Tensor integral reduction for non-physicists

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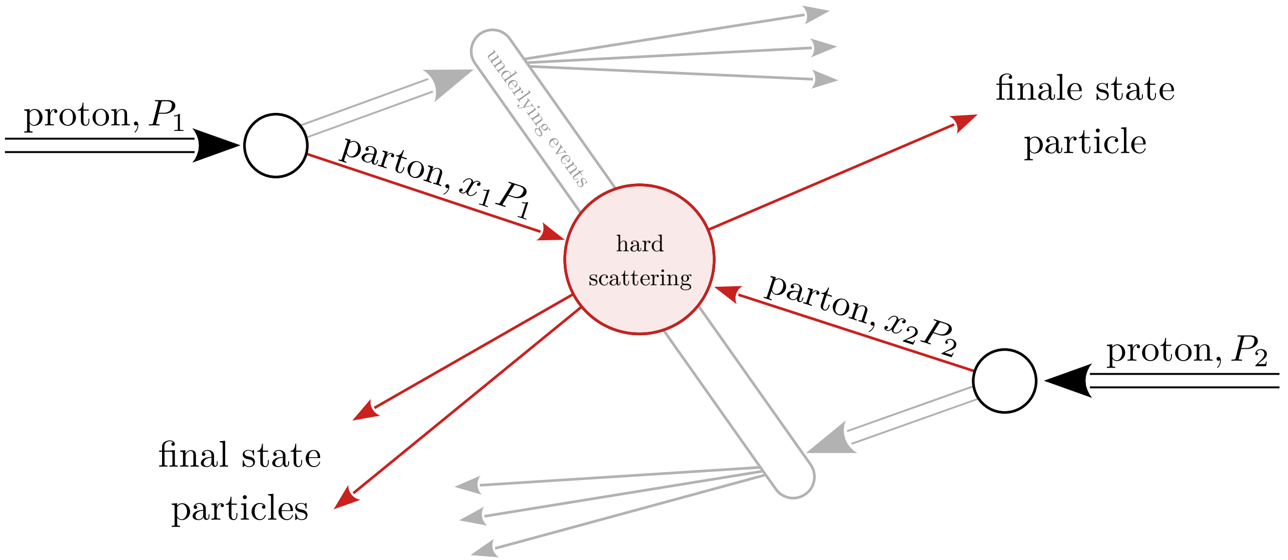
HET Lunch Discussions

Precise predictions for hard scattering at hadron colliders



- Hadronic cross section [Collins, Soper, Sterman, '1989]

$$\mathrm{d}\sigma_{\mathrm{H}} = \sum_{ij} \int_0^1 \mathrm{d}x_1 \mathrm{d}x_2 \; f_i(x_1) f_j(x_2) \underbrace{\boxed{\mathrm{d}\hat{\sigma}_{ij}(x_1, x_2)}}^{\textit{in the following}} \left[1 + \mathcal{O}\left(\frac{\Lambda_{\mathrm{QCD}}}{Q} \right) \right]$$

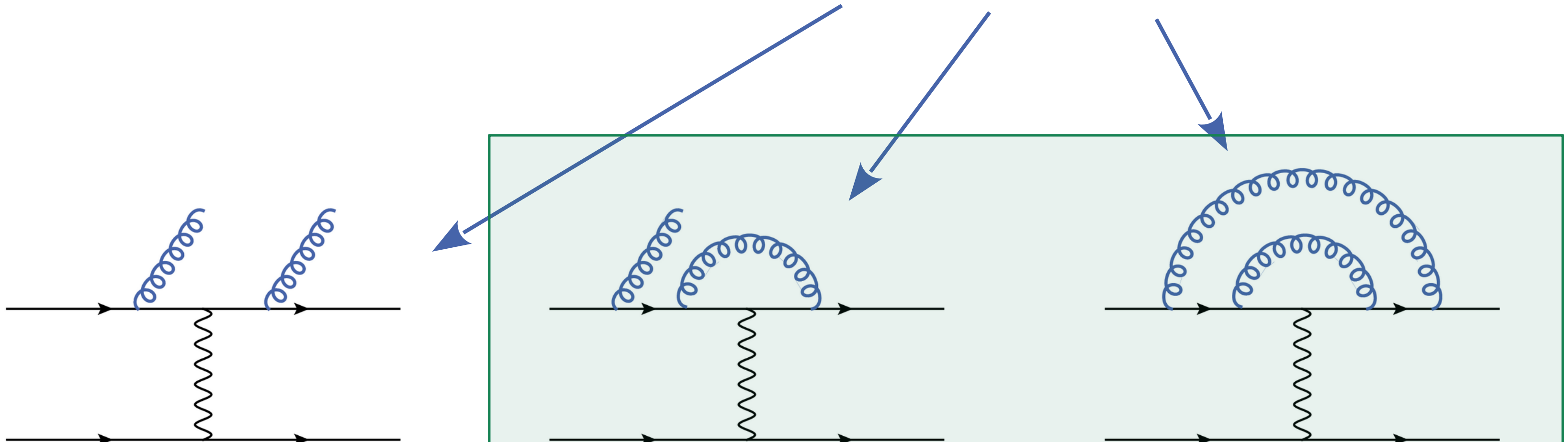
- Partonic cross section in perturbative QCD as expansion in the strong coupling constant α_s

$$\mathrm{d}\hat{\sigma}_{ij}(x_1, x_2) = \mathrm{d}\hat{\sigma}_{ij}^{\mathrm{lo}}(x_1, x_2) + \mathrm{d}\hat{\sigma}_{ij}^{\mathrm{nlo}}(x_1, x_2) + \mathrm{d}\hat{\sigma}_{ij}^{\mathrm{nnlo}}(x_1, x_2) + \mathcal{O}(\alpha_s^3)$$

Precise predictions for hard scattering at hadron colliders

$$\mathrm{d}\hat{\sigma}_{ij}(x_1,x_2) = \mathrm{d}\hat{\sigma}^{\mathrm{lo}}_{ij}(x_1,x_2) + \mathrm{d}\hat{\sigma}^{\mathrm{nlo}}_{ij}(x_1,x_2) + \underbrace{\mathrm{d}\hat{\sigma}^{\mathrm{nnlo}}_{ij}(x_1,x_2)}_{} + \mathcal{O}(\alpha_s^3)$$

$$= \mathrm{d}\hat{\sigma}_{\mathrm{rr}} + \mathrm{d}\hat{\sigma}_{\mathrm{rv}} + \mathrm{d}\hat{\sigma}_{\mathrm{vv}} + \mathrm{d}\hat{\sigma}_{\mathrm{pdf}} \quad \textit{(schematically)}$$



double real emission corrections — *real-virtual corrections* — *double virtual corrections*

in the following: computation of loop corrections

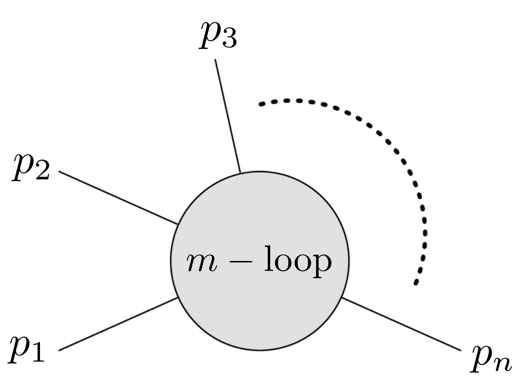
Integration-by-parts identities

- Integration-by-parts identities are an essential tool in the computation of multi-loop Feynman integrals [Chetyrkin, Tkachov '1981]

- These identities relate appearing integrals and therefore reduce the number of integrals that need to be computed

- Typical numbers at 2-loop: O(1000 – 100.000) integrals → O(10 – 100) **master** integrals

- However, they are only applicable to **scalar integral families** of the form



$$= \int \mathrm{d}^d l_1 \cdots \int \mathrm{d}^d l_m \, \frac{1}{d_1^{\alpha_1} d_2^{\alpha_2} d_3^{\alpha_3} \ldots d_m^{\alpha_m}}$$

where powers α_i can be positive and all scalar products $\{l_i \cdot l_j, l_i \cdot p_j\}$ can be expressed in terms of propagators

$$d_i = \left(\sum_j c^l_{ij} l_j - \sum_j c^p_{ij} p_i \right)^2 - m_i^2$$

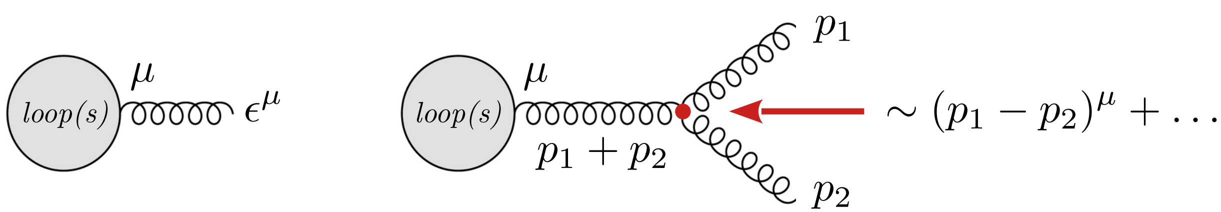
Tensor integrals in multi-loop computations

- In the computation of loop integrals we have to deal with tensor integrals of the form

$$\int \mathrm{d}^d l_1 \cdots \int \mathrm{d}^d l_n \, \frac{\prod (l_i \cdot l_j) \prod (l_i \cdot u_j)}{d_1 d_2 d_3 \ldots d_m}$$

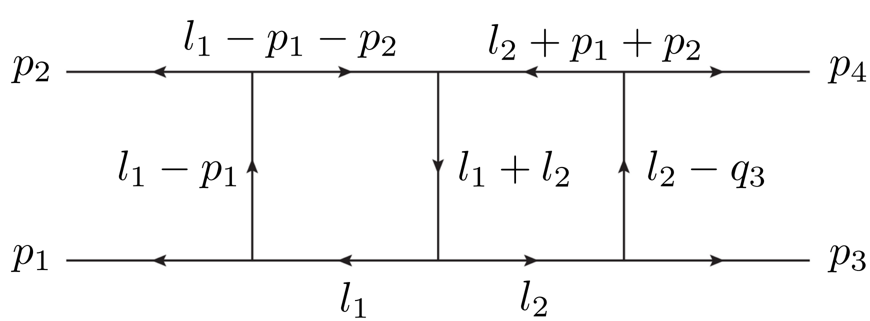
where u_j^μ are arbitrary physical vectors that do not have to appear in the propagators

- Possible source for such scalar products are polarization vectors, radiation from external legs, ...



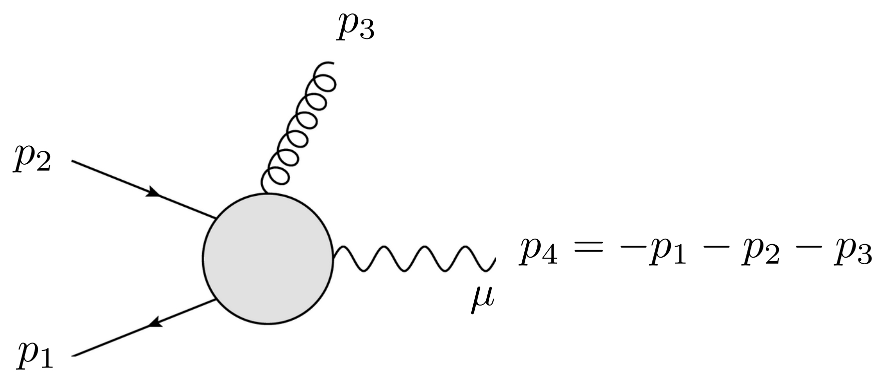
- **Need to rewrite tensor integral in terms of scalar integral to apply IBP relations**

- Starting from 2-loops we need to introduce additional propagators to be able to express all scalar products $\{l_i \cdot l_j, l_i \cdot p_j\}$ in terms of propagators



Projector method; the physicists approach

$$S_\mu(p_1, p_2, p_3) \; =$$



- Most general tensor structure (13 structures) [Garland, Gehrmann, Glover, Koukoutsakis, Remiddi ‘02]

$$\begin{aligned}
S_\mu(p_1,p_2,p_3) \;&=\; \bar{u}(p_1)\not{p}_3 u(p_2)\left(A_{11}\epsilon_3.p_1\; p_{1\mu} + A_{12}\epsilon_3.p_1\; p_{2\mu} + A_{13}\epsilon_3.p_1\; p_{3\mu}\right)\\
&+\; \bar{u}(p_1)\not{p}_3 u(p_2)\left(A_{21}\epsilon_3.p_2\; p_{1\mu} + A_{22}\epsilon_3.p_2\; p_{2\mu} + A_{23}\epsilon_3.p_2\; p_{3\mu}\right)\\
&+\; \bar{u}(p_1)\gamma_\mu u(p_2)\left(B_1\epsilon_3.p_1 + B_2\epsilon_3.p_2\right)\\
&+\; \bar{u}(p_1)\not{\epsilon}_3 u(p_2)\left(C_1 p_{1\mu} + C_2 p_{2\mu} + C_3 p_{3\mu}\right)\\
&+\; D_1\bar{u}(p_1)\not{\epsilon}_3\not{p}_3\gamma_\mu u(p_2) \qquad\qquad\qquad\qquad p_i u(p_i) = 0\\
&+\; D_2\bar{u}(p_1)\gamma_\mu\not{p}_3\not{\epsilon}_3 u(p_2)
\end{aligned}$$

- Already neglecting terms not contributing because of $\epsilon_3 \cdot p_3 = 0$ and $\not{p}_i u(p_i) = 0$. Further simplifications due to additional physical constraints:

$$S_\mu(p_1,p_2,p_3)p_4^\mu = 0 \qquad \textit{conserved current}$$

$$S_\mu(p_1,p_2,p_3)(\epsilon_3 \to p_3) = 0 \qquad \textit{Ward identity}$$

Projector method; the physicists approach

- 7 tensor structures remain

$$
\begin{aligned}
S_\mu(p_1,p_2,p_3) \quad = \quad & A_{11}(s_{13},s_{23},s_{123})T_{11\mu} + A_{12}(s_{13},s_{23},s_{123})T_{12\mu} + A_{13}(s_{13},s_{23},s_{123})T_{13\mu} \\
+ \quad & A_{21}(s_{13},s_{23},s_{123})T_{21\mu} + A_{22}(s_{13},s_{23},s_{123})T_{22\mu} + A_{23}(s_{13},s_{23},s_{123})T_{23\mu} \\
+ \quad & B(s_{13},s_{23},s_{123})T_\mu,
\end{aligned}
$$

with

$$
\begin{aligned}
T_{1J\mu} &= \bar{u}(p_1)\not{p}_3 u(p_2)\epsilon_3.p_1 p_{J\mu} - \frac{s_{13}}{2}\bar{u}(p_1)\not{\epsilon}_3 u(p_2)p_{J\mu} + \frac{s_{J4}}{4}\bar{u}(p_1)\not{\epsilon}_3\not{p}_3\gamma_\mu u(p_2) \\
T_{2J\mu} &= \bar{u}(p_1)\not{p}_3 u(p_2)\epsilon_3.p_2 p_{J\mu} - \frac{s_{23}}{2}\bar{u}(p_1)\not{\epsilon}_3 u(p_2)p_{J\mu} + \frac{s_{J4}}{4}\bar{u}(p_1)\gamma_\mu\not{p}_3\not{\epsilon}_3 u(p_2) \\
T_\mu &= s_{23}\left(\bar{u}(p_1)\gamma_\mu u(p_2)\epsilon_3.p_1 + \frac{1}{2}\bar{u}(p_1)\not{\epsilon}_3\not{p}_3\gamma_\mu u(p_2)\right) \\
&- s_{13}\left(\bar{u}(p_1)\gamma_\mu u(p_2)\epsilon_3.p_2 + \frac{1}{2}\bar{u}(p_1)\gamma_\mu\not{p}_3\not{\epsilon}_3 u(p_2)\right)
\end{aligned}
$$

- Coefficients of the tensor structures are **scalar functions** in s_{12}, s_{23}, s_{123}

- We can construct projectors such that

$$
\sum_{\text{spins}} \mathcal{P}(X)\epsilon_4^\mu S_\mu(p_1,p_2,p_3) = X(s_{12},s_{23},s_{123})
$$

where $X = A_{11}, A_{12}, \ldots, B$

Projector method; the physicists approach

$$
\begin{aligned}
S_\mu(p_1,p_2,p_3) \quad = \quad & A_{11}(s_{13},s_{23},s_{123})T_{11\mu} + A_{12}(s_{13},s_{23},s_{123})T_{12\mu} + A_{13}(s_{13},s_{23},s_{123})T_{13\mu} \\
+ \quad & A_{21}(s_{13},s_{23},s_{123})T_{21\mu} + A_{22}(s_{13},s_{23},s_{123})T_{22\mu} + A_{23}(s_{13},s_{23},s_{123})T_{23\mu} \\
+ \quad & B(s_{13},s_{23},s_{123})T_\mu,
\end{aligned}
$$

$$
\sum_{\text{spins}} \mathcal{P}(X)\epsilon_4^\mu S_\mu(p_1,p_2,p_3) = X(s_{12},s_{23},s_{123})
$$

- For instance to project on A_{11}

$$
\begin{aligned}
\mathcal{P}(A_{11}) \quad = \quad & \frac{(s_{23}s_{123}d + s_{13}s_{12}(d-2))}{2s_{13}^3 s_{12}^2(d-3)s_{123}} T_{11}^\dagger \cdot \epsilon_4^* - \frac{(s_{13}+s_{23})(d-2)}{2s_{13}^2 s_{12}^2 (d-3)s_{123}} T_{12}^\dagger \cdot \epsilon_4^* \\
& - \frac{((s_{23}+s_{12})d + 2s_{13})}{2s_{12}s_{13}^3(d-3)s_{123}} T_{13}^\dagger \cdot \epsilon_4^* - \frac{(s_{23}s_{123}(d-2) + s_{13}s_{12}(d-4))}{2s_{23}s_{12}^2 s_{13}^2 s_{123}(d-3)} T_{21}^\dagger \cdot \epsilon_4^* \\
& + \frac{(s_{13}+s_{23})(d-4)}{2(d-3)s_{12}^2 s_{123}s_{13}s_{23}} T_{22}^\dagger \cdot \epsilon_4^* + \frac{(s_{23}+s_{12})(d-4)}{2s_{23}s_{12}s_{13}^2 s_{123}(d-3)} T_{23}^\dagger \cdot \epsilon_4^* \\
& - \frac{1}{2s_{13}^2 s_{12}^2 (d-3)} T^\dagger \cdot \epsilon_4^*
\end{aligned}
$$

- **This projectors can be used at any fixed order in perturbation theory and remaining Feynman integral are then in the desired form to use integration-by-parts identities**

Origin of the new method

Passarino – Veltman reduction for one-loop diagrams [Passarino, Veltman ‘1979]
oversimplified: make an ansatz for the full tensor structure involving external momenta and metric tensor and fix coefficients

There might be problems (mainly connected with an increasing number of external particles):

- *many tensor structures*
- *huge intermediate expressions*
- *computation of high dimensional gram determinants*
- *...*

OPP reduction [Ossola, Papadopoulos, Pittau ‘2006]
oversimplified: reduction on the integrand level

NLO “revolution”: Unitarity methods that allow the efficient numerical computation of arbitrary 1-loop **amplitudes** [Bern, Dixon, Kosower ‘2007; Ellis, Giele, Kunszt ‘2008; Giele, Kunszt, Melnikov ‘2008; Berger, Bern, Dixon, Febres Cordero, Forde, Ita, Kosower, Maitre ‘2008; ...]

Recent effort to repeat the success of unitarity methods for 1-loop amplitudes for 2-loop amplitudes [Ita ‘2015; Mastrolia, Peraro, Primo ‘2016; ...]

not yet but as a by-product

new algebraic method to reduce tensor integrals to scalar integrals

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There might be problems (mainly connected with an increasing number of external particles):

- *n*
- *h*
- *c … erminants*
- *…*

***Time has shown:** these concerns were not justified*

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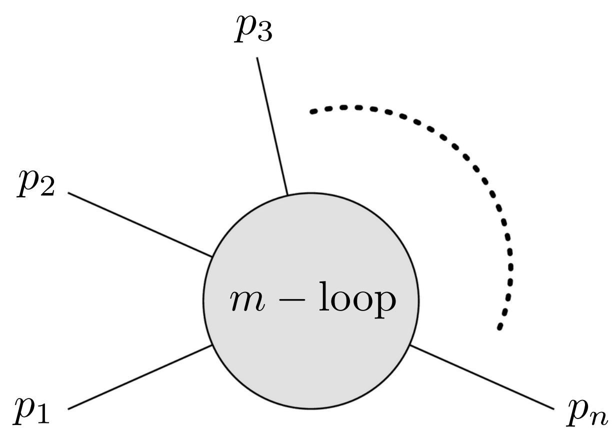
Physical and transverse space



- **Physical subspace:** subspace of d-dimensional space-time spanned by physical external momenta $\{p_1, p_2, p_3, \ldots, p_n\}$ … in other words: can be **expressed** in terms of the **propagators** of a well defined **integral family**

- Keeping external momenta in 4 dimensions, this space has dimension $D_{\|} = \min(4, n-1)$

- **Transversal subspace:** subspace that is orthogonal to all external momenta with dimension $D_{\perp} = d - D_{\|}$

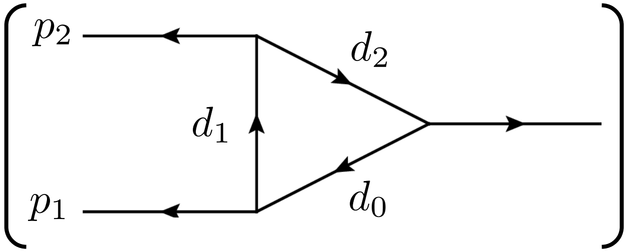
- If $n \geq 5$ all physical 4 dimensional vectors (momenta, polarization vectors, …) are contained in the physical space

⟶ **Only need to consider** $n \leq 4$

OPP reduction “lite” [Ossola, Papadopoulos, Pittau ‘2007]

- As example we consider a 1-loop tensor integral

$$\int \mathrm{d}^d l \, \frac{(l \cdot u)^2}{d_0 d_1 d_2} \longrightarrow \int \mathrm{d}^d l \, \frac{l^\mu l^\nu}{d_0 d_1 d_2}$$



with propagators

$$d_0 = l^2 - m_0^2 \qquad d_1 = (l - q_1)^2 - m_1^2 \qquad d_2 = (l - q_2)^2 - m_2^2$$

- **GOAL: rewrite the tensor integral in terms of scalar integrals**

- The integration over the loop momentum is split into **physical** and **transverse** space

$$\left.\begin{array}{r} \displaystyle\int \mathrm{d}^d l \\ l^\mu \\ d_i = (l - q_i)^2 - m_i^2 \end{array}\right\} \longrightarrow \left\{\begin{array}{l} \boxed{\displaystyle\int \mathrm{d}^{D_\parallel} l_{\parallel}} \boxed{\displaystyle\int \mathrm{d}^{D_\perp} l_{\perp}} \\ l^\mu = \boxed{l_{\parallel}^\mu} + \boxed{l_{\perp}^\mu} \\ d_i = \boxed{l_{\parallel}^2 - 2 l_{\parallel} \cdot q_i} + \boxed{l_{\perp}^2} - m_i^2 \end{array}\right.$$

OPP reduction "lite"

- The integral then becomes

$$\int \mathrm{d}^d l \; \frac{l^\mu l^\nu}{d_0 d_1 d_2} \quad \longrightarrow \quad \int \mathrm{d}^{D_\parallel} l_\parallel \int \mathrm{d}^{D_\perp} l_\perp \left[\frac{l_\parallel^\mu l_\parallel^\nu}{d_0 d_1 d_2} + \frac{l_\parallel^\mu l_\perp^\nu + l_\perp^\mu l_\parallel^\nu}{d_0 d_1 d_2} + \frac{l_\perp^\mu l_\perp^\nu}{d_0 d_1 d_2} \right]$$

- By construction **no dependence on the direction of the transverse component** in the propagators $d_i = l_\parallel^2 - 2 l_\parallel \cdot q_i + l_\perp^2 - m_i^2$
- Only object to describe the tensor structure: metric tensor of the transversal space $g_\perp^{\mu\nu}$

$$\int \mathrm{d}^{D_\perp} l_\perp \; \frac{l_\perp^\mu}{d_0 d_1 d_2} = 0 \quad \longrightarrow$$

true for any odd number of transversal components

$$\int \mathrm{d}^{D_\perp} l_\perp \; \frac{l_\perp^\mu l_\perp^\nu}{d_0 d_1 d_2} = g_\perp^{\mu\nu} A \quad \xrightarrow[\text{contract with } g_{\perp,\mu\nu}]{} \quad A = \frac{1}{D_\perp} \int \mathrm{d}^{D_\perp} l_\perp \; \frac{l_\perp^2}{d_0 d_1 d_2}$$

- The desired integral then becomes

$$\int \mathrm{d}^d l \; \frac{l^\mu l^\nu}{d_0 d_1 d_2} = \int \mathrm{d}^{D_\parallel} l_\parallel \int \mathrm{d}^{D_\perp} l_\perp \left[\frac{l_\parallel^\mu l_\parallel^\nu}{d_0 d_1 d_2} + \frac{g_\perp^{\mu\nu}}{D_\perp} \frac{l_\perp^2}{d_0 d_1 d_2} \right]$$

OPP reduction “lite”

$$\int \mathrm{d}^d l \; \frac{l^\mu l^\nu}{d_0 d_1 d_2} = \int \mathrm{d}^{D_\parallel} l_\parallel \int \mathrm{d}^{D_\perp} l_\perp \left[\frac{l_\parallel^\mu l_\parallel^\nu}{d_0 d_1 d_2} + \frac{g_\perp^{\mu\nu}}{D_\perp} \frac{\boxed{l_\perp^2}}{d_0 d_1 d_2} \right]$$

- All objects on the right-hand side can be rewritten in terms of $l^\mu = l_\parallel^\mu + l_\perp^\mu$
- **First:** scalar products of **transversal components** can be rewritten using l^2 and d_0

$$d_0 = l^2 - m_0^2 \longrightarrow \left. \begin{aligned} l^2 &= l_\parallel^2 + l_\perp^2 \\ l^2 &= d_0 + m_0^2 \end{aligned} \right\} \longrightarrow \boxed{l_\perp^2} = d_0 + m_0^2 - \boxed{l_\parallel^2}$$

we obtain

$$\int \mathrm{d}^d l \; \frac{l^\mu l^\nu}{d_0 d_1 d_2} = \int \mathrm{d}^{D_\parallel} l_\parallel \int \mathrm{d}^{D_\perp} l_\perp \left[\frac{\boxed{l_\parallel^\mu l_\parallel^\nu}}{d_0 d_1 d_2} + \frac{g_\perp^{\mu\nu}}{D_\perp} \frac{d_0 + m_0^2 - \boxed{l_\parallel^2}}{d_0 d_1 d_2} \right]$$

- **Second:** **Parallel components** can be **fully expressed** in terms of inverse **propagators**
- This can be achieved conveniently using the **van Neerven – Vermaseren basis**

Intermezzo: Van Neerven - Vermaseren basis [van Neerven, Vermaseren '1984]

- Consider a n-dimensional vector space spanned by a set of non-orthogonal, not normalized vectors $\{q_1, \ldots, q_n\}$
- To express a vector l^μ in this space we could make the ansatz $l^\mu = c_1 q_1^\mu + \cdots + c_n q_n^\mu$ but finding the coefficients c_i is complicated
- It is possible to find a basis where

$$l^\mu = (l \cdot q_1) b_1^\mu + \cdots + (l \cdot q_n) b_n^\mu \qquad \textit{with} \qquad q_i \cdot b_j = \delta_{ij} \qquad \textit{but} \qquad b_i \cdot b_j \neq \delta_{ij}$$

- The new basis vectors are defined as

$$b_i^\mu = \frac{\delta^{q_1 \cdots q_{i-1} \mu q_{i+1} q_n}_{q_1 \cdots q_n}}{\delta^{q_1 \cdots q_n}_{q_1 \cdots q_n}}$$

with

$$\delta^{\mu_1 \ldots \mu_n}_{\nu_1 \ldots \nu_n} = \det \begin{pmatrix} \delta^{\mu_1}_{\nu_1} & \delta^{\mu_1}_{\nu_2} & \cdots & \delta^{\mu_1}_{\nu_n} \\ \delta^{\mu_2}_{\nu_1} & \delta^{\mu_2}_{\nu_2} & \cdots & \delta^{\mu_2}_{\nu_n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{\mu_n}_{\nu_1} & \delta^{\mu_n}_{\nu_2} & \cdots & \delta^{\mu_n}_{\nu_n} \end{pmatrix} \qquad \textit{and} \qquad \delta^{k \ldots \mu_n}_{\nu_1 \ldots \nu_n} = k_{\mu_1} \delta^{\mu_1 \ldots \mu_n}_{\nu_1 \ldots \nu_n}$$

Back to: OPP reduction "lite"

$$\int \mathrm{d}^d l \; \frac{l^\mu l^\nu}{d_0 d_1 d_2} = \int \mathrm{d}^{D_\parallel} l_\parallel \int \mathrm{d}^{D_\perp} l_\perp \left[\frac{\boxed{l_\parallel^\mu l_\parallel^\nu}}{d_0 d_1 d_2} + \frac{g_\perp^{\mu\nu}}{D_\perp} \frac{d_0 + m_0^2 - \boxed{l_\parallel^2}}{d_0 d_1 d_2} \right]$$

- The van Neerven-Vermaseren basis for the space spanned by $\{q_1, q_2\}$ reads

$$b_1^\mu = \frac{q_2^2 q_1^\mu - (q_1 \cdot q_2) q_2^\mu}{q_1^2 q_2^2 - (q_1 \cdot q_2)^2} \qquad\qquad b_2^\mu = \frac{q_1^2 q_2^\mu - (q_1 \cdot q_2) q_1^\mu}{q_1^2 q_2^2 - (q_1 \cdot q_2)^2}$$

- Using this basis we can write the parallel part of the loop momentum as $l_\parallel^\mu = (l \cdot q_1) b_1^\mu + (l \cdot q_2) b_2^\mu$
- Scalar products $(l \cdot q_{i=1,2})$ can be expressed in terms of propagators, e.g.

$$\left.\begin{aligned} d_0 &= l^2 - m_0^2 \\ d_1 &= l^2 - 2 l \cdot q_1 + q_1^2 - m_1^2 \end{aligned}\right\} \longrightarrow \quad l \cdot q_1 = \frac{d_0 - d_1 + q_1^2 + m_0^2 - m_1^2}{2}$$

- The loop momentum can be therefore written as

$$\boxed{l_\parallel^\mu} = \left(\frac{d_0 - d_1 + q_1^2 + m_0^2 - m_1^2}{2} \right) b_1^\mu + \left(\frac{d_0 - d_2 + q_2^2 + m_0^2 - m_2^2}{2} \right) b_2^\mu$$

- **Finally:** integral only depends on $l_\parallel^\mu + l_\perp^\mu$: integration over physical and transverse components can be re-combined $\rightarrow$ integral is now in a form suitable for IBP reduction

Bonus: metric tensor of the transversal space

- It is convenient to rewrite the metric tensor of the transversal space through the d-dimensional metric tensor and momenta $\{q_1, q_2\}$
- Using the van Neerven – Vermaseren basis this is straightforward
- For an arbitrary d-dimensional vector v^μ we write

$$v_\mu g_\perp^{\mu\nu} = v_\perp^\nu = v_\mu g^{\mu\nu} - v_\parallel^\nu$$

- Using the van Neerven – Vermaseren basis we can write

$$v_\parallel^\nu = (v \cdot q_1) b_1^\nu + (v \cdot q_2) b_2^\nu = v_\mu (q_1^\mu b_1^\nu + q_2^\mu b_2^\nu)$$

- The metric tensor can be read off from this two equations

$$g_\perp^{\mu\nu} = g^{\mu\nu} - q_1^\mu b_1^\nu - q_2^\mu b_2^\nu \qquad \textit{with} \qquad b_1^\mu = \frac{q_2^2 q_1^\mu - (q_1 \cdot q_2) q_2^\mu}{q_1^2 q_2^2 - (q_1 \cdot q_2)^2} \qquad b_2^\mu = \frac{q_1^2 q_2^\mu - (q_1 \cdot q_2) q_1^\mu}{q_1^2 q_2^2 - (q_1 \cdot q_2)^2}$$

Some remarks

- The described procedure has **most but not all** of the ingredients of full OPP reduction
- OPP reduction follows the same lines but loop momenta are split into physical and transverse part iteratively

$$\int \mathrm{d}^d l \; \frac{l^\mu l^\nu}{d_0 d_1 d_2} \longrightarrow \int \mathrm{d}^d l \; \frac{l^\mu(\text{const}^\nu + c_0^\nu d_0 + c_1^\nu d_1 + c_2^\nu d_2 + l_\perp^\mu)}{d_0 d_1 d_2}$$

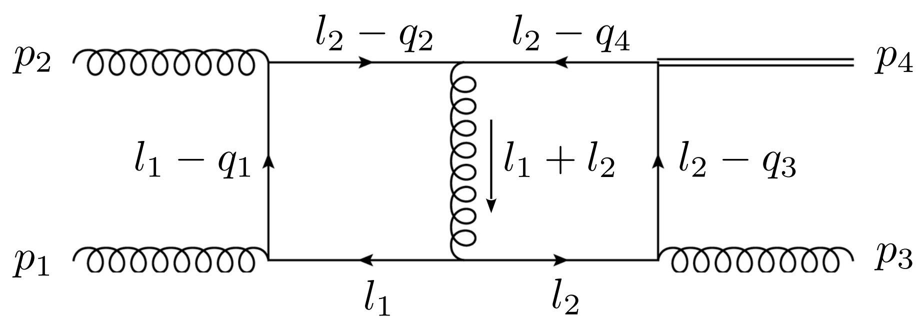
$$= \int \mathrm{d}^d l \left(\frac{l^\mu \text{const}^\nu}{d_0 d_1 d_2} + \frac{l^\mu c_0^\nu}{d_1 d_2} + \frac{l^\mu c_1^\nu}{d_0 d_2} + \frac{l^\mu c_2^\nu}{d_0 d_1} + \frac{l^\mu l_\perp^\mu}{d_0 d_1 d_2} \right)$$

- Physical and transverse space of the second loop momenta are then defined depending on which propagators are still present
- Similar to Passarino – Veltman reduction the final result is then "schematically"

$$\int \mathrm{d}^d l \; \frac{l^\mu l^\nu}{d_0 d_1 d_2} = C_3^{\mu\nu} \int \mathrm{d}^d l \frac{1}{d_0 d_1 d_2} + C_2^{\mu\nu} \int \mathrm{d}^d l \frac{1}{d_0 d_1} + C_1^{\mu\nu} \int \mathrm{d}^d l \frac{1}{d_0}$$

- **The OPP reduction method cannot be extended to multi-loop integrals; however, the previously presented "lite" version can**

2-loop example: Higgs + jet in gluon fusion



- As an example we consider the tensor integral

$$\int \mathrm{d}^d l_1 \int \mathrm{d}^d l_2 \; \frac{l_1^\mu l_1^\nu l_2^\rho l_2^\sigma}{d_{10} d_{11} d_{12} d_{20} d_{21} d_{22} d_{30}}$$

with propagators

$$d_{10} = l_1^2 - m^2 \qquad d_{20} = l_2^2 - m^2 \qquad \mathrm{d}_{30} = (l_1 + l_2)^2$$
$$d_{11} = (l_1 - p_1)^2 - m^2 \qquad d_{21} = (l_2 - p_3)^2 - m^2$$
$$d_{11} = (l_1 - p_1 - p_2)^2 - m^2 \qquad d_{21} = (l_2 + p_1 + p_2)^2 - m^2$$

- To close the algebra of inverse propagators we need to introduce two additional propagators, e.g.

$$d_{13} = (l_1 + p_3)^2$$
$$d_{23} = (l_2 + p_1)^2$$

2-loop example: Higgs + jet in gluon fusion

$$\int \mathrm{d}^d l_1 \int \mathrm{d}^d l_2 \; \frac{l_1^\mu l_1^\nu l_2^\rho l_2^\sigma}{d_{10} d_{11} d_{12} (d_{13})^0 d_{20} d_{21} d_{22} (d_{23})^0 d_{30}}$$

- We insert $l_{i=1,2}^\mu = l_{i\parallel} + l_{i\perp}$ and also split integration into physical and transversal components
- Terms with a odd number of $l_{1\perp}^\mu$ and $l_{2\perp}^\mu$ vanish
- Three structures that involve transversal components remain

$$\int \mathrm{d}^{D_\perp} l_{1\perp} \int \mathrm{d}^{D_\perp} l_{2\perp} \; \frac{l_{1\perp}^\mu l_{1\perp}^\nu l_{2\perp}^\rho l_{2\perp}^\sigma}{d_{10} d_{11} d_{12} (d_{13})^0 d_{20} d_{21} d_{22} (d_{23})^0 d_{30}}$$

$$\int \mathrm{d}^{D_\perp} l_{1\perp} \int \mathrm{d}^{D_\perp} l_{2\perp} \; \frac{l_{1\perp}^\mu l_{1\perp}^\nu}{d_{10} d_{11} d_{12} (d_{13})^0 d_{20} d_{21} d_{22} (d_{23})^0 d_{30}} \longrightarrow$$

no dependence on $l_{2\perp}^\mu$, already discussed in the previous 1-loop example

$$\int \mathrm{d}^{D_\perp} l_{1\perp} \int \mathrm{d}^{D_\perp} l_{2\perp} \; \frac{l_{1\perp}^\mu l_{2\perp}^\rho}{d_{10} d_{11} d_{12} (d_{13})^0 d_{20} d_{21} d_{22} (d_{23})^0 d_{30}} \longrightarrow$$

same as above but with $l_{1\perp} \cdot l_{2\perp}$ after averaging over transversal components

2-loop example: Higgs + jet in gluon fusion

$$\int \mathrm{d}^{D_\perp} l_{1\perp} \int \mathrm{d}^{D_\perp} l_{2\perp} \; \frac{l_{1\perp}^{\mu} l_{1\perp}^{\nu} l_{2\perp}^{\rho} l_{2\perp}^{\sigma}}{d_{10} d_{11} d_{12} (d_{13})^0 d_{20} d_{21} d_{22} (d_{23})^0 d_{30}}$$

- Using the $\mu \leftrightarrow \nu$ and $\rho \leftrightarrow \sigma$ symmetries we make the ansatz

$$\int \mathrm{d}^{D_\perp} l_{1\perp} \int \mathrm{d}^{D_\perp} l_{2\perp} \; \frac{l_{1\perp}^{\mu} l_{1\perp}^{\nu} l_{2\perp}^{\rho} l_{2\perp}^{\sigma}}{d_{10} d_{11} d_{12} (d_{13})^0 d_{20} d_{21} d_{22} (d_{23})^0 d_{30}} = g_\perp^{\mu\nu} g_\perp^{\rho\sigma} A + \big(g_\perp^{\mu\sigma} g_\perp^{\nu\rho} + g_\perp^{\mu\rho} g_\perp^{\nu\sigma} \big) B$$

- Upon contracting with $g_\perp^{\mu\nu} g_\perp^{\rho\sigma}$ and $g_\perp^{\mu\sigma} g_\perp^{\nu\rho}$ we obtain the system of linear equations

$$\int \mathrm{d}^{D_\perp} l_{1\perp} \int \mathrm{d}^{D_\perp} l_{2\perp} \; \frac{l_{1\perp}^{2} l_{2\perp}^{2}}{d_{10} d_{11} d_{12} (d_{13})^0 d_{20} d_{21} d_{22} (d_{23})^0 d_{30}} = D_\perp^2 A + 2 D_\perp B$$

$$\int \mathrm{d}^{D_\perp} l_{1\perp} \int \mathrm{d}^{D_\perp} l_{2\perp} \; \frac{(l_{1\perp} \cdot l_{2\perp})^2}{d_{10} d_{11} d_{12} (d_{13})^0 d_{20} d_{21} d_{22} (d_{23})^0 d_{30}} = D_\perp A + D_\perp \big(D_\perp + 1 \big) B$$

- Solving this system for A and B leads to ...

2-loop example: Higgs + jet in gluon fusion

$$\int \mathrm{d}^{D_\perp} l_{1\perp} \int \mathrm{d}^{D_\perp} l_{2\perp} \, \frac{l_{1\perp}^\mu l_{1\perp}^\nu l_{2\perp}^\rho l_{2\perp}^\sigma}{d_{10} d_{11} d_{12} (d_{13})^0 d_{20} d_{21} d_{22} (d_{23})^0 d_{30}}$$

$$= \frac{g_\perp^{\mu\nu} g_\perp^{\rho\sigma}}{D_\perp (D_\perp - 1)(D_\perp + 2)} \int \mathrm{d}^{D_\perp} l_{1\perp} \int \mathrm{d}^{D_\perp} l_{2\perp} \, \frac{(D_\perp + 1) \boxed{l_{1\perp}^2 l_{2\perp}^2} - 2 \boxed{(l_{1\perp} \cdot l_{2\perp})^2}}{d_{10} d_{11} d_{12} (d_{13})^0 d_{20} d_{21} d_{22} (d_{23})^0 d_{30}}$$

$$+ \frac{g_\perp^{\mu\sigma} g_\perp^{\nu\rho} + g_\perp^{\mu\rho} g_\perp^{\nu\sigma}}{D_\perp (D_\perp - 1)(D_\perp + 2)} \int \mathrm{d}^{D_\perp} l_{1\perp} \int \mathrm{d}^{D_\perp} l_{2\perp} \, \frac{D_\perp \boxed{(l_{1\perp} \cdot l_{2\perp})^2} - \boxed{l_{1\perp}^2 l_{2\perp}^2}}{d_{10} d_{11} d_{12} (d_{13})^0 d_{20} d_{21} d_{22} (d_{23})^0 d_{30}}$$

- Scalar products of transversal components can be expressed again using propagators

$$\boxed{l_{1\perp}^2} = d_{10} + m^2 - \boxed{l_{1\parallel}^2}$$

$$\boxed{l_{2\perp}^2} = d_{20} + m^2 - \boxed{l_{2\parallel}^2}$$

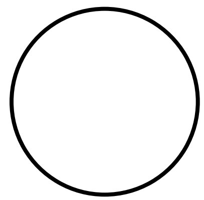
$$\boxed{l_{1\perp} \cdot l_{2\perp}} = \frac{1}{2} \left(d_{30} - d_{10} - d_{20} - 2m^2 - \boxed{2 l_{1\parallel} \cdot l_{2\parallel}} \right)$$

- The transversal metric tensor and the **parallel components** can be expressed in terms of propagators using the van Neerven – Vermaseren basis; for the latter: having introduced the additional propagators is crucial for this step

Topologies and independent families

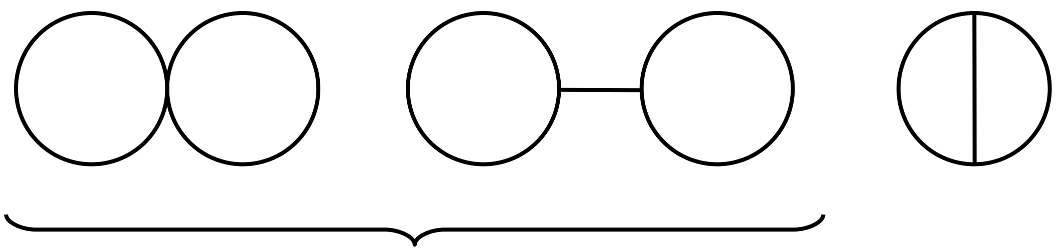
- Ignoring tadpoles and propagator corrections: To obtain a full set of identities we only need to consider 3 and 4 external momenta

- **1-loop:** 1 topology



→ ***2 diagrams***

- **2-loop:** 2 sets of topologies



→ ***5 additional diagrams***

Integration over loop momenta factorizes

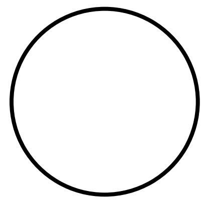
$$\int \mathrm{d}^d l_1 \;\; f_1(l_1, \{p_i\}) \times \int \mathrm{d}^d l_2 \;\; f_2(l_2, \{p_i\})$$

- **3-loop:** 3 sets of topologies, partially overlapping → ***O(25) additional diagrams***

Topologies and independent families

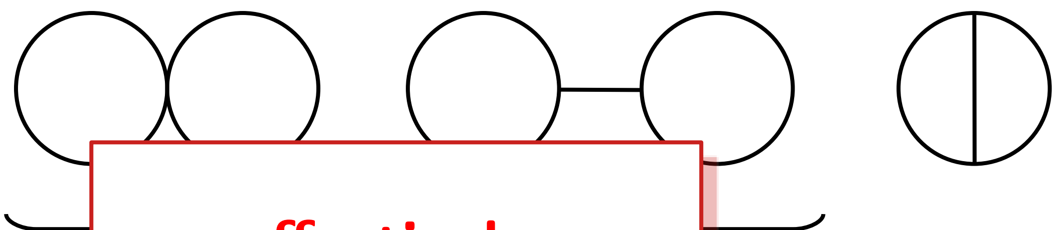
- Ignoring tadpoles and propagator corrections: To obtain a full set of identities we only need to consider 3 and 4 external momenta

- **1-loop:** 1 topology



2 diagrams

- **2-loop:** 2 sets of topologies



7 additional diagrams

Int ... orizes

effectively

1-loop x 1-loop

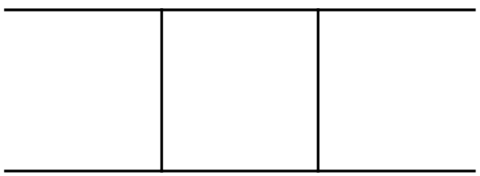
$\int d \ldots \times \int \ldots p_i\})$

- **3-loop:** 3 sets of topologies, partially overlapping

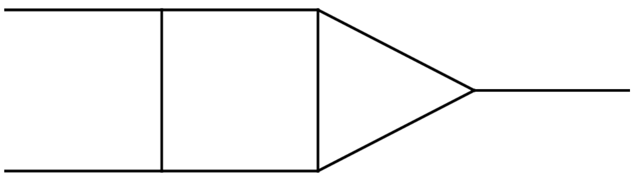
O(25) additional diagrams

Highest ranks

- Each of this dirgams needs to be considered up to a finite rank
- In renormalizable theories the highest necessary rank can be easily read of by counting vertices



1-loop box
→ up to rank 4



2-loop triangle
→ up to rank 5

- Doing the combinatorics:

1-loop: 3-point → 3 identities / **4-point → 4 identities** ⟶ 7 **identities**

2-loop: **3-point → 15 identities** / 4-point → 24 identities ⟶ 39 **identities**

3-loop: ...

Final remarks

- Orthogonal approach to currently used projector method

  Projector method: is based on physical properties of the external particles; has to be done once for each process and can then be used at each order in perturbation theory

  “Averaging” method: purely algebraic approach; linear algebra no physics; has to be done once for each order in perturbation theory and can then be used for any process

- Due to physical constraints (highest rank, external vectors are 4-dimensional) only a “small” number of identities needed
- Never used to replace projector method
- Used in the context of generalized unitary methods for 2-loop computations
  [Ita et al., Mastrolia et al., ...]
- Can be used to automatize 2-loop computations up to master integrals

Notes

Do we need a new method?

- Projector are complicated objects and require a lot of computation time
- Need to be computed for each process but can be used at any order in perturbation theory
- Projectors spoil the origin of tensor structures → not suitable if one wants to understand the tensor structure

The new method:

- Algebraic method → complete understanding of tensor structure
- Needs to be computed for each tensor integral but can be applied to any process
- Is used in the context of generalized unitary methods [leading colour planar 5 parton, 2-loop Ita et al. ‘2018 - 19]