} \boldsymbol{\mathcal{M}}^*_{24}(\eta) + \ldots$$

$\widetilde{\pi}_{-\boldsymbol{q}} \to$ pole at $\eta = q^2 - M_\pi^2 = E(E - 2\omega_q)$ (virtuality)

black blobs $=$ off-shell scattering matrix $\boldsymbol{\mathcal{M}}_{2n}(\eta)$

$\lim_{\eta \to 0} \boldsymbol{\mathcal{M}}_{2n}(\eta) = \mathcal{M}_{2n}$ on-shell $2 \to n$ scattering matrix

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Bottom line: $\boldsymbol{q}$ fixed, hence energy E is our knob
for finding unique on-shell point $\eta = 0$!

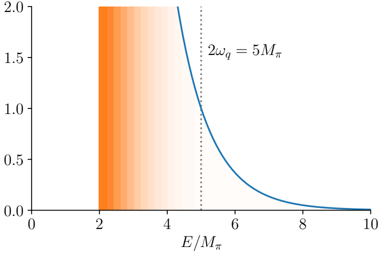
Maiani-Testa - III

Key points: in any given channel n, out

1. unique physical point in energy E
2. phase integral $\to$ integral over energy, $E \in [2M_\pi, \infty)$
3. enhancement at $E = 2M_\pi$ from $e^{-(E-2\omega_q)t_2}$

Conclusion:
we can not extract physical amplitude!

contamination from off-shell physics
$\to \widetilde{\pi}$ dependence



We can solve the problem by considering $\boldsymbol{q} = 0$ [Maiani, Testa '90]
points **1.** and **2.** still valid, but
enhancement at $E = 2M_\pi$ corresponds to physical point

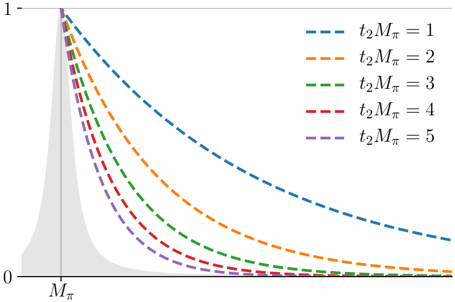
Maiani-Testa - IV

Final key concept: use time-dependence to focus integral at $E = 2M_\pi$

$e^{-(E-2M_\pi)t_2}$ like "half-δ"

localization at $E = 2M_\pi$

expansion about large t_2
$\rightarrow$ amplitude at threshold
$\rightarrow$ scattering length a



sketch derivation:

$$t_2 \int_0^\infty dy e^{-yt_2} G(y) = \int_0^\infty dx e^{-x} G(x/t_2) = \sum_n g_n \int dx \frac{e^{-x} x^n}{t_2^n}$$

$g_0 = G(y=0)$: how Maiani-Testa get $\mathcal{M}_{2n} = \boldsymbol{\mathcal{M}}_{2n}(\eta = 0)$

Maiani-Testa - Final

Conclusion: physical scattering only at $\boldsymbol{q} = 0$ [Maiani, Testa '90]

$$\langle \pi, \mathbf{0} | \widetilde{\pi}_{\mathbf{0}}(t_2) J(0) | 0 \rangle \overset{t_2 \gg 0}{\to} \mathcal{F}_2(4M_\pi^2) \left[1 + a \sqrt{\frac{M_\pi}{\pi t_2}} + O(t_2^{-3/2}) \right]$$

at threshold: $\mathcal{F}_2$+scatt.length a

No-go theorem for $\boldsymbol{q} \neq 0$

Maiani-Testa: at threshold there is no inverse problem!

analytic control over inverse problem thanks to t_2!

$t_2 \Delta E \ll 1$, with ΔE level spacing

What about $K \to \pi\pi$?

final two pions relative non-zero momentum

$\to$ Maiani-Testa say can not use lattice (euclidean) correlators

Intermezzo - Finite volume

QCD in a box [Lüscher '85, '86]

- spectrum distorted by finite volume
- use distorsion to extract interactions $\to$ amplitudes

Also matrix elements distorted by finite volume

- well-defined mapping to infinite volume [Lellouch-Lüscher '01]

Main limitation: $1 \to 2$, $2 \to 2$

- 3-particle quantization completed [Hansen, Sharpe '19]
  - 4-particles? generalization of formalism? [Blanton, Sharpe '20]
- at phys. pions we cannot (yet) reach the ρ mass!

With our work we hope to:

- extract $1, 2 \to 2$ beyond 4-particle threshold
- generalize to n particles in final states

Intermezzo - ε'/ε

+ G-parity boundary conditions
  lowest 2π state has relative non-zero momentum
+ Domain-wall formulation, retains good chiral symmetry
  $N_f = 2+1$ ensemble with physical masses
  full operator basis, including QCD and EW penguins
+ Large operator basis
  GEVP analysis to control excited states
  scattering phase at M_K in agreement with dispersive prediction

= Most precise prediction of CP violation in $K \to \pi\pi$ from LQCD

[RBC/UKQCD, MB et al. '20]

$$\mathrm{Re}(\varepsilon'/\varepsilon) = 0.00217(26)_{\mathrm{stat.}}(62)_{\mathrm{syst.}}(50)_{\mathrm{iso.breaking}}$$

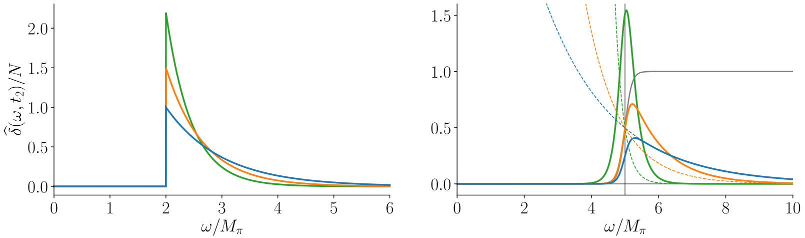
[experiment]

$$\mathrm{Re}(\varepsilon'/\varepsilon) = 0.00166(23)$$

Our proposal

$\langle \pi, \boldsymbol{q}_1 | \widetilde{\pi}_{\boldsymbol{q}_2}(0) \qquad\qquad\qquad e^{-(\hat{H}-2\omega_{\boldsymbol{q}})t_2} J(0)|0\rangle$ [Maiani-Testa '90]

$\langle \pi, \boldsymbol{q}_1 | \widetilde{\pi}_{\boldsymbol{q}_2}(0)\, \Theta(\hat{H}-2\omega_{\boldsymbol{q}}, \Delta) e^{-(\hat{H}-2\omega_{\boldsymbol{q}})t_2} J(0)|0\rangle$ [Bruno-Hansen, '20]



smooth Θ, smearing width Δ

- tames growing exponentials in $2M_\pi < E < 2\omega_{\boldsymbol{q}}$

combination of Θ and exponential $\rightarrow$ localization in energy ω

- like Maiani and Testa we want analytic control by expanding t_2

Generalized Maiani-Testa - I

$$\langle \pi, \mathbf{0} | \widetilde{\pi}_{\mathbf{0}}(0) e^{-(\hat{H}-2\omega_{\mathbf{0}})t_2} J(0)|0\rangle$$ [Maiani-Testa '90]

$$\overset{t_2 \gg 0}{\to} \mathcal{F}_2(4M_\pi^2)\left[1 + a\sqrt{\frac{M_\pi}{\pi t_2}} + O(t_2^{-3/2})\right]$$

$$\langle \pi, \boldsymbol{q} | \widetilde{\pi}_{-\boldsymbol{q}} \Theta(\hat{H} - 2\omega_{\boldsymbol{q}}, \Delta) e^{-(\hat{H}-2\omega_{\boldsymbol{q}})t_2} J(0)|0\rangle$$ [Bruno-Hansen '20]

$$\to \ \mathrm{Re}\left[\mathcal{F}_2(4\omega_{\boldsymbol{q}}^2)\right] + \textstyle\sum_{n=0} g_n \mathcal{J}^{(n)}(t_2, \omega_{\boldsymbol{q}}, \Delta)$$

Generalization of Maiani-Testa work

$\mathcal{J}^{(n)}$ pure analytic functions

we have full control over large t_2 expansion

unitarity relations imply $g_0 \simeq \mathrm{Im}\left[\mathcal{F}_2\right]$

$$2i\mathrm{Im}\left[\mathcal{F}_2\right] = \mathrm{disc}[\mathcal{F}_2] = 2\pi i \sum_n \int d\Phi_n \mathcal{M}^*_{2n} \mathcal{F}_n \delta(2\omega_{\boldsymbol{q}} - \sum_i \omega_{\boldsymbol{p}_i})$$

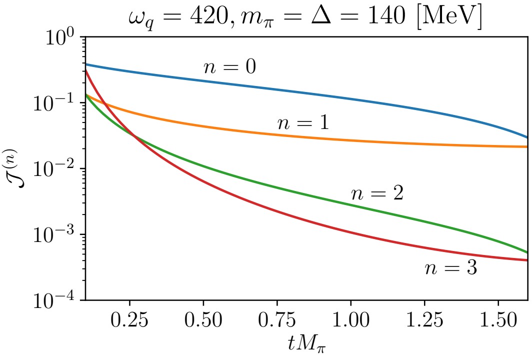
we go beyond the two-pion intermediate channel

Generalized Maiani-Testa - II

$$\mathcal{N}\langle\pi,\boldsymbol{q}|\widetilde{\pi}_{-\boldsymbol{q}}\Theta(\hat{H}-\sqrt{s},\Delta)e^{-(\hat{H}-2\omega_{\boldsymbol{q}})t_2}J(0)|0\rangle$$ [Bruno-Hansen '20]

$$= e^{-\sqrt{m_\pi^2+\boldsymbol{q}^2}t_2}\left[\Theta(0,\Delta)\mathrm{Re}\left[\mathcal{F}_2(s)\right]-2\mathrm{Im}\left[\mathcal{F}_2(s)\right]\mathcal{J}^{(0)}(t_2,s,\Delta)+\cdots\right]$$

$\mathcal{J}^{(n)}$ pure analytic functions



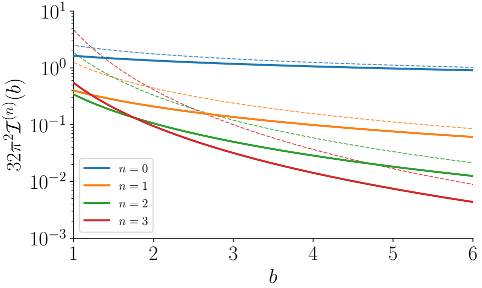
large hierarchy $\rightarrow$ hope for reliable extraction of $\mathrm{Re}\left[\mathcal{F}_2\right]$ and $\mathrm{Im}\left[\mathcal{F}_2\right]$

Generalized Maiani-Testa - III

Threshold: take $\boldsymbol{q} = 0$

$\langle \pi, \mathbf{0} | \widetilde{\pi}_{\mathbf{0}}(0) e^{-(\hat{H} - 2\omega_{\mathbf{0}}) t_2} J(0) | 0 \rangle$ [Bruno, Hansen '20]

$$\to \mathcal{F}_2\big(4M_\pi^2\big) \left[1 - \frac{32\pi M_\pi a}{\sqrt{2M_\pi t_2}} \mathcal{I}^{(0)}(2M_\pi t_2) - \frac{g_1 \mathcal{I}^{(1)}(2M_\pi t_2)}{\sqrt{2M_\pi t_2}} + \mathcal{O}\big(\mathcal{I}^{(2)}\big) \right]$$

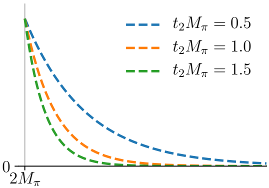


large t_2 reproduce Maiani-Testa result

go beyond with full integrals $\mathcal{I}^{(n)}$

improved hierarchy

Inverse problem

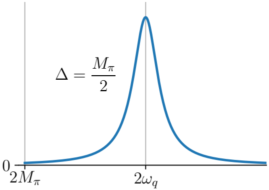


[Maiani-Testa '90]

$$\langle \pi, \boldsymbol{q}_1 | \widetilde{\pi}_{\boldsymbol{q}_2}(0)\, e^{-(\hat{H}-2\omega_{\boldsymbol{q}})t_2}\, J(0)|0\rangle$$

physical scattering at $\boldsymbol{q}_1 = \boldsymbol{q}_2 = 0$

exponentials mimick "half" $\delta(E - 2M_\pi)$

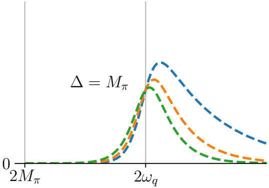


[Bulava-Hansen '18]

$$\langle \pi, \boldsymbol{q}_1 | \widetilde{\pi}_{\boldsymbol{q}_2}(0)\, \delta(\hat{H} - 2\omega_{\boldsymbol{q}}, \Delta)\, J(0)|0\rangle$$

physical scattering at $E = 2\omega_{\boldsymbol{q}}$

ordered double-limit $\lim_{\Delta\to 0} \lim_{V\to\infty}$



[Bruno-Hansen '20]

$$\langle \pi, \boldsymbol{q}_1 | \widetilde{\pi}_{\boldsymbol{q}_2}(0)\, \Theta(\hat{H} - 2\omega_{\boldsymbol{q}}, \Delta) e^{-(\hat{H}-2\omega_{\boldsymbol{q}})t_2}\, J(0)|0\rangle$$

physical scattering at pole $E = 2\omega_{\boldsymbol{q}}$

physical scattering at fixed Δ

Practical implementations

Finite but large volume, say $M_\pi L \simeq 5$

1. exact reconstruction

large basis of operators $\to$ GEVP

$O(30)$ energy levels possible [HadSpec]

$\pi\pi$ $I = 1$, P-wave, in context of $(g-2)_\mu$

4/5 levels at physical pions [MB, Izubuchi, Meyer, Lehner '18]

2. approx. reconstruction

Backus-Gilbert the Θ is possible [Hansen-Lupo-Tantalo '19]

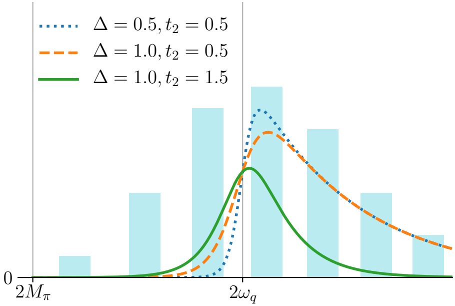
less severe inverse problem than δ

likely more suited for higher-energies

Finite Volume errors

$$\langle \pi, \boldsymbol{q}_1 | \widetilde{\pi}_{\boldsymbol{q}_2}(0) \Theta(\hat{H} - 2\omega_{\boldsymbol{q}}, \Delta) e^{-(\hat{H}-2\omega_{\boldsymbol{q}})t_2} J(0)|0\rangle = \int d\omega K(\omega, t_2) \rho_L(\omega)$$

Spectral-function $\rho_L(\omega) = \sum_n \delta(\omega - E_n) c_n$



$\Delta \approx M_\pi$ we expect
$O(e^{-M_\pi L})$ FV errors

Large $t_2 \simeq$ narrow δ-function

Maiani-Testa: $t_2 \Delta E \ll 1$, ΔE level spacing
window in t_2 where method $O(e^{-M_\pi L})$

Conclusions

Generalization of the Maiani-Testa result [Bruno-Hansen '20]

- away from threshold
- time-like form factor (real and imaginary)
- resummed all intermediate channels
- understood the connection with the inverse problem
- extension to $2 \to 2$ processes
  - novel method to extract $N\pi$ scattering length

Next steps

1. numerical tests
2. improve understanding of FV errors
3. extend to neutral meson oscillations, QED semileptonic etc..

Thanks for your attention

More Maiani-Testa

$$\langle \pi, \boldsymbol{q} | \widetilde{\pi}_{-\boldsymbol{q}}(0) | n, \text{out} \rangle = \text{discon} \times \delta_{2n} + \sqrt{Z_\pi} \boldsymbol{\mathcal{M}}^*_{2n}(\eta)[\eta + i\epsilon]^{-1}$$

1. disconnected part isolates $\mathcal{F}_2$, complex?

$$Z_\pi^{-1/2}[2\omega_{\boldsymbol{q}}]\langle \pi, \boldsymbol{q} | \widetilde{\pi}_{-\boldsymbol{q}}(0) e^{-(\hat{H}-2\omega_{\boldsymbol{q}})t_2} J(0)|0\rangle =$$
$$\mathcal{F}_2[4\omega_{\boldsymbol{q}}^2] + 2\omega_{\boldsymbol{q}} \frac{1}{2} \sum_n \int d\Phi_n \frac{e^{-(E-2\omega_{\boldsymbol{q}})t_2}}{\eta + i\epsilon} \boldsymbol{\mathcal{M}}^*_{2n}(\eta)\, \mathcal{F}_n$$

2. Maiani and Testa clever observation: separate the absorptive part

$$2\omega_{\boldsymbol{q}} \tfrac{1}{2} \textstyle\sum_n \int d\Phi_n (-2\pi i)\delta(\eta) e^{-(E-2\omega_{\boldsymbol{q}})t_2} \boldsymbol{\mathcal{M}}^*_{2n} \mathcal{F}_n = -i\mathrm{Im}\,[\mathcal{F}_2]$$

3. $\mathcal{F}_2 - i\mathrm{Im}\,[\mathcal{F}_2] = \mathrm{Re}\,[\mathcal{F}_2] \to$ real ✓, time-like ✓

Let's turn to principal value part $\mathcal{P}\frac{1}{\eta}$

More Maiani-Testa

$$2\omega_{\boldsymbol{q}} \frac{1}{2} \sum_n \int d\Phi_n \mathcal{P} \frac{1}{\eta} e^{-(E-2\omega_{\boldsymbol{q}})t_2} \boldsymbol{\mathcal{M}}^*_{2n} \mathcal{F}_n$$

$$[\eta(E) = E(E - 2\omega_{\boldsymbol{q}})]$$

a. integration $E \in [2M_\pi, \infty)$
b. $\boldsymbol{\mathcal{M}}$ on-shell only at pole

$\rightarrow$

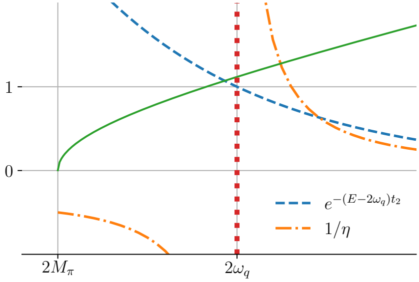
for large t_2,
$E \approx 2M_\pi$
dominates integral

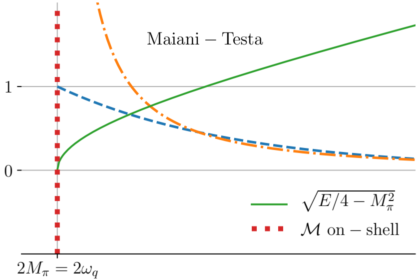
Visual inspection

restrict to $n = 2\pi$

$d\Phi_2 \to dE\sqrt{E^2/4 - M_\pi^2}$

$$\int d\Phi_2\, \mathcal{P}\frac{1}{\eta} e^{-(E-2\omega_{\boldsymbol{q}})t_2} \boldsymbol{\mathcal{M}}^*_{2n}\mathcal{F}_n$$





phase space $d\Phi_2$

$\boldsymbol{\mathcal{M}}$ on shell only at pole

principal value pole $1/\eta$

exponentials