Weak decays of hadrons from lattice QCD+QED simulations

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OUTLINE

- Motivations
- Isospin-breaking effects on the lattice: the RM123 method
- Light meson leptonic decays

In collaboration with:

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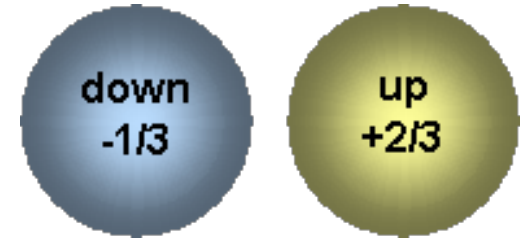
F. Sanfilippo, S. Simula and N. Tantalo

ISOSPIN-BREAKING EFFECTS

Isospin symmetry is an almost exact property of the strong interactions



Isospin-breaking effects are induced by:

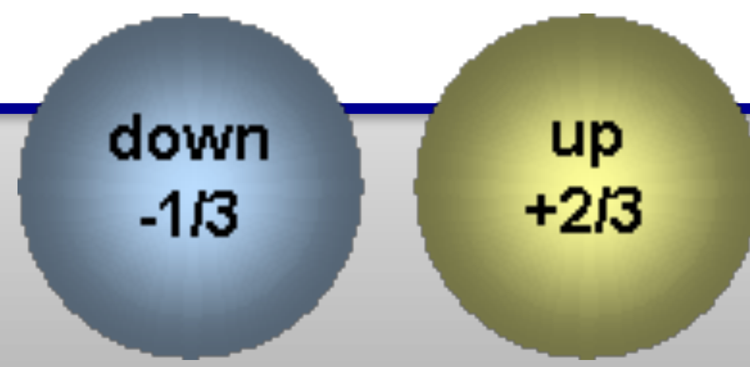
$m_u \neq m_d$: $O[(m_d - m_u)/\Lambda_{QCD}] \approx 1/100$ "Strong"

$Q_u \neq Q_d$: $O(\alpha_{em}) \approx 1/100$ "Electromagnetic"

Since electromagnetic interactions renormalize quark masses the two corrections are intrinsically related

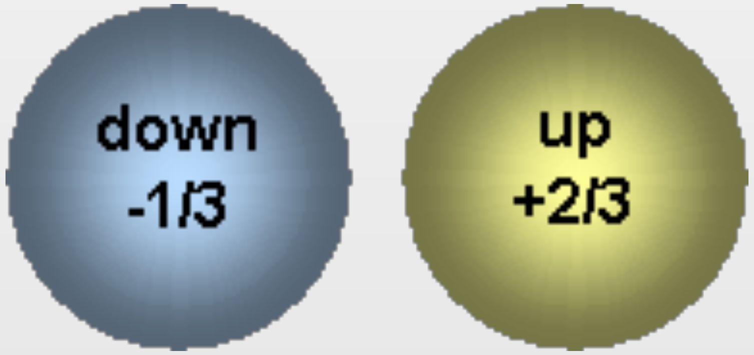
Though small, IB effects play often a very important role (quark masses, Mn - Mp, leptonic decay constants, vector form factor)

Phenomenological motivations



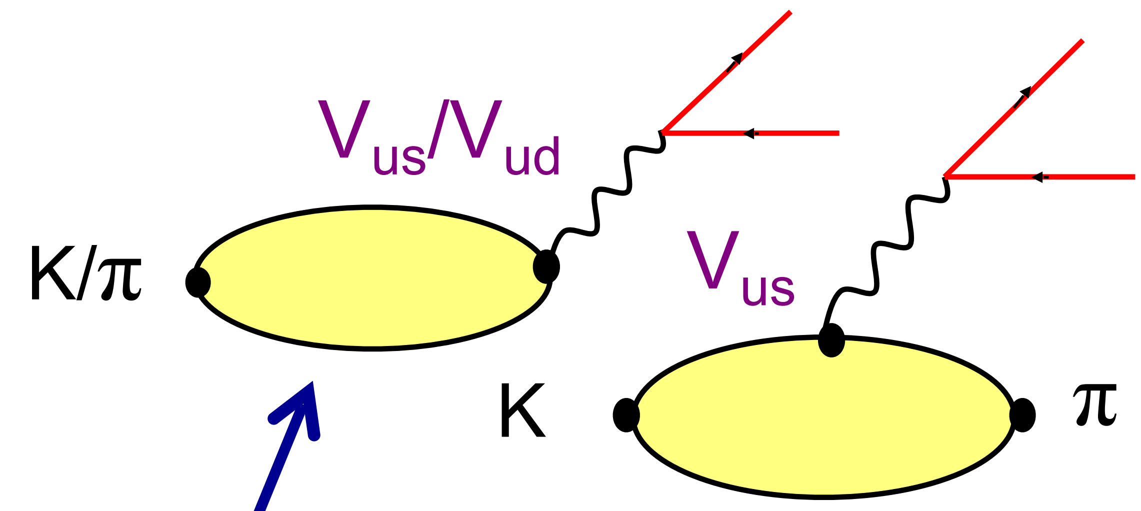
Phenomenological motivations

The determination of some hadronic observables in flavor physics has reached such an accurate degree of experimental and theoretical precision that electromagnetic and strong isospin-breaking effects cannot be neglected anymore



The determination of Vus and Vud

The relevant processes are leptonic and semileptonic K and π decays

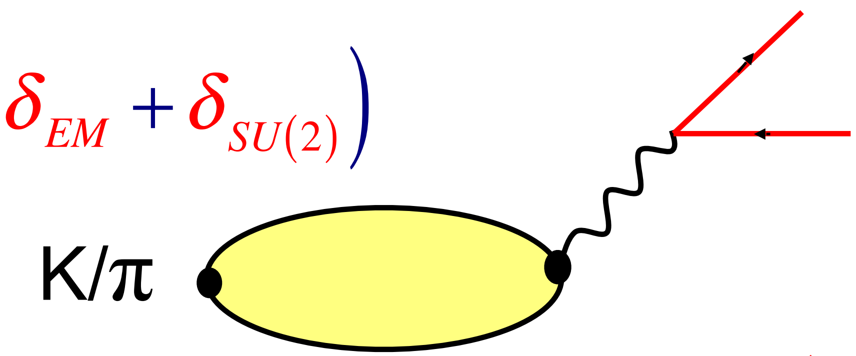


$$\frac{\Gamma\left(K^+ \to \ell^+ \nu_\ell(\gamma)\right)}{\Gamma\left(\pi^+ \to \ell^+ \nu_\ell(\gamma)\right)} = \left(\frac{|V_{us}|}{|V_{ud}|}\frac{f_K}{f_\pi}\right)^2 \frac{M_{K^+}\left(1-m_\ell^2/M_{K^+}^2\right)^2}{M_{\pi^+}\left(1-m_\ell^2/M_{\pi^+}^2\right)^2}\left(1+\delta_{EM}+\delta_{SU(2)}\right)$$

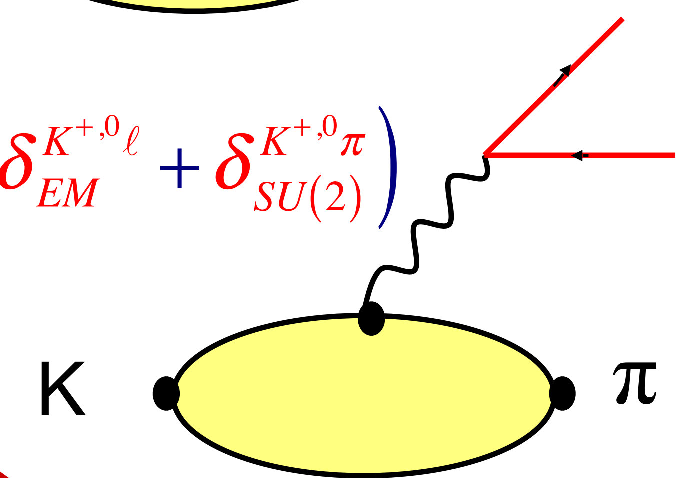
$$\Gamma\left(K^{+,0} \to \pi^{0,-} \ell^+ \nu_\ell(\gamma)\right) = \frac{G_F^2 M_{K^{+,0}}^5}{192\pi^3} C_{K^{+,0}}^2 \left|V_{us} f_+^{K^0\pi^-}(0)\right|^2 I_{K\ell}^{(0)} S_{EW}\left(1+\delta_{EM}^{K^{+,0}\ell}+\delta_{SU(2)}^{K^{+,0}\pi}\right)$$

Vus and Vud: experimental results

$$\frac{\Gamma\left(K^+ \to \ell^+ \nu_\ell(\gamma)\right)}{\Gamma\left(\pi^+ \to \ell^+ \nu_\ell(\gamma)\right)} = \left(\frac{|V_{us}|}{|V_{ud}|}\frac{f_K}{f_\pi}\right)^2 \frac{M_{K^+}\left(1 - m_\ell^2/M_{K^+}^2\right)^2}{M_{\pi^+}\left(1 - m_\ell^2/M_{\pi^+}^2\right)^2}\left(1 + \delta_{EM} + \delta_{SU(2)}\right)$$



$$\Gamma\left(K^{+,0} \to \pi^{0,-}\ell^+ \nu_\ell(\gamma)\right) = \frac{G_F^2 M_{K^{+,0}}^5}{192\pi^3} C_{K^{+,0}}^2 \left|V_{us} f_+^{K^0\pi^-}(0)\right|^2 I_{K\ell}^{(0)} S_{EW}\left(1 + \delta_{EM}^{K^{+,0}\ell} + \delta_{SU(2)}^{K^{+,0}\pi}\right)$$



$$\frac{|V_{us}|}{|V_{ud}|}\frac{f_K}{f_\pi} = 0.27599(38)$$

$$|V_{us}|\, f_+(0) = 0.21654(41)$$

< 0.2%

FlaviA net Kaon WG

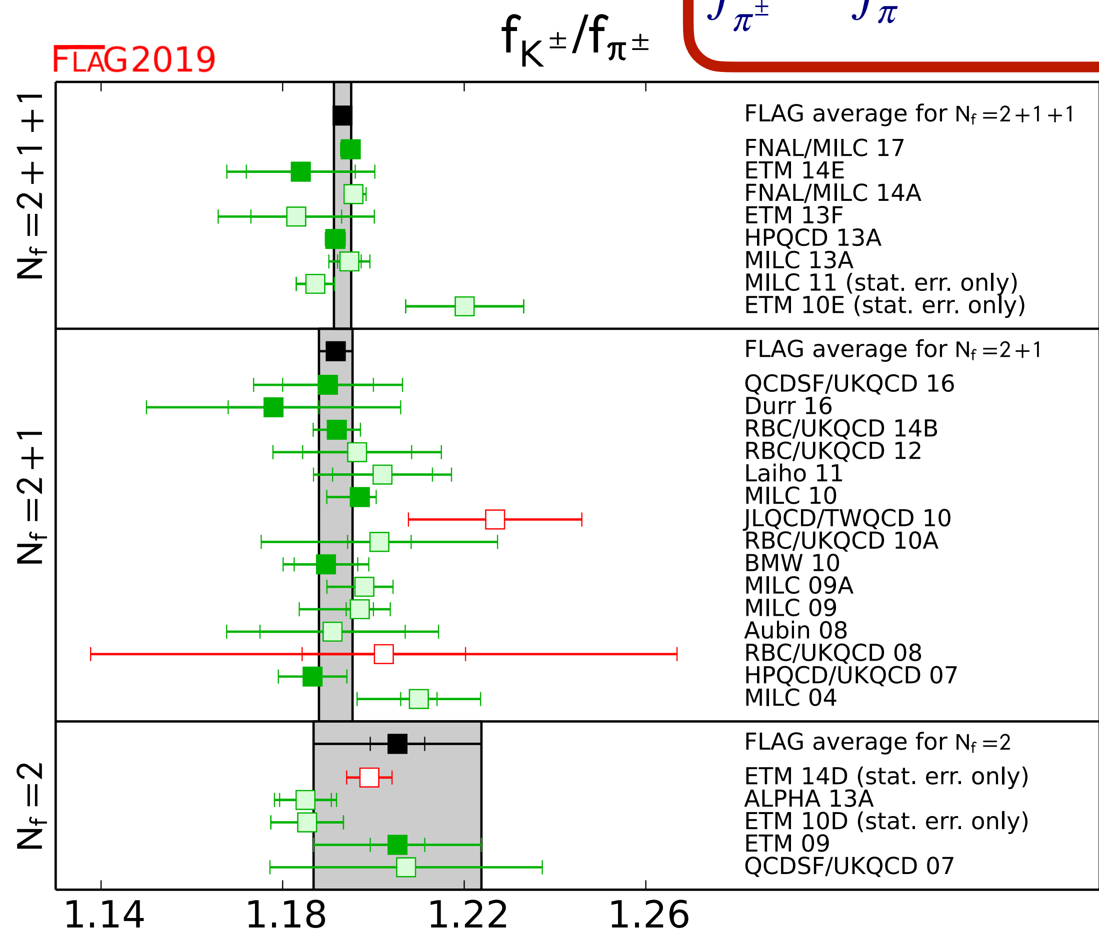
PDG

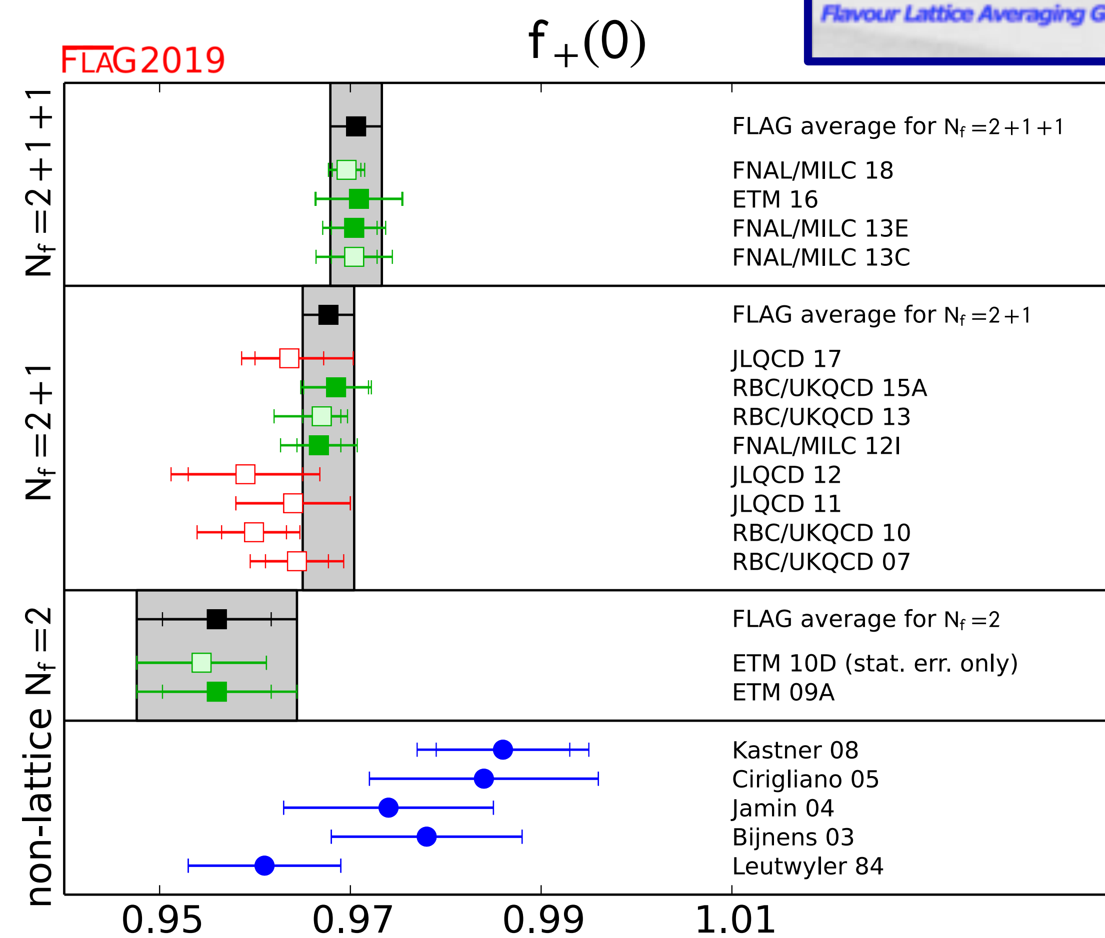
M. Moulson, arXiv:1704.04104

Vus and Vud: results from lattice QCD

$$\frac{f_{K^\pm}}{f_{\pi^\pm}} = \frac{f_K}{f_\pi}\sqrt{1+\delta_{SU(2)}}$$

FLAG 2019
Flavour Lattice Averaging Group





$f_{K^\pm} / f_{\pi^\pm} = 1.1932(19)$ Nf=2+1+1

$f_{K^\pm} / f_{\pi^\pm} = 1.1917(37)$ Nf=2+1

0.2%

$f_+(0) = 0.9706(27)$ Nf=2+1+1

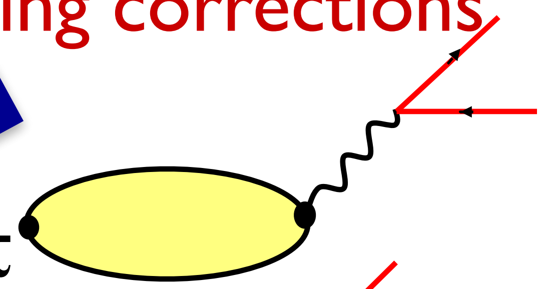
$f_+(0) = 0.9677(27)$ Nf=2+1

0.3%

Electromagnetic and isospin-breaking effects

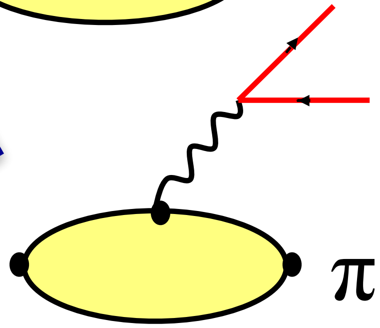
Given the present exper. and theor. (LQCD) accuracy, an important source of uncertainty are long distance electromagnetic and SU(2)-breaking corrections

$$\frac{\Gamma\left(K^+ \to \ell^+ \nu_\ell(\gamma)\right)}{\Gamma\left(\pi^+ \to \ell^+ \nu_\ell(\gamma)\right)} = \left(\frac{|V_{us}|}{|V_{ud}|}\frac{f_K}{f_\pi}\right)^2 \frac{M_{K^+}\left(1 - m_\ell^2/M_{K^+}^2\right)^2}{M_{\pi^+}\left(1 - m_\ell^2/M_{\pi^+}^2\right)^2}\boxed{\left(1 + \delta_{EM} + \delta_{SU(2)}\right)}$$



K/π

$$\Gamma\left(K^{+,0} \to \pi^{0,-} \ell^+ \nu_\ell(\gamma)\right) = \frac{G_F^2 M_{K^{+,0}}^5}{192\pi^3} C_{K^{+,0}}^2 \left|V_{us} f_+^{K^0\pi^-}(0)\right|^2 I_{K\ell}^{(0)} S_{EW} \boxed{\left(1 + \delta_{EM}^{K^{+,0}\ell} + \delta_{SU(2)}^{K^{+,0}\pi}\right)}$$

K  π

For $\Gamma_{Kl2}/\Gamma_{\pi l2}$ At leading order in ChPT both δ_{EM} and $\delta_{SU(2)}$ can be expressed in terms of physical quantities (e.m. pion mass splitting, f_K/f_π, …)

- $\boxed{\delta_{EM} = -0.0069(17)}$ 25% of error due to higher orders ➡ 0.2% on $\Gamma_{Kl2}/\Gamma_{\pi l2}$

  M.Knecht *et al.*, 2000; V.Cirigliano and H.Neufeld, 2011

- $\boxed{\delta_{SU(2)} = \left(\frac{f_{K^+}/f_{\pi^+}}{f_K/f_\pi}\right)^2 - 1 = -0.0044(12)}$ 25% of error due to higher orders ➡ 0.1% on $\Gamma_{Kl2}/\Gamma_{\pi l2}$

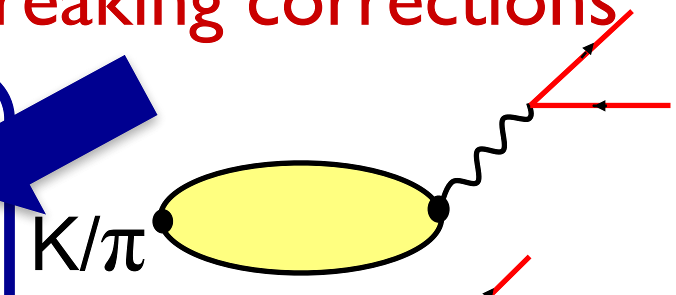
  J.Gasser and H.Leutwyler, 1985; V.Cirigliano and H.Neufeld, 2011

ChPT is not applicable to D and B decays

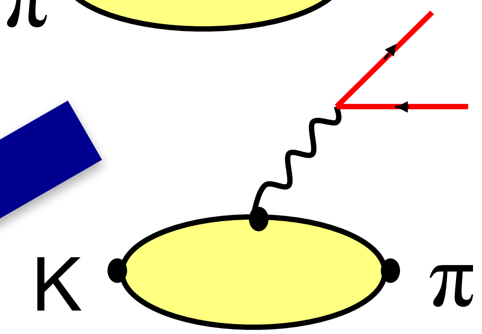
Electromagnetic and isospin-breaking effects

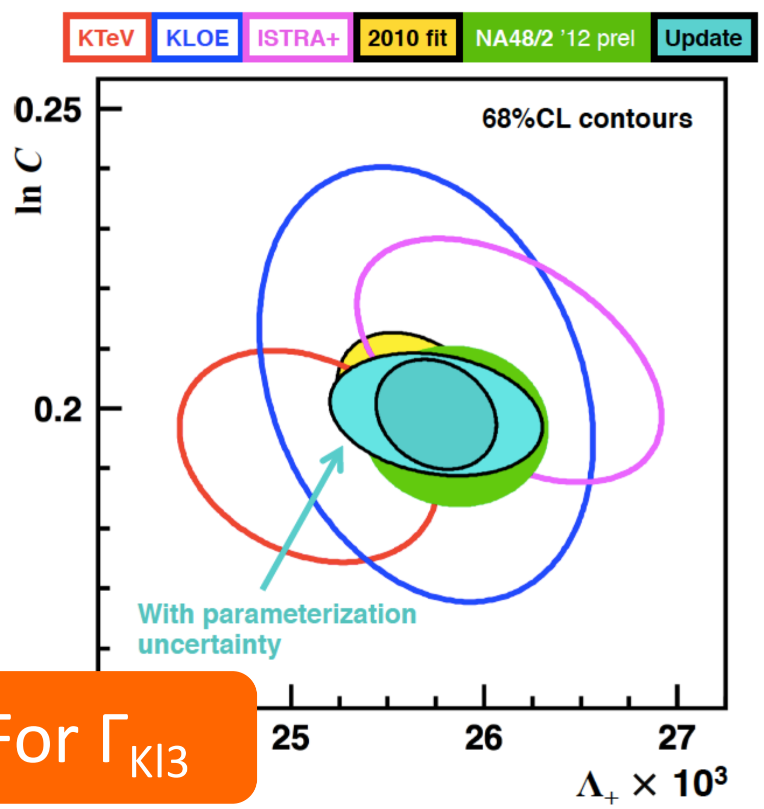
Given the present exper. and theor. (LQCD) accuracy, an important source of uncertainty are long distance electromagnetic and SU(2)-breaking corrections

$$\frac{\Gamma\left(K^+ \to \ell^+ \nu_\ell(\gamma)\right)}{\Gamma\left(\pi^+ \to \ell^+ \nu_\ell(\gamma)\right)} = \left(\frac{|V_{us}|}{|V_{ud}|}\frac{f_K}{f_\pi}\right)^2 \frac{M_{K^+}\left(1 - m_\ell^2/M_{K^+}^2\right)^2}{M_{\pi^+}\left(1 - m_\ell^2/M_{\pi^+}^2\right)^2}\boxed{\left(1 + \delta_{EM} + \delta_{SU(2)}\right)}$$



$$\Gamma\left(K^{+,0} \to \pi^{0,-}\ell^+\nu_\ell(\gamma)\right) = \frac{G_F^2 M_{K^{+,0}}^5}{192\pi^3} C_{K^{+,0}}^2 \left|V_{us} f_+^{K^0\pi^-}(0)\right|^2 I_{K\ell}^{(0)} S_{EW} \boxed{\left(1 + \delta_{EM}^{K^{+,0}\ell} + \delta_{SU(2)}^{K^{+,0}\pi}\right)}$$





For Γ_{Kl3}

Mode	$V_{us} f_+(0)$	% err	Approx contrib to % err: BR	τ	Δ	I
K_{Le3}	0.2163(6)	0.25	0.09	0.20	0.11	0.05
$K_{L\mu3}$	0.2166(6)	0.28	0.15	0.18	0.11	0.06
K_{Se3}	0.2155(13)	0.61	0.60	0.02	0.11	0.05
$K_{e3}^{\pm}$	0.2171(8)	0.36	0.27	0.06	0.22	0.05
$K_{\mu3}^{\pm}$	0.2170(11)	0.51	0.45	0.06	0.22	0.06

M. Moulson, arXiv:1704.04104

Isospin-breaking effects on the lattice

RM123 method

A strategy for Lattice QCD:

The isospin-breaking part of the Lagrangian is treated as a perturbation

Expand in: $m_d - m_u$ + α_{em}

JHEP

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Isospin breaking effects due to the up-down mass difference in lattice QCD

RM123 collaboration

G.M. de Divitiis,[a,b] P. Dimopoulos,[c,d] R. Frezzotti,[a,b] V. Lubicz,[e,f] G. Martinelli,[g,d] R. Petronzio,[a,b] G.C. Rossi,[a,b] F. Sanfilippo,[c,d] S. Simula,[f] N. Tantalo[a,b] and C. Tarantino[e,f]

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Leading isospin breaking effects on the lattice

G. M. de Divitiis,[1,2] R. Frezzotti,[1,2] V. Lubicz,[3,4] G. Martinelli,[5,6] R. Petronzio,[1,2] G. C. Rossi,[1,2] F. Sanfilippo,[7] S. Simula,[4] and N. Tantalo[1,2]

(RM123 Collaboration)

arXiv:1303.4896

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RM123 Collaboration

① The (md-mu) expansion

- Identify the isospin-breaking term in the QCD action

$$S_m = \sum_x \left[m_u \bar{u}u + m_d \bar{d}d \right] = \sum_x \left[\frac{1}{2}(m_u + m_d)(\bar{u}u + \bar{d}d) - \frac{1}{2}(m_d - m_u)(\bar{u}u - \bar{d}d) \right] =$$

$$= \sum_x \left[m_{ud}(\bar{u}u + \bar{d}d) - \Delta m(\bar{u}u - \bar{d}d) \right] = S_0 - \Delta m \hat{S}$$

$\hat{S} = \Sigma_x(\bar{u}u - \bar{d}d)$

- Expand the functional integral in powers of Δm

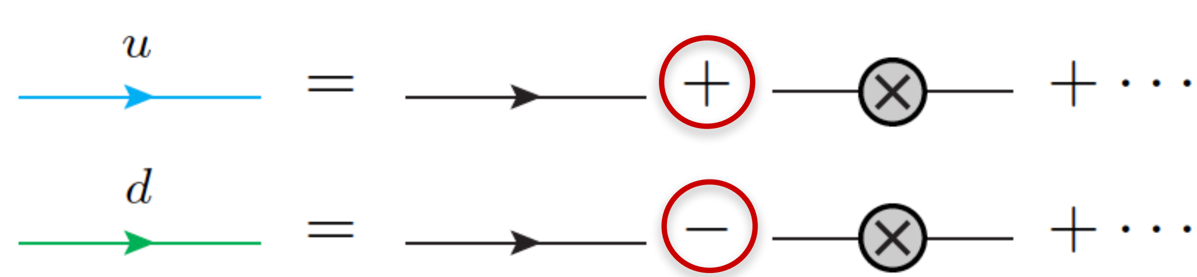
Advantage: factorized out

$$\langle O \rangle = \frac{\int D\phi\, O e^{-S_0 + \Delta m \hat{S}}}{\int D\phi\, e^{-S_0 + \Delta m \hat{S}}} \overset{1st}{\simeq} \frac{\int D\phi\, O e^{-S_0}(1 + \Delta m \hat{S})}{\int D\phi\, e^{-S_0}(1 + \Delta m \hat{S})} \simeq \frac{\langle O \rangle_0 + \Delta m \langle O\hat{S} \rangle_0}{1 + \Delta m \cancel{\langle \hat{S} \rangle_0}} = \langle O \rangle_0 + \Delta m \langle O\hat{S} \rangle_0$$

for isospin symmetry

- At leading order in Δm the corrections only appear in the valence-quark propagators:

(disconnected contractions of ūu and d̄d vanish due to isospin symmetry)



② The QED expansion

- Non-compact QED: the dynamical variable is the gauge potential $A_\mu(x)$ in a fixed covariant gauge ($\nabla^-_\mu A_\mu(x) = 0$)

$$S_{QED} = \frac{1}{2}\sum_{x;\mu\nu} A_\nu(x)\left(-\nabla^-_\mu \nabla^+_\mu\right) A_\nu(x) \overset{(p.b.c.)}{=} \frac{1}{2}\sum_{k;\mu\nu} \tilde{A}^*_\nu(k)\left(2\sin(k_\mu/2)\right)^2 \tilde{A}_\nu(k)$$

- The photon propagator is IR divergent → subtract the zero momentum mode

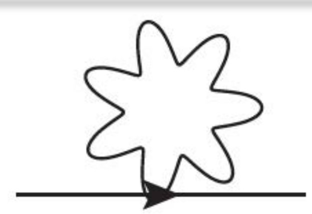
- Full covariant derivatives are defined introducing QED and QCD links:

$$A_\mu(x) \to E_\mu(x) = e^{-iaeA_\mu(x)}$$

$$D^+_\mu q_f(x) = \underbrace{\left[E_\mu(x)\right]^{e_f}}_{\text{QED}} \underbrace{U_\mu(x)}_{\text{QCD}} q_f(x+\hat{\mu}) - q_f(x)$$

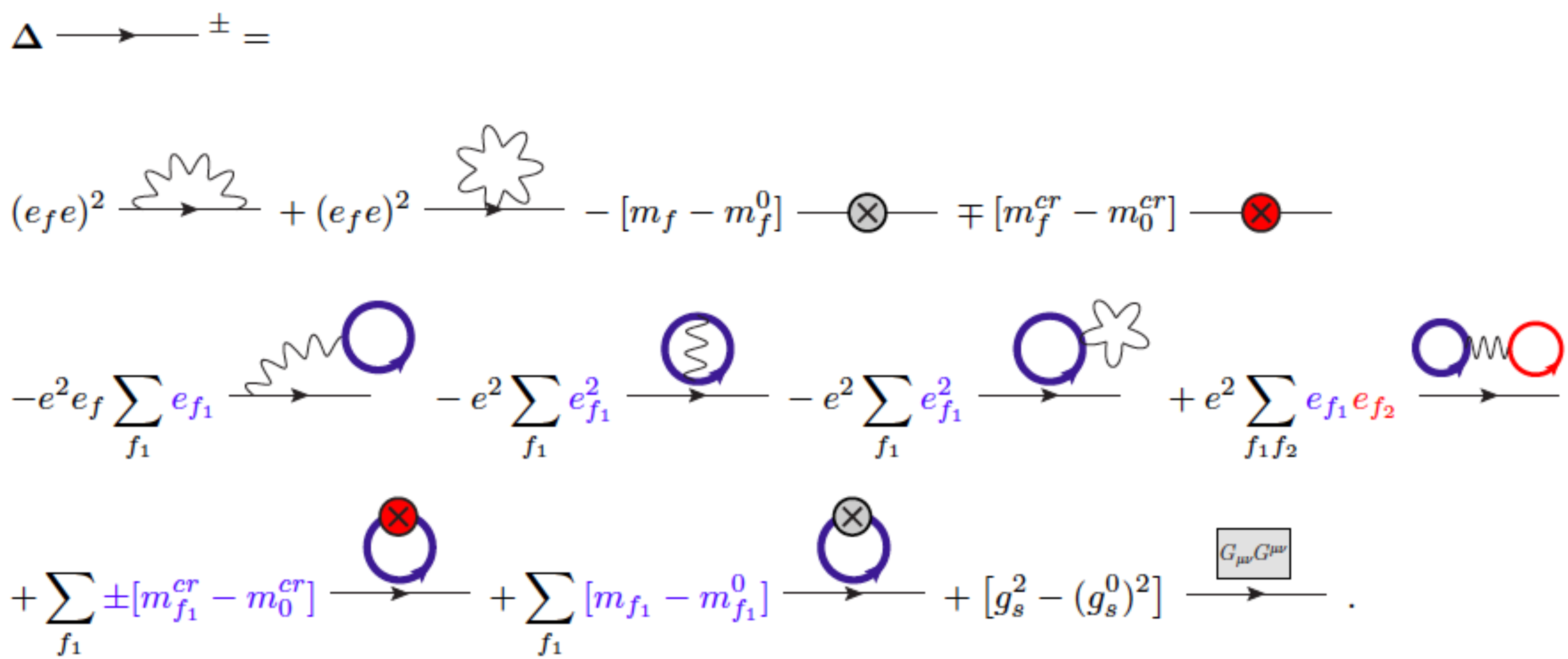
- Since $E_\mu(x) = e^{-ieA_\mu(x)} = 1 - ieA_\mu(x) - 1/2\, e^2 A^2_\mu(x) + \ldots$ the expansion leads to:

$(e_f e)^2$ 

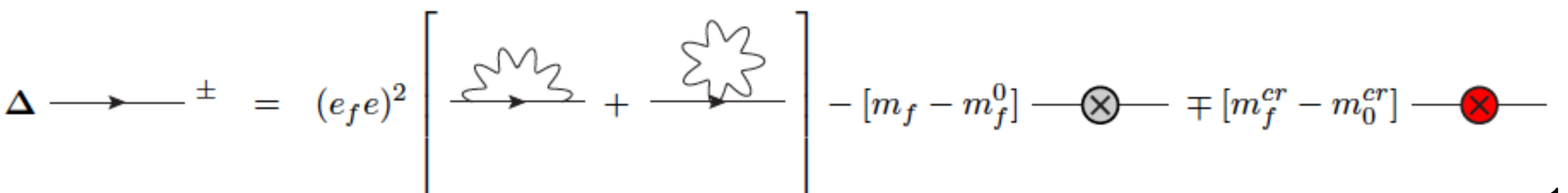
$(e_f e)^2$ 

+ counterterms

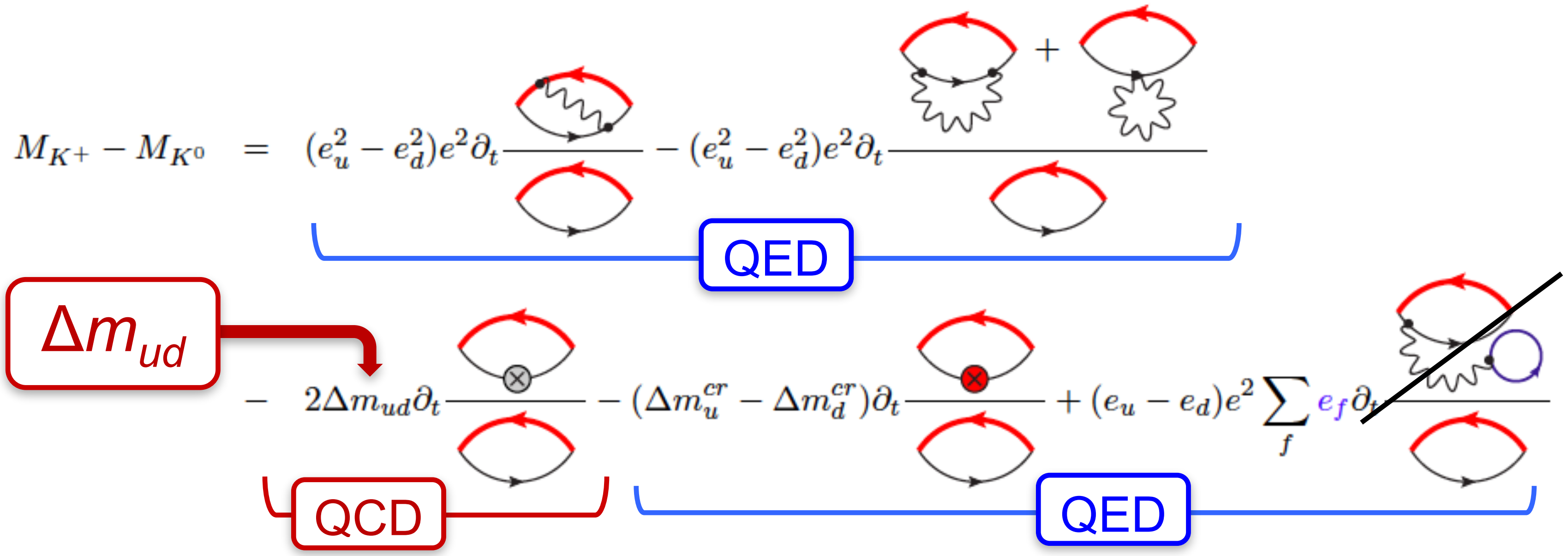
The QED expansion for the quark propagator

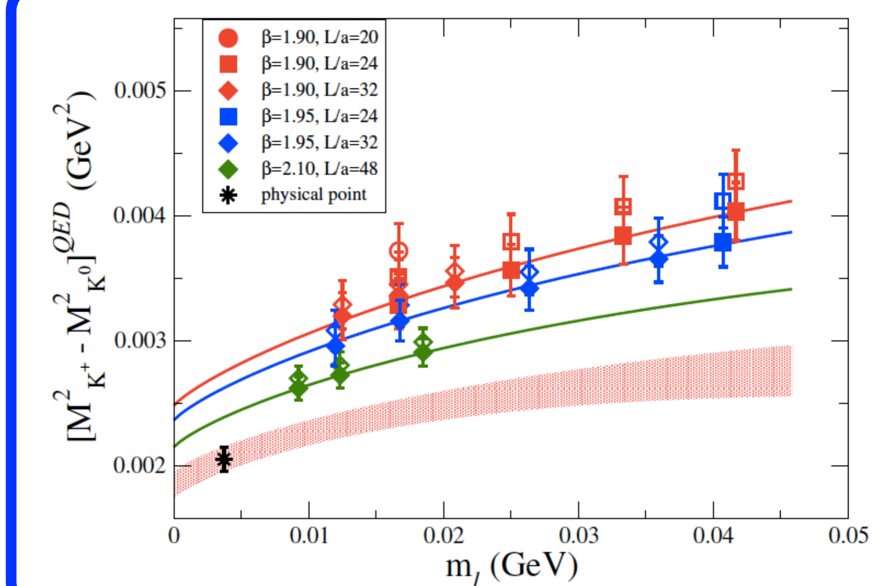


In the electro-quenched approximation:



The down- and up-quark mass difference

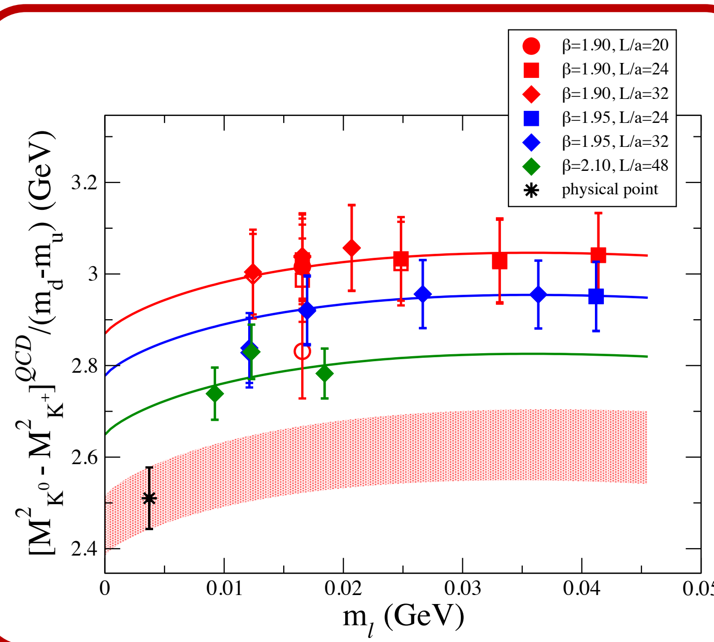


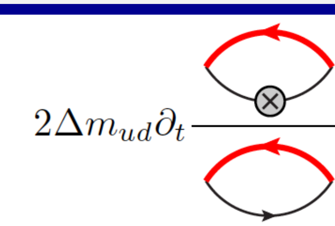


$$\left[M_{K^+} - M_{K^0}\right]^{QED} = 2.07(15)\ \text{MeV}$$

and from the experimental value

$$\left[M_{K^+} - M_{K^0}\right]^{QCD} = -6.00(15)\ \text{MeV}$$





$$\frac{\left[M^2_{K^0} - M^2_{K^+}\right]^{QCD}}{m_d - m_u} = 2.51(18)\ \text{GeV}$$

All masses in MSbar at 2 GeV

electro-quenched approximation

$$m_d - m_u = 2.38(18)\ \text{MeV}$$

$$m_u = 2.50(17)\ \text{MeV}$$
$$m_d = 4.88(20)\ \text{MeV}$$

RM123 Collaboration, arXiv:1704.06561

QED corrections to hadronic decays

QED corrections to hadronic decays

In general the amplitudes are infrared divergent.

On the lattice, a natural infrared cutoff is provided by the finite volume.

But a delicate procedure to remove it is needed.

A method to solve this problem is presented

We consider the leptonic decay of a charged pseudoscalar meson, but the method is general

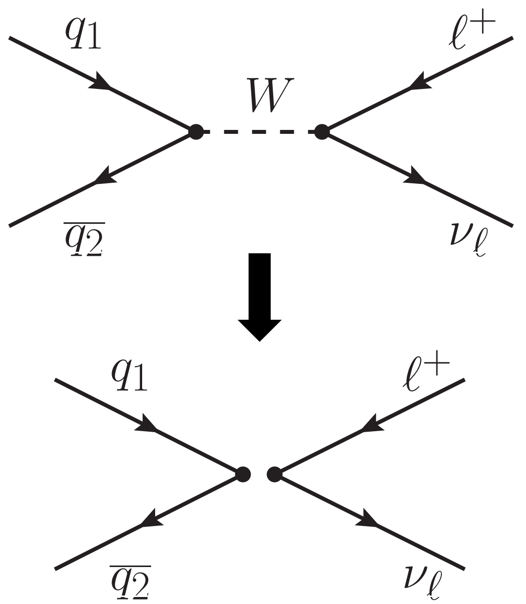
(it can be used for semileptonic decays)

N. Carrasco, V. Lubicz, G. Martinelli, C.T. Sachrajda, N. Tantalo, C. Tarantino, M. Testa

PRD 91 (2015) 074506, arXiv:1502.00257

Leptonic decays at tree level

Since the masses of the pion and kaon are much smaller than M_W we use the effective Hamiltonian



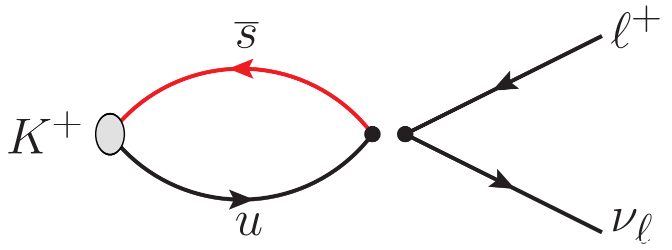
$$H_{eff} = \frac{G_F}{\sqrt{2}} V^*_{q_1 q_2} \left(\overline{q_2}\gamma^\mu (1-\gamma_5) q_1\right)\left(\overline{\nu_\ell}\gamma_\mu (1-\gamma_5)\ell\right)$$

This replacement is necessary in a lattice calculation, since $1/a \ll M_W$

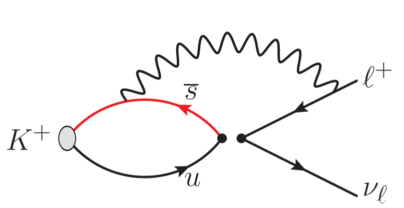
The rate is:

$$\Gamma^{(tree)}_{P^\pm}\left(P^\pm \to \ell^\pm \nu_\ell\right) = \frac{G_F^2}{8\pi}\left|V_{q_1 q_2}\right|^2 \left[f_P^{(0)}\right]^2 M_{P^\pm} m_\ell^2 \left(1 - \frac{m_\ell^2}{M_{P^\pm}^2}\right)^2$$

In the absence of electromagnetism, the non-perturbative QCD effects are contained in a single number, the pseudoscalar decay constant



$$A_P^{(0)} \equiv \left\langle 0 \middle| \overline{q}_2 \gamma_4 \gamma_5 q_1 \middle| P^{(0)} \right\rangle = f_P^{(0)} M_P^{(0)}$$



In the presence of electromagnetism it is not even possible to give a physical definition of f_P

J. Gasser and G.R.S. Zarnauskas, PLB 693 (2010) 122

Leptonic decays at O(α): matching

- When including the O(α) corrections, the UV contributions in the effective theory are different from those in the Standard Model:

➡ A MATCHING BETWEEN THE TWO THEORIES IS REQUIRED

$$H_{eff} = \frac{G_F}{\sqrt{2}} V^*_{q_1 q_2} \left(1 + \frac{\alpha}{\pi} \ln\left(\frac{M_Z}{M_W}\right)\right) \underline{\left(\overline{q_2} \gamma^\mu \left(1-\gamma^5\right) q_1\right)\left(\overline{\nu_\ell} \gamma_\mu \left(1-\gamma^5\right) \ell\right)}$$

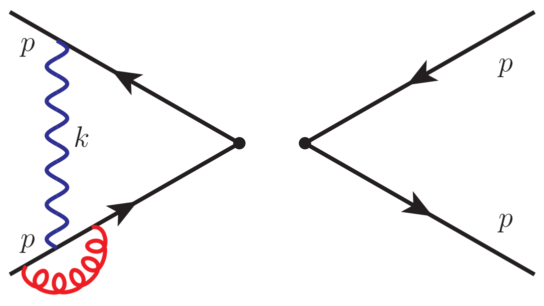
$$\frac{1}{k^2} = \frac{1}{k^2 - M_W^2} + \frac{M_W^2}{M_W^2 - k^2}\frac{1}{k^2}$$

A. Sirlin, NPB 196 (1982) 83; E. Braaten and C.S. Li, PRD 42 (1990) 3888

W-regularization

This factor provides the matching between the SM and the local Fermi theory

- The W-regularization cannot be implemented directly in lattice simulations since: $1/a \ll M_W$
- The lattice 4-fermion operator is renormalized in RI'-MOM and then perturbatively matched to the one in the W-regularization



$$O_1^{\text{W-reg}}\left(M_W\right) = Z^{\text{W-RI'}}\left(\alpha_s\left(M_W\right), \alpha\right) U^{\text{RI'}}\left(M_W, \mu, \alpha\right) O_1^{\text{RI'}}(\mu)$$

At first order in α and up to terms of $O\left(\alpha\alpha_s\left(M_W\right)\right)$, the result is

$$O_1^{\text{W-reg}}\left(M_W\right) = \left\{1 + \frac{\alpha}{4\pi}\left[2\left(1 - \frac{\alpha_s(\mu)}{4\pi}\right)\right] \ln\left(\frac{M_W^2}{\mu^2}\right) + C^{\text{W-RI'}}\right\} O_1^{\text{RI'}}(\mu)$$

$$C^{\text{W-RI'}} = -5.7825$$

RI'-MOM in QCD+QED

For Wilson and Twisted-Mass fermions

$$O_1^{\mathrm{RI'}}(\mu) = \sum_{i=1}^{5} (Z_O)_{1i}(a\mu) O_i^{\mathrm{bare}}(a)$$

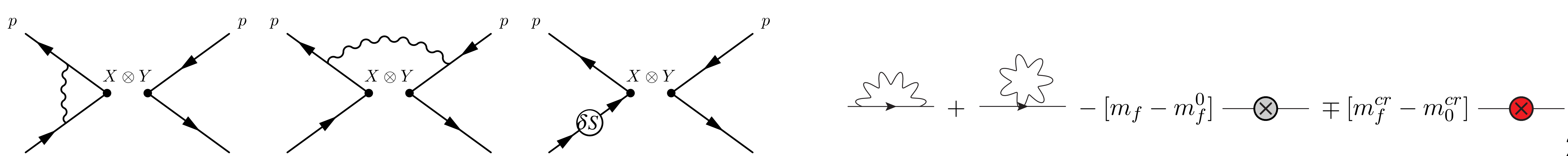
$$\begin{bmatrix} O_1 = (\bar{d}\gamma^\mu(1-\gamma^5)u)(\bar{\nu}_\ell\gamma_\mu(1-\gamma^5)\ell), \\ O_2 = (\bar{d}\gamma^\mu(1+\gamma^5)u)(\bar{\nu}_\ell\gamma_\mu(1-\gamma^5)\ell), \\ O_3 = (\bar{d}(1-\gamma^5)u)(\bar{\nu}_\ell(1+\gamma^5)\ell), \\ O_4 = (\bar{d}(1+\gamma^5)u)(\bar{\nu}_\ell(1+\gamma^5)\ell), \\ O_5 = (\bar{d}\sigma^{\mu\nu}(1+\gamma^5)u)(\bar{\nu}_\ell\sigma_{\mu\nu}(1+\gamma^5)\ell) \end{bmatrix}$$

The weak 4-fermion operator is renormalized non-perturbatively on the lattice to all orders in α_s and up to first order in α by imposing the RI'-MOM condition:

$$Z_{\Gamma_O}(a\mu)\Gamma_O(ap)\Big|_{p^2=\mu^2} = \hat{1} \qquad \begin{bmatrix} Z_{\Gamma_O} = Z_O \prod_f Z_f^{-1/2} \\ \Gamma_O = \mathrm{Tr}[\Lambda_O P_O] \end{bmatrix}$$

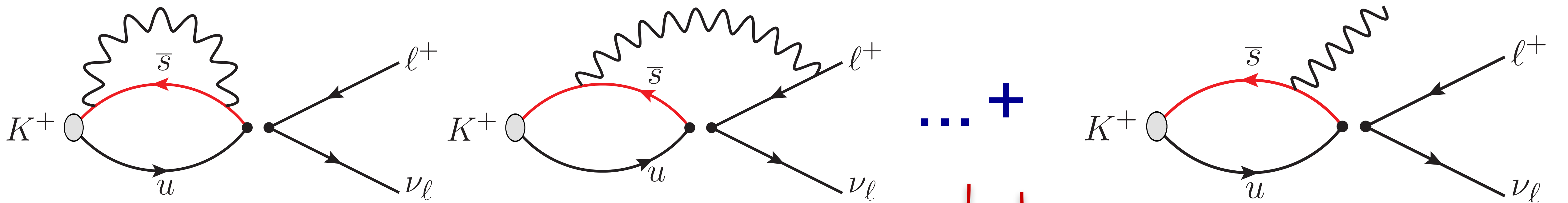
By decomposing the RCs as $Z_O = \left(\hat{1} + \frac{\alpha}{4\pi}\delta Z_O\right) Z_O^{(0)}$, it follows:

$$\delta Z_O = -Z_{\Gamma_O}^{(0)}\delta\Gamma_O + \frac{1}{2}\left(\delta Z_{q_1} + \delta Z_{q_2} + \delta Z_\ell\right)$$



Leptonic decays at O(α): the IR problem

At O(α), Γ_0 contains **infrared divergences**. One has to consider:



Γ_0 $\Gamma_1(\Delta E)$

$$\boxed{\Gamma\left(P^{\pm}_{\ell 2}\right) = \Gamma\left(P^{\pm} \to \ell^{\pm}\nu_{\ell}\right) + \Gamma\left(P^{\pm} \to \ell^{\pm}\nu_{\ell}\gamma(\Delta E)\right) \equiv \Gamma_0 + \Gamma_1(\Delta E)}$$

with $0 \le E_\gamma \le \Delta E$. The sum is infrared finite

F. Bloch and A. Nordsieck, PR 52 (1937) 54

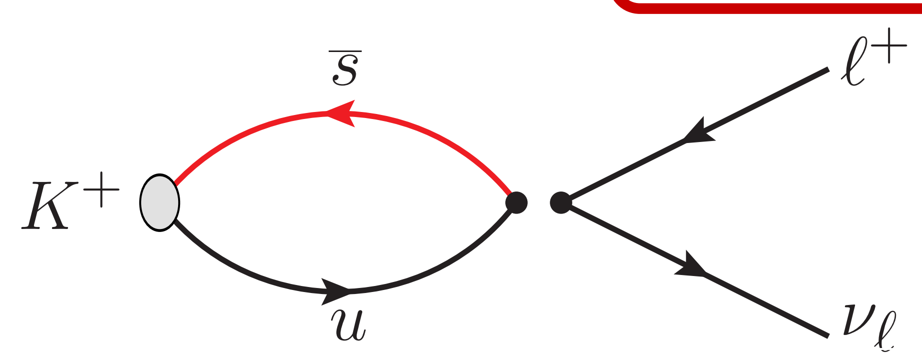
Both Γ_0 and $\Gamma_1(\Delta E)$ can be evaluated in a fully non-perturbative way in lattice simulations.

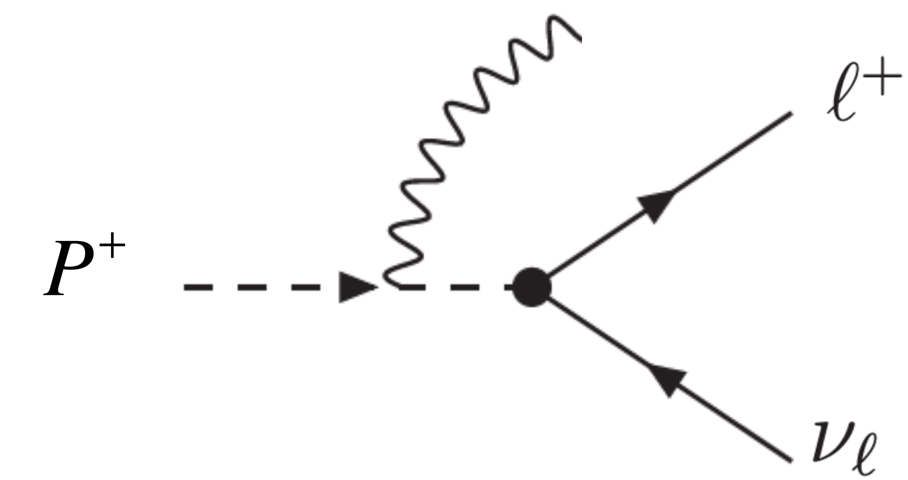
However, as a first approach to the the problem, we have considered a different strategy, applicable to both pion and kaon decays

The strategy

At first we considered sufficiently soft photons (i.e. they do not resolve the internal structure of the pion (kaon)), so that the pointlike approximation can be used to compute $\Gamma_1(\Delta E)$ in perturbation theory, but hard with respect to the experimental resolution

$$\Delta E \ll \Lambda_{\mathrm{QCD}}$$





➡ A cut-off $\boxed{\Delta E \sim O(20\ \mathrm{MeV})}$ appears to be appropriate, both experimentally and theoretically

F. Ambrosino *et al.*, KLOE Collaboration, PLB 632 (2006) 76; EPJC 64 (2009) 627; 65 (2010) 703(E)

J. Bijnens *et al.*, NPB 396 (1993) 81; V.Cirigliano and I.Rosell, JHEP 0710 (2007) 005

The strategy

$$\Gamma\left(P^{\pm}_{\ell 2}\right) = \Gamma_0 + \Gamma_1^{\mathrm{pt}}(\Delta E) \qquad \Delta E \sim O(20\ \mathrm{MeV})$$

$\Gamma\left(P^{\pm} \to \ell^{\pm}\nu_{\ell}\right)$

Montecarlo simulation
Lattice QCD

$\Gamma\left(P^{\pm} \to \ell^{\pm}\nu_{\ell}\gamma(\Delta E)\right)$

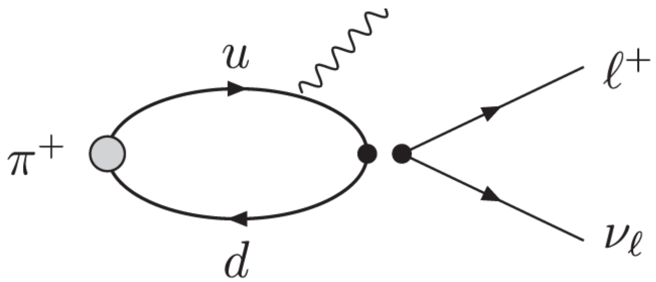
Perturbation theory
with pointlike pion

In order to ensure the cancellation of IR divergences with good numerical precision, we rewrite:

$$\Gamma\left(P^{\pm}_{\ell 2}\right) = \underbrace{\left(\Gamma_0 - \Gamma_0^{pt}\right)} + \underbrace{\left(\Gamma_0^{pt} + \Gamma_1^{pt}(\Delta E)\right)}$$

Γ_0^{pt} is an unphysical quantity

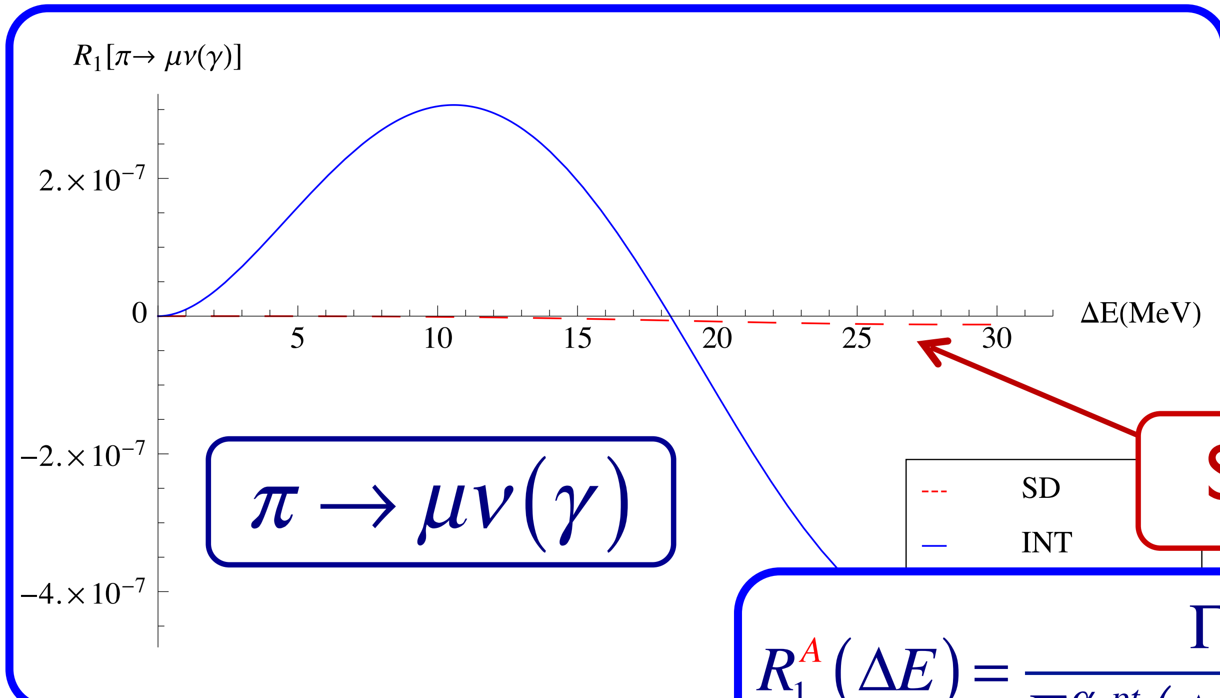
Estimates of SD contributions to $\Gamma_1(\Delta E)$

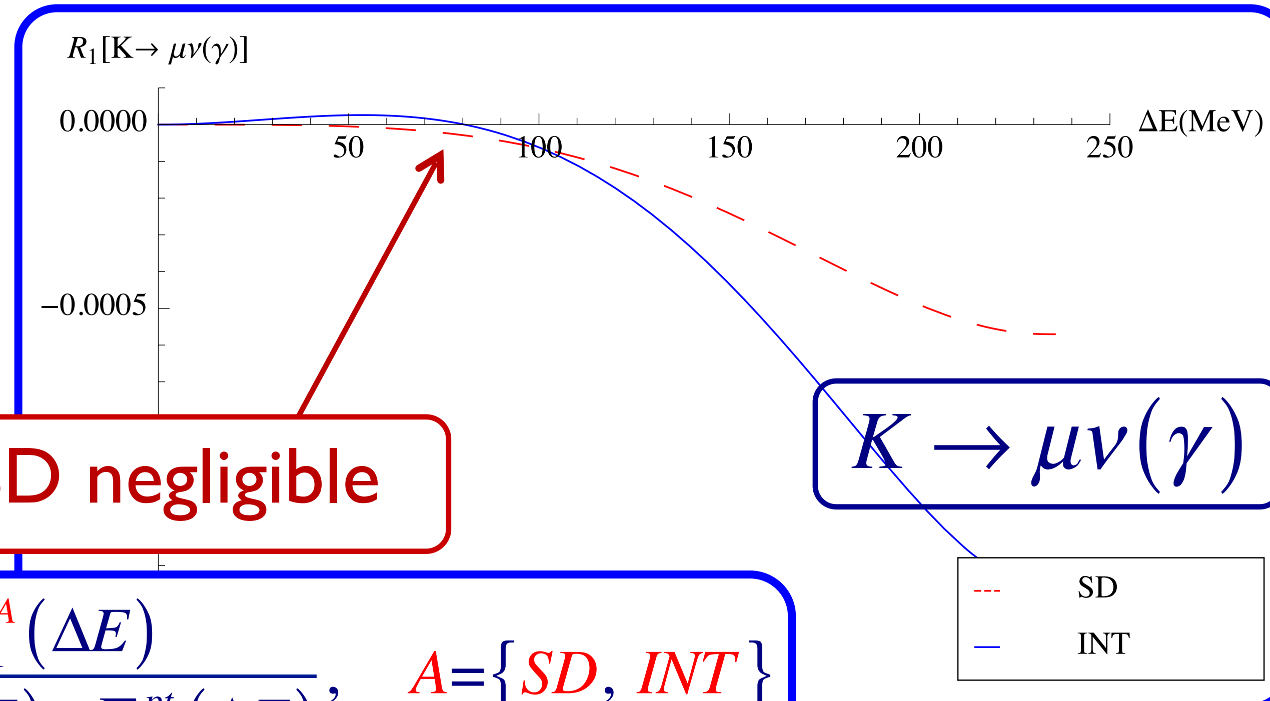


$$H^{\nu}(k,p_{\pi}) = \varepsilon_{\mu}^{*}(k)\int d^4x\, e^{ikx}\, T\,\langle 0|\, j^{\mu}(x) J_W^{\nu}(0)\, |\pi(p_{\pi})\rangle \qquad k^2=0,\ \varepsilon^{*}\cdot k=0$$

expressed in terms of two hadronic form-factors $F_{V,A}$

- For π and K decays, the size of the neglected structure-dependent contributions can be estimated, as a function of ΔE, using ChPT at O(p⁴)





SD negligible

$$R_1^{A}(\Delta E) = \frac{\Gamma_1^{A}(\Delta E)}{\Gamma_0^{\alpha,pt}(\Delta E) + \Gamma_1^{pt}(\Delta E)}, \qquad A = \{SD,\ INT\}$$

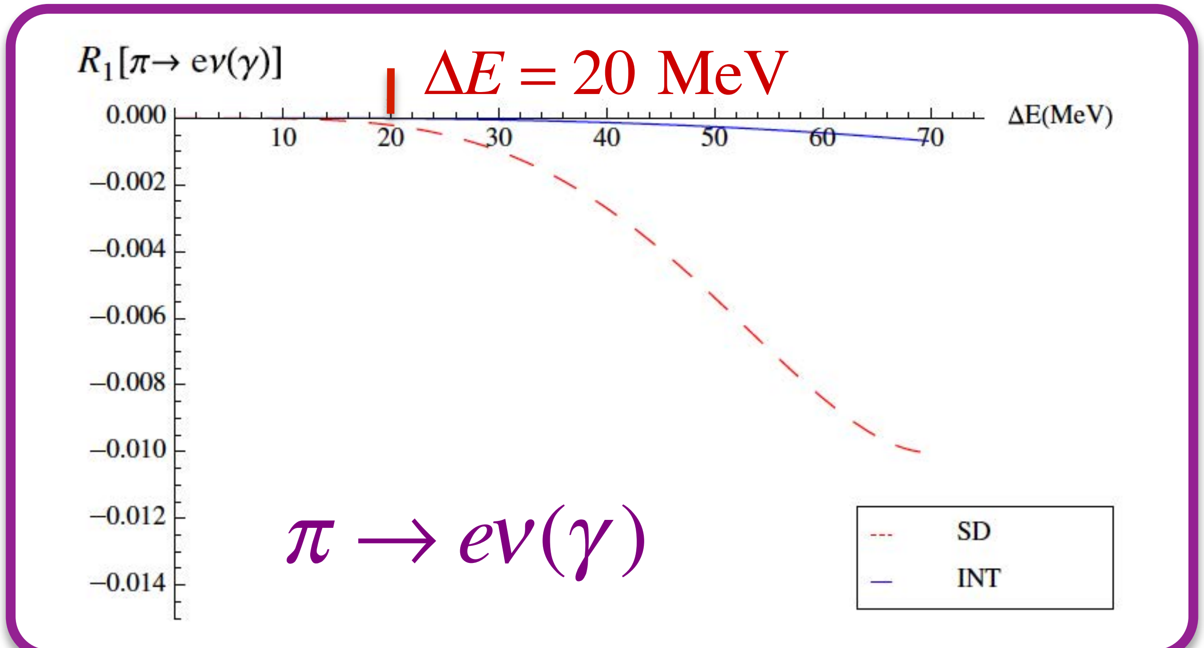
- For B decays, due to the presence of the small scale, $m_{B^*} - m_B \simeq 45\ \text{MeV}$, the radiation of a soft photon may still induce sizeable SD effects and a full non-perturbative calculation of real emission is likely necessary

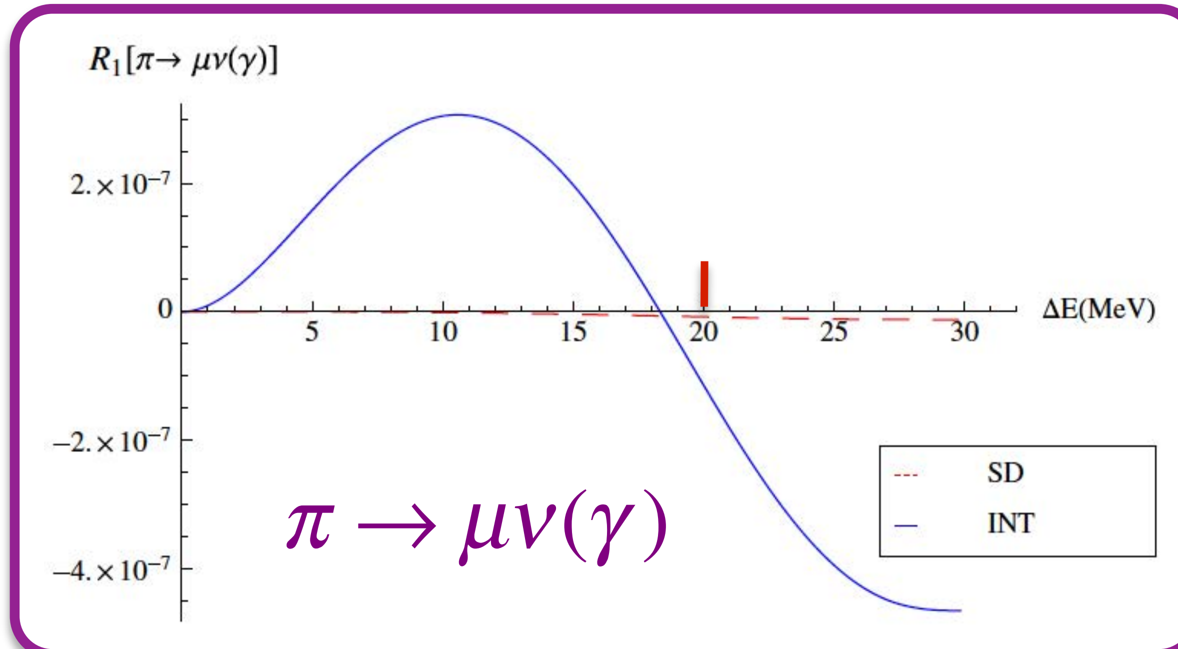
D. Becirevic, B. Haas, E. Kou, PLB 681 (2009) 257

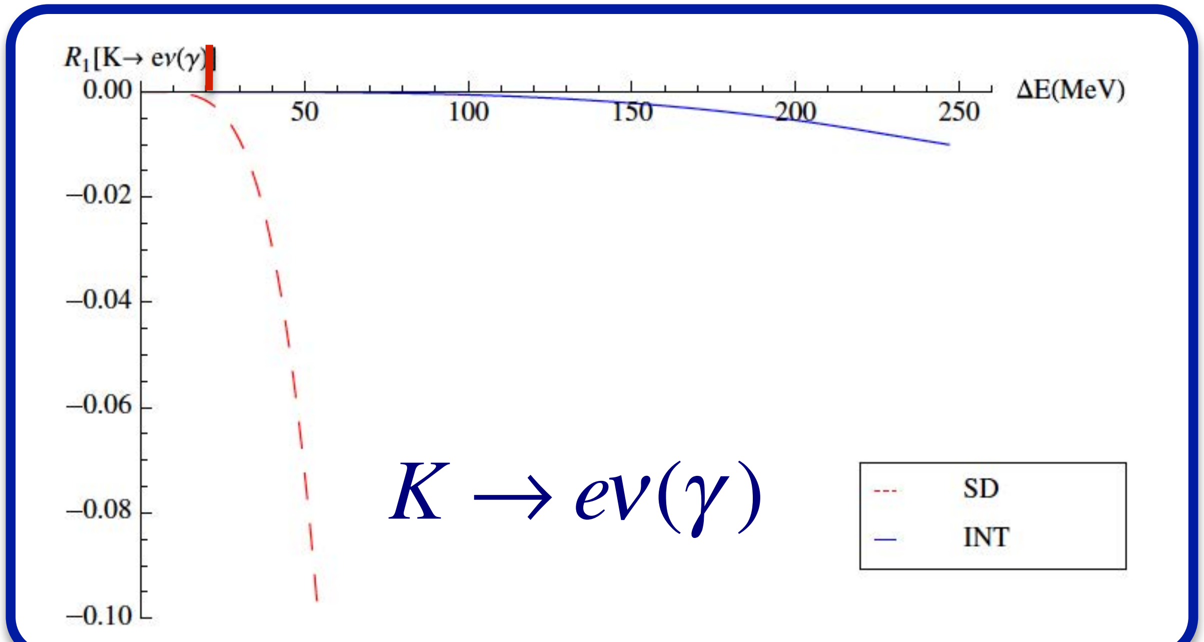
$$R_1^A(\Delta E) = \frac{\Gamma_1^A(\Delta E)}{\Gamma_0^{\alpha,\mathrm{pt}} + \Gamma_1^{\mathrm{pt}}(\Delta E)} \quad , \quad \mathrm{A} = \{\mathrm{SD}, \mathrm{INT}\}$$

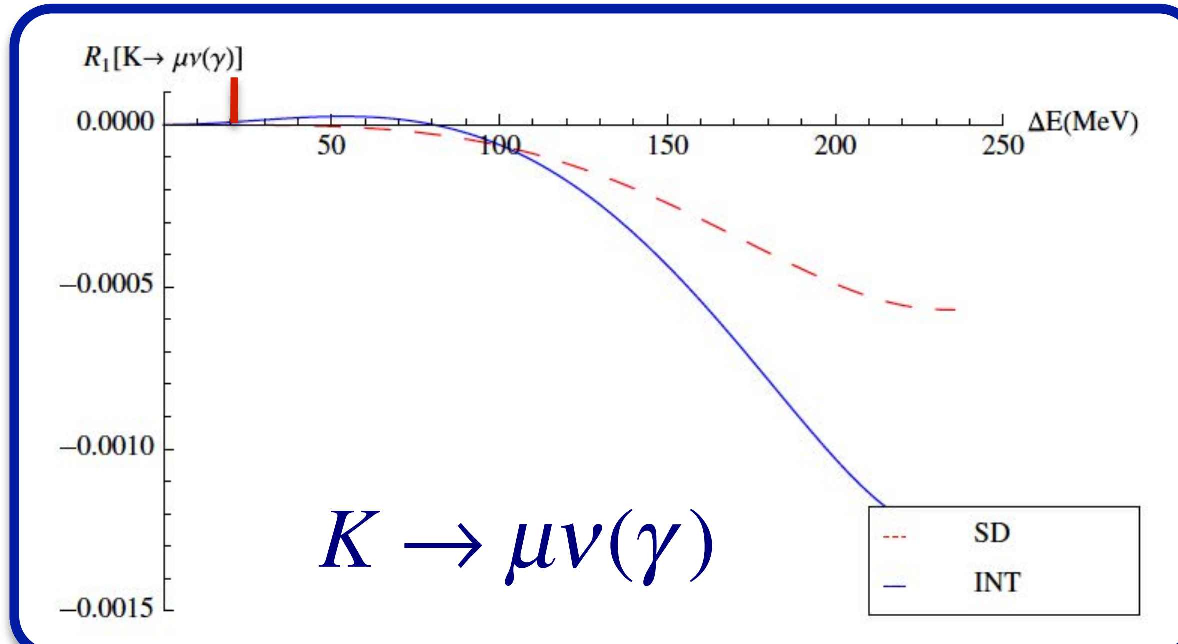
SD = structure dependent

INT = interference









- Interference contributions are negligible in all the decays
- Structure-dependent contributions can be sizable for $K \to e\nu(\gamma)$ but they are negligible for $\Delta E < 20$ MeV (which is experimentally accessible)

The strategy: a substantial improvement

In order to ensure the cancellation of IR divergences with good numerical precision, we rewrite:

$$\Gamma\left(P_{\ell 2}^{\pm}\right) = \left(\Gamma_0 - \Gamma_0^{pt}\right) + \left(\Gamma_0^{pt} + \Gamma_1^{pt}(\Delta E)\right) + \left(\Gamma_1(\Delta E) - \Gamma_1^{pt}(\Delta E)\right)$$

PRACE

- Both the quantities Γ_0 and $\Gamma_1(\Delta E)$ are now evaluated on the lattice

$$\frac{d^2\Gamma_1}{dx_\gamma dx_\ell} = \frac{\alpha_{em}\Gamma^{(tree)}}{4\pi}\left\{\frac{d^2\Gamma_{pt}}{dx_\gamma dx_\ell} + \frac{d^2\Gamma_{SD}}{dx_\gamma dx_\ell} + \frac{d^2\Gamma_{INT}}{dx_\gamma dx_\ell}\right\}$$

$$x_\gamma = \frac{2p\cdot k}{m_P^2} \qquad x_\ell = \frac{2p\cdot p_\ell - m_\ell^2}{m_P^2}$$

- The contribution $\Gamma_1 - \Gamma_1^{pt} = \Gamma_{SD} + \Gamma_{INT}$ can be computed in the infinite-volume limit requiring the knowledge of the structure dependent form factors $F_{A,V}(x_\gamma)$ and of f_P

The strategy

$$\Gamma\left(P_{\ell 2}^{\pm}\right)=\left(\Gamma_0-\Gamma_0^{pt}\right)+\left(\Gamma_0^{pt}+\Gamma_1^{pt}(\Delta E)\right)$$
$$+\left(\Gamma_1(\Delta E)-\Gamma_1^{pt}(\Delta E)\right)$$

- The contributions from soft virtual photon to Γ_0 and Γ_0^{pt} in the first term are exactly the same and the IR divergence cancels in the difference $\Gamma_0-\Gamma_0^{\mathrm{pt}}$.
- The sum $\Gamma_0^{\mathrm{pt}}+\Gamma_1^{\mathrm{pt}}(\Delta E)$ in the second term is IR finite since it is a physically well defined quantity. This term can be thus calculated in perturbation theory with a different IR cutoff.
- The difference $\Gamma_1-\Gamma_1^{pt}$ in the third term is also IR finite.
- The three terms are also separately gauge invariant.

$$\Delta\Gamma_0(L)=\Gamma_0(L)-\Gamma_0^{pt}(L) \qquad \Gamma^{pt}(\Delta E)=\lim_{m_\gamma\to 0}\left[\Gamma_0^{pt}(m_\gamma)+\Gamma_1^{pt}(\Delta E,m_\gamma)\right]$$

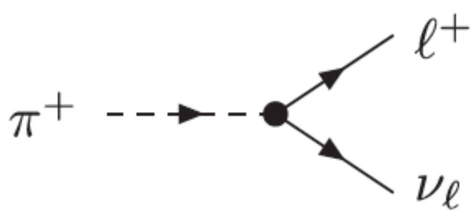
Calculation of $\Gamma^{\text{pt}}(\Delta E) = \Gamma_0^{\text{pt}} + \Gamma_1^{\text{pt}}(\Delta E)$

- $\Gamma^{\text{pt}}(\Delta E)$ is calculated in perturbation theory with a pointlike pion

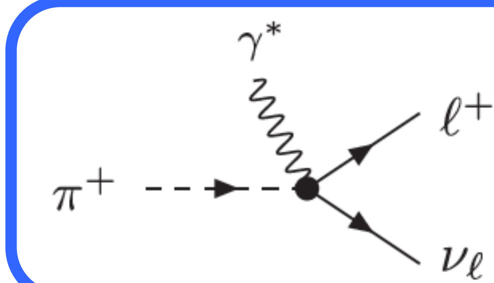
$$\mathcal{L}_{\pi-\ell-\nu_\ell} = iG_F f_\pi V^*_{ud}\{(\partial_\mu - ieA_\mu)\pi\}\left\{\bar{\psi}_{\nu_\ell}\frac{1+\gamma_5}{2}\gamma^\mu\psi_\ell\right\}$$

+ QED

for π and l^+

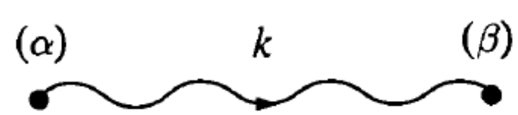


$$= -iG_F f_\pi V^*_{ud}\, p^\mu_\pi \frac{1+\gamma^5}{2}\gamma_\mu$$



$$= ie\, G_F f_\pi V^*_{ud}\, g^{\mu\nu} \frac{1+\gamma^5}{2}\gamma_\mu$$

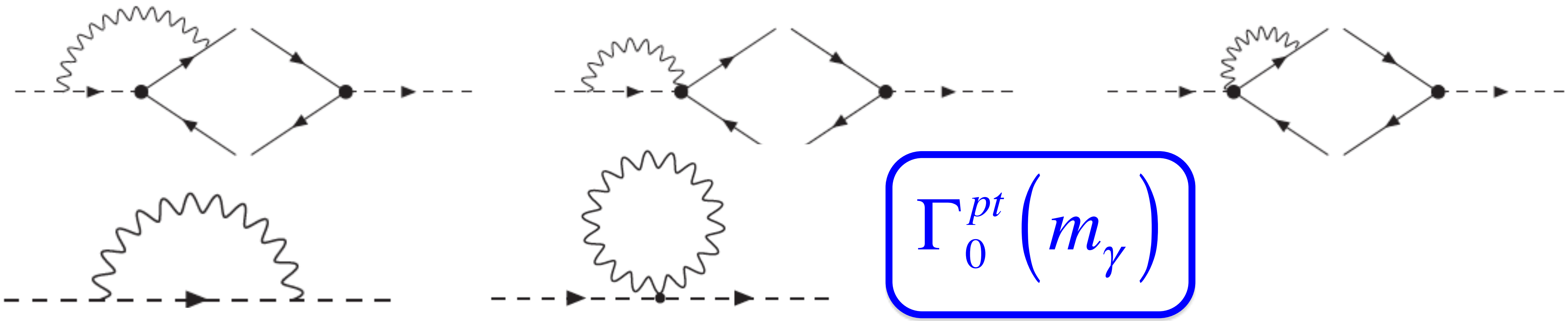
- UV divergences in Γ_0^{pt} are regularized with the W-regularization



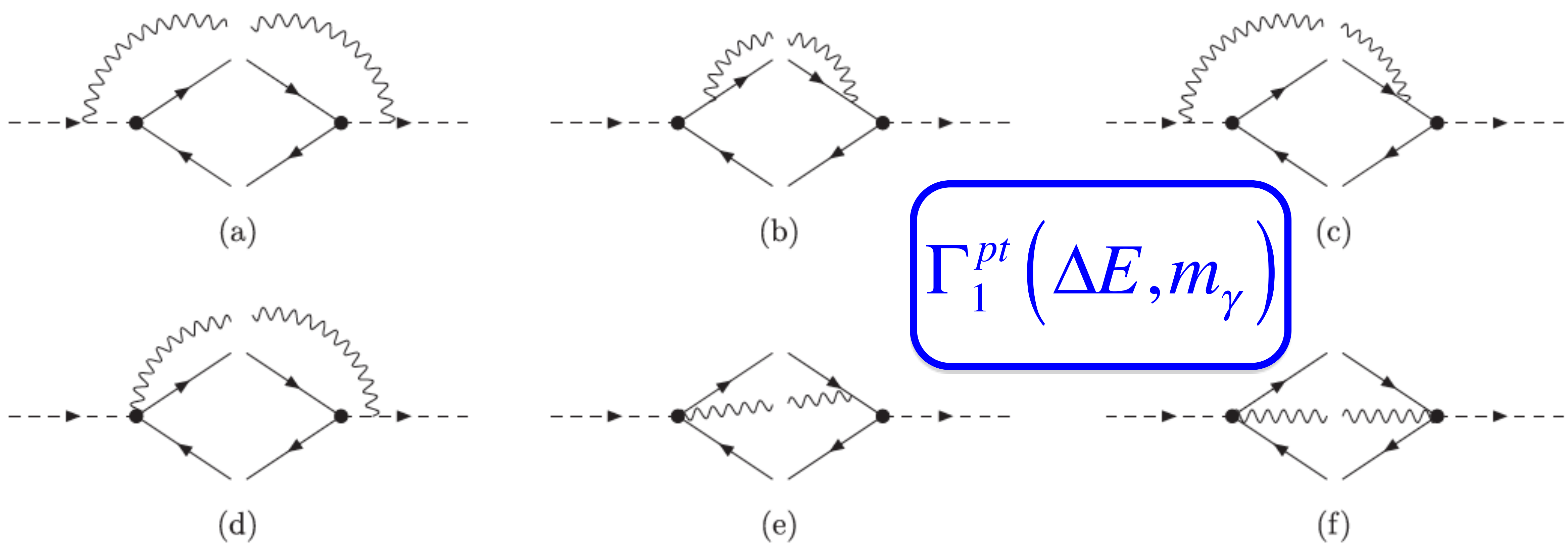
$$\frac{1}{k^2} \rightarrow \frac{M_W^2}{M_W^2 - k^2}\frac{1}{k^2}$$

- IR divergences are regularized with the a photon mass

$$\Gamma^{pt}(\Delta E) = \lim_{m_\gamma \to 0}\left[\Gamma_0^{pt}(m_\gamma) + \Gamma_1^{pt}(\Delta E, m_\gamma)\right]$$



$$\boxed{\Gamma_0^{pt}(m_\gamma)}$$



$$\boxed{\Gamma_1^{pt}(\Delta E, m_\gamma)}$$

$$\Gamma^{pt}(\Delta E) = \lim_{m_\gamma \to 0}\left[\Gamma_0^{pt}(m_\gamma) + \Gamma_1^{pt}(\Delta E, m_\gamma)\right]$$

The result is:

$$\Gamma(\Delta E) = \quad \Gamma_0^{\rm tree} \times \left(1 + \frac{\alpha}{4\pi}\left\{3\log\left(\frac{m_\pi^2}{M_W^2}\right) + \log\left(r_\ell^2\right) - 4\log(r_E^2) + \frac{2-10r_\ell^2}{1-r_\ell^2}\log(r_\ell^2)\right.\right.$$

$$-2\frac{1+r_\ell^2}{1-r_\ell^2}\log(r_E^2)\log(r_\ell^2) - 4\frac{1+r_\ell^2}{1-r_\ell^2}\,{\rm Li}_2(1-r_\ell^2) - 3$$

$$+\left[\frac{3+r_E^2-6r_\ell^2+4r_E(-1+r_\ell^2)}{(1-r_\ell^2)^2}\log(1-r_E) + \frac{r_E(4-r_E-4r_\ell^2)}{(1-r_\ell^2)^2}\log(r_\ell^2)\right.$$

$$\left.\left.\left. - \frac{r_E(-22+3r_E+28r_\ell^2)}{2(1-r_\ell^2)^2} - 4\frac{1+r_\ell^2}{1-r_\ell^2}\,{\rm Li}_2(r_E)\right]\right\}\right)$$

$$\boxed{r_E = 2\Delta E/m_\pi}$$

$$\boxed{r_\ell = m_\ell/m_\pi}$$

NEW

IMPORTANT CHECK: For $\Delta E=\Delta E_{MAX}$ the well known result for the total rate as in S. M. Berman, PRL 1 (1958) 468 and T. Kinoshita, PRL 2 (1959) 477 is reproduced

$\Delta\Gamma_0(L) = \Gamma_0(L) - \Gamma_0^{pt}(L)$

- $\Delta\Gamma_0(L)$ is the first term in the master formula

$$\Gamma(P_{\ell 2}^{\pm}) = \lim_{L\to\infty}\left[\Gamma_0(L) - \Gamma_0^{pt}(L)\right] + \left[\Gamma^{pt} + \Gamma_{SD} + \Gamma_{INT}\right](\Delta E)$$

Montecarlo simulation Lattice QCD

PHYSICAL REVIEW LETTERS **120,** 072001 (2018)

1

arXiv:1711.06537

First Lattice Calculation of the QED Corrections to Leptonic Decay Rates

D. Giusti,[1] V. Lubicz,[1] G. Martinelli,[2] C.T. Sachrajda,[3] F. Sanfilippo,[4] S. Simula,[4] N. Tantalo,[5] and C. Tarantino[1]

Light-meson leptonic decay rates in lattice QCD+QED

arXiv:1904.08731

M. Di Carlo,[1] D. Giusti,[2] V. Lubicz,[2] G. Martinelli,[1] C.T. Sachrajda,[3] F. Sanfilippo,[4] S. Simula,[4] and N. Tantalo[5]

2

Perturbation theory with pointlike pion in finite volume

PHYSICAL REVIEW D **95,** 034504 (2017)

Finite-volume QED corrections to decay amplitudes in lattice QCD

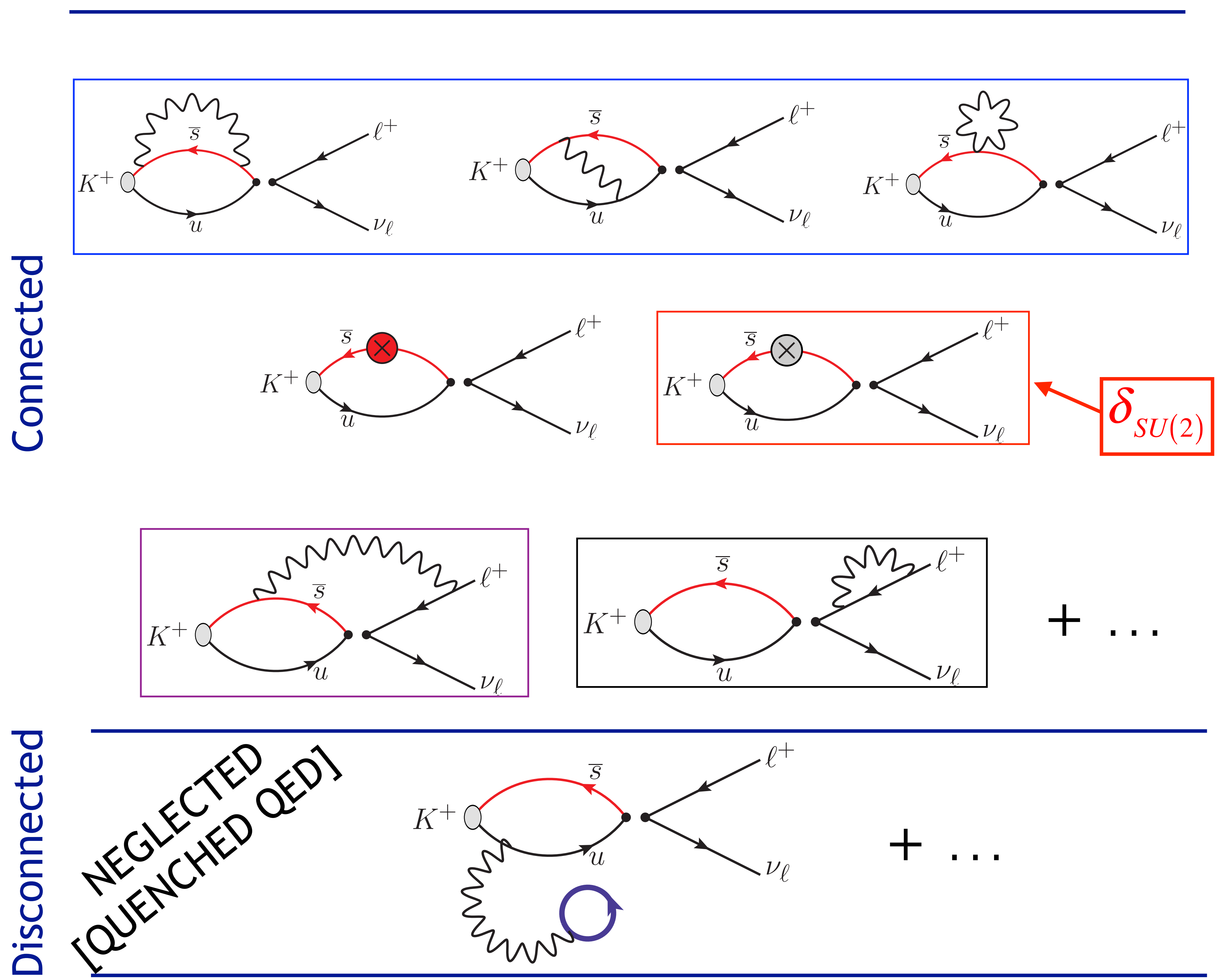
V. Lubicz,[1] G. Martinelli,[2,3] C. T. Sachrajda,[4] F. Sanfilippo,[4] S. Simula,[5] and N. Tantalo[6]

- IR divergences (Log(L)) cancel in the difference.

  Also 1/L corrections are universal and cancel in the difference

Lattice calculation of $\Gamma_0(L)$ at $O(\alpha)$

The Feynman diagrams at O(α) can be divided in 3 classes



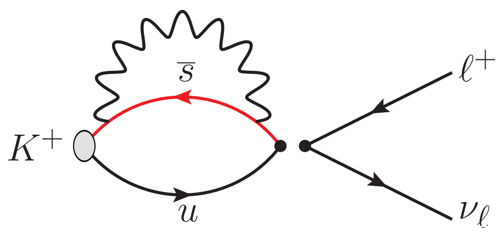
① The photon is attached to quark lines

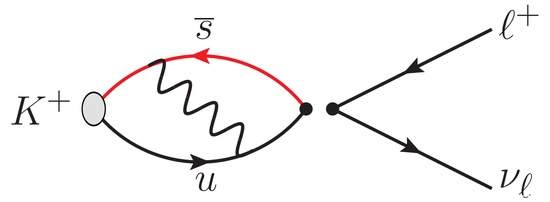
② The photon connects one quark and one charged lepton line

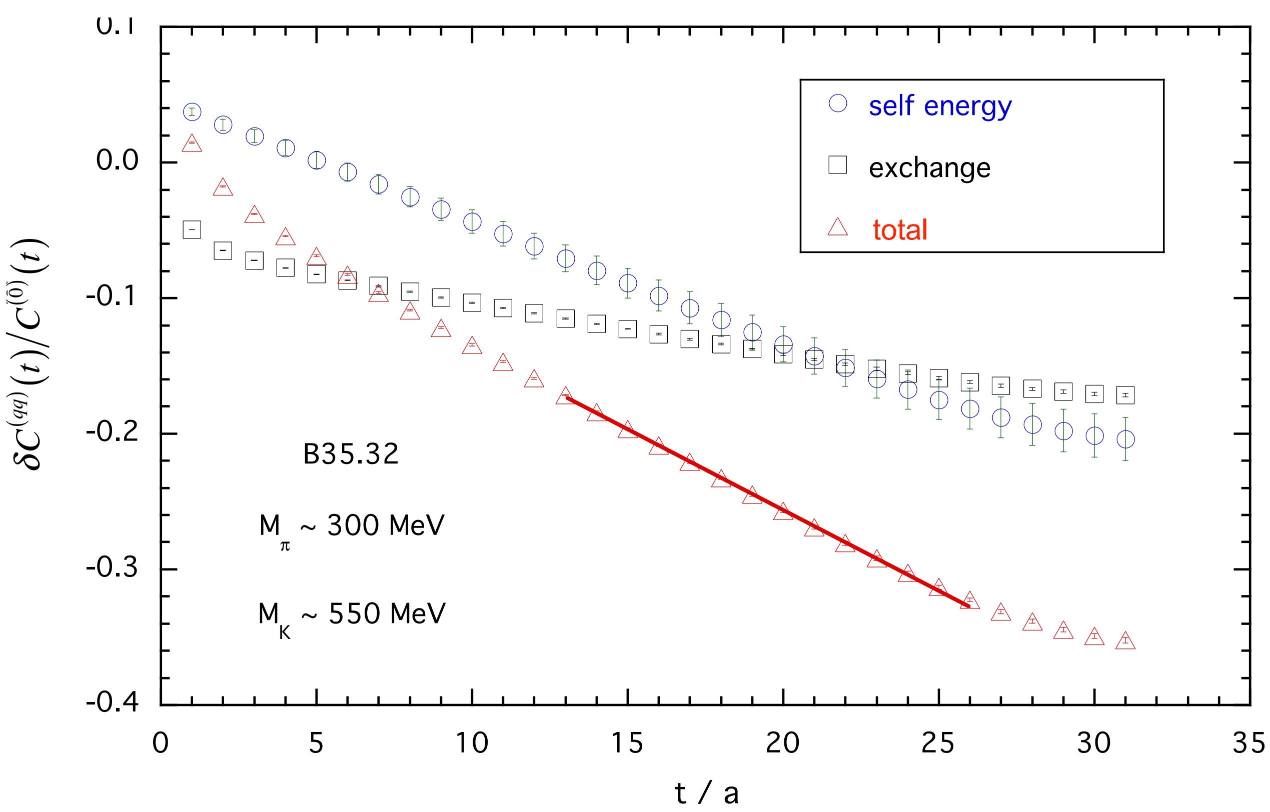
③ Leptonic wave function renormalization. It cancels in

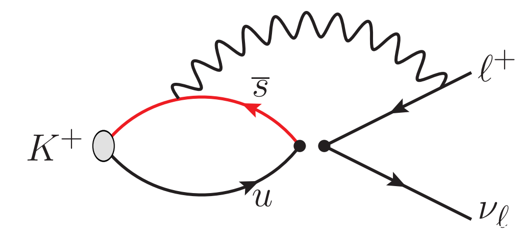
$$\Gamma_0(L) - \Gamma_0^{\mathrm{pt}}(L)$$

Lattice calculation of $\Gamma_0(L)$ at $O(\alpha)$









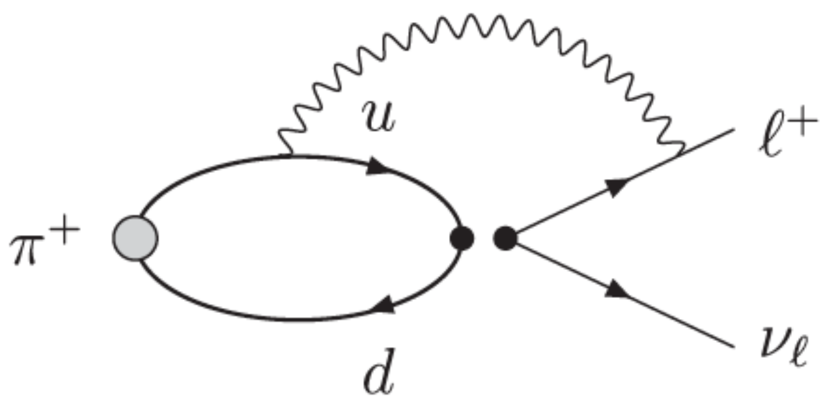
$$\frac{\delta C^{(qq)}(t)}{C^{(0)}(t)} \xrightarrow[t>>a]{} \frac{\delta\left[Z_P A_P^{(qq)}\right]}{Z_P^{(0)} A_P^{(0)}} - \frac{\delta M_P}{M_P^{(0)}}\left(1+M_P^{(0)}t\right)$$

$$\frac{\delta C^{(q\ell)}(t)}{C^{(0)}(t)} \xrightarrow[t>>a]{} \frac{\delta A_P^{(q\ell)}}{A_P^{(0)}}$$

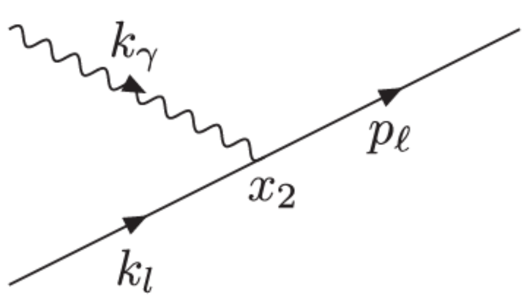
Lattice calculation of $\Gamma_0(L)$ at $O(\alpha)$

- A technical but important point:

$$\delta C^{(q\ell)}(t)_{\alpha\beta} = -\int d^3\vec{x}\, d^4x_1\, d^4x_2 \langle 0| T\left\{ J_W^{\nu}(0)\, j_\mu(x_1)\, \phi^\dagger(\vec{x},-t) \right\} |0\rangle$$
$$\times\, \Delta(x_1,x_2) \left(\gamma_\nu (1-\gamma^5) S(0,x_2) \gamma_\mu \right)_{\alpha\beta} e^{E_\ell t_2 - i\, \vec{p}_\ell \cdot \vec{x}_2}$$



We need to ensure that the t_2 integration converges as $t_2 \to \infty$. The large t_2 behavior is given by the factor $\exp\left[\left(E_\ell - \omega_\ell - \omega_\gamma\right) t_2\right]$



$$E_\ell = \sqrt{\vec{p}_\ell^{\,2} + m_\ell^2} \qquad \omega_\ell = \sqrt{\vec{k}_\ell^{\,2} + m_\ell^2} \qquad \omega_\gamma = \sqrt{\vec{k}_\gamma^{\,2} + m_\gamma^2} \qquad \vec{k}_\ell + \vec{k}_\gamma = \vec{p}_\ell$$

$$\Rightarrow \quad \boxed{\left(\omega_\ell + \omega_\gamma\right)_{\min} = \sqrt{\left(m_\ell^2 + m_\gamma^2\right) + \vec{p}_\ell^{\,2}} > E_\ell}$$

The integral is convergent and the continuation from Minkowski to Euclidean space can be performed (same if we set m_γ=0 but remove the photon zero mode in FV).

CONDITIONS: - mass gap between the decaying particle and the intermediate states
- absence of lighter intermediate states

Calculation of $\Gamma_0^{\rm pt}(L)$

$$\Gamma(\Delta E) = \boxed{\lim_{V\to\infty}\left(\Gamma_0(L) - \Gamma_0^{\rm pt}(L)\right)} + \left[\Gamma^{pt} + \Gamma_{SD} + \Gamma_{INT}\right](\Delta E)$$

- $\Gamma_0^{\rm pt}(L)$ is calculated in perturbation theory with a pointlike pion

- IR divergences are regularized by the finite volume (same of $\Gamma_0(L)$)

$$\boxed{\int d^3q\ldots \;\to\; \left(\frac{2\pi}{L}\right)^3 \sum_{\vec q} \ldots} \qquad \text{with} \quad \begin{cases} \vec q = \dfrac{2\pi}{L}(n_x, n_y, n_z) \\ \vec q \neq (0,0,0) \end{cases}$$

- The result has the form:

$$\boxed{\Gamma_0^{\rm pt}(L) = \tilde C_0(r_\ell)\log(m_P L) + C_0(r_\ell) + \frac{C_1(r_\ell)}{m_P L} + O\!\left(\frac{1}{L^2}\right)} \qquad r_\ell = m_\ell / m_P$$

Calculation of $\Gamma_0^{\mathrm{pt}}(L)$

$$\Gamma_0^{\mathrm{pt}}(L) = \tilde{C}_0(r_\ell)\log(m_P L) + C_0(r_\ell) + \frac{C_1(r_\ell)}{m_P L} + O\left(\frac{1}{L^2}\right)$$

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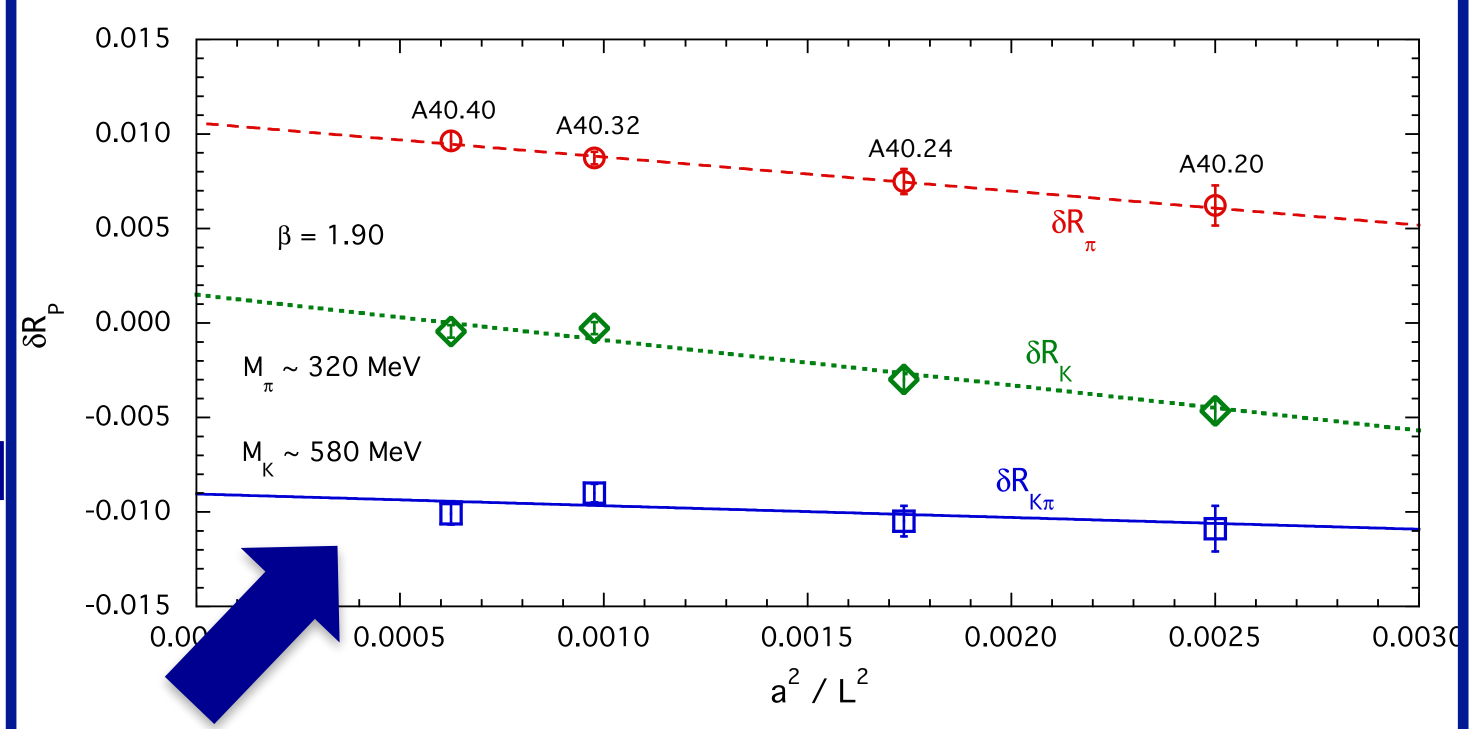
PRD 95 (2017) 034504

- Both the leading [$\log(m_P L)$] and next-to-leading [$O(1/L)$] volume dependence cancel in $\Gamma_0(L) - \Gamma_0^{\mathrm{pt}}(L)$.
- The remaining, structure dependent, $O(1/L^2)$ finite volume effects are milder and can be fitted from the lattice data evaluated at different volumes.



Details of the lattice simulation

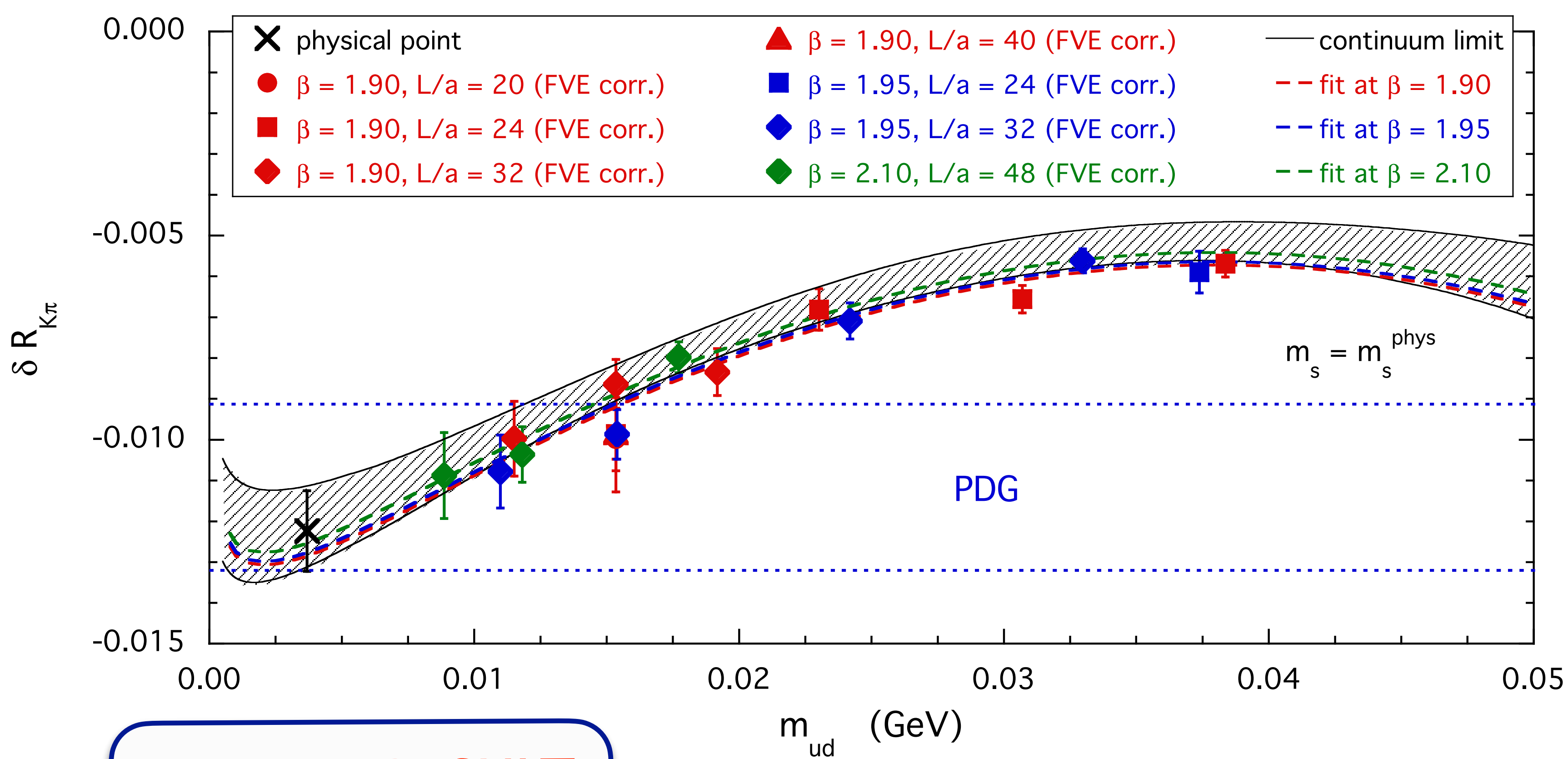
We have used the gauge field configurations generated by ETMC, European Twisted Mass Collaboration, in the pure isosymmetric QCD theory with Nf=2+1+1 dynamical quarks

ensemble	β	V/a^4	$a\mu_{ud}$	$a\mu_\sigma$	$a\mu_\delta$	N_{cf}	$a\mu_s$	M_π (MeV)	M_K (MeV)
$A40.40$	1.90	$40^3 \cdot 80$	0.0040	0.15	0.19	100	0.02363	317(12)	576(22)
$A30.32$		$32^3 \cdot 64$	0.0030			150		275(10)	568(22)
$A40.32$			0.0040			100		316(12)	578(22)
$A50.32$			0.0050			150		350(13)	586(22)
$A40.24$		$24^3 \cdot 48$	0.0040			150		322(13)	582(23)
$A60.24$			0.0060			150		386(15)	599(23)
$A80.24$			0.0080			150		442(17)	618(14)
$A100.24$			0.0100			150		495(19)	639(24)
$A40.20$		$20^3 \cdot 48$	0.0040			150		330(13)	586(23)
$B25.32$	1.95	$32^3 \cdot 64$	0.0025	0.135	0.170	150	0.02094	259 (9)	546(19)
$B35.32$			0.0035			150		302(10)	555(19)
$B55.32$			0.0055			150		375(13)	578(20)
$B75.32$			0.0075			80		436(15)	599(21)
$B85.24$		$24^3 \cdot 48$	0.0085			150		468(16)	613(21)
$D15.48$	2.10	$48^3 \cdot 96$	0.0015	0.1200	0.1385	100	0.01612	223 (6)	529(14)
$D20.48$			0.0020			100		256 (7)	535(14)
$D30.48$			0.0030			100		312 (8)	550(14)

- Gluon action: Iwasaki
- Quark action: twisted mass at maximal twist (automatically O(a) improved)
  OS for s and c valence quarks
- Scale setting: $f_\pi^{(0)} = 130.41(20)$ MeV

European Twisted Mass Collaboration

Leptonic decays at O(α): RESULTS



$$\delta R_{K\pi} = C_0 + C_\chi \log(m_{ud}) + C_1 m_{ud} + C_2 m_{ud}^2 + Da^2$$
$$+\frac{K_2}{L^2}\left[\frac{1}{M_K^2} - \frac{1}{M_\pi^2}\right] + \frac{K_2^\mu}{L^2}\left[\frac{1}{\left(E_\mu^K\right)^2} - \frac{1}{\left(E_\mu^\pi\right)^2}\right]$$
$$+\delta\Gamma^{pt}\left(\Delta E_\gamma^{\max,K}\right) - \delta\Gamma^{pt}\left(\Delta E_\gamma^{\max,\pi}\right)$$

LATTICE RESULT

$$\delta R_{K^\pm\pi^\pm} = -0.0126(10)_{stat}(2)_{input}(5)_{chir}(5)_{FVE}(4)_{disc}(6)_{qQED}$$
$$= -0.0126(14)$$

ChPT

$$\delta R_K - \delta R_\pi = -0.0112(21)$$

V.Cirigliano and H.Neufeld, PLB 700 (2011) 7

$$\left|\frac{V_{us}}{V_{ud}}\right| \frac{f_K^{(0)}}{f_\pi^{(0)}} = 0.27683(29)_{\exp}(20)_{th}$$

$$\left|\frac{V_{us}}{V_{ud}}\right| = 0.23135(24)_{\exp}(39)_{th}$$

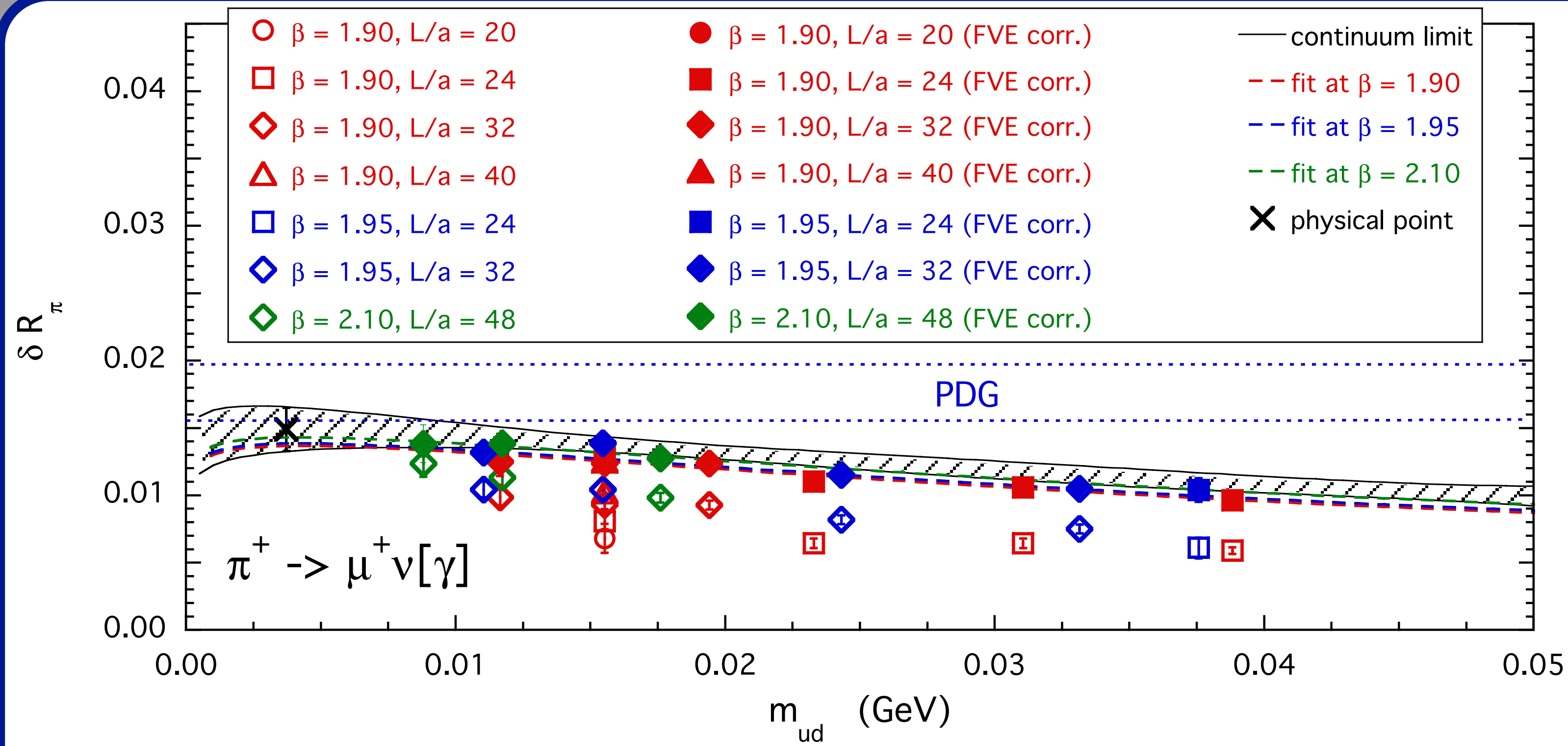
FLAG(2019) N_f=2+1+1 $\frac{f_K^{(0)}}{f_\pi^{(0)}} = 1.1966(18)$

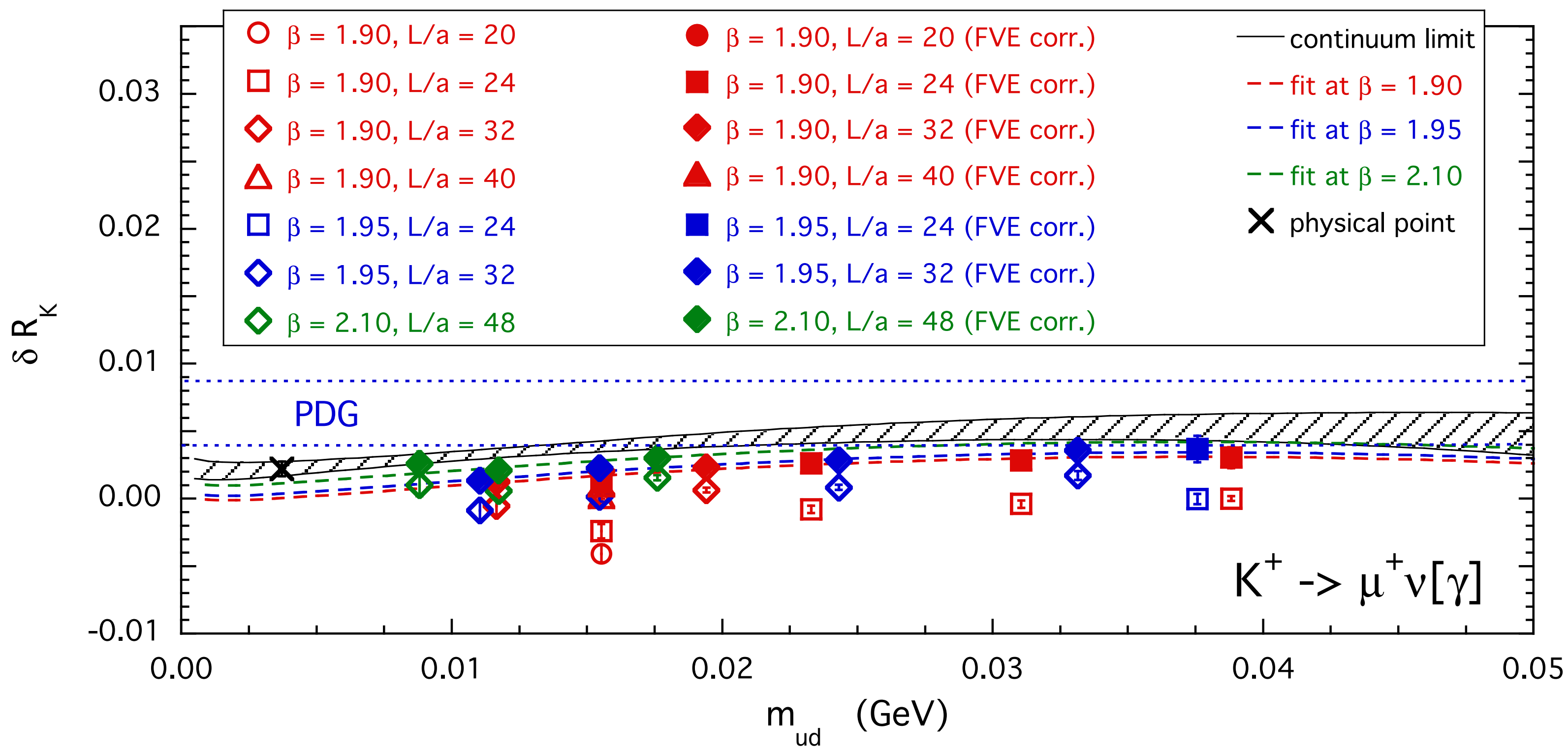
$|V_{ud}|$ from

$|V_{us}| = 0.22538(46)$ Hardy and Towner, 2016

$|V_{us}| = 0.22526(46)$ Seng *et al.*, 2018

Leptonic decays at O(α): RESULTS





$$\Gamma\left(P^{\pm} \to \mu^{\pm}\nu_{\mu}[\gamma]\right) = \Gamma^{(tree)}\left[1+\delta R_{P^{\pm}}\right]$$

$$\boxed{\delta R_{\pi^{\pm}} = 0.0153(16)_{stat}(10)_{syst} = 0.0153(19)}$$

ChPT/PDG $\quad \delta \mathrm{R}_{\pi} = 0.0176(21)$

$$\boxed{\delta R_{K^{\pm}} = 0.0024(6)_{stat}(8)_{syst} = 0.0024(10)}$$

ChPT/PDG $\quad \delta \mathrm{R}_{\mathrm{K}} = 0.0064(24)$

V.Cirigliano, H.Neufeld; PLB 700 (2011) 7

$$f_K^{(0)}|V_{us}| = 35.23(4)_{exp}(2)_{th} \text{ MeV}$$

$$f_{\pi}^{(0)}|V_{ud}| = 127.28(2)_{exp}(12)_{th} \text{ MeV}$$

$f_K^{(0)} = 156.11(21)$ MeV

FLAG(2019) N_f=2+1+1

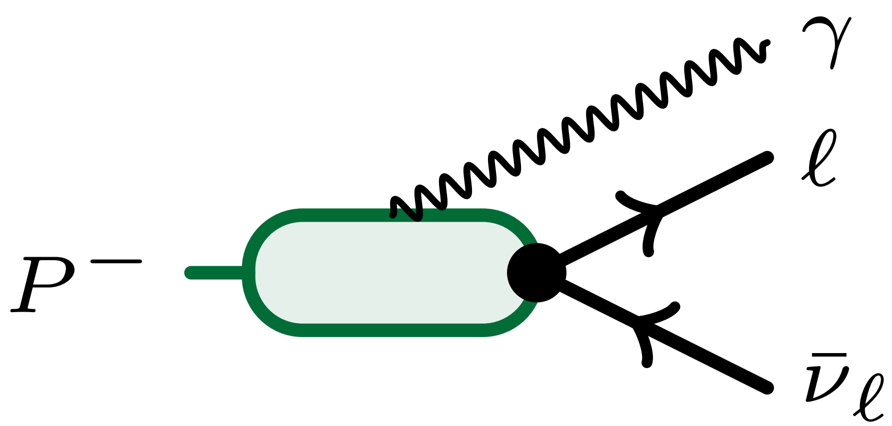
$$\boxed{|V_{us}| = 0.22567(42)}$$

$|V_{us}| = 0.2253(7)$ PDG

$|V_{ud}|$ cannot be predicted (scale setting: $f_{\pi}^{(0)}$)

Real photon emission amplitude

NEW



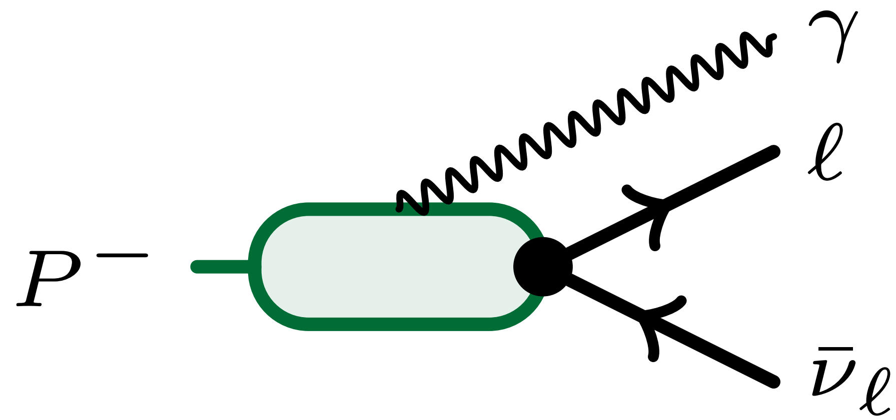
$$\boxed{H_W^{\alpha r}(k,p) = \epsilon_\mu^r(k)\, H_W^{\alpha\mu}(k,p) = \epsilon_\mu^r(k) \int d^4y\, e^{ik\cdot y}\, \mathrm{T}\langle 0|j_W^\alpha(0) j_{em}^\mu(y)|P(\boldsymbol{p})\rangle}$$

$$H_W^{\alpha r}(k,p) = \epsilon_\mu^r(k)\Bigg\{ \underbrace{H_1 \left[k^2 g^{\mu\alpha} - k^\mu k^\alpha\right] + H_2 \left[(p\cdot k - k^2)k^\mu - k^2(p-k)^\mu\right](p-k)^\alpha - i\frac{F_V}{m_P}\varepsilon^{\mu\alpha\gamma\beta}k_\gamma p_\beta + \frac{F_A}{m_P}\left[(p\cdot k - k^2)g^{\mu\alpha} - (p-k)^\mu k^\alpha\right]}_{H_{SD}^{\alpha\mu}(k,p)}$$

$$\underbrace{+ \boxed{f_P \left[g^{\mu\alpha} + \frac{(2p-k)^\mu (p-k)^\alpha}{2p\cdot k - k^2}\right]}}_{H_{pt}^{\alpha\mu}(k,p)} \Bigg\}$$

$$k_\mu\, H_W^{\alpha\mu}(k,p) = k_\mu\, H_{pt}^{\alpha\mu}(k,p) = i\langle 0|j_W^\alpha(0)|P(\boldsymbol{p})\rangle = f_P\, p^\alpha$$

Real photon emission amplitude



$$H_W^{\alpha r}(k,p) = \epsilon_\mu^r(k)\, H_W^{\alpha\mu}(k,p) = \epsilon_\mu^r(k) \int d^4y\, e^{ik\cdot y}\, \mathtt{T}\langle 0|j_W^\alpha(0) j_{em}^\mu(y)|P(\boldsymbol{p})\rangle$$

By setting $k^2 = 0$, at fixed meson mass, the form factors depend on $p \cdot k$ only. Moreover, by choosing a *physical* basis for the polarization vectors, *i.e.* $\epsilon_r(\mathbf{k}) \cdot k = 0$, one has

$$H_W^{\alpha r}(k,p) = \epsilon_\mu^r(\boldsymbol{k}) \left\{ -i\frac{F_V}{m_P}\varepsilon^{\mu\alpha\gamma\beta}k_\gamma p_\beta + \left[\frac{F_A}{m_P} + \frac{f_P}{p\cdot k}\right](p\cdot k\, g^{\mu\alpha} - p^\mu k^\alpha) + \frac{f_P}{p\cdot k}p^\mu p^\alpha \right\}$$

Euclidean correlators

$$C_W^{\alpha r}(t,\boldsymbol{p},\boldsymbol{k}) = -i\,\varepsilon_\mu^r(\boldsymbol{k}) \int d^4y \int d^3\boldsymbol{x}\, \langle 0|\mathrm{T}\{j_W^\alpha(t,\boldsymbol{0})\, j_{em}^\mu(y)\}P(0,\boldsymbol{x})|0\rangle\, e^{E_\gamma t_y - i\boldsymbol{k}\cdot\boldsymbol{y} + i\boldsymbol{p}\cdot\boldsymbol{x}}$$

The convergence of the integral over t_y is ensured by the safe analytic continuation from Minkowsky to Euclidean spacetime, because of the absence of intermediate states lighter than the pseudoscalar meson

$$H_W^{\alpha r}(k,p) = \int d^4y\, e^{ik\cdot y}\, \epsilon_\mu^r(k)\mathrm{T}\langle 0|j_W^\alpha(0) j_{em}^\mu(y)|P(p)\rangle$$

$$\Downarrow$$

$$H_W^{\alpha r}(k,p) = H_{W,1}^{\alpha r}(k,p) + H_{W,2}^{\alpha r}(k,p) \qquad j^r(\boldsymbol{k}) = \int d^3y\, e^{-i\boldsymbol{k}\cdot\boldsymbol{y}}\, \epsilon_\mu^r(k)\, j_{em}^\mu(0,\boldsymbol{y})$$

$$H_{W,1}^{\alpha r}(k,p) = -\,i\int_{-\infty}^{0} dt_y\, \langle 0\,|\,j_W^\alpha(0)e^{(H+E_\gamma-E_P)t_y}\, j^r(\mathbf{k})\,|\,P(p)\rangle$$

$$H_{W,2}^{\alpha r}(k,p) = -\,i\int_{0}^{\infty} dt_y\, \langle 0\,|\,j^r(\mathbf{k})e^{-(H-E_\gamma)t_y}\, j_W^\alpha(0)\,|\,P(p)\rangle$$

$$\sqrt{m_P^2 + (\boldsymbol{p}-\boldsymbol{k})^2} + E_\gamma > \sqrt{m_P^2 + \boldsymbol{p}^2}\,, \qquad |\boldsymbol{k}| \neq 0$$

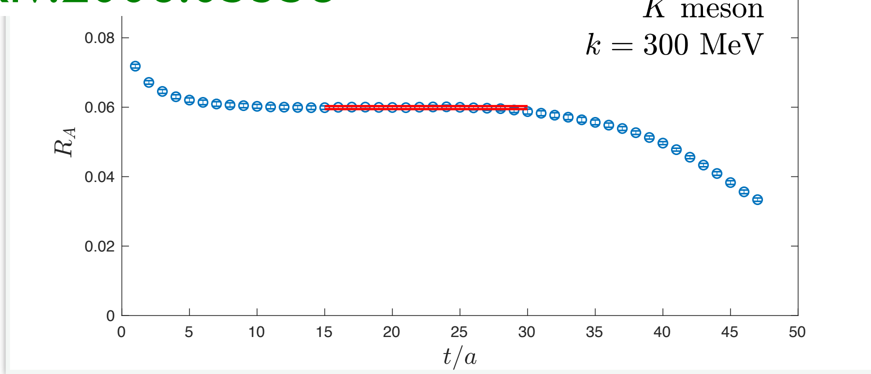
Form factors from Euclidean correlators

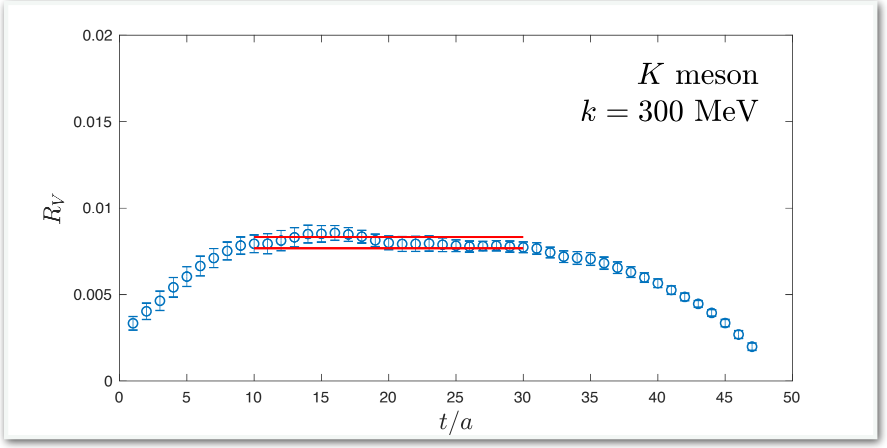
The physical form factors can be extracted directly from the Euclidean correlation functions (in the infinite-*T* limit)

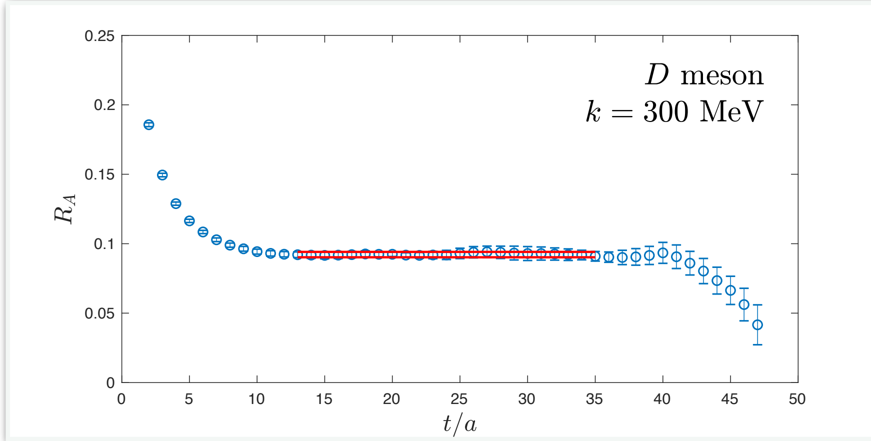
$$R_W^{\alpha r}(t;\boldsymbol{p},\boldsymbol{k}) = \frac{2E_P}{e^{-t(E_P-E_\gamma)}\langle P(\boldsymbol{p})|P|0\rangle} C_W^{\alpha r}(t;\boldsymbol{p},\boldsymbol{k})$$

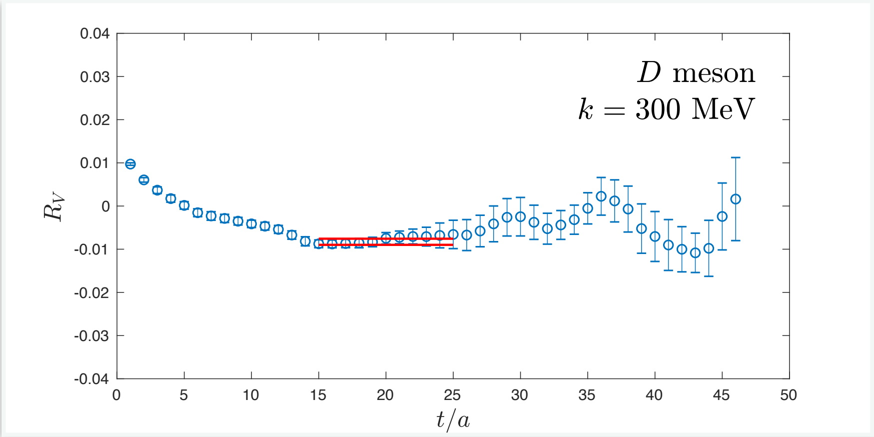
The numerical ratios $R_W^{\alpha r}(t;\mathbf{p},\mathbf{k})$ are expected to exhibit plateaux for $0 \ll t \ll T/2$, where exponentially-suppressed contributions can be neglected. In that region the above ratios give access to matrix elements $H_W^{\alpha r}(k,p)$

arXiv:2006.05358



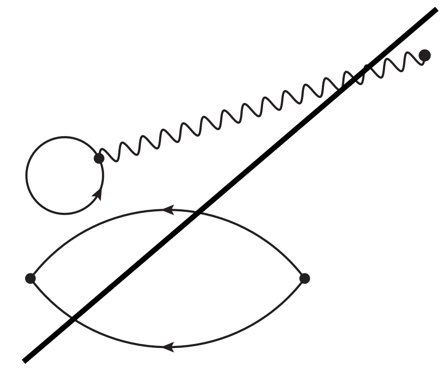






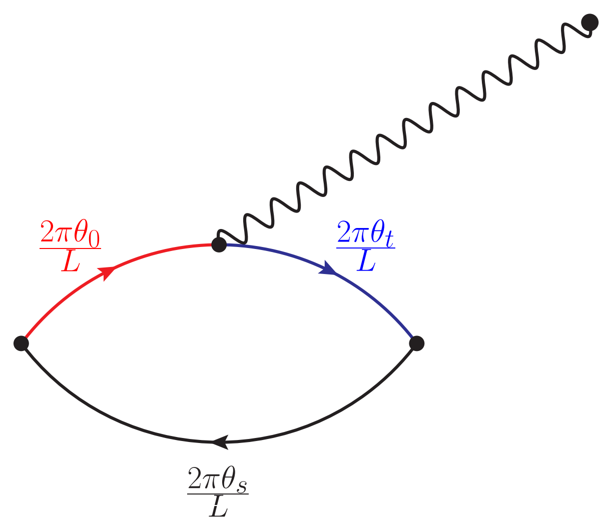
Form factors from Euclidean correlators

RM123 & Soton Coll., arXiv:2006.05358

Within the electro-quenched approximation  it is possible to

choose arbitrary values of the spatial momenta by using different spatial b.c. for the quark fields

$$\psi(x+\hat{\boldsymbol{k}}L) = \exp(2\pi i \hat{\boldsymbol{k}}\cdot\boldsymbol{\theta}_s/L)\psi(x)$$



$$\boldsymbol{p} = \frac{2\pi}{L}\left(\boldsymbol{\theta}_0 - \boldsymbol{\theta}_s\right)$$

$$\boldsymbol{k} = \frac{2\pi}{L}\left(\boldsymbol{\theta}_0 - \boldsymbol{\theta}_t\right)$$

We set: $\boldsymbol{p} = (0,0,|\boldsymbol{p}|)\ ,\qquad \boldsymbol{k} = (0,0,E_\gamma)\qquad \epsilon_1^\mu = \left(0,-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0\right)\ ,\qquad \epsilon_2^\mu = \left(0,\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0\right)$

Thus

$$H_A^{ir}(k,p) = \frac{\epsilon_r^i\, m_P}{2}\, x_\gamma \left[F_A + \frac{2f_P}{m_P x_\gamma}\right]\ ,\qquad H_V^{ir}(k,p) = \frac{i\left(E_\gamma\, \boldsymbol{\epsilon}_r \wedge \boldsymbol{p} - E\, \boldsymbol{\epsilon}_r \wedge \boldsymbol{k}\right)^i}{m_P \quad {}^{P}} F_V$$

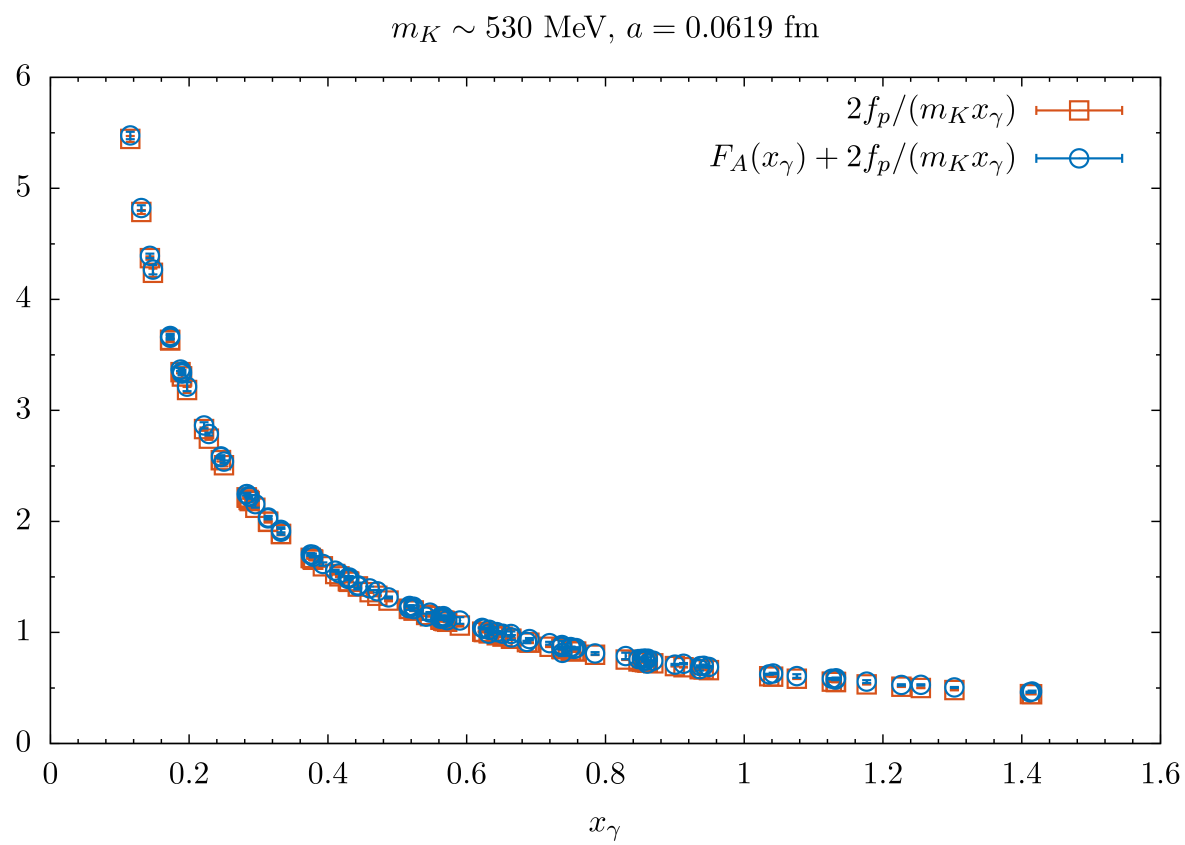
Form factors from Euclidean correlators

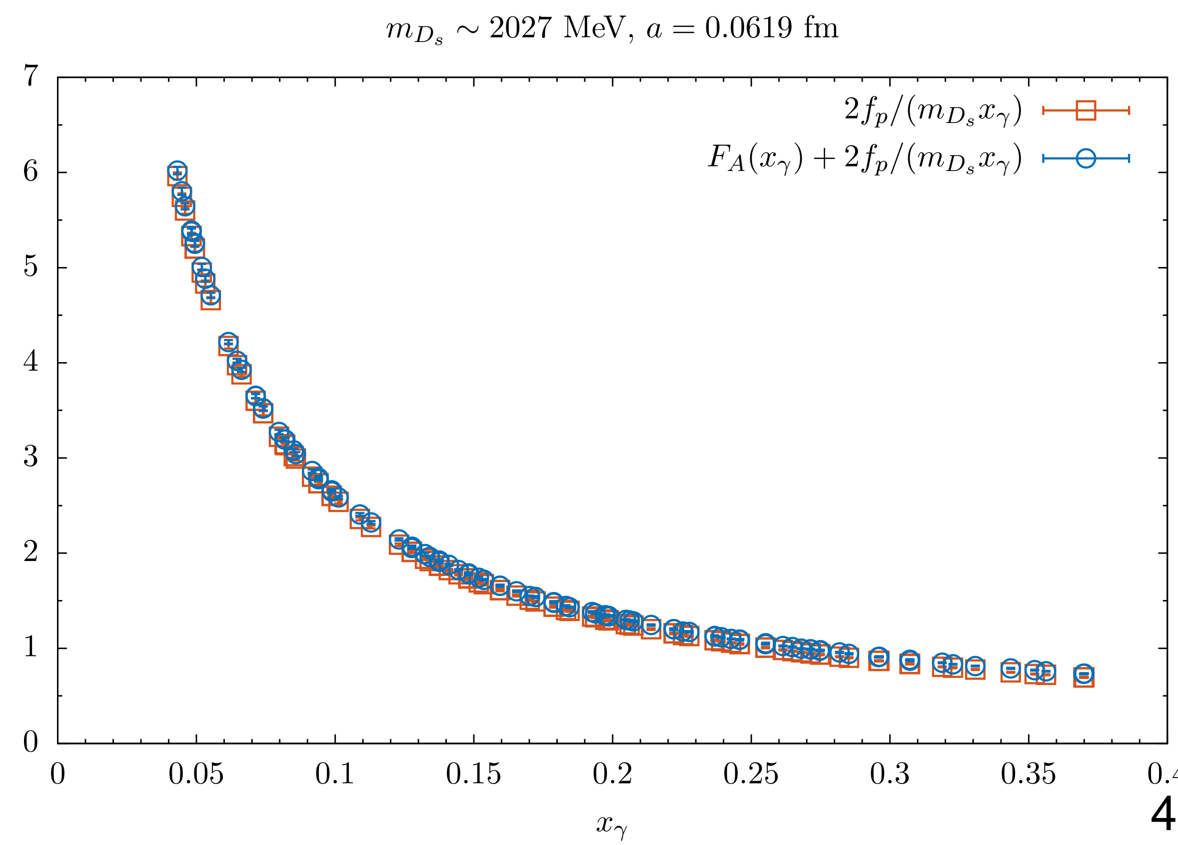
We build the following estimators

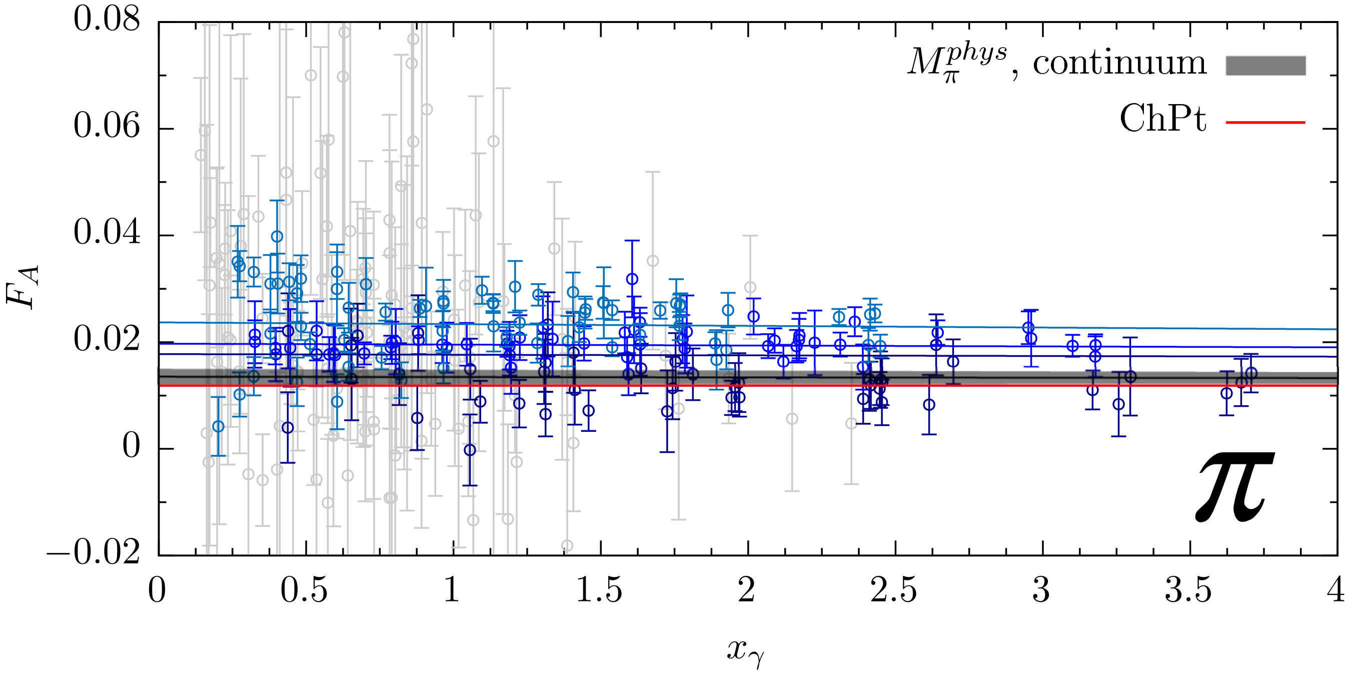
$$R_A(t) = \frac{1}{2m_P} \sum_{r=1,2} \sum_{j=1,2} \frac{R_A^{jr}(t;\boldsymbol{p},\boldsymbol{k})}{\epsilon_r^j} \quad \to \quad x_\gamma F_A(x_\gamma) + \frac{2f_P}{m_P}$$

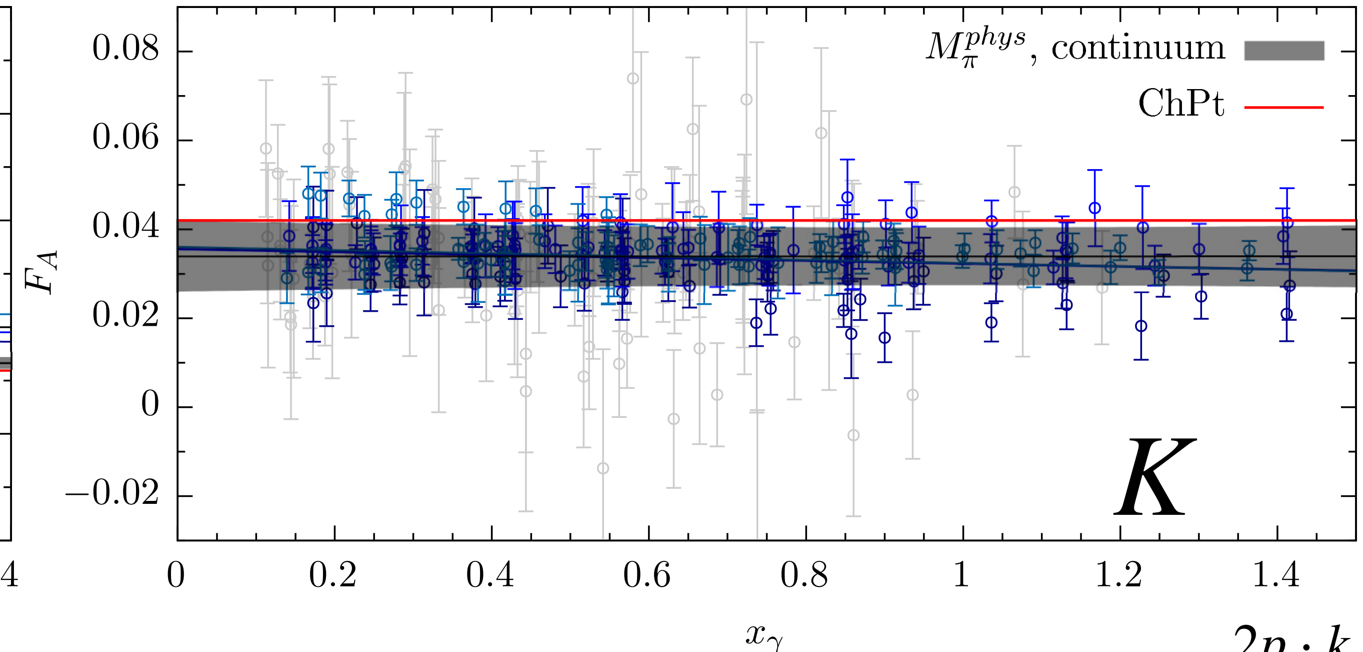
$$R_V(t) = \frac{m_P}{4} \sum_{r=1,2} \sum_{j=1,2} \frac{R_V^{jr}(t;\boldsymbol{p},\boldsymbol{k})}{i\left(E_\gamma\, \boldsymbol{\epsilon}_r \wedge \boldsymbol{p} - E_P \boldsymbol{\epsilon}_r \wedge \boldsymbol{k}\right)^j} \quad \to F_V(x_\gamma)$$

In our numerical calculation the above ratios are built in terms of finite-*T* correlators and time-reversal symmetries are exploited









$$x_\gamma = \frac{2p \cdot k}{m_P^2}$$

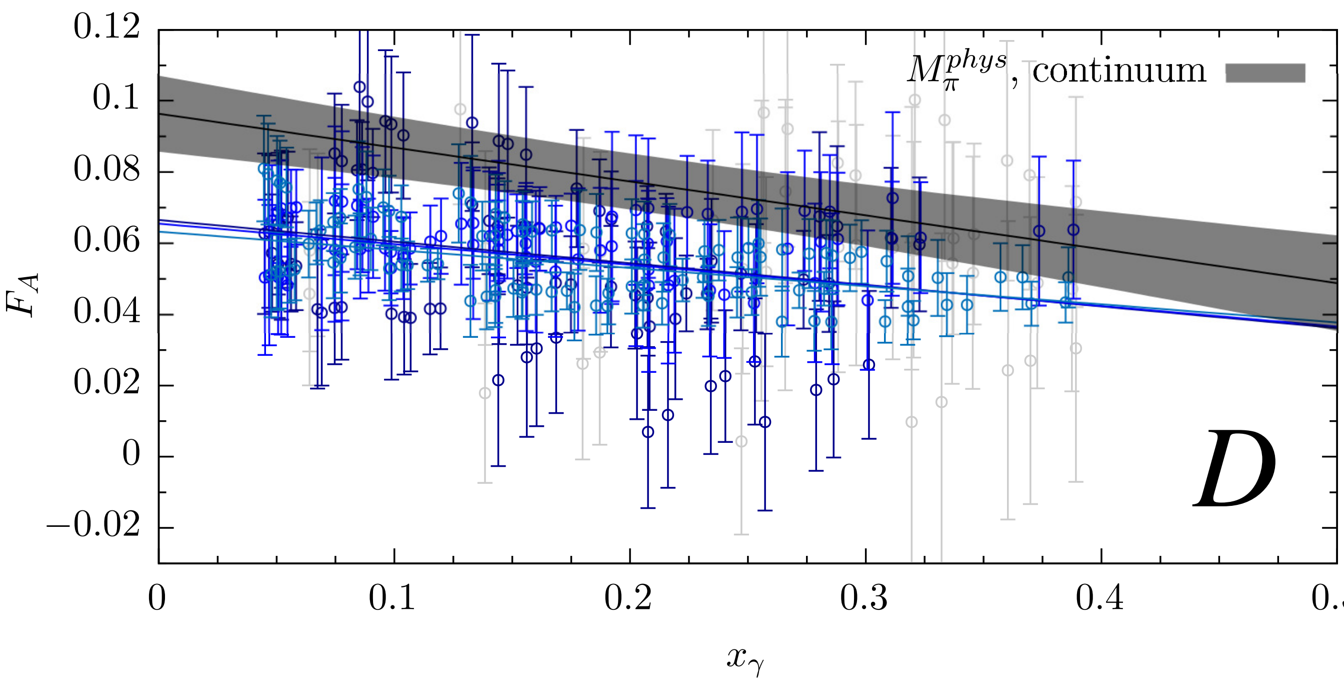
ChPt : $8 m_P (L_9^r + L_{10}^r)/f_P$

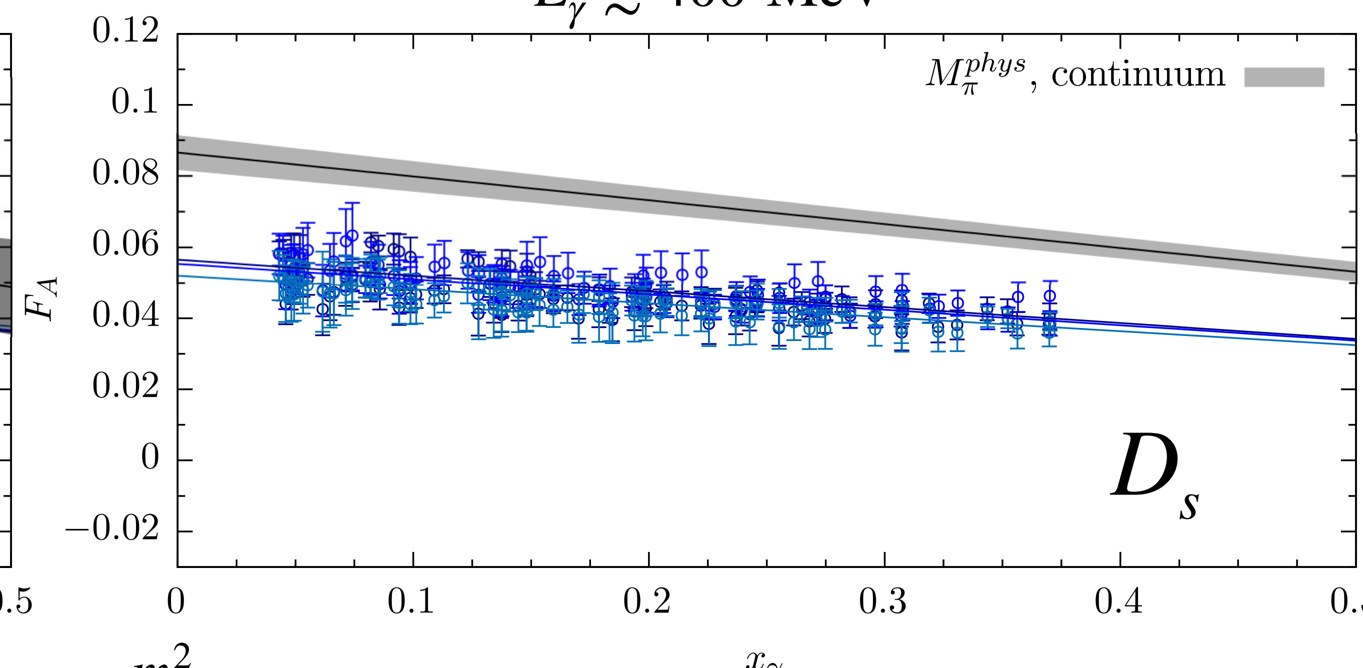
J. Bijnens *et al.*, 1993

$$F_{A,V}(x_\gamma) = \frac{m_P}{f_P}\left[\left(c_0 + c_0'\xi + \tilde{c}_0 \frac{a^2}{r_0^2}\right) + \left(c_1 + c_1'\xi + \tilde{c}_1 \frac{a^2}{r_0^2}\right) x_\gamma + O(a^2 m_P^2)\right]$$

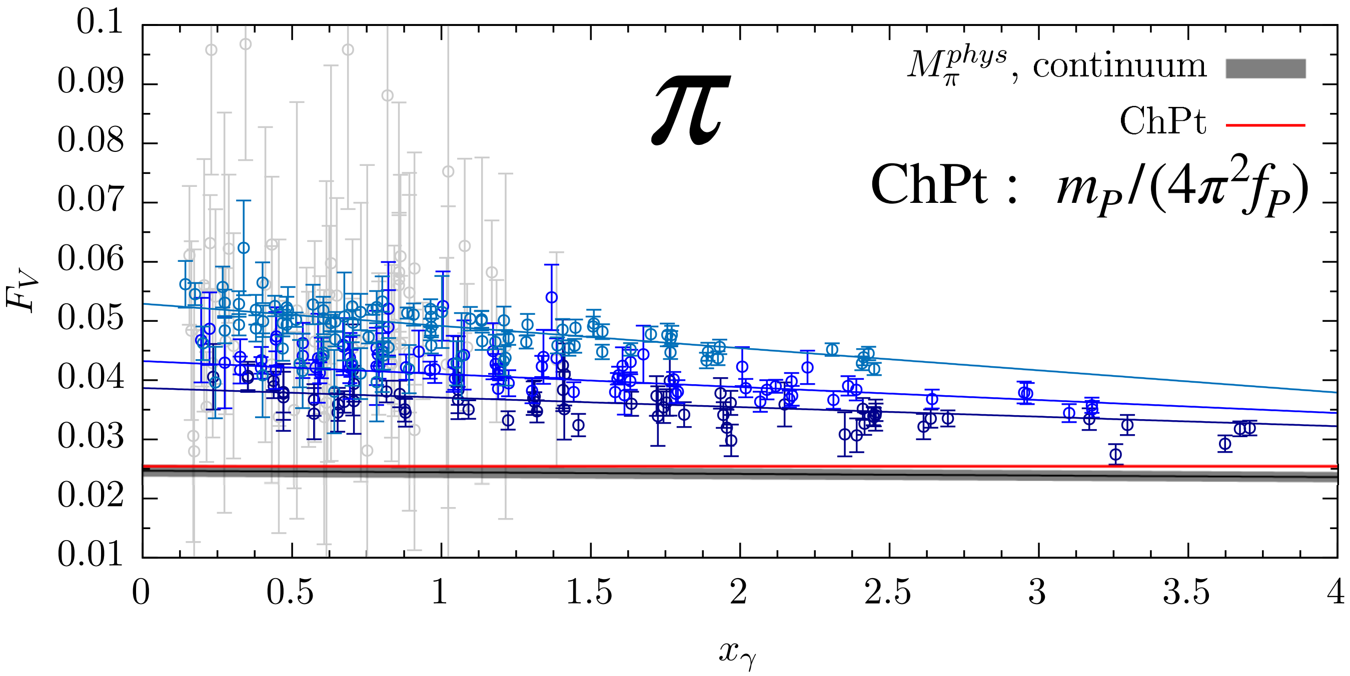
$$\xi = \frac{m_P^2}{(4\pi f_P)^2}$$

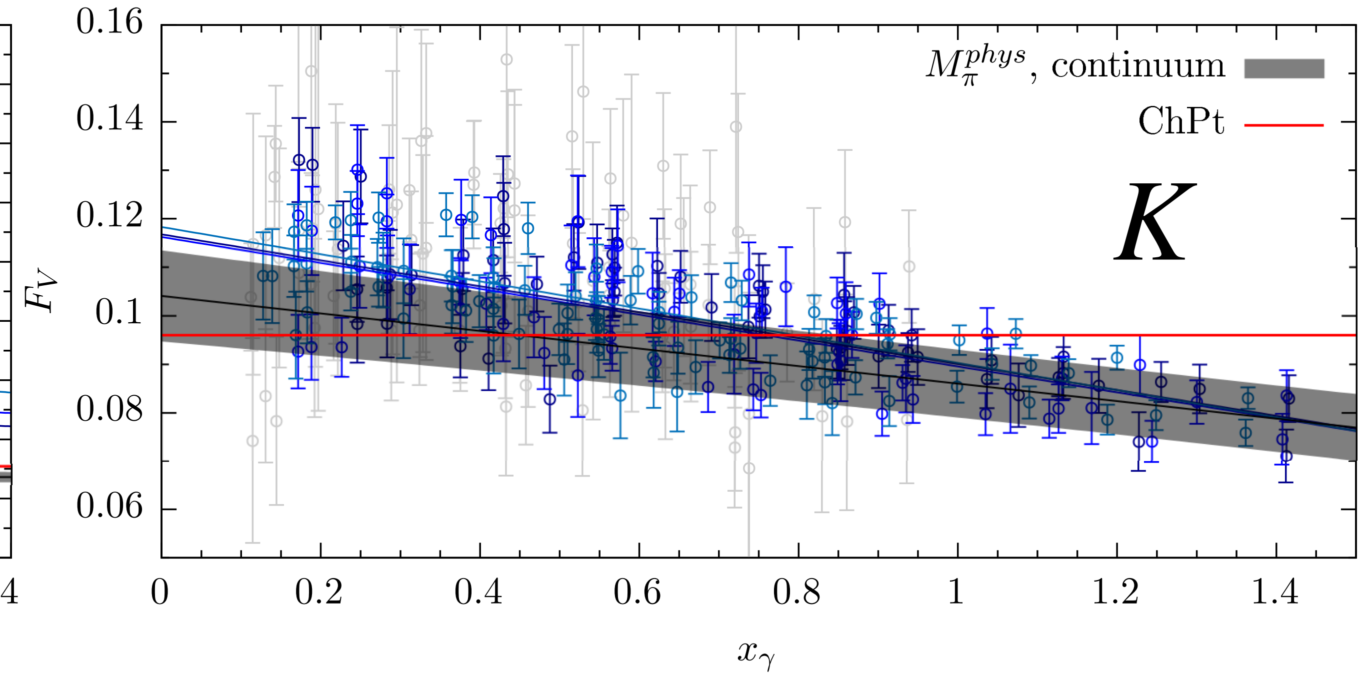
$E_\gamma \lesssim 400$ MeV





$$0 \le x_\gamma \le 1 - \frac{m_\ell^2}{m_P^2}$$



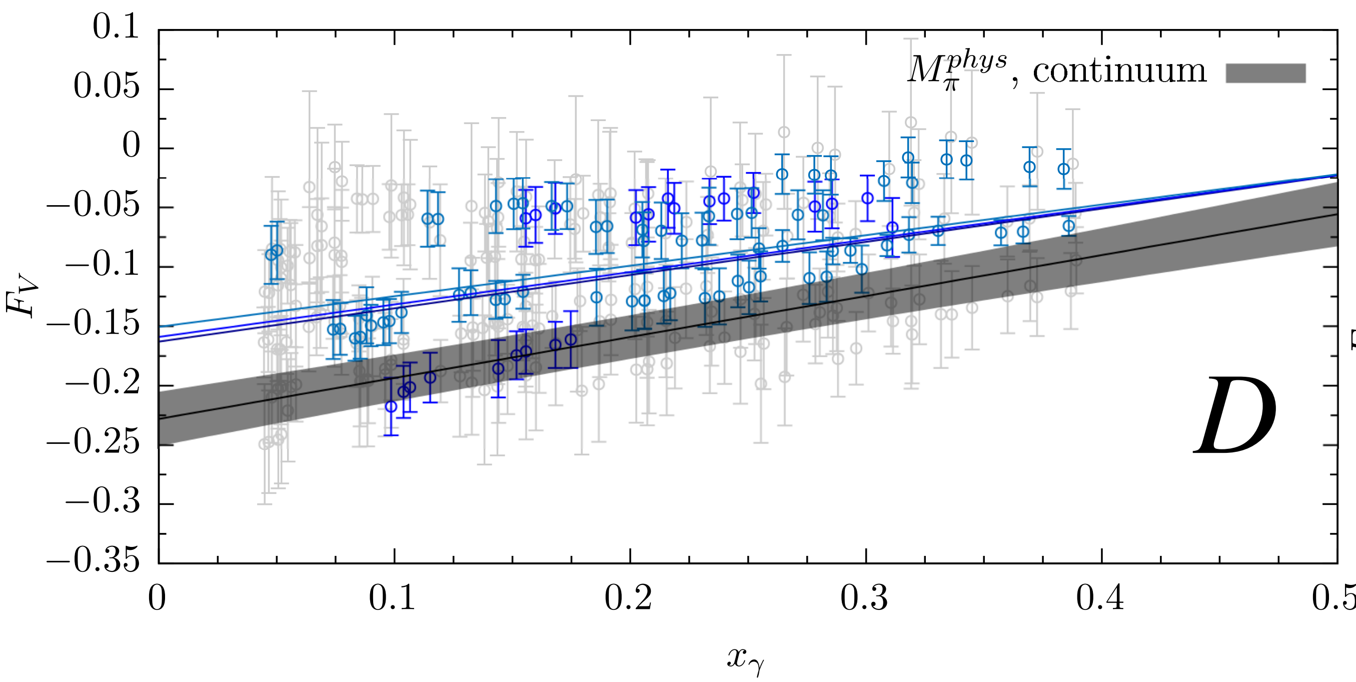


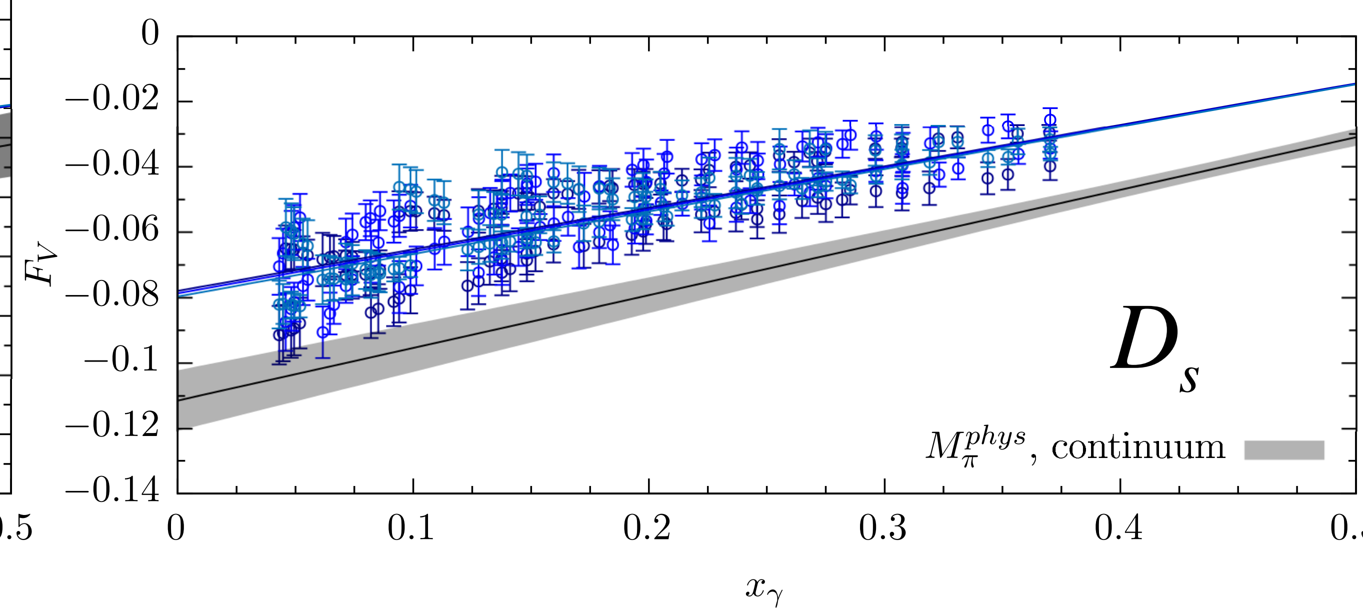
polynomial Ansatz

$$F_{A,V}(x_\gamma) = d_0 + d_0'\xi + \tilde{d}_0\frac{a^2}{r_0^2} + \left(d_1 + d_1'\xi + \tilde{d}_1\frac{a^2}{r_0^2}\right)x_\gamma$$

pole-like Ansatz

$$F_{A,V}(x_\gamma) = \frac{d_0 + d_0'\xi}{1 + (\Delta_1 + \Delta_1'\xi)x_\gamma} + \tilde{d}_0\frac{a^2}{r_0^2} + \tilde{d}_1\frac{a^2}{r_0^2}x_\gamma$$





Form factors: results

PHYSICAL REVIEW D **103,** 014502 (2021)

arXiv:2006.05358

First lattice calculation of radiative leptonic decay rates of pseudoscalar mesons

A. Desiderio,[1] R. Frezzotti,[1] M. Garofalo,[2] D. Giusti,[3,4] M. Hansen,[5] V. Lubicz,[2] G. Martinelli,[6] C. T. Sachrajda,[7] F. Sanfilippo,[4] S. Simula,[4] and N. Tantalo[1]

$$F^P_{A,V}(x_\gamma) = C^P_{A,V} + D^P_{A,V} x_\gamma$$

F_A

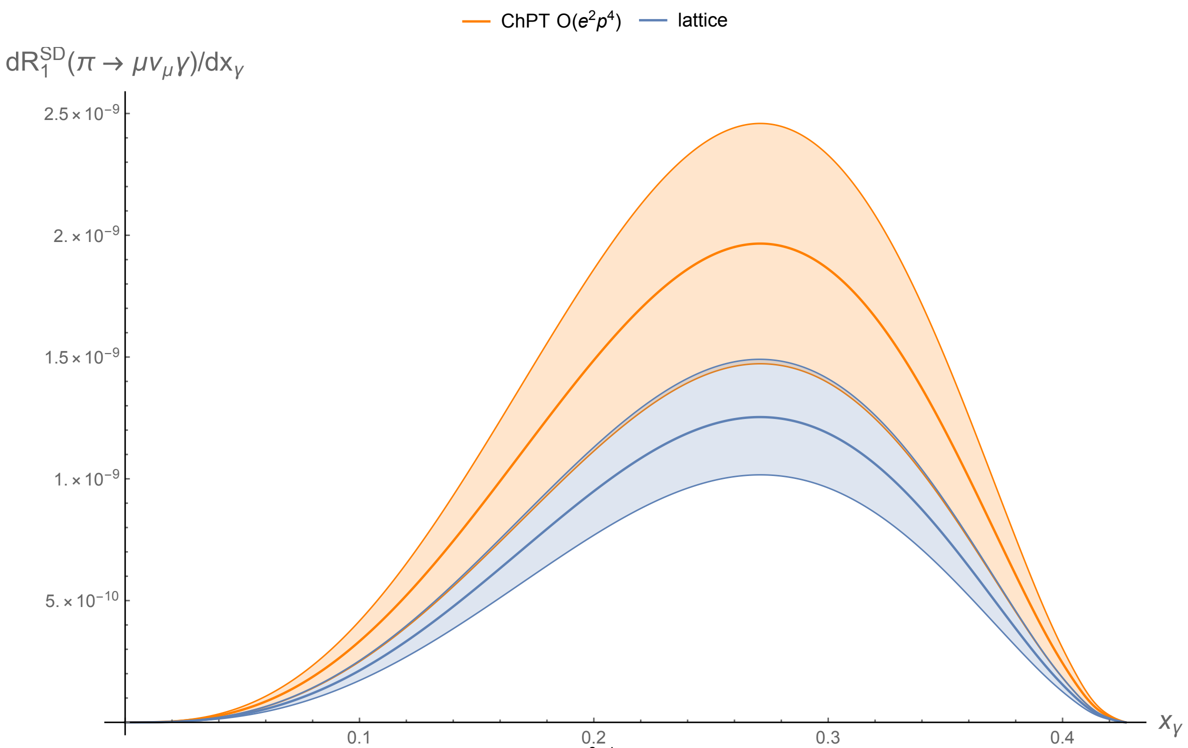
$$C_A^\pi = 0.010 \pm 0.003\,; \quad D_A^\pi = 0.0004 \pm 0.0006\,; \quad \rho_{C_A^\pi, D_A^\pi} = -0.419\,;$$
$$C_A^K = 0.037 \pm 0.009\,; \quad D_A^K = -0.001 \pm 0.007\,; \quad \rho_{C_A^K, D_A^K} = -0.673\,;$$
$$C_A^D = 0.109 \pm 0.009\,; \quad D_A^D = -0.10 \pm 0.03\,; \quad \rho_{C_A^D, D_A^D} = -0.557\,;$$
$$C_A^{D_s} = 0.092 \pm 0.006\,; \quad D_A^{D_s} = -0.07 \pm 0.01\,; \quad \rho_{C_A^{D_s}, D_A^{D_s}} = -0.745\,.$$

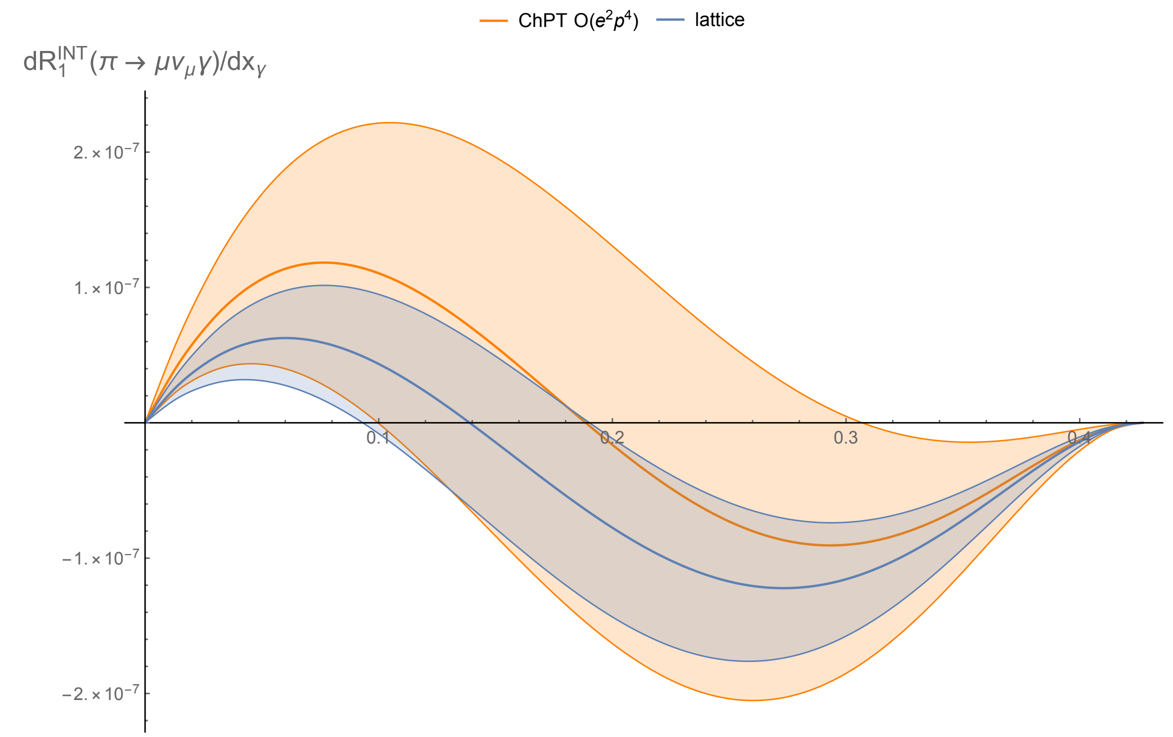
F_V

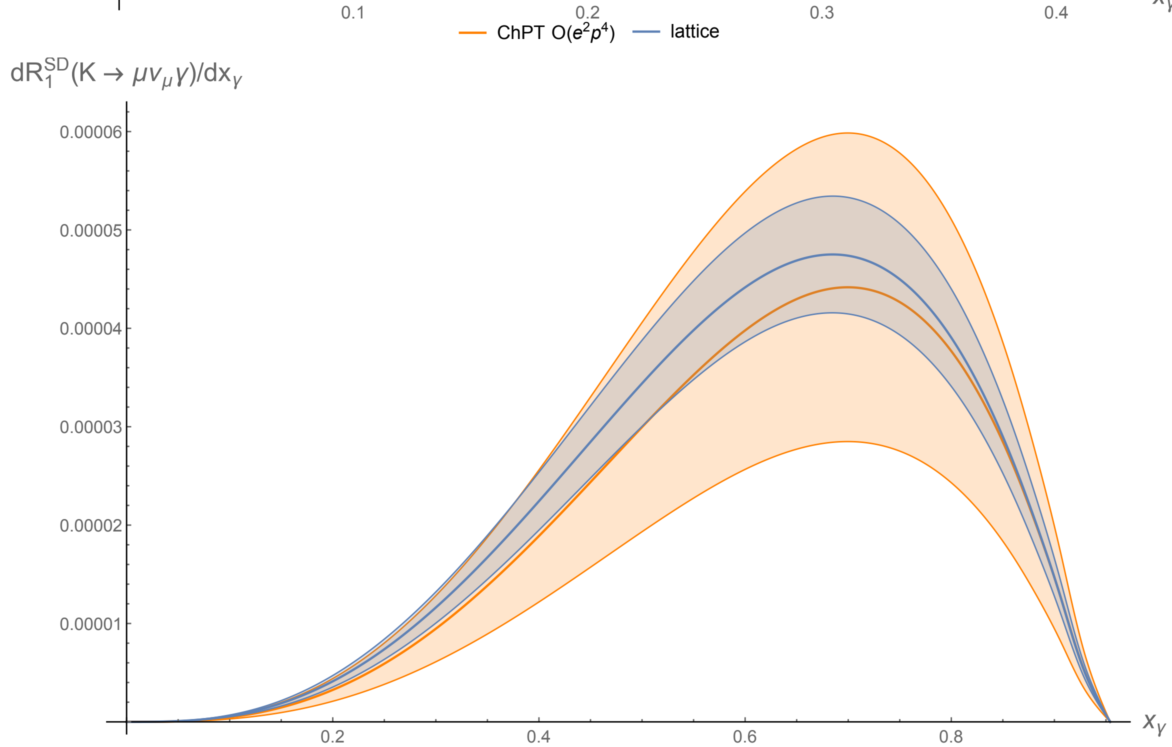
$$C_V^\pi = 0.023 \pm 0.002\,; \quad D_V^\pi = -0.0003 \pm 0.0003\,; \quad \rho_{C_V^\pi, D_V^\pi} = -0.570\,;$$
$$C_V^K = 0.12 \pm 0.01\,; \quad D_V^K = -0.02 \pm 0.01\,; \quad \rho_{C_V^K, D_V^K} = -0.714\,;$$
$$C_V^D = -0.15 \pm 0.02\,; \quad D_V^D = 0.12 \pm 0.04\,; \quad \rho_{C_V^D, D_V^D} = -0.580\,;$$
$$C_V^{D_s} = -0.12 \pm 0.02\,; \quad D_V^{D_s} = 0.16 \pm 0.03\,; \quad \rho_{C_V^{D_s}, D_V^{D_s}} = -0.900\,.$$

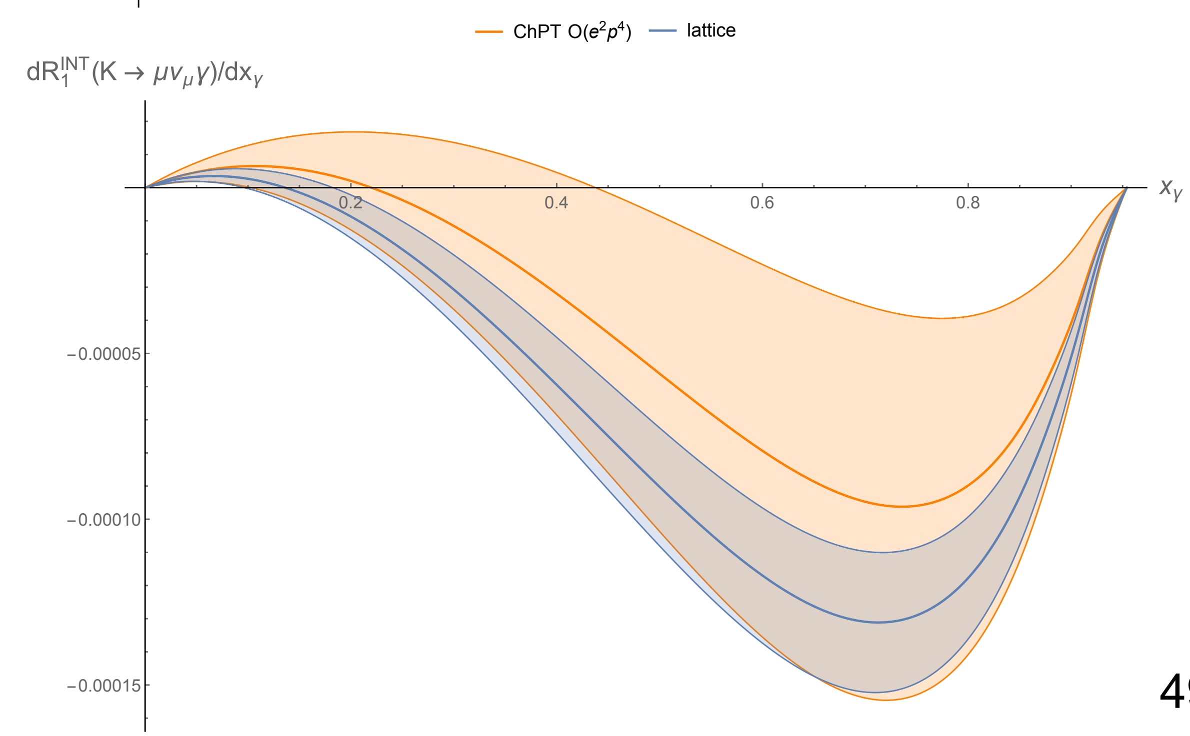
$$\frac{4\pi}{\alpha\,\Gamma_0^{\rm tree}}\frac{d\Gamma_1^{\rm SD}}{dx_\gamma} = \frac{m_P^2}{6 f_P^2\, r_\ell^2\,(1-r_\ell^2)^2}\left[F_V(x_\gamma)^2 + F_A(x_\gamma)^2\right] f^{\rm SD}(x_\gamma)$$

$$\frac{4\pi}{\alpha\,\Gamma_0^{\rm tree}}\frac{d\Gamma_1^{\rm INT}}{dx_\gamma} = -\frac{2\, m_P}{f_P\,(1-r_\ell^2)^2}\left[F_V(x_\gamma)\, f_V^{\rm INT}(x_\gamma) + F_A(x_\gamma)\, f_A^{\rm INT}(x_\gamma)\right]$$









Leptonic decays at $O(\alpha)$: RESULTS

$$\Gamma(\Delta E) = \Gamma^{(tree)} \left[1 + \delta R_0 + \delta R_{pt}(\Delta E) + \delta R_1^{SD}(\Delta E) + \delta R_1^{INT}(\Delta E) \right]$$

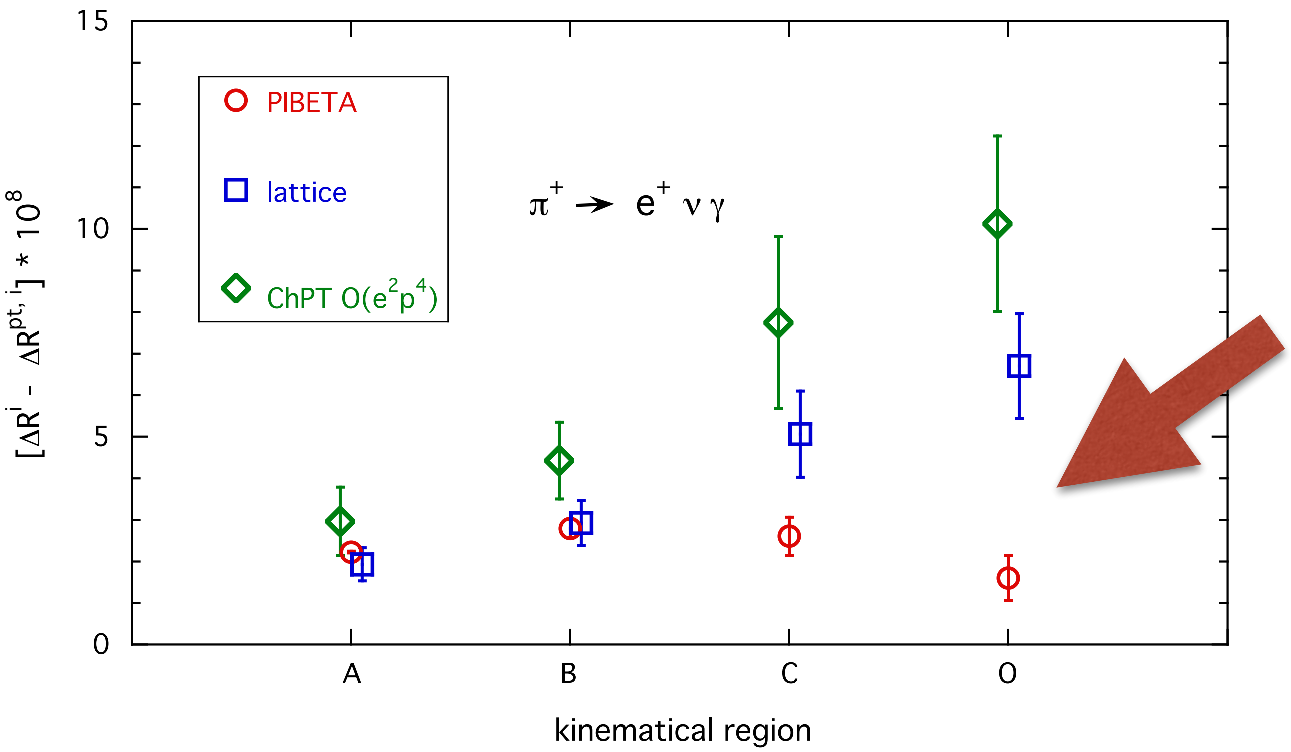
	$\pi_{e2[\gamma]}$	$\pi_{\mu 2[\gamma]}$	$K_{e2[\gamma]}$	$K_{\mu 2[\gamma]}$
δR_0	(*)	0.0411 (19)	(*)	0.0341 (10)
$\delta R_{\mathrm{pt}}(\Delta E_\gamma^{max})$	-0.0651	-0.0258	-0.0695	-0.0317
$\delta R_1^{\mathrm{SD}}(\Delta E_\gamma^{max})$	$5.4\ (1.0) \times 10^{-4}$	$2.6\ (5) \times 10^{-10}$	1.19 (14)	$2.2\ (3) \times 10^{-5}$
$\delta R_1^{\mathrm{INT}}(\Delta E_\gamma^{max})$	$-4.1\ (1.0) \times 10^{-5}$	$-1.3\ (1.5) \times 10^{-8}$	$-9.2\ (1.3) \times 10^{-4}$	$-6.1\ (1.1) \times 10^{-5}$
ΔE_γ^{max} (MeV)	69.8	29.8	246.8	235.5

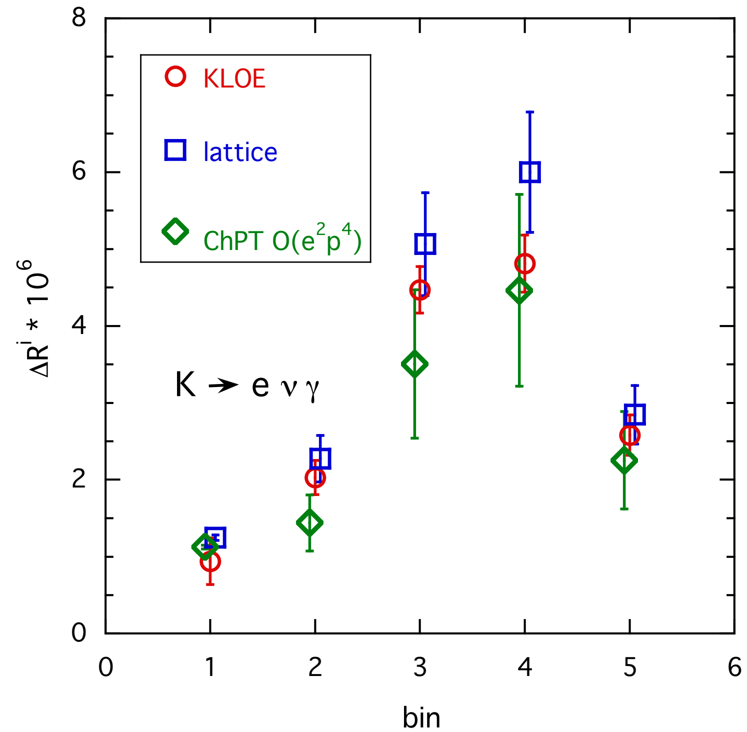
(*) *Not yet evaluated by numerical lattice QCD+QED simulations.*

$$\delta R_{\pi_{\mu 2}} = 0.0153\,(19)$$

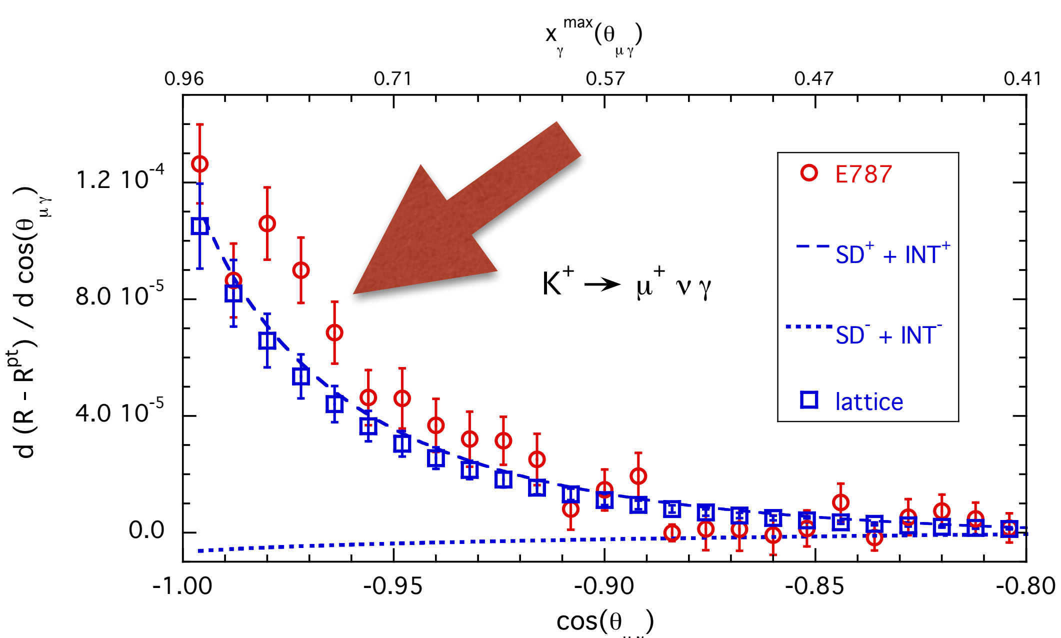
$$\delta R_{K_{\mu 2}} = 0.0024\,(10)$$

Comparison with experimental data





region	E_γ	E_e	$\theta_{e\gamma}$	$\Delta R^{\exp,i}$	$\Delta R^{\mathrm{pt},i}$	$(\Delta R^{\exp,i} - \Delta R^{\mathrm{pt},i})$	$\Delta R^{\mathrm{SD},i}$	$(\Delta R^{\mathrm{th},i} - \Delta R^{\mathrm{pt},i})$	ChPT
A	> 50	> 50	$> 40^\circ$	2.614 ± 0.021	0.385	2.229 ± 0.021	1.94 ± 0.40	1.93 ± 0.40	2.97 ± 0.82
B	> 50	> 10	$> 40^\circ$	14.46 ± 0.22	11.66	2.80 ± 0.22	3.01 ± 0.54	2.93 ± 0.54	4.43 ± 0.92
C	> 10	> 50	$> 40^\circ$	37.69 ± 0.46	35.08	2.61 ± 0.46	5.07 ± 1.03	5.07 ± 1.04	7.75 ± 2.07
O	> 10	$> m_e$	$> 40^\circ$	73.86 ± 0.54	72.26	1.60 ± 0.54	6.87 ± 1.26	6.70 ± 1.26	10.13 ± 2.11

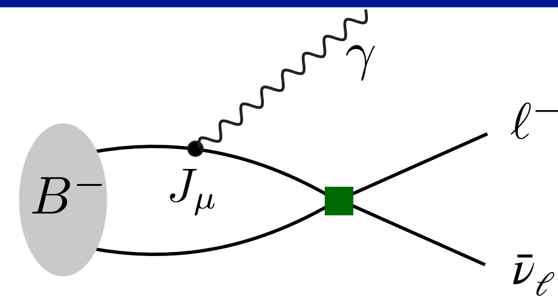


Tensions may be due to the presence of NP, such as flavour changing interactions beyond the V-A couplings and non-universal corrections to lepton couplings

R. Frezzotti *et al.*, arXiv:2012.02120

Radiative corrections to leptonic B-meson decays

$B^- \to \ell^- \bar{\nu}_\ell \gamma$



- The emission of a real hard photon removes the $(m_\ell/M_B)^2$ helicity suppression
- This is the simplest process that probes (for large E_γ) the first inverse moment of the B-meson LCDA

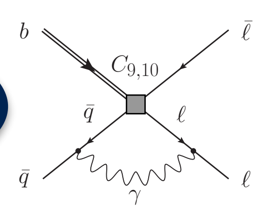
$$\frac{1}{\lambda_B(\mu)} = \int_0^\infty \frac{d\omega}{\omega} \Phi_{B+}(\omega, \mu)$$

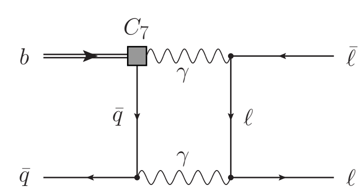
λ_B is an important input in QCD-factorization predictions for non-leptonic B decays but is poorly known

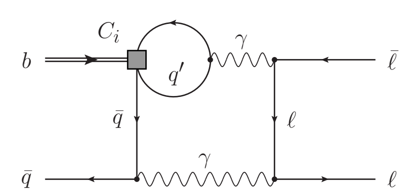
M. Beneke, V. M. Braun, Y. Ji, Y.-B. Wei, 2018

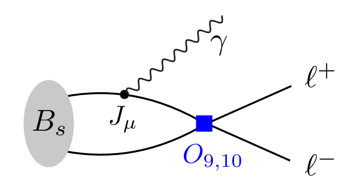
- Belle 2018: $\mathscr{B}(B^- \to \ell^- \bar{\nu}_\ell \gamma, E_\gamma > 1 \text{ GeV}) < 3.0 \cdot 10^{-6}$ $\longrightarrow$ $\lambda_B > 0.24$ GeV
- QCD sum rules in HQET: $\lambda_B(1 \text{ GeV}) = 0.46\,(11)$ GeV

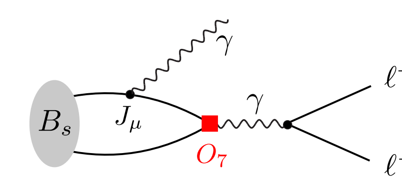
$B_q \to \ell^+ \ell^- (\gamma)$











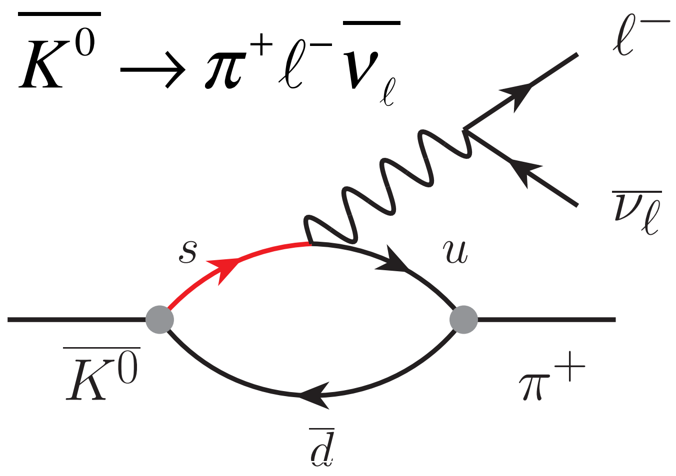
- Enhancement of the virtual corrections by a factor M_B/Λ_{QCD} and by large logarithms

M. Beneke, C. Bobeth, R. Szafron, 2019

- The real photon emission process is a clean probe of NP: sensitiveness to $F_{V,A,TV,TA}(E_\gamma)$

Semileptonic decay amplitudes

$\overline{K^0} \to \pi^+ \ell^- \overline{\nu}_\ell$



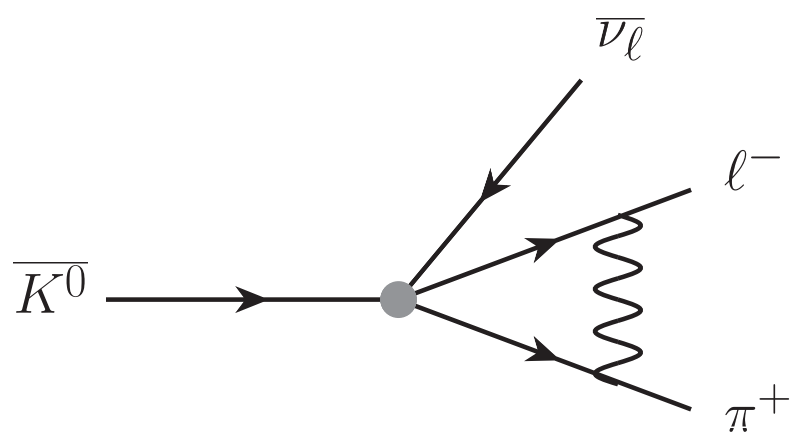
Without QED

$$\langle \pi(p_\pi) | \bar{s}\gamma_\mu u | K(p_K) \rangle = f_0(q^2) \frac{M_K^2 - M_\pi^2}{q^2} q_\mu + f_+(q^2) \left[(p_\pi + p_K)_\mu - \frac{M_K^2 - M_\pi^2}{q^2} q_\mu \right]$$

$q = p_K - p_\pi = p_\ell + p_\nu$

$s_{\pi\ell} = (p_\pi + p_\ell)^2$

$$\frac{d^2\Gamma}{dq^2 ds_{\pi\ell}} = G_F^2 |V_{us}|^2 \left[a_+(q^2, s_{\pi\ell}) |f_+(q^2)|^2 + a_0(q^2, s_{\pi\ell}) |f_0(q^2)|^2 + a_{0+}(q^2, s_{\pi\ell}) f_0(q^2) f_+(q^2) \right]$$

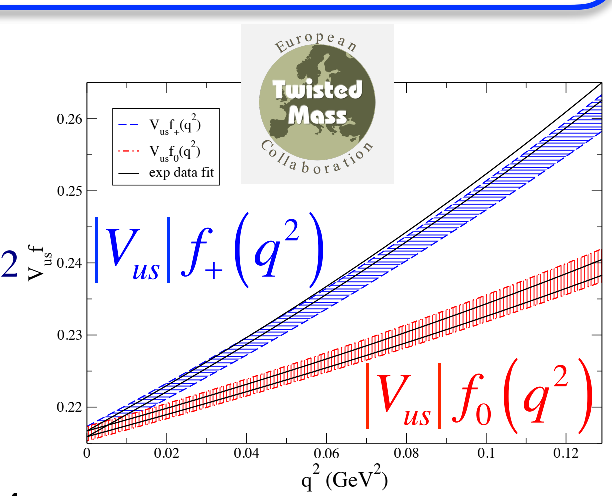


$$\frac{d^2\Gamma}{dq^2 ds_{\pi\ell}} = \lim_{L\to\infty} \left(\frac{d^2\Gamma_0(L)}{dq^2 ds_{\pi\ell}} - \frac{d^2\Gamma_0^{pt}(L)}{dq^2 ds_{\pi\ell}} \right) + \lim_{m_\gamma \to 0} \left(\frac{d^2\Gamma_0^{pt}(m_\gamma)}{dq^2 ds_{\pi\ell}} + \frac{d^2\Gamma_1^{pt}(\Delta E, m_\gamma)}{dq^2 ds_{\pi\ell}} \right)$$

lighter intermediate states $\pi\ell(\gamma)$

- IR divergences cancel out
- *1/L* corrections depend on $df_\pm / dq^2$

D. Giusti *et al.*, arXiv:1811.06364



N. Carrasco *et al.*, 2016

<u>To be addressed:</u> FV corrections due to em rescattering

Conclusions and future perspectives

- We have performed **the FIRST lattice calculation** (arXiv:1711.06537, arXiv:1904.08731, arXiv:2006.05358) of isospin-breaking corrections to light-meson leptonic decay rates

- Setting the lattice scale of our simulations with an hadron mass (*e.g.* M_{Ω}) allows to predict $|V_{ud}|$

- The inclusion of disconnected diagrams is mandatory for removing the qQED (quenched-QED) approximation

- Extensions to leptonic heavy-light meson decays and semileptonic Kl3 decays are being targeted