EIGENVECTORS FROM EIGENVALUES

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Abstract. We present and prove a method of succinctly determining eigenvectors from eigenvalues. Specifically, we relate the norm squared of the elements of eigenvectors to the eigenvalues and the submatrix eigenvalues for Hermitian matrices.

1. Expanded Version of the Determinant Proof

We present here a lengthier version of the determinant proof for the interested reader. The approach we will take is as follows. We begin by stating the general result for any size Hermitian matrix. Then we shift the eigenvalues. We prove a useful Cauchy-Binet style theorem along the way and then apply that to our particular case.

Let A be a $n \times n$ Hermitian matrix with eigenvalues $\lambda_i(A)$ and unit eigenvectors v_i. The elements of each eigenvector are denoted $v_{i,j}$ for $j \in [1, n]$. Let M_j be the $n-1 \times n-1$ submatrix of A that results from deleting the j^{th} row and column, with $n-1$ eigenvalues $\lambda_i(M_j)$. The generalized result can be simply stated as,

$$|v_{i,j}|^2 \prod_{k=1;k\neq i}^{n} (\lambda_i(A) - \lambda_k(A)) = \prod_{k=1}^{n-1} (\lambda_i(A) - \lambda_k(M_j)) \,, \tag{1}$$

which we note is well defined even if two of the eigenvalues are equal.

We now present a proof of this result. Take $j = 1$ and $i = n$. Next, we make the transformation $A \to A - \lambda_n(A) I_n$ which sets $\lambda_n(A) = 0$; this also shifts the other eigenvalues $\lambda_i(A)$ and $\lambda_i(M_j)$ down by the same amount, but clearly does not modify eq. 1 which is only a function of the differences of eigenvalues. After this shift, eq. 1 is,

$$|v_{n,1}|^2 \prod_{k=1}^{n-1} \lambda_k(A) = \prod_{k=1}^{n-1} \lambda_k(M_1) \,. \tag{2}$$

Note that the RHS of eq. 2 is $\det(M_1)$.

Next, we state and prove a useful intermediate result for a Hermitian $n \times n$ matrix A where $\lambda_n(A) = 0$:

$$\det(B^\dagger A B) = |\det(B \quad v_n)|^2 \prod_{i=1}^{n-1} \lambda_i(A) \,, \tag{3}$$

where B is an arbitrary $n \times n-1$ matrix and $(B \quad v_n)$ is a $n \times n$ matrix where B is the left $n \times n-1$ component. This result is similar to the Cauchy-Binet formula.

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We make the useful definition,

$$B = \begin{pmatrix} b \\ X \end{pmatrix}, \tag{4}$$

where b is an $n-1 \times n-1$ matrix and X is a $1 \times n-1$ vector that will be annihilated. We will show that both sides of eq. 3 are equal to

$$|\det(b)|^2 \prod_{i=1}^{n-1} \lambda_i(A)\,. \tag{5}$$

First, we define V such that V diagonalizes A. That is, $A = VDV^\dagger$ where $D \equiv \mathrm{diag}(\lambda_1(A), \ldots, \lambda_{n-1}(A), 0)$. Then eq. 3 is

$$\det(B^\dagger VDV^\dagger B) = |\det(B \quad v_n)|^2 \prod_{i=1}^{n-1} \lambda_i(A)\,. \tag{6}$$

If we define a new matrix B' given by $V^\dagger B$ and a new vector v_n' given by $V^\dagger v_n$, then we have,

$$\det(B'^\dagger DB') = |\det[V(B' \quad v_n')]|^2 \prod_{i=1}^{n-1} \lambda_i(A)\,. \tag{7}$$

Since V is unitary (since A is Hermitian) and $\det(AB) = \det(A)\det(B)$, we get

$$\det(B'^\dagger DB') = |\det(B' \quad v_n')|^2 \prod_{i=1}^{n-1} \lambda_i(A)\,. \tag{8}$$

We now drop the primes and have the same expression as eq. 3 except that A is diagonal. In addition, since the matrix in question is diagonal, the normed eigenvector is just $v_n = e_n$ where $e_n = (0, \ldots, 0, 1)^T$. That is, the equation we wish to prove is

$$\det(B^\dagger DB) = |\det(B \quad e_n)|^2 \prod_{i=1}^{n-1} \lambda_i(A)\,, \tag{9}$$

where $D = \mathrm{diag}(\lambda_1(A), \ldots, \lambda_{n-1}(A), 0)$.

Using eq. 4 we find that

$$B^\dagger DB = \begin{pmatrix} b^\dagger & X^\dagger \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b \\ X \end{pmatrix} = b^\dagger db\,, \tag{10}$$

where $d \equiv \mathrm{diag}(\lambda_1(A), \ldots, \lambda_{n-1}(A))$. Then since b and d are both $n-1 \times n-1$ matrices, we use the fact that $\det(AB) = \det(A)\det(B)$ and $\det(B^\dagger) = \det((\overline{B})^T) = \det(\overline{B}) = \overline{\det(B)}$ to see that

$$\det(B^\dagger DB) = \det(b^\dagger)\det(d)\det(b) = |\det(b)|^2 \prod_{i=1}^{n-1} \lambda_i(A)\,, \tag{11}$$

which shows that the LHS of eq. 10 (and thus the LHS of eq. 3) is equal to the desired quantity, eq. 5.

Next, for the RHS of eq. 9 we note that

$$\det(B \quad e_n) = \begin{vmatrix} & 0 \\ b & \vdots \\ & 0 \\ X & 1 \end{vmatrix} = \det(b)\,. \tag{12}$$

This shows that the RHS of eq. 9 (and thus the RHS of eq. 3) is equal to the desired quantity, eq. 5. This completes the proof of eq. 3.

To complete the proof of eq. 2, we apply eq. 3 for a specific B,

$$B = \begin{pmatrix} 0 & \cdots & 0 \\ & I_{n-1} & \end{pmatrix}\,. \tag{13}$$

We notice that $B^\dagger AB = M_1$ since the zeros in B kill the first row and the first column of A which leaves just M_1, so the LHS of eq. 3 is the RHS of eq. 2. To complete the proof it suffices to show that

$$|v_{n,1}|^2 = |\det(B \quad v_n)|^2\,. \tag{14}$$

Since

$$(B \quad v_n) = \begin{pmatrix} 0 & v_{n,1} \\ & v_{n,2} \\ I_{n-1} & \vdots \\ & v_{n,n} \end{pmatrix}\,, \tag{15}$$

the determinant of this is $(-1)^n v_{n,1} \det(I_{n-1}) = (-1)^n v_{n,1}$ which completes the proof.

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