Abstract

The oscillation of neutrinos is a distinct phenomenon in particle physics; it is also the best probe we have to the remaining unknown parameters of particle physics. This new phenomenon adds at least 7 parameters to the Standard Model and oscillations can probe 6 of them. In this talk I will discuss the impact of the matter effect and the difficulty of exact expressions. I will show how we stumbled into an interesting math identity and then derive a proof.

Neutrino Oscillations in Matter and Linear Algebra

Peter B. Denton

HET Lunch Discussion

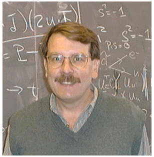
July 23, 2021

Brookhaven National Laboratory

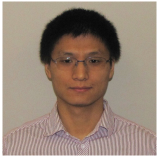
Brookhaven NDI
Neutrino Discovery Initiative

Speaking from Setauket land

Analytic Oscillation Probability Collaborators



Stephen Parke



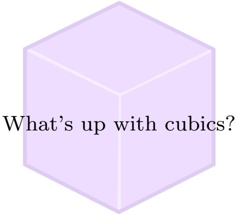
Xining Zhang

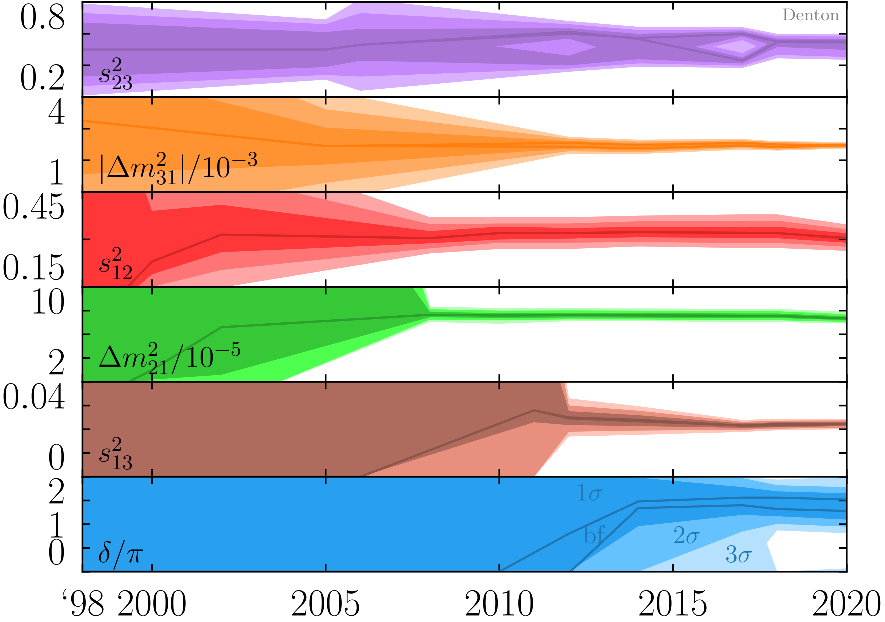
Terence Tao

Path

1. **Statement** of oscillation question
2. Get the eigen**values**

3. Get the eigen**vectors**
4. One **proof**





Denton
0.8
0.2
s^2_{23}
4
1
$|\Delta m^2_{31}|/10^{-3}$
0.45
0.15
s^2_{12}
10
2
$\Delta m^2_{21}/10^{-5}$
0.04
0
s^2_{13}
2
1
0
δ/π
1σ
bf
2σ
3σ
'98 2000
2005
2010
2015
2020

Schrödinger Equation

Neutrinos propagate in eigenstates of the Hamiltonian

$$i\frac{d}{dt}|\nu\rangle = H|\nu\rangle$$

In the absence of any interactions $H_{\rm vac}|\nu_i\rangle = E_i|\nu_i\rangle$.

$$|\nu_i(L)\rangle = e^{-iE_iL}|\nu_i(0)\rangle \to e^{-im_i^2L/2E}|\nu_i(0)\rangle$$

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We don't produce neutrinos in eigenstates of the Hamiltonian in vacuum, e.g. mass eigenstates

$$|\nu_\alpha\rangle = \sum_{i=1}^{3} U_{\alpha i}^*|\nu_i\rangle \qquad\qquad \alpha \in \{e, \mu, \tau\}$$

U is a unitary 3×3 matrix which has four degrees of freedom

Unitarity $\Rightarrow$ 9 dofs, rephasing $\Rightarrow 9 - 5 = 4$

Matter Effects Matter

$$\mathcal{A}(\nu_\alpha \to \nu_\beta) = \sum_{i=1}^{3} U_{\alpha i}^* \, e^{-im_i^2 L/2E} \, U_{\beta i} \qquad\qquad P = |\mathcal{A}|^2$$

Matter Effects Matter

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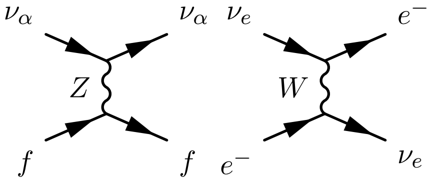
In matter ν's propagate in a new basis that depends on $a \propto N_e E_\nu$.



L. Wolfenstein PRD 17 (1978)

Eigenvalues: $m_i^2 \to \widehat{m^2}_i(a)$

Eigenvectors are given by $\theta_{ij} \to \widehat{\theta}_{ij}(a)$ $\quad\Leftarrow\quad$ Unitarity

Hamiltonian Dynamics

$$H_{\rm flav} = \frac{1}{2E}\left[U\begin{pmatrix}0 & & \\ & \Delta m^2_{21} & \\ & & \Delta m^2_{31}\end{pmatrix}U^\dagger + \begin{pmatrix}a & & \\ & 0 & \\ & & 0\end{pmatrix}\right]$$

$a = 2\sqrt{2}G_F N_e E$

$$U = \begin{pmatrix}1 & & \\ & c_{23} & s_{23} \\ & -s_{23} & c_{23}\end{pmatrix}\begin{pmatrix}c_{13} & & s_{13}e^{-i\delta} \\ & 1 & \\ -s_{13}e^{i\delta} & & c_{13}\end{pmatrix}\begin{pmatrix}c_{12} & s_{12} & \\ -s_{12} & c_{12} & \\ & & 1\end{pmatrix}$$

$s_{ij} = \sin\theta_{ij}$, $c_{ij} = \cos\theta_{ij}$

Find eigenvalues and eigenvectors:

PBD, R. Pestes 2006.09384

$$H_{\rm flav} = \frac{1}{2E}\widehat{U}\begin{pmatrix}0 & & \\ & \Delta\widehat{m^2}_{21} & \\ & & \Delta\widehat{m^2}_{31}\end{pmatrix}\widehat{U}^\dagger$$

J. Kopp physics/0610206

Computationally works, but we can do better than a **black box**...

Analytic expression?

Analytic Oscillation Probabilities in Matter

- ☑ Solar: $P_{ee} \simeq \sin^2\theta_\odot$

Approx: S. Mikheev, A. Smirnov Nuovo Cim. C9 (1986) 17-26

Exact: S. Parke PRL 57 (1986) 2322

- ☑ Long-baseline: All three flavors

Exact: H. Zaglauer, K. Schwarzer Z.Phys. C40 (1988) 273

Approx: PBD, H. Minakata, S. Parke, 1604.08167

Review: G. Barenboim, PBD, S. Parke, C. Ternes 1902.00517

- ☑ ν_e disappearance (neutrino factory): $\Delta\widehat{m^2}_{ee} = \widehat{m^2}_3 - (\widehat{m^2}_1 + \widehat{m^2}_2 - \Delta m^2_{21} c^2_{12})$

PBD, S. Parke, 1808.09453

- ☐ Atmospheric

Get the eigen**values**

Eigenvalues Analytically: The Exact Solution

Solve the cubic characteristic equation: eigen**values**

$$(\widehat{m^2}_i)^3 - A(\widehat{m^2}_i)^2 + B\widehat{m^2}_i - C = 0$$

$$A \equiv \sum_i \widehat{m^2}_i = \Delta m^2_{31} + \Delta m^2_{21} + a$$

$$B \equiv \sum_{i>j} \widehat{m^2}_i \widehat{m^2}_j = \Delta m^2_{31}\Delta m^2_{21} + a(\Delta m^2_{ee} c^2_{13} + \Delta m^2_{21})$$

$$C \equiv \prod_i \widehat{m^2}_i = a\Delta m^2_{31}\Delta m^2_{21} c^2_{13} c^2_{12}$$

G. Cardano *Ars Magna* 1545

V. Barger, et al. PRD 22 (1980) 2718

H. Zaglauer, K. Schwarzer Z.Phys. C40 (1988) 273

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Then write down eigen**vectors** (mixing angles)

Eigenvectors form a unitary matrix; parameterize by effective mixing angles

H. Zaglauer, K. Schwarzer Z.Phys. C40 (1988) 273

K. Kimura, A. Takamura, H. Yokomakura `hep-ph/0205295`

PBD, S. Parke, X. Zhang `1907.02534`

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PBD, S. Parke, X. Zhang 1907.02534

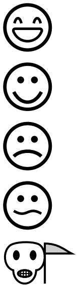
"Unfortunately, the algebra is rather impenetrable."

V. Barger, et al.

The Cubic

Math history aside

Linear:	$ax + b = 0$	
Quadratic:	$ax^2 + bx + c = 0$	
Cubic:	$ax^3 + bx^2 + cx + d = 0$	
Quartic:	$ax^4 + bx^3 + cx^2 + dx + e = 0$	
Qunitic+:	$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$	



Abel-Ruffini theorem, 1824

The Cubic

Math history aside

1. **Ancients** (20-16C BC)
   Babylonians, Greeks, Chinese, Indians, Egyptians:
   thought about cubics, calculated cube roots

$$x^3 = a$$

2. Chinese **Wang Xiaotong** (7C AD):
   numerically solved 25 general cubics
3. Persian **Omar Khayyam** (11C AD):
   realized there are multiple solutions
4. Italian **Fibonacci** (12C AD):
   Approximate solution to one cubic

Math history aside: (16C AD)

5. Scipione del **Ferro**:
   Secret solution, nearly all (didn't know negative numbers)
   $$x^3 + mx = n$$
6. Antonio **Fiore**: Ferro's student, from just before his death
7. Niccol **Tartaglia**: Claimed a solution, was challenged by Fiore
8. Gerolamo **Cardano**: Gets Tartaglia's (winner) solution, promises to keep it secret. Later publishes Ferro's solution via Fiore
9. Tartaglia challenges Cardano who denies it. Cardano's student **Ferrari** accepted, Tartaglia lost along with prestige and income
10. Cardano almost discovered complex numbers

Quartic was (nearly) solved around the same time by Ferrari,
before the cubic solution was published

(François Viète independently solved the cubic in France a few years later)

Back to neutrinos

Eigenvalues Analytically: The Exact Solution

The cubic solution (in neutrino terms)

$$\widehat{m^2}_1 = \frac{A}{3} - \frac{1}{3}\sqrt{A^2-3B}S - \frac{\sqrt{3}}{3}\sqrt{A^2-3B}\sqrt{1-S^2}$$

$$\widehat{m^2}_2 = \frac{A}{3} - \frac{1}{3}\sqrt{A^2-3B}S + \frac{\sqrt{3}}{3}\sqrt{A^2-3B}\sqrt{1-S^2}$$

$$\widehat{m^2}_3 = \frac{A}{3} + \frac{2}{3}\sqrt{A^2-3B}S$$

$$A = \Delta m^2_{21} + \Delta m^2_{31} + a$$

$$B = \Delta m^2_{21}\Delta m^2_{31} + a\left[c^2_{13}\Delta m^2_{31} + (c^2_{12}c^2_{13} + s^2_{13})\Delta m^2_{21}\right]$$

$$C = a\Delta m^2_{21}\Delta m^2_{31}c^2_{12}c^2_{13}$$

$$S = \cos\left\{\frac{1}{3}\cos^{-1}\left[\frac{2A^3 - 9AB + 27C}{2\left(A^2-3B\right)^{3/2}}\right]\right\}$$

H. Zaglauer, K. Schwarzer Z.Phys. C40 (1988) 273

Get the eigen**vectors**

Values and Vectors

Probability amplitude:

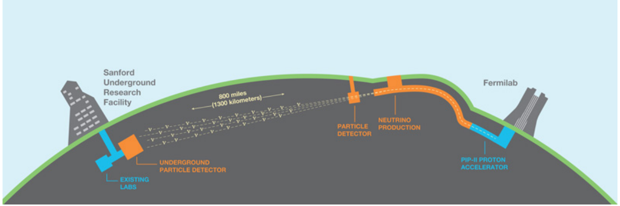
$$\mathcal{A}_{\alpha\beta} = \sum_i \widehat{U}^*_{\alpha i}\, e^{-i\widehat{m^2}_i L/2E}\, \widehat{U}_{\beta i}$$

- Eigen**values** give the frequencies of the oscillations

Where should DUNE be?

- Eigen**vectors** give the amplitudes of the oscillations

How many events will DUNE see?



Exact Neutrino Oscillations in Matter: Mixing Angles

$$s^2_{\widehat{12}} = \frac{-\left[(\widehat{m^2}_2)^2 - \alpha\widehat{m^2}_2 + \beta\right]\Delta\widehat{m^2}_{31}}{\left[(\widehat{m^2}_1)^2 - \alpha\widehat{m^2}_1 + \beta\right]\Delta\widehat{m^2}_{32} - \left[(\widehat{m^2}_2)^2 - \alpha\widehat{m^2}_2 + \beta\right]\Delta\widehat{m^2}_{31}}$$

$$s^2_{\widehat{13}} = \frac{(\widehat{m^2}_3)^2 - \alpha\widehat{m^2}_3 + \beta}{\Delta\widehat{m^2}_{31}\Delta\widehat{m^2}_{32}}$$

$$s^2_{\widehat{23}} = \frac{s^2_{23}E^2 + c^2_{23}F^2 + 2c_{23}s_{23}c_\delta EF}{E^2 + F^2}$$

$$e^{-i\widehat{\delta}} = \frac{c_{23}s_{23}\left(e^{-i\delta}E^2 - e^{i\delta}F^2\right) + \left(c^2_{23} - s^2_{23}\right)EF}{\sqrt{\left(s^2_{23}E^2 + c^2_{23}F^2 + 2EFc_{23}s_{23}c_\delta\right)\left(c^2_{23}E^2 + s^2_{23}F^2 - 2EFc_{23}s_{23}c_\delta\right)}}$$

$$\alpha = c^2_{13}\Delta m^2_{31} + \left(c^2_{12}c^2_{13} + s^2_{13}\right)\Delta m^2_{21}\,, \;\; \beta = c^2_{12}c^2_{13}\Delta m^2_{21}\Delta m^2_{31}$$

$$E = c_{13}s_{13}\left[\left(\widehat{m^2}_3 - \Delta m^2_{21}\right)\Delta m^2_{31} - s^2_{12}\left(\widehat{m^2}_3 - \Delta m^2_{31}\right)\Delta m^2_{21}\right]$$

$$F = c_{12}s_{12}c_{13}\left(\widehat{m^2}_3 - \Delta m^2_{31}\right)\Delta m^2_{21}$$

H. Zaglauer, K. Schwarzer Z.Phys. C40 (1988) 273

The Rosetta Stone

Eigenvalues to Eigenvectors

KTY pushed calculating the eigenvectors from the eigenvalues.

K. Kimura, A. Takamura, H. Yokomakura hep-ph/0205295

$$\widehat{U}_{\alpha i}\widehat{U}^*_{\beta i} = \frac{\widehat{p}_{\alpha\beta}\widehat{m^2}_i + \widehat{q}_{\alpha\beta} - \delta_{\alpha\beta}\widehat{m^2}_i(\widehat{m^2}_j + \widehat{m^2}_k)}{\Delta\widehat{m^2}_{ji}\Delta\widehat{m^2}_{ki}}$$

$$\widehat{p}_{\alpha\beta} = (2E)H_{\alpha\beta}$$

$$\widehat{q}_{\alpha\beta} = (2E)^2(H_{\gamma\beta}H_{\alpha\gamma} - H_{\alpha\beta}H_{\gamma\gamma})$$

valid for $\alpha \neq \beta$.

Wanted to preserve phase information for $\widehat{\delta}$.

Eigenvalues: the Rosetta Stone

We realized:

$$|\widehat{U}_{\alpha i}|^2 = \frac{(\widehat{m^2}_i - \xi_\alpha)(\widehat{m^2}_i - \chi_\alpha)}{\Delta\widehat{m^2}_{ij}\Delta\widehat{m^2}_{ik}}$$

PBD, S. Parke, X. Zhang 1907.02534

where ξ_α and χ_α are the submatrix eigenvalues

$$H = \begin{pmatrix} H_{\alpha\alpha} & H_{\alpha\beta} & H_{\alpha\gamma} \\ H_{\beta\alpha} & H_{\beta\beta} & H_{\beta\gamma} \\ H_{\gamma\alpha} & H_{\gamma\beta} & H_{\gamma\gamma} \end{pmatrix} \quad \to \quad H_\alpha = \begin{pmatrix} H_{\beta\beta} & H_{\beta\gamma} \\ H_{\gamma\beta} & H_{\gamma\gamma} \end{pmatrix}$$

e.g.

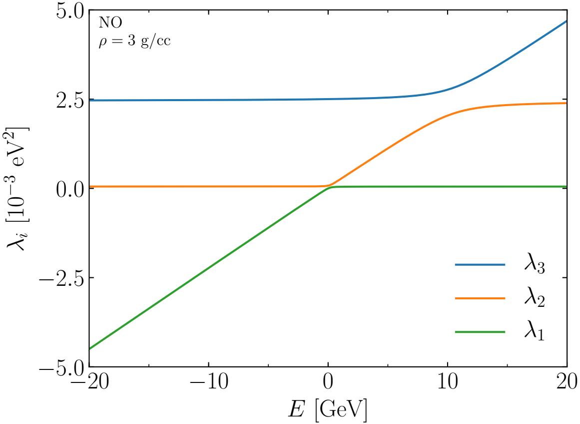
$$\xi_e + \chi_e = \Delta m^2_{21} + \Delta m^2_{ee} c^2_{13}$$

$$\xi_e \chi_e = \Delta m^2_{21}[\Delta m^2_{ee} c^2_{13} c^2_{12} + \Delta m^2_{21}(s^2_{12}c^2_{12} - s^2_{13}s^2_{12}c^2_{12})]$$

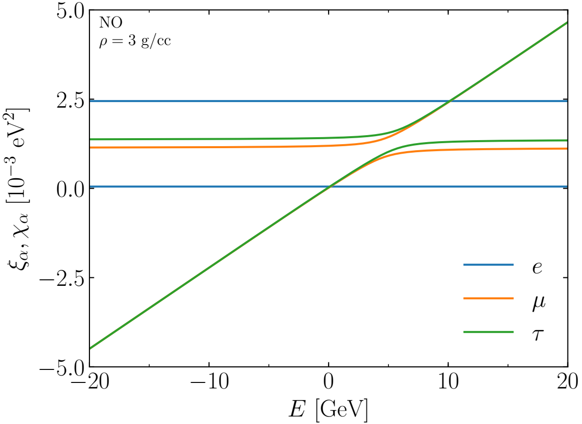
$$\Delta m^2_{ee} = c^2_{12}\Delta m^2_{31} + s^2_{12}\Delta m^2_{32}$$

H. Nunokawa, S. Parke, R. Z. Funchal hep-ph/0503283

Submatrix Eigenvalues



Eigenvalues



Submatrix Eigenvalues

Eigenvalues: the Rosetta Stone

$$s^2_{\widehat{13}} = |\widehat{U}_{e3}|^2 = \frac{(\widehat{m^2}_3 - \xi_e)(\widehat{m^2}_3 - \chi_e)}{\Delta\widehat{m^2}_{31}\Delta\widehat{m^2}_{32}}$$

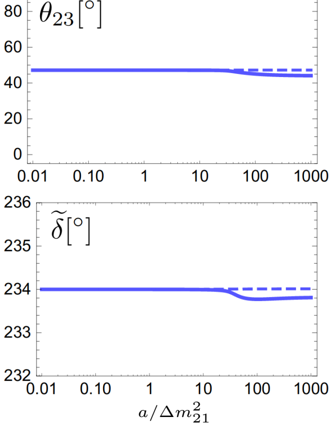
$$s^2_{\widehat{12}}c^2_{\widehat{13}} = |\widehat{U}_{e2}|^2 = -\frac{(\widehat{m^2}_2 - \xi_e)(\widehat{m^2}_2 - \chi_e)}{\Delta\widehat{m^2}_{32}\Delta\widehat{m^2}_{21}}$$

$$s^2_{\widehat{23}}c^2_{\widehat{13}} = |\widehat{U}_{\mu 3}|^2 = \frac{(\widehat{m^2}_3 - \xi_\mu)(\widehat{m^2}_3 - \chi_\mu)}{\Delta\widehat{m^2}_{31}\Delta\widehat{m^2}_{32}}$$

What about $\widehat{\delta}$?

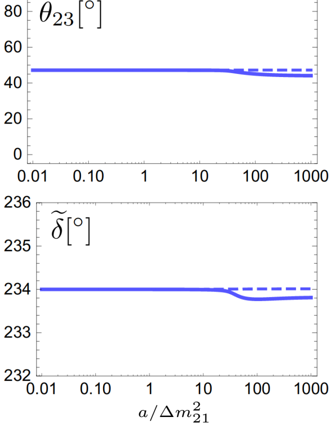
CPV From Rosetta

$\widehat{\delta}$ nearly constant, but have to get it right



Z-z. Xing, S. Zhou
Y-L. Zhou 1802.00990

CPV From Rosetta



Z-z. Xing, S. Zhou
Y-L. Zhou 1802.00990

$\widehat{\delta}$ nearly constant, but have to get it right

Toshev identity:

$$\sin\widehat{\delta} = \frac{\sin 2\theta_{23}}{\sin 2\widehat{\theta}_{23}} \sin\delta$$

S. Toshev MPL A6 (1991) 455

Get the sign of $\cos\widehat{\delta}$ from e.g. $|\widehat{U}_{\mu 1}|^2$.

In General

Two flavor:

$$|\widehat{U}_{\alpha i}|^2 = \frac{\widehat{m^2}_i - \xi_\alpha}{\Delta\widehat{m^2}_{ij}}$$

leads to

$$\sin^2\widehat{\theta} = |\widehat{U}_{e2}|^2 = \frac{\widehat{m^2}_2 - \xi_e}{\widehat{m^2}_2 - \widehat{m^2}_1}$$

$$= \frac{1}{2}\left(1 - \frac{\Delta m^2 \cos 2\theta - a}{\sqrt{(\Delta m^2 \cos 2\theta - a)^2 + (\Delta m^2 \sin 2\theta)^2}}\right)$$

In General

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$$= \frac{1}{2}\left(1 - \frac{\Delta m^2\cos 2\theta - a}{\sqrt{(\Delta m^2\cos 2\theta - a)^2 + (\Delta m^2\sin 2\theta)^2}}\right)$$

Numerically checked for $N = 4, 5$.

True for all N?

A Cheery Firehose

1. Terry posted on the same question with a different answer
2. We emailed our result, < 2 hours later:
   - "Very nice identity!"
   - 3 distinct proofs
3. 6d later, we've sorted 1 proof, send a draft, < 1 hr later:
   - Agrees to a paper
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   - Adds several new observations
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"He's famously like a cheery firehose of mathematics guess he's power-washing you today" - D. McGady

EIGENVECTORS FROM EIGENVALUES

PETER B. DENTON, STEPHEN J. PARKE, TERENCE TAO, AND XINING ZHANG

ABSTRACT. We present a new method of succinctly determining eigenvectors from eigenvalues. Specifically, we relate the norm squared of the elements of eigenvectors to the eigenvalues and the submatrix eigenvalues.

$$|v_{i,j}|^2 = \frac{\prod_{k=1}^{n-1}(\lambda_i - \xi_{j,k})}{\prod_{k=1;k\neq i}^{n}(\lambda_i - \lambda_k)}$$

Historical survey

Dubbed it the: **Eigenvector-Eigenvalue Identity**

- Added many lemmas
- Reviewed existing proofs
- Discussed sociology of the disconnected citation graph

Eigenvector-eigenvalue identity [edit]

For a Hermitian matrix, the norm squared of the *j*th component of a normalized eigenvector can be calculated using only the matrix eigenvalues and the eigenvalues of the corresponding minor matrix,

$$|v_{i,j}|^2 = \frac{\prod_k \left(\lambda_i - \lambda_k(M_j)\right)}{\prod_{k \neq i} \left(\lambda_i - \lambda_k\right)},$$

where M_j is the submatrix formed by removing the *j*th row and column from the original matrix.[35][36][37]

Eigenvector-Eigenvalue Identity Impact

It now appears in papers, theses, and textbooks about:

1. Neutrino oscillations
2. Radiofrequency resonators/MRI
3. Singular matrices
4. Computational methods
5. Regression analysis
6. Methods of medical research
7. Condensed matter
8. Graph theory
9. Economics of technology
10. Statistics
11. Many other math/computer science things

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11. Many other math/computer science things

Even in a field as old as linear algebra,
there are still gems hiding out there

Proofs

1. From previous result with $n-1$ subvectors using derivatives

L. Erdos, B. Schlein, H-T. Yau 0711.1730

T. Tao, V. Vu 0906.0510

2. Geometric formulation with exterior algebra
3. Using determinants and a Cauchy-Binet variant
4. Adjugate matrices

Can get off-diagonal elements, thus CP phase

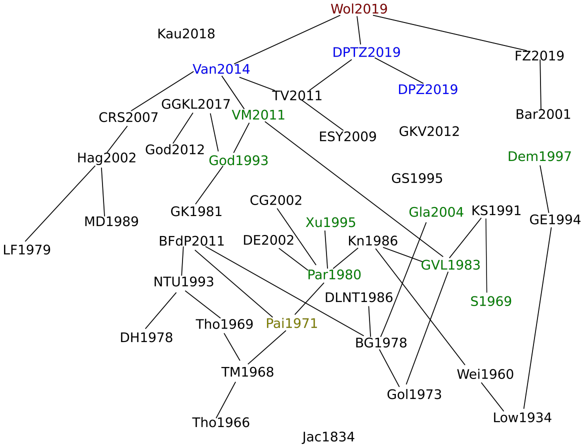
5. Cramer's rule
6. Two other mathematicians provided other proofs
7. Another mathematician generalized it to all square matrices

https://terrytao.wordpress.com/2019/08/13/eigenvectors-from-eigenvalues/

History of the Identity

Same result previously appeared

R. Thompson, P. McEnteggert Lin. Alg. App. 1 211-243 (1968)



Derivation

Key Points

- Long-baseline experiments are best way to probe remaining unknowns
- Matter effect requires diagonalizing Hamiltonian
- Calculate eigenvalues first
- Get eigenvectors from eigenvalues
- Don't assume something is known just because it seems like it should be

Thanks!

Juneteenth (June Nineteenth)

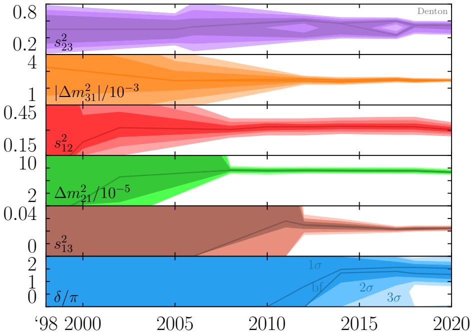
US history of emancipation aside

1. Lincoln's Emancipation Proclamation:
   Outlawed slavery in rebelling states, only enforced as union soldiers arrived
2. Confederate soldiers surrender in VA
3. **Jun 19**, 1865: General Granger announced and enforced emancipation in TX
4. 13th amendment passed outlaws slavery across US
   This outlawed slavery in Delaware, Maryland, and Kentucky

Juneteenth was first celebrated the next year in 1866
It has been a state holiday in most states for several decades
As of last month, "Juneteenth National Independence Day" is a federal holiday
Popular songs: Lift Every Voice and Sing & Swing Low, Sweet Chariot

Backups

References



SK hep-ex/9807003

M. Gonzalez-Garcia, et al. hep-ph/0009350

M. Maltoni, et al. hep-ph/0207227

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SK hep-ex/0604011

T. Schwetz, M. Tortola, J. Valle 0808.2016

M. Gonzalez-Garcia, M. Maltoni, J. Salvado 1001.4524

T2K 1106.2822

D. Forero, M. Tortola, J. Valle 1205.4018

D. Forero, M. Tortola, J. Valle 1405.7540

P. de Salas, et al. 1708.01186

CP violation in matter

The CPV Term in Matter

The amount of CPV is

$$P_{\alpha\beta} - \bar{P}_{\alpha\beta} = \pm 16 J \sin\Delta_{21} \sin\Delta_{31} \sin\Delta_{32} \qquad \alpha \neq \beta$$

where the Jarlskog is

$$J \equiv \Im[U_{\alpha i} U_{\beta j} U^*_{\alpha j} U^*_{\beta i}] \qquad \alpha \neq \beta,\ i \neq j$$
$$J = c_{12} s_{12} c^2_{13} s_{13} c_{23} s_{23} \sin\delta$$

C. Jarlskog PRL 55 (1985)

The exact term in matter is known to be

$$\frac{\widehat{J}}{J} = \frac{\Delta m^2_{21} \Delta m^2_{31} \Delta m^2_{32}}{\Delta\widehat{m^2}_{21} \Delta\widehat{m^2}_{31} \Delta\widehat{m^2}_{32}}$$

V. Naumov IJMP 1992

P. Harrison, W. Scott hep-ph/9912435

CPV in Matter

CPV in matter can be written sans $\cos(\frac{1}{3}\cos^{-1}(\cdots))$ term.

$$\frac{\widehat{J}}{J}=\frac{\Delta m^2_{21}\Delta m^2_{31}\Delta m^2_{32}}{\Delta\widehat{m^2}_{21}\Delta\widehat{m^2}_{31}\Delta\widehat{m^2}_{32}}$$

$$\left(\Delta\widehat{m^2}_{21}\Delta\widehat{m^2}_{31}\Delta\widehat{m^2}_{32}\right)^2=(A^2-4B)(B^2-4AC)+(2AB-27C)C$$

$$A\equiv\sum_j\widehat{m^2}_j=\Delta m^2_{31}+\Delta m^2_{21}+a$$

$$B\equiv\sum_{j>k}\widehat{m^2}_j\widehat{m^2}_k=\Delta m^2_{31}\Delta m^2_{21}+a(\Delta m^2_{ee}c^2_{13}+\Delta m^2_{21})$$

$$C\equiv\prod_j\widehat{m^2}_j=a\Delta m^2_{31}\Delta m^2_{21}c^2_{13}c^2_{12}$$

This is the only oscillation quantity in matter that can be written exactly without $\cos(\frac{1}{3}\cos^{-1}(\cdots))$!

H. Yokomakura, K. Kimura, A. Takamura hep-ph/0009141

CPV Factorizes

Thus $\widehat{J}^{-2}$ is fourth order in matter potential:
only two matter corrections are needed.

$$\frac{\widehat{J}}{J}=\frac{1}{|1-(a/\alpha_1)e^{i2\theta_1}||1-(a/\alpha_2)e^{i2\theta_2}|}$$

CPV Factorizes

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$$\frac{\widehat{J}}{J}=\frac{1}{|1-(a/\alpha_1)e^{i2\theta_1}||1-(a/\alpha_2)e^{i2\theta_2}|}$$

CPV in matter can be well approximated:

$$\frac{\widehat{J}}{J}\approx\frac{1}{|1-(a/\Delta m^2_{ee})e^{i2\theta_{13}}||1-(c^2_{13}a/\Delta m^2_{21})e^{i2\theta_{12}}|}$$

PBD, Parke 1902.07185

See also X. Wang, S. Zhou 1901.10882

Precise at the $<0.04\%$ level!

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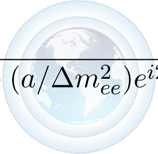

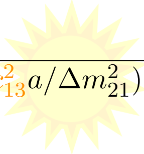

PBD, Parke 1902.07185

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Precise at the $< 0.04\%$ level!

CPV Factorizes Part II

- Option 1: Use NHS identity and $\Delta\widehat{m^2}$'s
- Option 2: Use the angles?

$$\frac{\widehat{J}}{J}=\frac{s_{\widehat{12}}c_{\widehat{12}}s_{\widehat{13}}c_{\widehat{13}}^2s_{\widehat{23}}c_{\widehat{23}}\sin\widehat{\delta}}{s_{12}c_{12}s_{13}c_{13}^2s_{23}c_{23}\sin\delta}$$

Toshev: θ_{23}, δ:

S. Toshev MPL A6 (1991) 455

$$\frac{\widehat{J}}{J}=\frac{s_{\widehat{12}}c_{\widehat{12}}s_{\widehat{13}}c_{\widehat{13}}^2}{s_{12}c_{12}s_{13}c_{13}^2}$$

How to split up?

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S. Toshev MPL A6 (1991) 455

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How to split up?

From DMP:

PBD, H. Minakata, S. Parke 1604.08167

$$\frac{1}{|1-(a/\Delta m^2_{ee})e^{i2\theta_{13}}|} \approx \frac{s_{\widehat{13}}c_{\widehat{13}}}{s_{13}c_{13}}$$

Hopefully:

$$\frac{1}{|1-(c^2_{13}a/\Delta m^2_{21})e^{i2\theta_{12}}|} \approx \frac{s_{\widehat{12}}c_{\widehat{12}}c_{\widehat{13}}}{s_{12}c_{12}c_{13}}$$

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PBD, H. Minakata, S. Parke 1604.08167

$$\frac{1}{|1-(a/\Delta m^2_{ee})e^{i2\theta_{13}}|}\approx\frac{s_{\widehat{13}}c_{\widehat{13}}}{s_{13}c_{13}}\qquad\sim 0.4\%$$

Hopefully:

$$\frac{1}{|1-(c^2_{13}a/\Delta m^2_{21})e^{i2\theta_{12}}|}\approx\frac{s_{\widehat{12}}c_{\widehat{12}}c_{\widehat{13}}}{s_{12}c_{12}c_{13}}\qquad\sim 0.4\%$$

Key Cancellation

Expect s_{13}^2 or $\Delta m_{21}^2/\Delta m_{ee}^2 \sim 2-3\%$ precision

The atmospheric term:

$$\Delta m_{31}^2 \quad \Delta m_{ee}^2 \quad \Delta m_{32}^2$$

Solar correction:

$$1 \quad c_{13}^2 \quad \cos 2\theta_{13}$$

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✗	✓	✗

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✗	✓	✗

Key Cancellation

Expect s_{13}^2 or $\Delta m_{21}^2/\Delta m_{ee}^2 \sim 2-3\%$ precision

The atmospheric term:

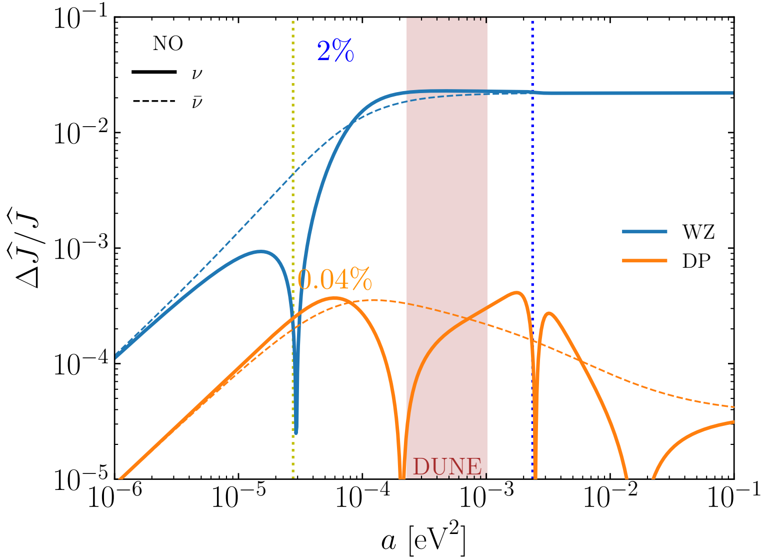
Δm_{31}^2 ✗ Δm_{ee}^2 ✓ Δm_{32}^2 ✗

Solar correction:

1 ✗ c_{13}^2 ✓ $\cos 2\theta_{13}$ ✗

$$\frac{\Delta\widehat{J}}{\widehat{J}} \sim \mathcal{O}\left(s_{13}^2\frac{\Delta m_{21}^2}{\Delta m_{ee}^2}\right) + \mathcal{O}\left[\left(\frac{\Delta m_{21}^2}{\Delta m_{ee}^2}\right)^2\right] \sim 0.04\%$$

CPV In Matter Approximation Precision



Factorization Conditions