MIXED EW-QCD TWO-LOOP AMPLITUDES FOR DRELL-YAN LEPTON PAIR PRODUCTION

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[1907.00491 + 2012.05918]

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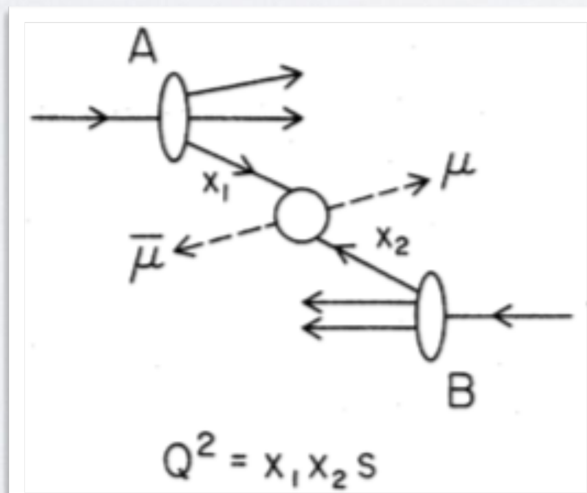
THE DRELL YAN PROCESS

Phys. Rev. Lett. 25 (1970) 316

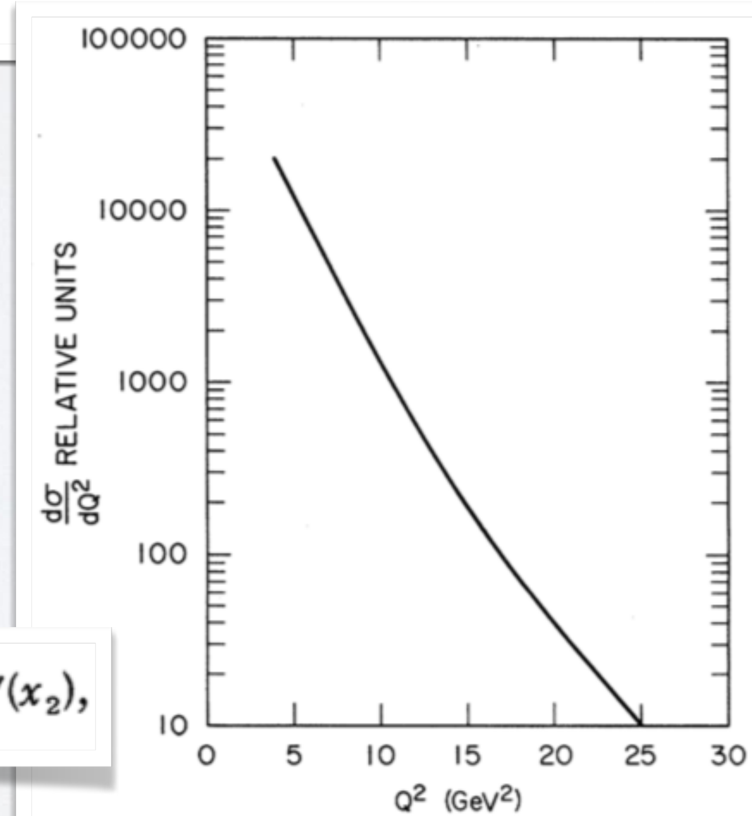
MASSIVE LEPTON-PAIR PRODUCTION IN HADRON-HADRON COLLISIONS AT HIGH ENERGIES*

Sidney D. Drell and Tung-Mow Yan
Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305
(Received 25 May 1970)

On the basis of a parton model studied earlier we consider the production process of large-mass lepton pairs from hadron-hadron inelastic collisions in the limiting region, $s\to\infty$, Q^2/s finite, Q^2 and s being the squared invariant masses of the lepton pair and the two initial hadrons, respectively. General scaling properties and connections with deep inelastic electron scattering are discussed. In particular, a rapidly decreasing cross section as $Q^2/s\to 1$ is predicted as a consequence of the observed rapid falloff of the inelastic scattering structure function νW_2 near threshold.



$$\frac{d\sigma}{dQ^2} = \left(\frac{4\pi\alpha^2}{3Q^2}\right)\left(\frac{1}{Q^2}\right)\mathcal{F}(\tau) = \left(\frac{4\pi\alpha^2}{3Q^2}\right)\left(\frac{1}{Q^2}\right)\int_0^1 dx_1\int_0^1 dx_2\,\delta(x_1x_2-\tau)\sum_a \lambda_a^{-2}F_{2a}(x_1)F_{2\bar{a}}'(x_2),$$

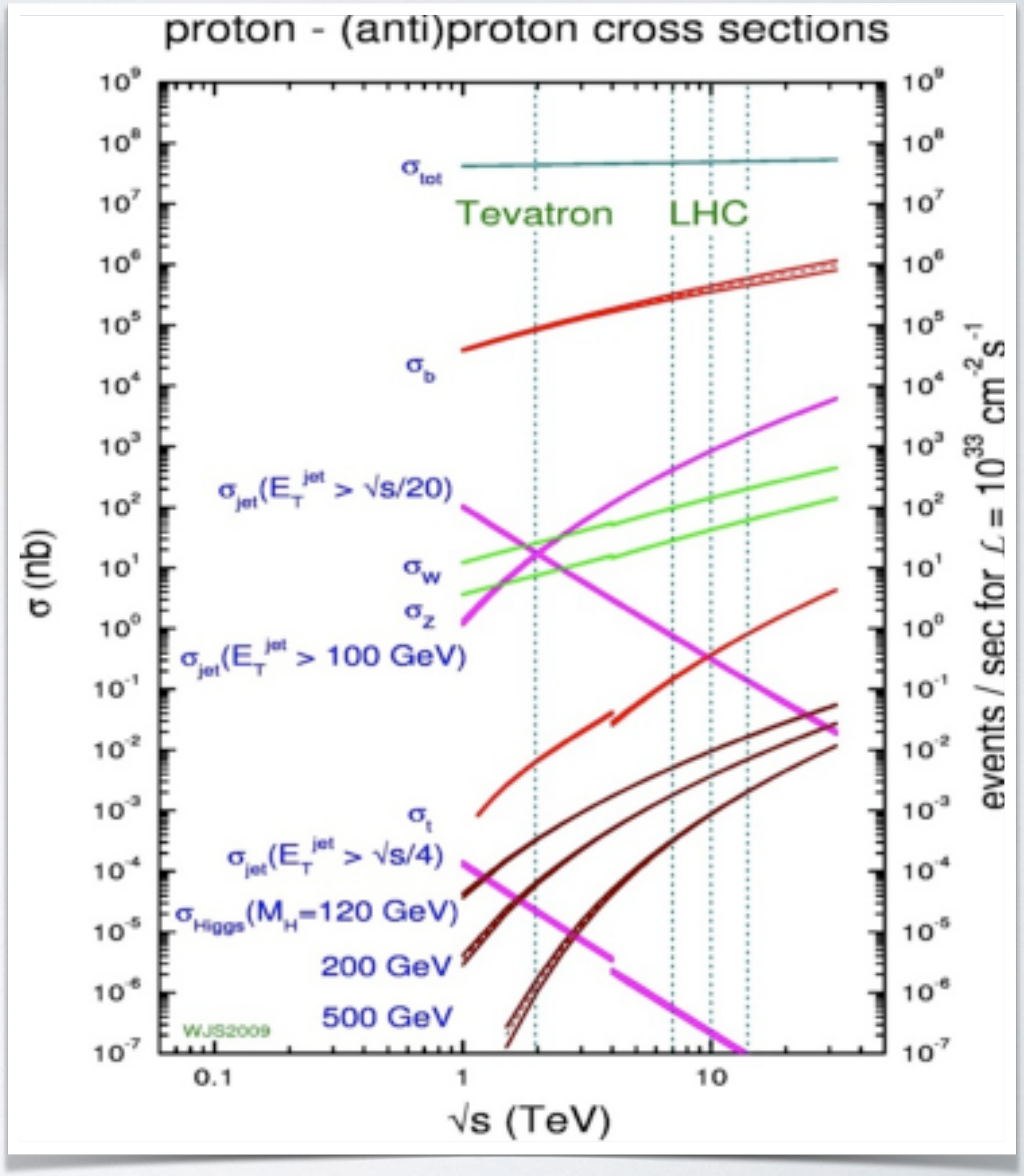


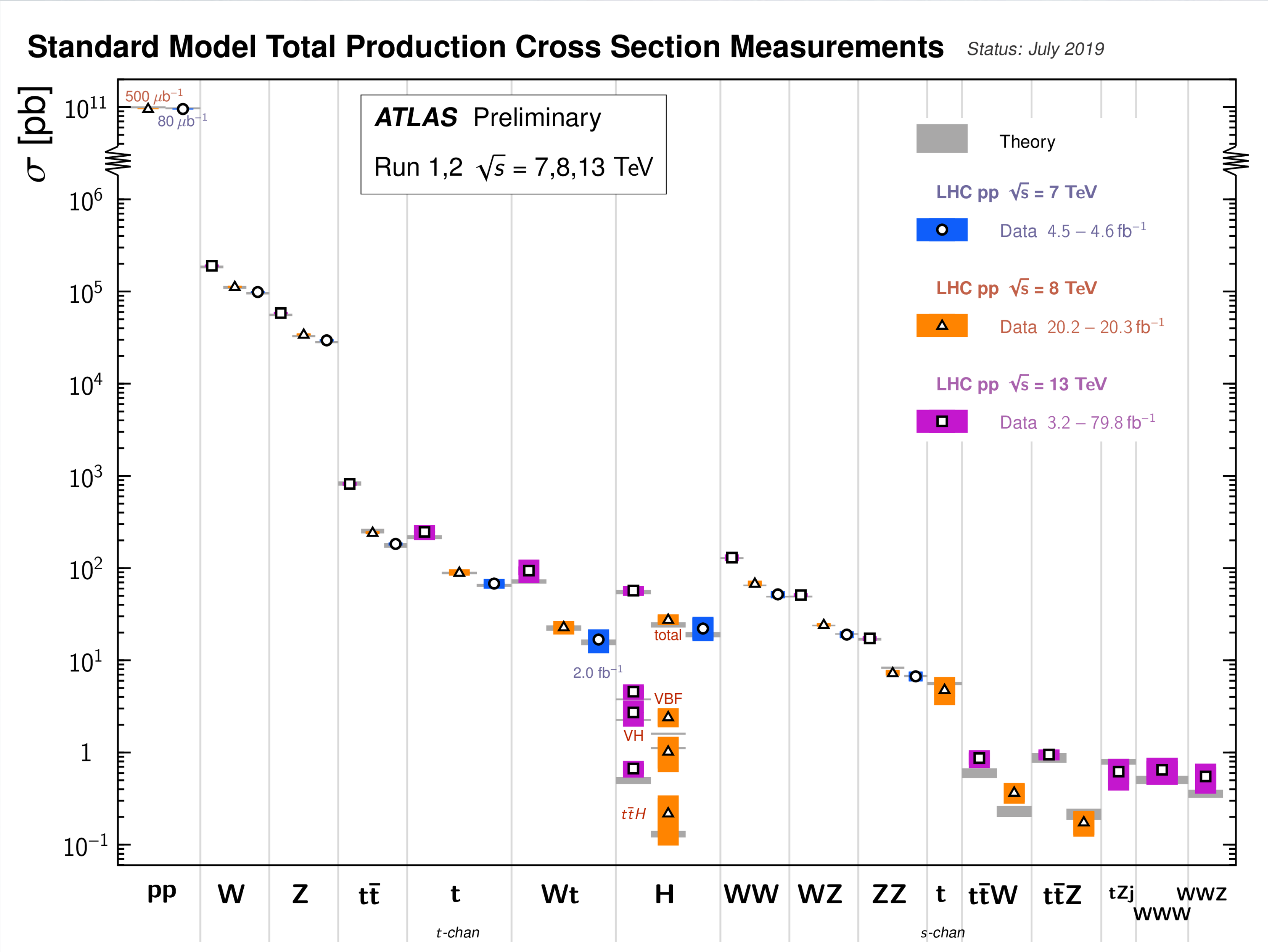
DRELL-YAN PROCESS @ LHC

$$pp \to l^- l^+ + X, \quad l = e, \mu \quad (\mathrm{NC})$$

$$pp \to l^\mp \overset{(-)}{\nu_l} + X, \quad l = e, \mu \quad (\mathrm{CC})$$

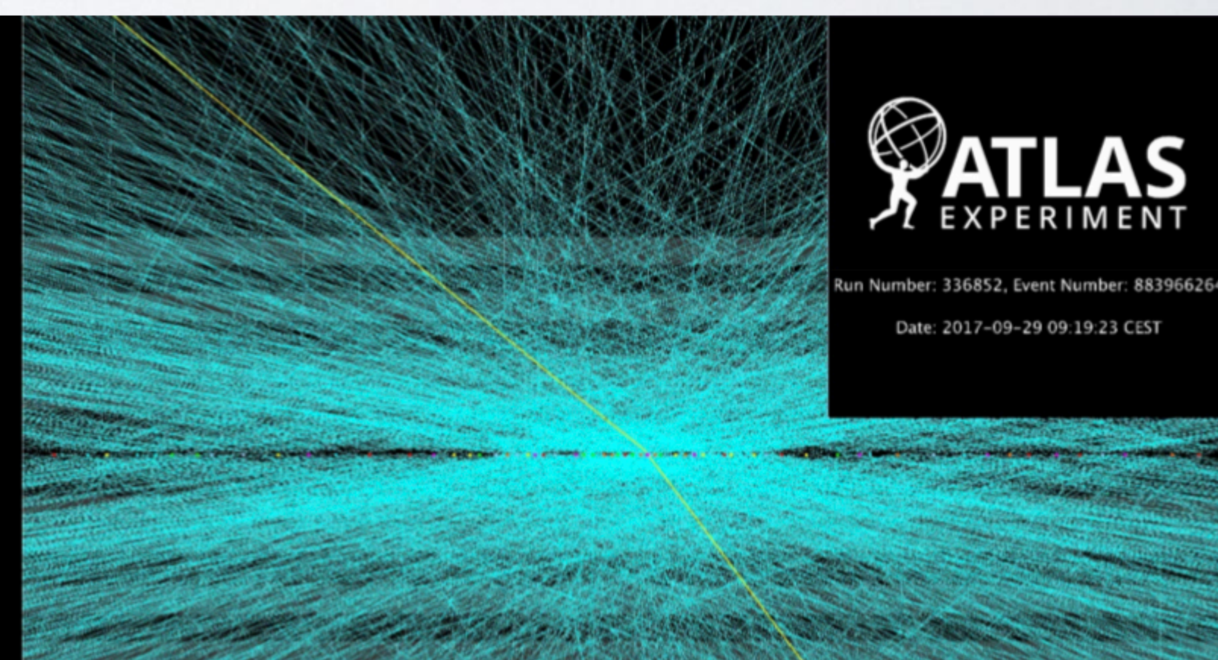
- ***EW precision measurements:*** W mass, Z mass, weak mixing (resonant W/Z)
- ***New physics searches*** direct or via SMEFT (large dilepton mass)
- ***PDF determinations***





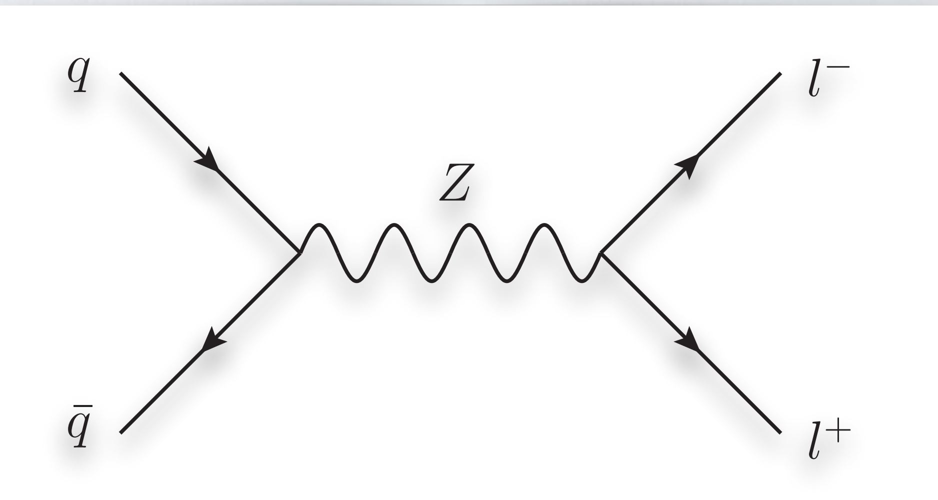
- Measurements with impressive precision
- More data to be analyzed, in particular for high-mass
- “Easy” signature (but see below)

Z ⟶ μμ candidate event, with 65 **additional** reconstructed primary vertices.



PERTURBATIVE CORRECTIONS

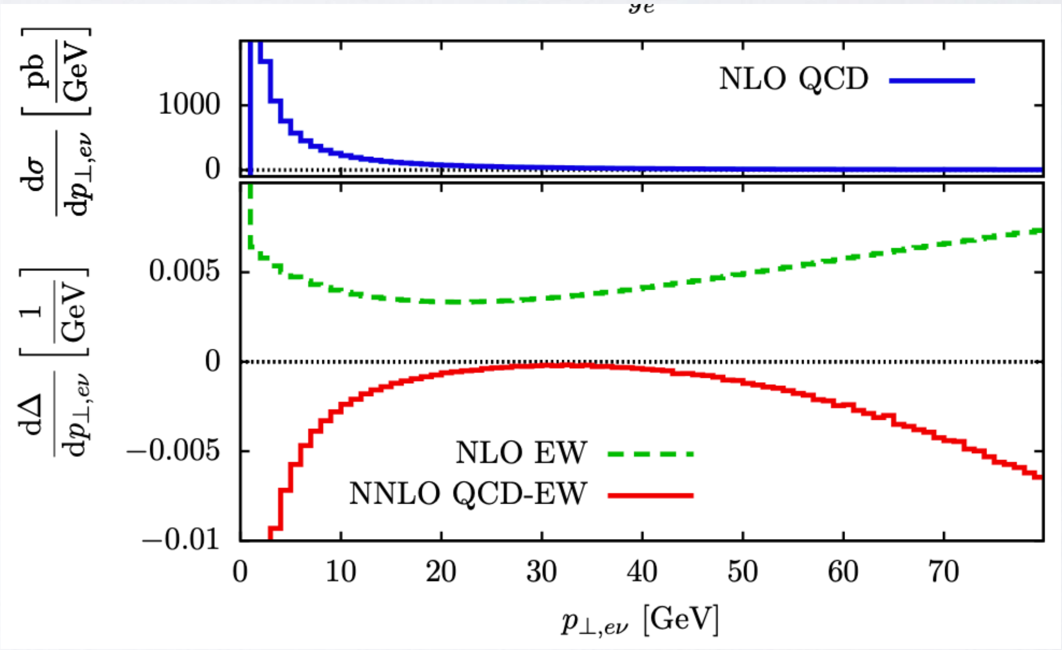
- LO: quark-pair initiated s-channel
- QCD on-shell: NNLO, recently: N3LO
  [Hamberg, van Neerven, Matsuura '91, Harlander, Kilgore '02, Anastasiou, Dixon, Melnikov, Petriello '03, Melnikov, Petriello '06], [Duhr, Dulat, Mistlberger '20,'20]
- QED on-shell NC: NNLO
  [Berends, van Neerven, Burgers '88,
  discrepancies: Blümlein, De Freitas, Raab, Schönwald '19]
- EW: NLO
  [Baur, Brein, Holllik, Schappacher, Wackeroth '01, Dittmaier, Krämer '01, Baur, Wackeroth '04], …
- Mixed EW-QCD resonance region:
  [Dittmaier, Huss, Schwinn '14,'15]
- Mixed QED-QCD:
  [Kilgore, Sturm '11] off-shell; [de Florian, Der, Fabre '18] on-shell



EW-QCD ON-SHELL Z/W

- N_f terms for on-shell Z/W *[Dittmaier, Schmidt, Schwarz '20]*
- Total Z+X cross section *[Bonciani, Buccioni, Rana, Vicini '20]*
- Differential W+X cross section (nested subtraction) *[Behring, Buccioni, Caola, Delta, Jaquier, Melnikov, Röntsch '20]*

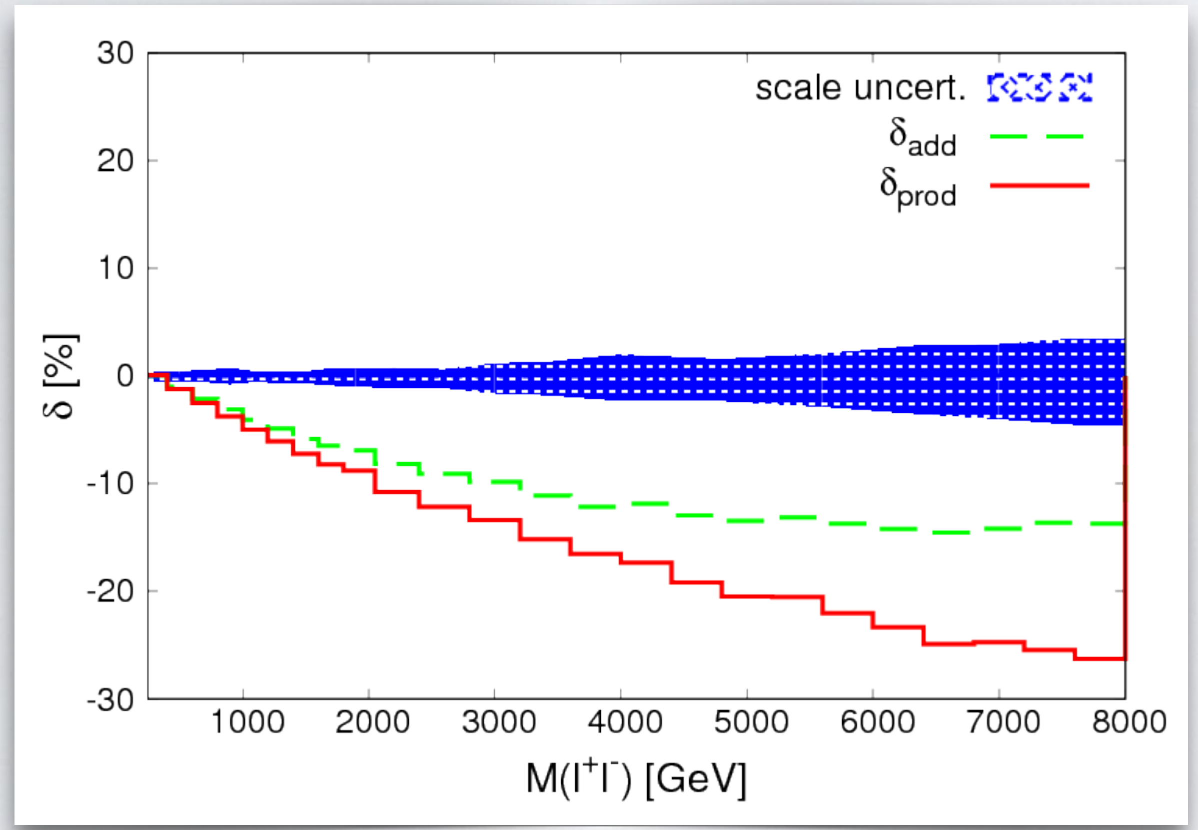
σ[pb]	channel	$\mu = M_W$	$\mu = M_W/2$	$\mu = M_W/4$
σ_{LO}		6007.6	5195.0	4325.9
$\Delta\sigma_{\mathrm{NLO},\alpha_s}$	all ch.	508.8	1137.0	1782.2
	$q\bar{q}'$	1455.2	1126.7	839.2
	qg/gq	-946.4	10.3	943.0
$\Delta\sigma_{\mathrm{NLO},\alpha}$	all ch.	2.1	-1.0	-2.6
	$q\bar{q}'$	-2.2	-5.2	-6.7
	$q\gamma/\gamma q$	4.2	4.2	4.04
$\Delta\sigma_{\mathrm{NNLO},\alpha_s\alpha}$	all ch.	-2.4	-2.3	-2.8
	$q\bar{q}'/qq'$	-1.0	-1.2	-1.0
	qg/gq	-1.4	-1.2	-2.1
	$q\gamma/\gamma q$	0.06	0.03	-0.04
	$g\gamma/\gamma g$	-0.12	0.04	0.30



- Note: α and $\alpha\alpha_s$ can be of similar order
- Note: cancellations between partonic channels (also for N3LO)

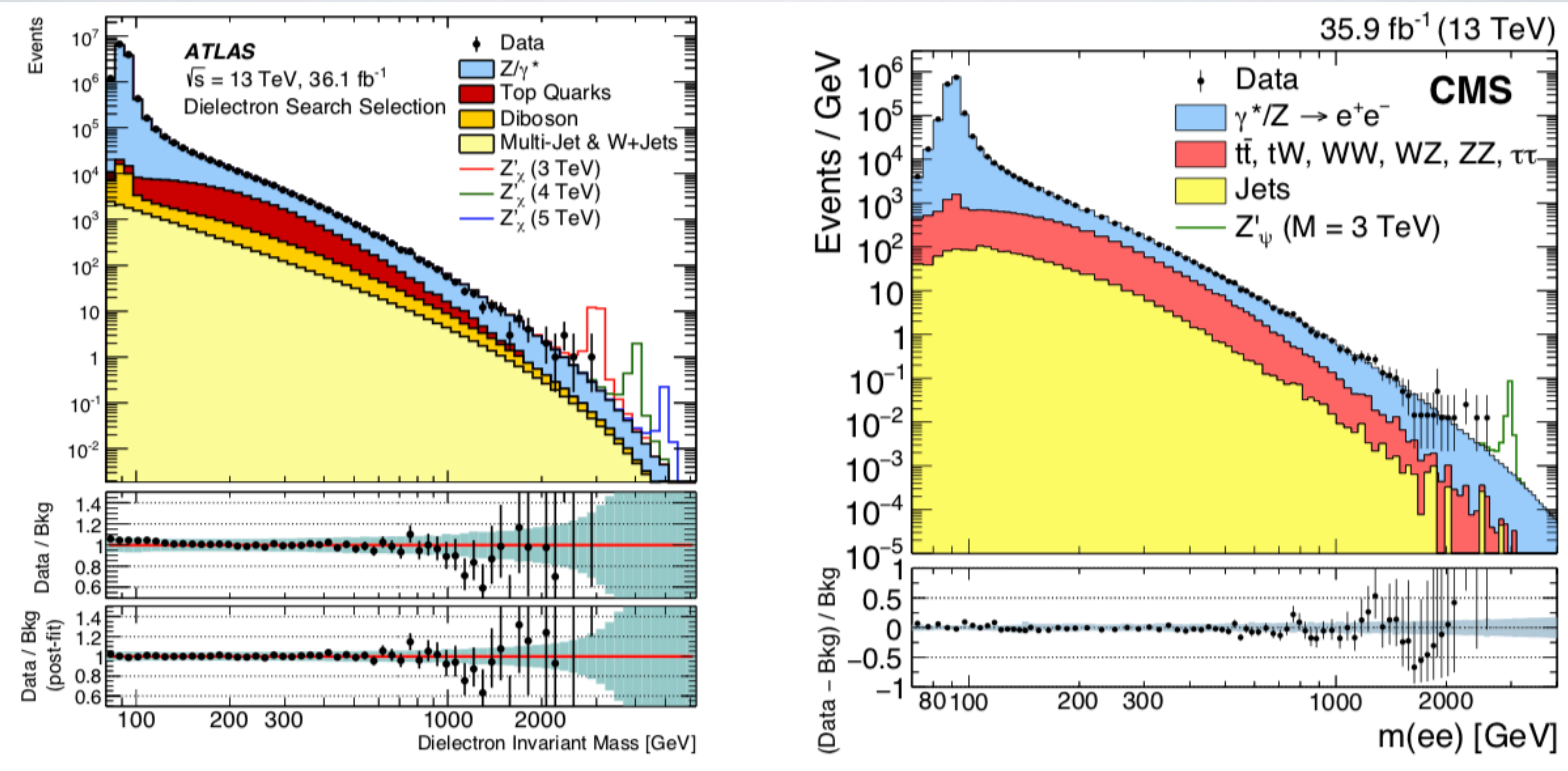
LARGE ENERGIES

- ***Mixed EW-QCD important at higher invariant masses*** uncertainties considered in *[Campbell, Wackeroth, Zhou '16]:*

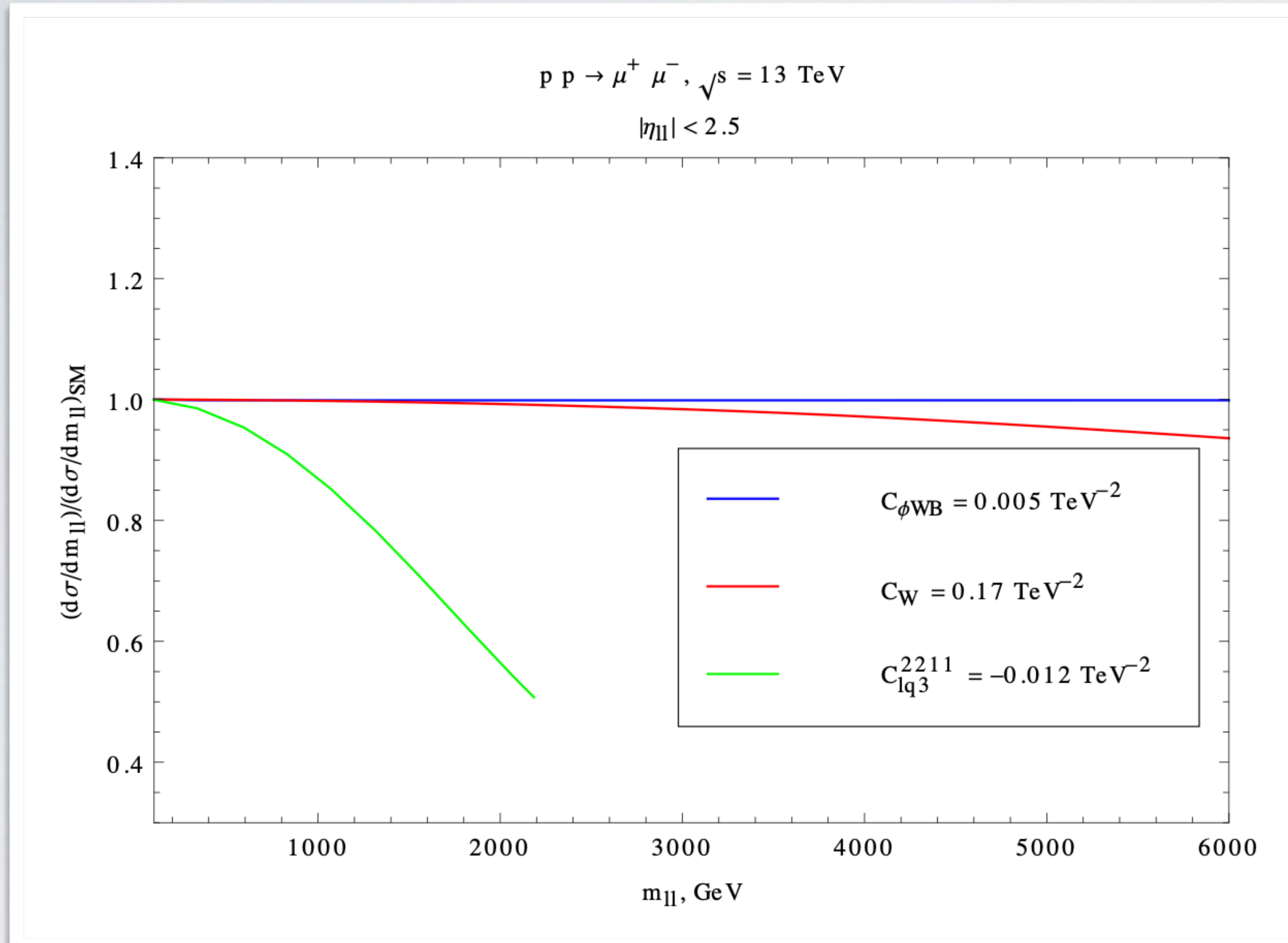


- In that region, no reason to restrict to single W/Z resonance

DILEPTON INVARIANT MASS

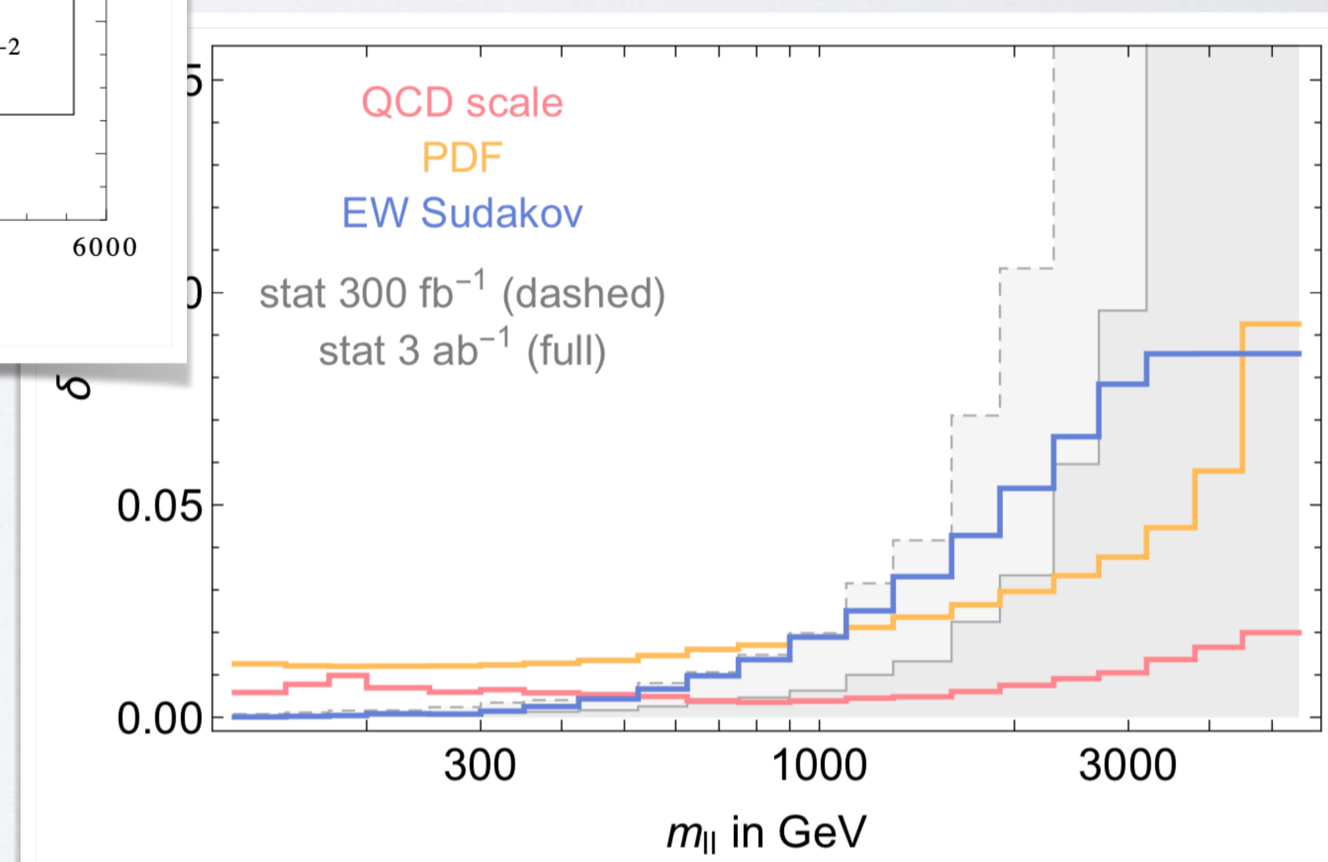


NEW RESONANCES AND EFTS



[Dawson, Giardino, Ismail '19]

- Want sensitivity to $>$ 5 TeV resonances
- Motivates non-bump searches
- Interferences important (diff. Xsec)
- EFT useful

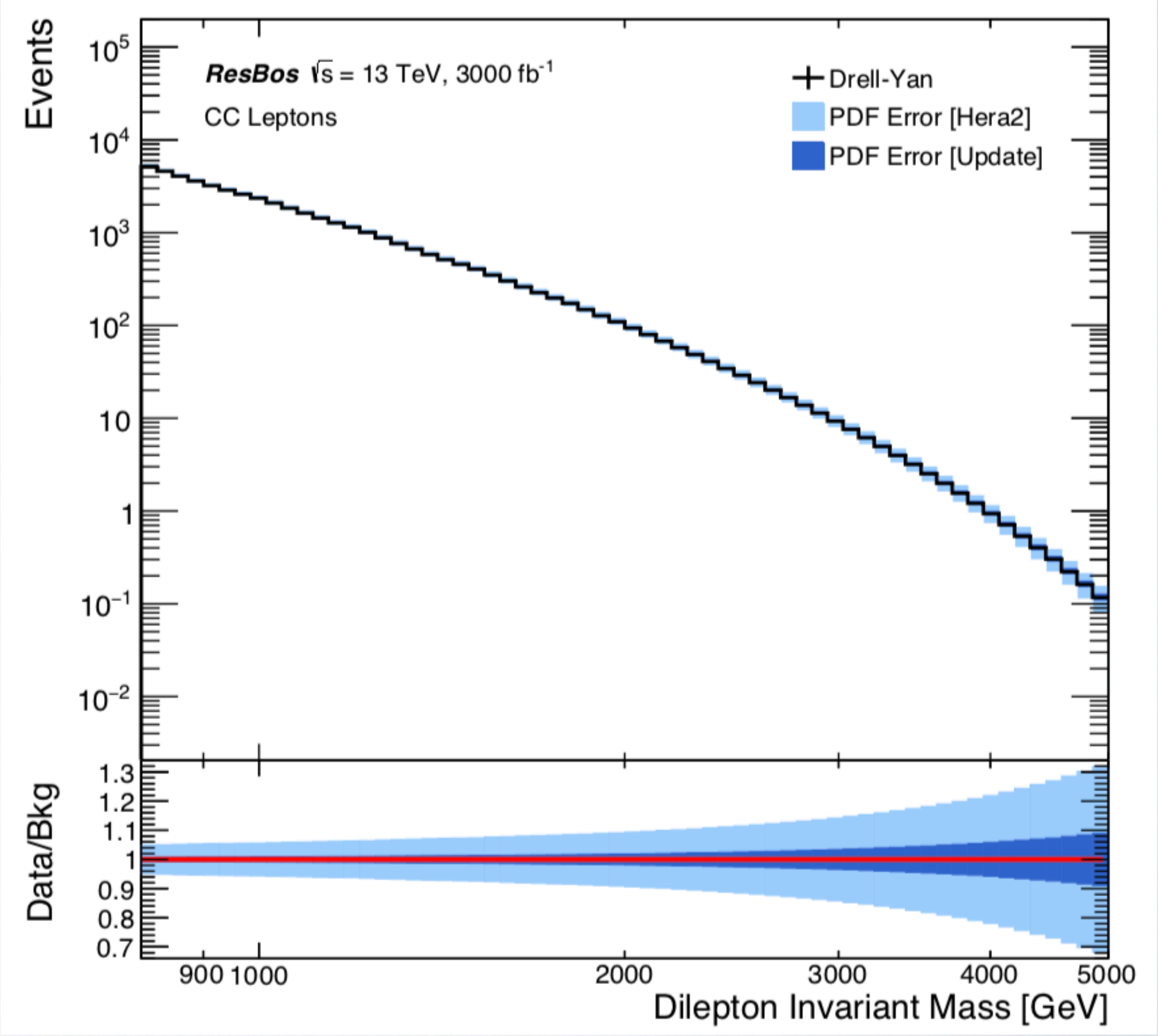


[Alioli, Farina, Pappadopulo, Ruderman '17]

PDF UNCERTAINTIES
(and how to improve them)

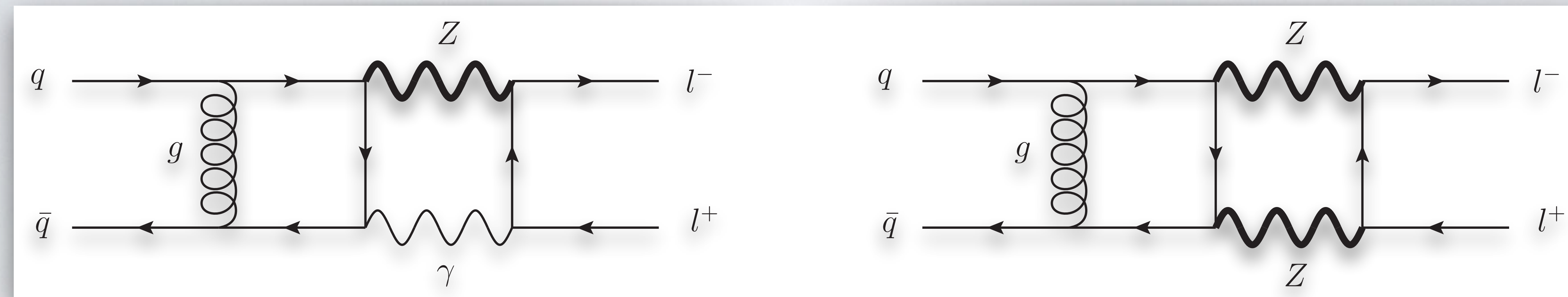
[Willis, Brock, Hayden, Hou, Isaacson, Schmidt, Yuan '19]

- Large PDF uncertainties when extrapolating over large range
- Idea: use control region < 1 TeV to improve multi-TeV signal region
- Make sure radiative corrections under control !

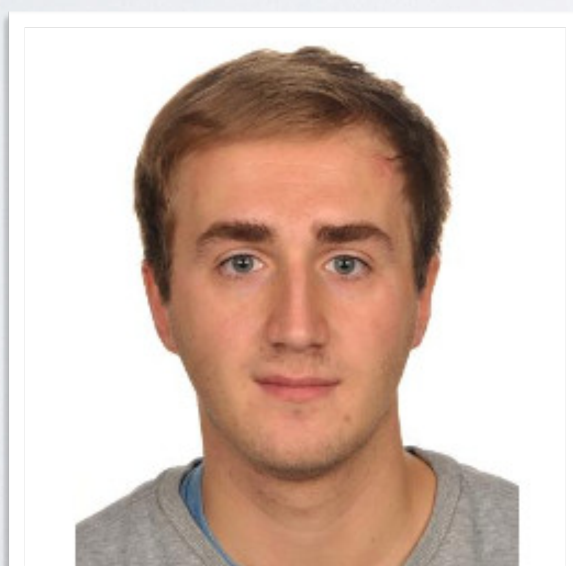


EW-QCD CORRECTIONS TO DILEPTON PRODUCTION

- Don’t restrict to singly resonant V production, consider ***dilepton*** final state
- ***Representative Feynman diagrams***

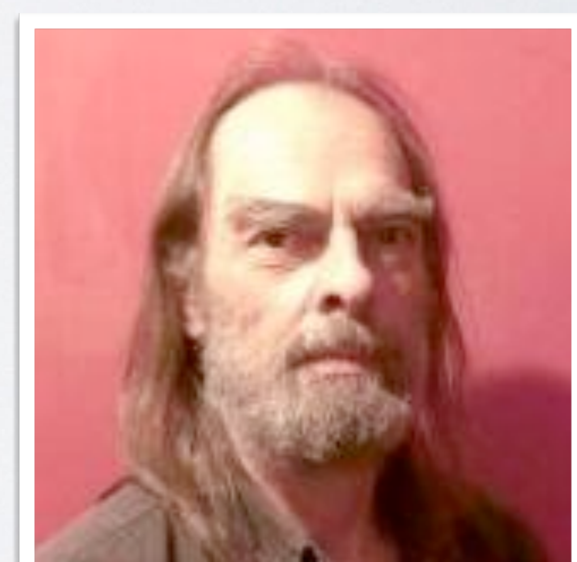


- Collaboration with:



Matthias Heller (Mainz)

Robert Schabinger (MSU)

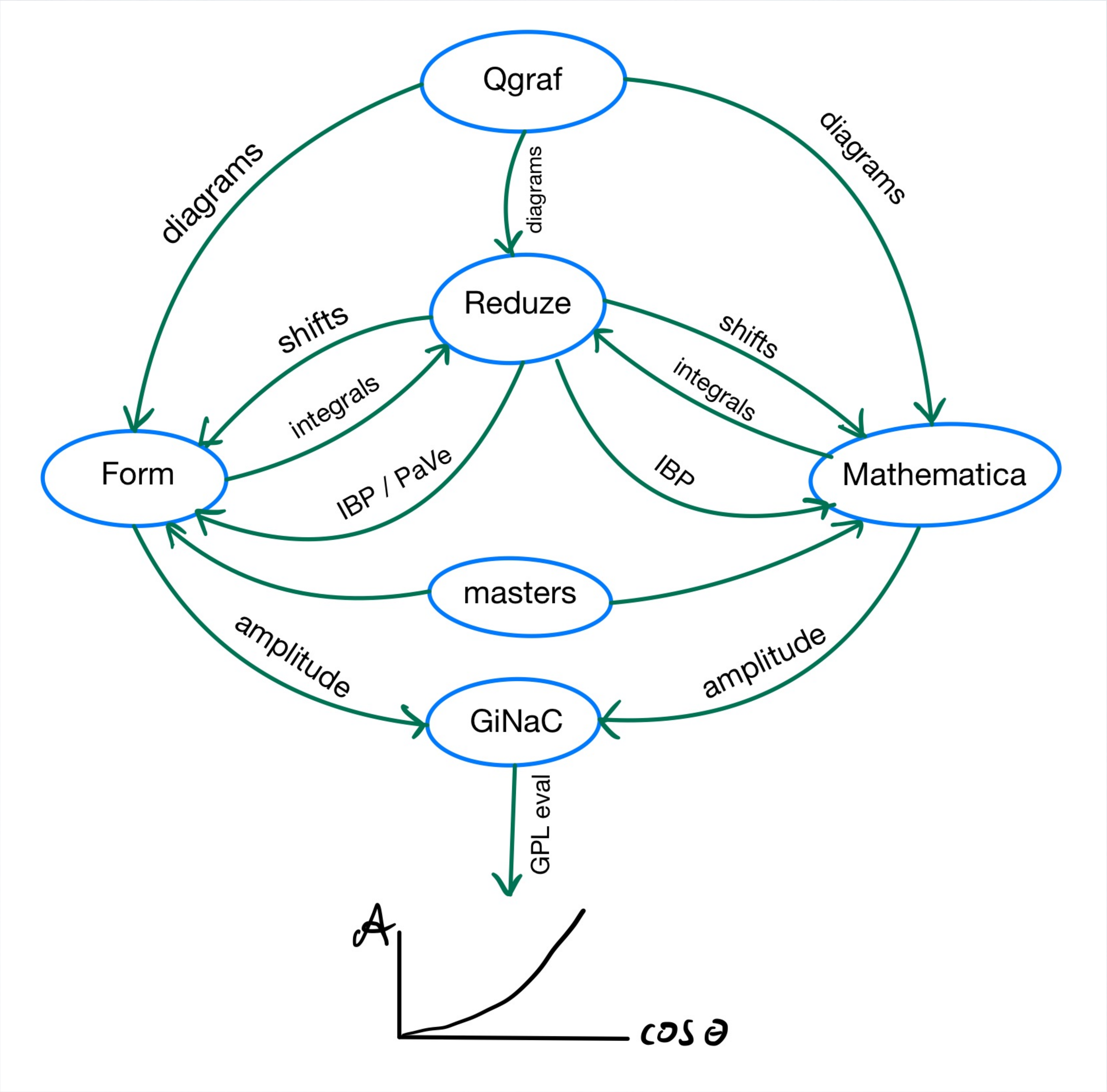


Hubert Spiesberger (Mainz)

WORKFLOW TO CALCULATE AMPLITUDE

1. Generate Feynman diagrams
2. Insert Feynman rules, perform Lorentz, Dirac, Gauge algebra
3. Reduce loop integrals to a set of basis integrals
4. Evaluate the basis integrals analytically or numerically
5. Ultraviolet renormalization, infrared subtractions
6. Write Monte-Carlo code for phase phase integration

TOOLS


Qgraf
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A fast univariate solver

rational solver: reduce matrix $I_{\mathbb{Q}}$ of rational numbers

$$\begin{array}{ccccccccc}
I_{\mathbb{Q}} & \xrightarrow[\text{image}]{\text{homomorphic}} & I_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{reduction}]{\text{Gauss row}} & O_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{remaindering}]{\text{Chinese}} & O_{\mathbb{Z}_{p_1\cdot p_2\cdot p_3\cdots}} & \xrightarrow[\text{reconstruction}]{\text{rational}} & O_{\mathbb{Q}} \\
 & \longrightarrow & I_{\mathbb{Z}_{p_2}} & \longrightarrow & O_{\mathbb{Z}_{p_2}} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_{p_3}} & \longrightarrow & O_{\mathbb{Z}_{p_3}} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}$$

univariate solver: reduce matrix $I_{\mathbb{Q}[x]}$ of rational functions in x

$$\begin{array}{ccccccccc}
I_{\mathbb{Q}[x]} & \xrightarrow[\text{img.}]{\text{hom.}} & I_{\mathbb{Z}_{p_1}[x]} & \xrightarrow[\text{(see below)}]{\text{aux solver}} & O_{\mathbb{Z}_{p_1}[x]} & \xrightarrow[\text{remaind.}]{\text{Chinese}} & O_{\mathbb{Z}_{p_1\cdot p_2\cdot p_3\cdots}[x]} & \xrightarrow[\text{rec.}]{\text{rat.}} & O_{\mathbb{Q}[x]} \\
 & \longrightarrow & I_{\mathbb{Z}_{p_2}[x]} & \longrightarrow & O_{\mathbb{Z}_{p_2}[x]} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_{p_3}[x]} & \longrightarrow & O_{\mathbb{Z}_{p_3}[x]} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}$$

aux solver: reduce matrix $I_{\mathbb{Z}_p[x]}$ of polynomials in x with finite field coefficients

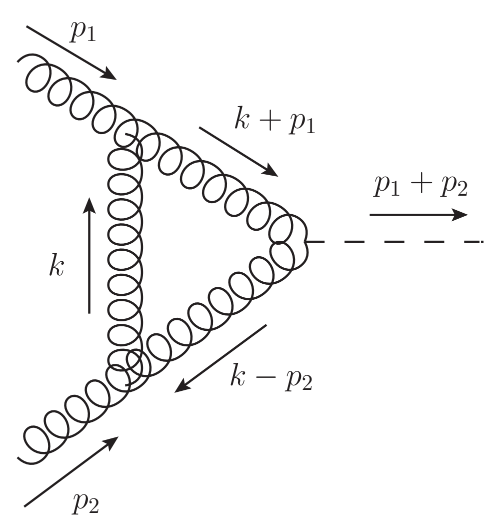
$$\begin{array}{ccccccccc}
I_{\mathbb{Z}_p}[x] & \xrightarrow[\text{by number } x_i]{\text{sample } x} & I_{\mathbb{Z}_p,x_1} & \xrightarrow[\text{reduction}]{\text{row}} & O_{\mathbb{Z}_p,x_1} & \xrightarrow[\text{interpolation}]{\text{polynomial}} & O_{\mathbb{Z}_p}[x] & \xrightarrow[\text{reconstruction}]{\text{rational function}} & O_{\mathbb{Z}_p}[x] \\
 & \longrightarrow & I_{\mathbb{Z}_p,x_2} & \longrightarrow & O_{\mathbb{Z}_p,x_2} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_p,x3} & \longrightarrow & O_{\mathbb{Z}_p,x3} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}$$

note: multivariate case by iteration

- Reduce loop integrals with finite-field sampling + rational reconstruction (works also for huge matrices) *[AvM, Schabinger '2014, …]*

LOOP AMPLITUDES AND INTEGRALS

- ***Feynman diagrams*** describe ***scattering amplitudes***
- Quantum principle: ***sum coherently*** over ***unobserved*** possibilities



$\rightarrow$

"loop integral"

$$\int d^4k \frac{1}{k^2(k+p_1)^2(k-p_2)^2}$$

- ***Dimensional regularization*** of infrared and ultraviolet divergences

$$\int d^4k \frac{1}{k^2(k+p_1)^2(k-p_2)^2} \quad \rightarrow \quad \int d^dk \frac{1}{k^2(k+p_1)^2(k-p_2)^2}$$

- ***Observables*** in terms of observables finite for $d \rightarrow 4$

γ_5 AND DIM. REG.

- Conventional dimensional regularization (CDR):

$$\int \frac{\mathrm{d}^4 k_i}{(2\pi)^4} \longrightarrow \left(\mu^2\right)^{\epsilon} \int \frac{\mathrm{d}^{4-2\epsilon} k_i}{(2\pi)^{4-2\epsilon}}$$

$$g^{\mu\nu} g_{\mu\nu} = 4 - 2\epsilon\,,$$
$$\{\gamma_\mu, \gamma_\nu\} = 2 g_{\mu\nu}\,\mathbf{1}\,,$$
$$\text{and} \quad \gamma^\mu \gamma_\mu = \frac{1}{2} g^{\mu\nu} \{\gamma_\mu, \gamma_\nu\} = g^{\mu\nu} g_{\mu\nu} = 4 - 2\epsilon$$

  all vectors d-dim (internal+external), fermions 2 pol (enters only via $\sum u(p)\bar{u}(p) = p^\mu \gamma_\mu$)

- Problem: γ_5 really a ***4-dimensional*** object

$$\mathrm{tr}\,\{\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_5\} = -4i\,\varepsilon_{\mu\nu\rho\sigma}$$

- ***Split*** d dimensional space into 4 dimensional one (bar) + $\epsilon = (4 - d)/2$ dimensional one (hat)

$$k^\mu = \bar{k}^\mu + \hat{k}^\mu$$
$$\gamma^\mu = \bar{\gamma}^\mu + \hat{\gamma}^\mu$$

  in particular: $\varepsilon^{\mu\nu\rho}{}_{\alpha}\, \varepsilon_{\mu\nu\rho\beta} = -6\, \bar{g}_{\alpha\beta}$ etc

- To give meaning to γ_5 in d dims:
  - give up ***anti-commutativity***: *'t Hooft, Veltman, Breitenlohner, Maison* (HVBM)

    $$\{\bar{\gamma}_\mu, \gamma_5\} = 0$$
    $$[\hat{\gamma}_\mu, \gamma_5] = 0$$

    (violates chiral symmetry)
  - give up ***cyclicity*** of Dirac trace: *Kreimer*

    $$\{\gamma_\mu, \gamma_5\} = 0$$

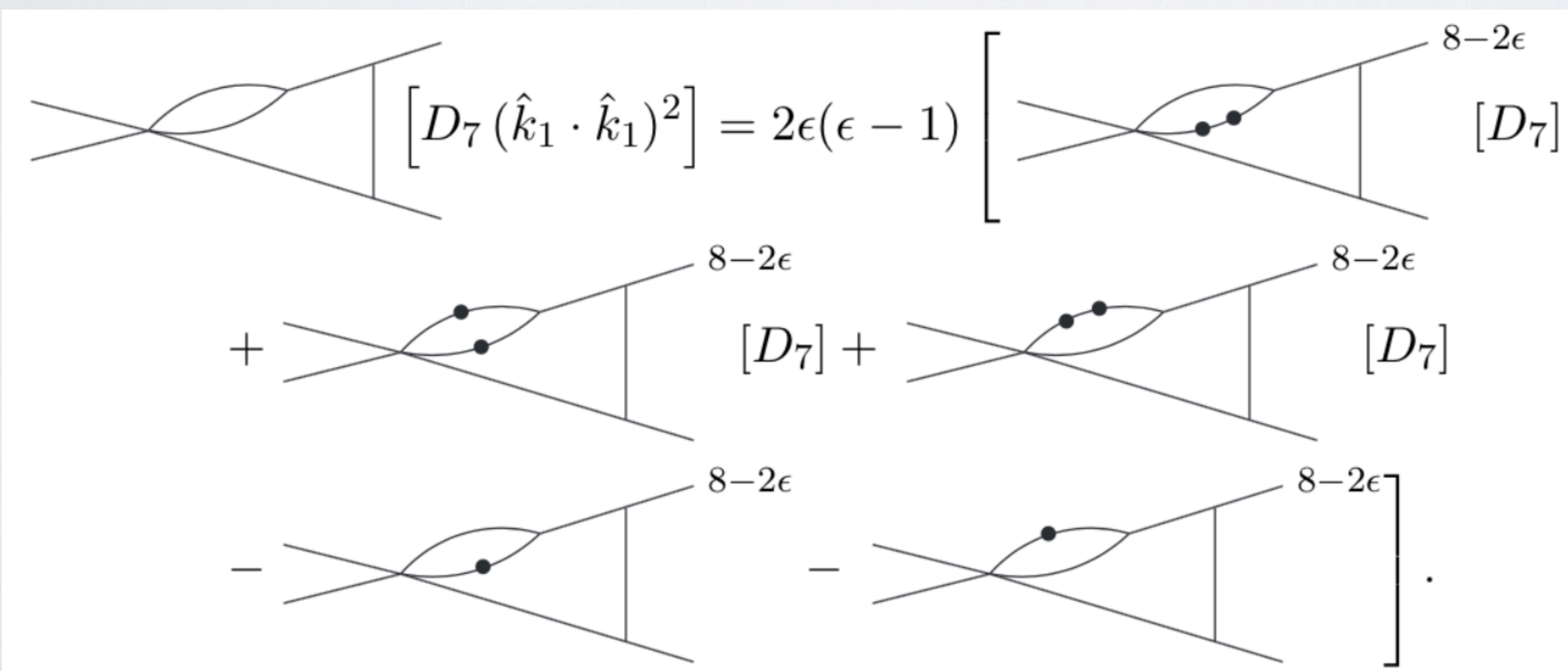
    (requires reading point prescription)

SETUP AND μ TERMS

- We perform the calculation using ***3 different setups***:

    1. HVBM scheme + projectors + mu-terms
    2. Kreimer's scheme + projectors + mu-terms (boxes)
    3. Kreimer's scheme + PaVe reduction

- HVMB requires ***split of indices*** right away, Kreimer's scheme allows to perform Dirac traces without
- Tensor integrals with ϵ ***dimensional loop momenta*** (μ ***terms***) treated using dimension shifts



CHIRAL SYMMETRY

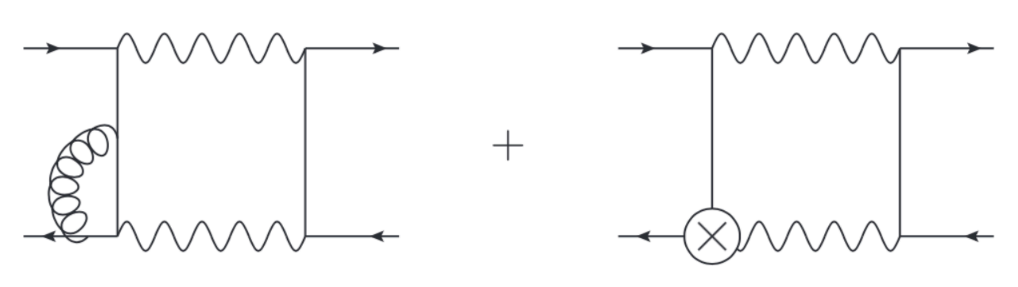
- In HVBM, corrections to ***vector and axial-vector currents*** differ
- Restore ***chiral symmetry*** by adding counter terms
- For vertex, ***require***

$$\bar{\mathcal{A}}^{(0,1)}_{Z\bar{q}q}(s) = -\frac{a_q}{v_q}\,\bar{\mathcal{V}}^{(0,1)}_{Z\bar{q}q}(s)$$

- Implement by adding ***counter terms***

$$\begin{aligned}\delta Z^{(0,1)}_{Z\bar{q}q} &= \bar{\mathcal{A}}^{(0,1)}_{Z\bar{q}q}(s) - \mathcal{A}^{(0,1)}_{Z\bar{q}q}(s) \\ &= 2\,a_q \frac{(2-\epsilon)\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)e^{\gamma_E \epsilon}}{(1-\epsilon)\Gamma(2-2\epsilon)} C_F e^{i\pi\epsilon}\left(\frac{\mu^2}{s}\right)^{\epsilon}\end{aligned}$$

- Note: need also symmetry restoring counter terms in boxes



- We keep also the ***higher order ϵ terms*** for the counter term

TENSOR INTEGRALS

- Consider treatment of tensor integrals

$$I^{\mu_1 \dots \mu_n} = \int d^d k_1 \dots d^d k_m \frac{k_{i_1}^{\mu_1} \dots k_{i_n}^{\mu_n}}{D_1 \dots D_N} = \sum_j I_j p_{j_1}^{\mu_1} \dots g^{\mu_k \mu_l} \dots$$

- One option: compute interference terms
- Here: ***Lorentz decomposition***
  - at level of ***integrals***: Passarino-Veltman reduction
  - at level of ***amplitude***: form factor decomposition

FORM FACTORS

- Start with CDR building blocks *[Glover]*

$$\bar{\mathcal{T}}_1 = \bar{v}(p_2)\gamma^\mu u(p_1)\ \bar{u}(p_3)\gamma_\mu v(p_4),$$
$$\bar{\mathcal{T}}_2 = \bar{v}(p_2)\not{p}_3 u(p_1)\ \bar{u}(p_3)\not{p}_1 v(p_4),$$
$$\bar{\mathcal{T}}_3 = \bar{v}(p_2)\gamma^\mu\gamma^\nu\gamma^\rho u(p_1)\ \bar{u}(p_3)\gamma_\mu\gamma_\nu\gamma_\rho v(p_4),$$
$$\bar{\mathcal{T}}_4 = \bar{v}(p_2)\gamma^\mu\not{p}_3\gamma^\nu u(p_1)\ \bar{u}(p_3)\gamma_\mu\not{p}_1\gamma_\nu v(p_4),$$
$$\bar{\mathcal{T}}_5 = \bar{v}(p_2)\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\tau u(p_1)\ \bar{u}(p_3)\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\tau v(p_4),$$
$$\bar{\mathcal{T}}_6 = \bar{v}(p_2)\gamma^\mu\gamma^\nu\not{p}_3\gamma^\rho\gamma^\sigma u(p_1)\ \bar{u}(p_3)\gamma_\mu\gamma_\nu\not{p}_1\gamma_\rho\gamma_\sigma v(p_4),$$

- Insert γ_5 , for example in $\mathcal{T}_1$:

$$\bar{\mathcal{T}}_{\mathrm{VV}} = \bar{v}(p_2)\gamma^\mu u(p_1)\ \bar{u}(p_3)\gamma_\mu v(p_4),$$
$$\bar{\mathcal{T}}_{\mathrm{AA}} = \bar{v}(p_2)\gamma^\mu\gamma_5 u(p_1)\ \bar{u}(p_3)\gamma_\mu\gamma_5 v(p_4),$$
$$\bar{\mathcal{T}}_{\mathrm{AV}} = \bar{v}(p_2)\gamma^\mu\gamma_5 u(p_1)\ \bar{u}(p_3)\gamma_\mu v(p_4),$$
$$\bar{\mathcal{T}}_{\mathrm{VA}} = \bar{v}(p_2)\gamma^\mu u(p_1)\ \bar{u}(p_3)\gamma_\mu\gamma_5 v(p_4),$$

(use $\frac{1}{2}[\gamma_\mu, \gamma_5] = \bar{\gamma}_\mu\gamma_5$ in HVBM)

- Amplitude is now

$$i\mathcal{A}_{\mathrm{DY}} = i\sum_{\alpha=1}^{16}\mathbf{C}_\alpha\, T_\alpha,$$

$$T_\alpha = (\bar{\mathcal{T}}_{1,\mathrm{VV}}, \bar{\mathcal{T}}_{1,\mathrm{AA}}, \bar{\mathcal{T}}_{2,\mathrm{VV}}, \bar{\mathcal{T}}_{2,\mathrm{AA}}, \ldots, \bar{\mathcal{T}}_{1,\mathrm{AV}}, \bar{\mathcal{T}}_{1,\mathrm{VA}}, \bar{\mathcal{T}}_{2,\mathrm{AV}}, \bar{\mathcal{T}}_{2,\mathrm{VA}}, \ldots)$$

FORM FACTORS

- Computation of projectors to extract form factors:

$$i\mathcal{A}_{\rm DY} = i\sum_{\alpha=1}^{16} \mathbf{C}_\alpha T_\alpha, \qquad M_{\alpha\beta} = \sum_{\rm spin,color} T_\alpha^\dagger T_\beta \qquad \mathbf{C}_\alpha = \sum_{\rm spin,color} \mathcal{P}_\alpha i\mathcal{A}_{\rm DY} \qquad \mathcal{P}_\alpha = -i\sum_{\beta=1}^{16} M_{\alpha\beta}^{-1} T_\beta^\dagger$$

- Only ***4 tensors independent*** in $d = 4$, equal to number of ***helicity amplitudes***.
- Would like to ignore other directions, but how ? Note: M^{-1} ***diverges*** for $d \to 4$!
- ***Change basis*** of tensors *[see also Peraro,Tancredi '19,'20; Chen, Ravindran et al '19,'19]*

$$T_1' = T_1, \qquad T_2' = T_2, \qquad T_\alpha' = T_\alpha + \sum_{\beta=1}^{2} R_{\alpha\beta} T_\beta \quad \text{for } \alpha = 3\ldots 8$$

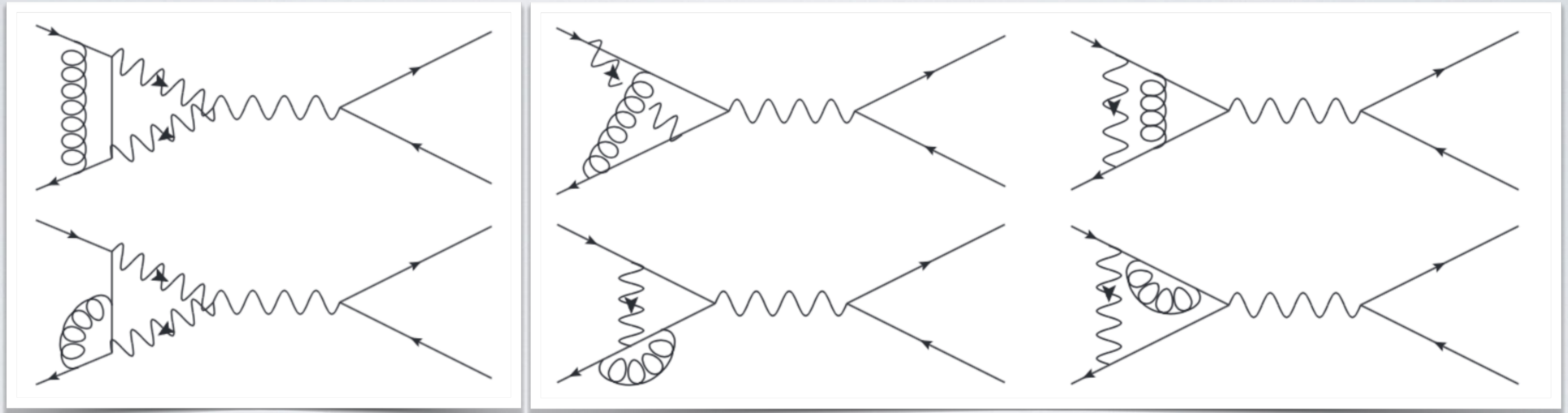
  such that irrelevant directions ***decouple*** exactly in d dimensions

$$(M_{\alpha\beta}') = \sum_{\rm spin,color} (T_\alpha'^\dagger T_\beta') = (RMR^\dagger) = \begin{pmatrix} M_{2\times 2}' & 0 \\ 0 & M_{6\times 6}' \end{pmatrix}$$

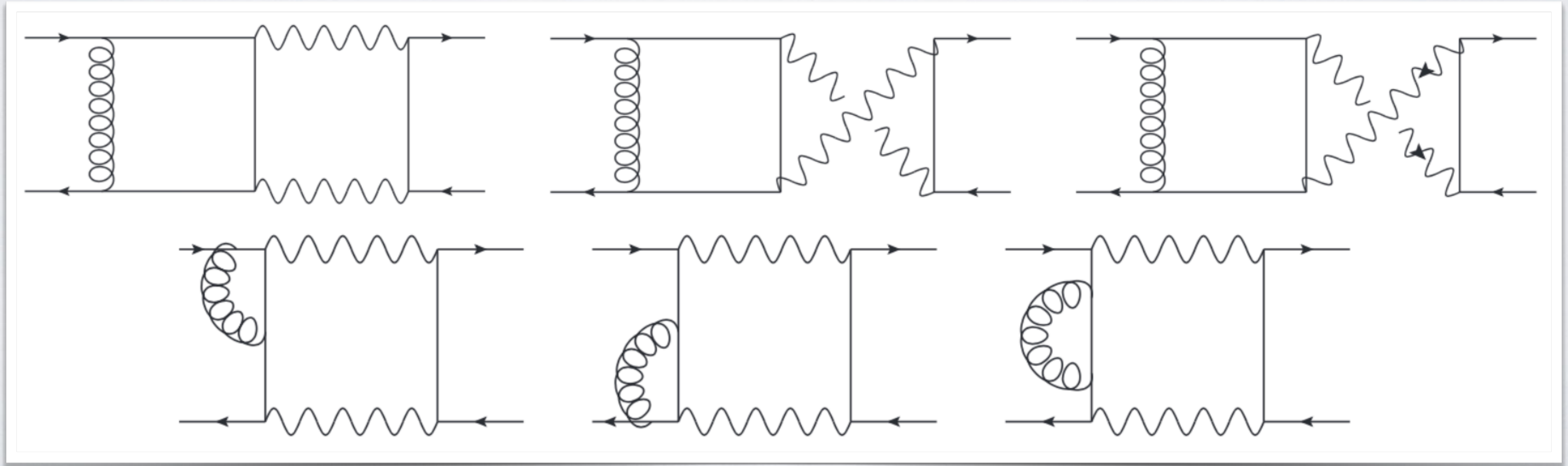
- M_{2x2}^{-1} is ***regular*** for $d \to 4$, irrelevant directions contribute only at order $d - 4$!
- **Result:** exact d dim. projectors for relevant form factors and subtraction terms (γ_5 scheme dep.), irrelevant ones not needed for finite remainder

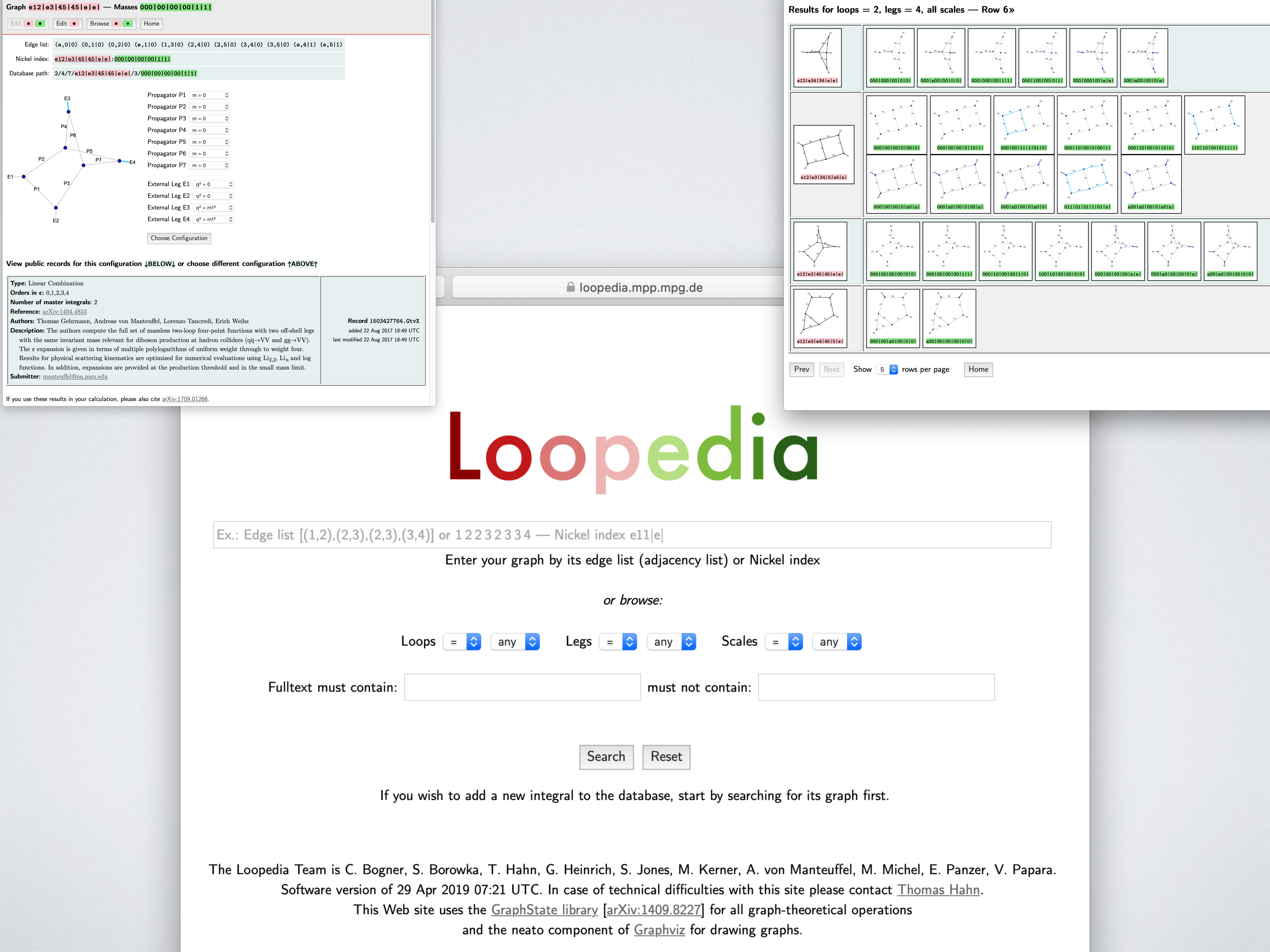
FEYNMAN DIAGRAMS

Examples for vertex corrections at two loops:



Examples for box corrections at two-loops:





Graph e12|e3|45|45|e|e| — Masses 000|00|00|00|1|1|
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Edit
Browse
Home
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View public records for this configuration ↓BELOW↓ or choose different configuration ↑ABOVE↑
Type: Linear Combination
Orders in ε: 0,1,2,3,4
Number of master integrals: 2
Reference: arXiv:1404.4853
Authors: Thomas Gehrmann, Andreas von Manteuffel, Lorenzo Tancredi, Erich Weihs
Description: The authors compute the full set of massless two-loop four-point functions with two off-shell legs with the same invariant mass relevant for diboson production at hadron colliders (qq̄→VV and gg→VV). The ε expansion is given in terms of multiple polylogarithms of uniform weight through to weight four. Results for physical scattering kinematics are optimized for numerical evaluations using Li2,2, Lin and log functions. In addition, expansions are provided at the production threshold and in the small mass limit.
Submitter: manteuffel@pa.msu.edu
Record 1503427764.GtvX
added 22 Aug 2017 18:49 UTC
last modified 22 Aug 2017 18:49 UTC
If you use these results in your calculation, please also cite arXiv:1709.01266.
Results for loops = 2, legs = 4, all scales — Row 6»
e12|e34|34|e|e|
e12|e3|34|5|e5|e|
e12|e3|45|45|e|e|
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Ex.: Edge list [(1,2),(2,3),(2,3),(3,4)] or 1 2 2 3 2 3 3 4 — Nickel index e11|e|
Enter your graph by its edge list (adjacency list) or Nickel index
or browse:
Loops = any
Legs = any
Scales = any
Fulltext must contain:
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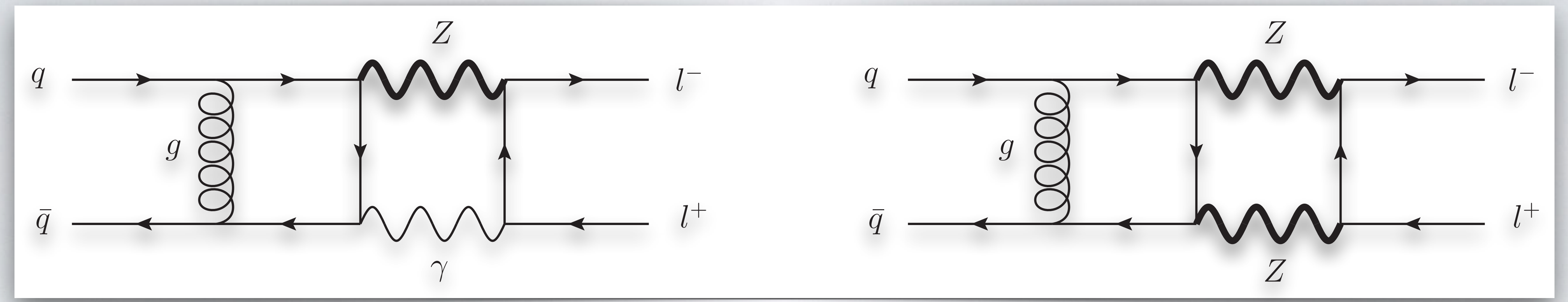
If you wish to add a new integral to the database, start by searching for its graph first.
The Loopedia Team is C. Bogner, S. Borowka, T. Hahn, G. Heinrich, S. Jones, M. Kerner, A. von Manteuffel, M. Michel, E. Panzer, V. Papara.
Software version of 29 Apr 2019 07:21 UTC. In case of technical difficulties with this site please contact Thomas Hahn.
This Web site uses the GraphState library [arXiv:1409.8227] for all graph-theoretical operations
and the neato component of Graphviz for drawing graphs.

MASTER INTEGRALS

- ***Feynman diagrams*** with one and two masses:

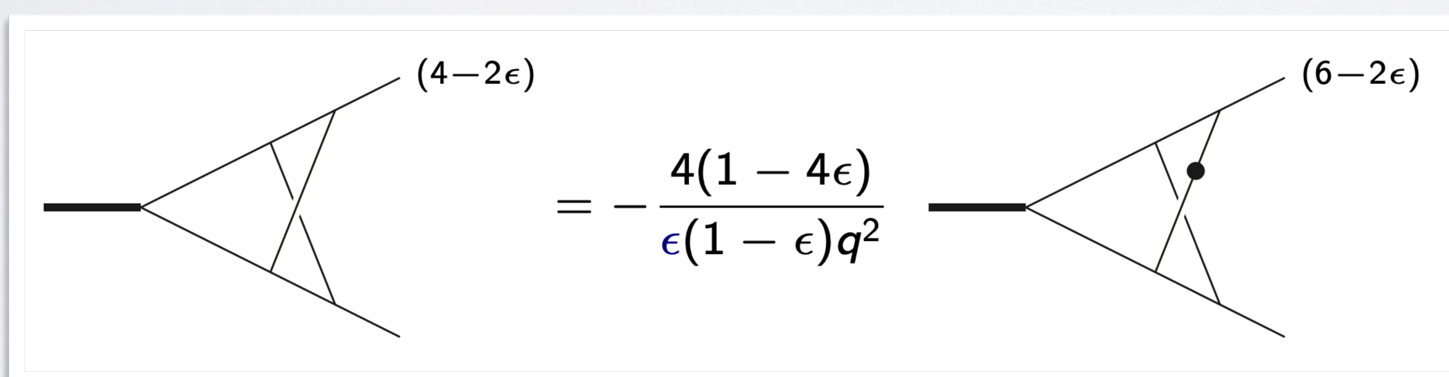


- Master integrals:
  - 1-fold integral over polylogarithms (Euclidean region)
    [Bonciani, Di Vita, Mastrolia, Schubert '16]
  - One-mass: real-valued multiple polylogarithms (physical region)
    [AvM, Schabinger '17]
  - Two-mass: ***optimized representation*** in physical region
    [Heller, AvM, Schabinger '19]

HOW TO SOLVE MASTER INTEGRALS

- Feynman/Schwinger parameters (analytic + numeric)
- Differential equations (analytic, numeric, expansions)
- Loop momentum integration, Mellin-Barnes, …

- Note *[Panzer 2014; AvM, Panzer, Schabinger 2014]*:
    - any ***divergent*** loop integral expressible in terms of **finite** ones !



    - expand integrands of finite integrals around $\epsilon = (4-d)/2 \approx 0$

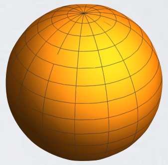
      see e.g. HH *[Heinrich et al]*, Hj *[Jones et al]*

DIFFERENTIAL EQUATIONS

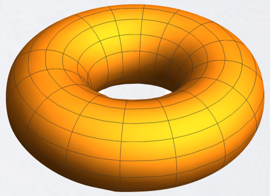
- Need to solve ***master integrals***, use method of differential equations
- Aim: analytical integration of differential equations *[Kotikov '91, Remiddi '97]*:

$$\boxed{\partial_x \vec{I}(x;\epsilon) = A(x;\epsilon)\vec{I}(x;\epsilon)} \quad \text{where} \;\; \epsilon = (4-d)/2$$

- Homogeneous solutions for $\epsilon = 0$ (leading singularities):
  - ***Rational number***, e.g. $1/2$



  - ***Rational functions***, e.g. $1/x$
  - ***Algebraic functions***, e.g. $\sqrt{x(x-4)}$
  - ***Elliptic integrals***, e.g. $K(x) = \int_0^1 \frac{\mathrm{d}z}{\sqrt{(1-z^2)(1-xz^2)}}, \ldots$

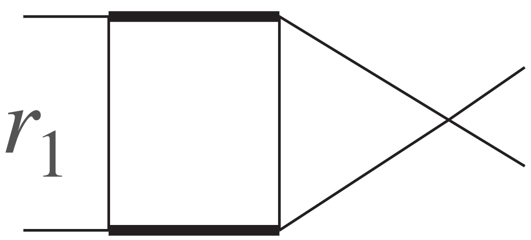
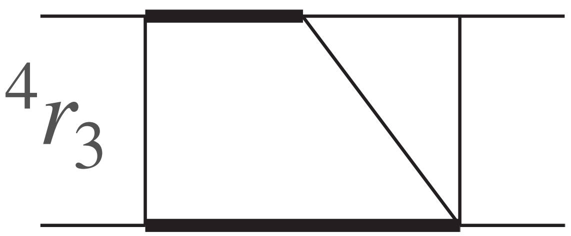
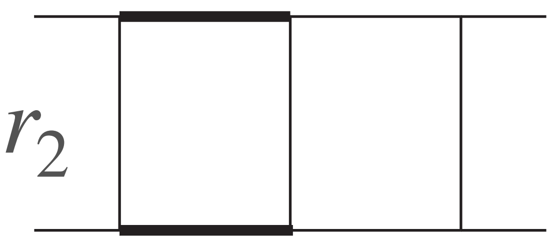


- Basis change involving homogenous solutions may allow to find ***ϵ-form***:

$$\boxed{d\vec{m} = \epsilon \, \mathrm{dln}(l_a(x)) \, A^{(a)}(x) \, \vec{m}}$$

[Kotikov '10, Henn '13, Remiddi, Tancredi '16, AvM, Tancredi '17, Adams, Weinzierl '18]

NON RATIONALIZABLE ROOTS

$\epsilon^4 s r_1$  $\epsilon^4 r_3$  $\epsilon^4 s r_2$  etc.

- $r_1 = \sqrt{s(s-4m^2)}, \;\; r_2 = \sqrt{-st\big(4m^2(t+m^2)-st\big)}, \;\; r_3 = \sqrt{s\big(t^2(s-4m^2)+sm^2(m^2-2t)\big)}$

- Reparametrization $s = -m^2\frac{(1-w)^2}{w}, \;\; t = -m^2\frac{w(1+z)^2}{z(1+w)^2}$ rationalizes 2 out of 3 roots, but

$$r = \sqrt{4(1-w)^2wz^2 + (w+z)^2(1+wz)^2}$$

not rationalizable !

- birationally equivalent to K3 *[van Straten ’14, Besier, Festi, Harrison, Naskrecki ’19]*

DIFFERENTIAL EQUATION

Diff. eq. $\boxed{\mathrm{d}\vec{m} = \epsilon \sum_a \mathrm{d}\ln(l_a)\, A^{(a)}\, \vec{m}}$ with root-valued letters l_i

e.g. $l_{13} = -(1-w)(z-w)(1-w\,z) + (1+w)\sqrt{4(1-w)^2 wz^2 + (w+z)^2(1+wz)^2}$

- ***How to solve ?***
  - Can't integrate in terms of GPLs in ***filtration basis***
  - Match against ***multiple polylog*** ansatz ? *[Duhr, Gangl, Rhodes 2011]*
  - Need ***elliptic polylogarithms*** ?

    $r = \sqrt{4(1-w)^2 wz^2 + (w+z)^2(1+wz)^2}$ elliptic curve ?

SYMBOL CALCULUS

- ***How to solve the differential equation ?***

$$\mathrm{d}\vec{m} = \epsilon \sum_a \mathrm{d}\ln(l_a)\, A^{(a)}\, \vec{m}$$

- If letters l_i simple: iterated integration gives ***multiple polylogarithms***

  e.g. $l_1 = x, \quad l_2 = x - 1, \quad Li_2(x) = -\int_0^x \frac{dt}{t} \int_0^t \frac{dt'}{t' - 1}$

- ***Symbol calculus***: $S(Li_2(x)) = -\left(\ln(x) \otimes \ln(x-1)\right)$

- ***Letters*** form ***words*** with grammar (log law):

$$\ln(l_1 l_2) \otimes \ln(l_3) = \ln(l_1) \otimes \ln(l_3) + \ln(l_2) \otimes \ln(l_3)$$

- C***omputer algebra*** allows to automatically derive functional identities, take limits, …

CONSTRUCTING LETTERS

- ***Rational letters:***

$$\mathscr{L}_R = \{1-w, -w, 1+w, 1-w+w^2, 1-z, -z, 1+z, \\ 1-wz, 1+w^2z, -z-w^2, z-w\}$$

- ***Initial algebraic letters:***

$$\mathscr{L}_A = \{r, -(1-w)(z-w)(1-wz)+r(1+w), \\ -(1-w)\big(4wz+(w+z)(1+wz)\big)-r(1+w), \\ r^2-2wz^2(1-w)^2+r(w+z)(1+wz), \\ r^2(1-z)^2+2z^2(z+w^2)(1+w^2z)+r(1-z)(1+z)\big(2wz-(w+z)(1+wz)\big)$$

with $r = \sqrt{4(1-w)^2wz^2+(w+z)^2(1+wz)^2}$

- ***New algorithm to construct improved algebraic letters:***

$$\mathscr{L}_{\tilde{A}} = \Big\{r, \tfrac{1}{2}\big(2+z-w+wz(w+z)+r\big), \tfrac{1}{2}\big(2w^2+z-w+wz(w+z)+r\big), \\ \tfrac{1}{2}\big(-(w+z)(1-wz)+r\big), \tfrac{1}{2}\big(-(z-w)(1+wz)+r\big)\Big\}$$

SUCCESS WITH NEW LETTERS

- ***Factorization*** of old letters w.r.t. new alphabet:

$$-(1-w)(z-w)(1-w\,z)+\mathbf{r}\,(1+w) = \frac{2(-w)(1+z)(-z-w^2)(2+z-w+w\,z(w+z)+\mathbf{r})}{2w^2+z-w+w\,z(w+z)+\mathbf{r}}$$

$$-(1-w)\big(4w\,z+(w+z)(1+w\,z)\big)-\mathbf{r}\,(1+w) = \frac{8(-w)^2(-z)(1+z)^3(1+w^2z)(2w^2+z-w+w\,z(w+z)+\mathbf{r})}{(2+z-w+w\,z(w+z)+r)(-(w+z)(1-w\,z)+\mathbf{r})(-(z-w)(1+w\,z)+r)}$$

$$r^2-2w\,z^2(1-w)^2+\mathbf{r}\,(w+z)(1+w\,z) = \frac{(-z)^2(2+z-w+w\,z(w+z)+\mathbf{r})^2(2w^2+z-w+w\,z(w+z)+r)^2}{8(1+z)^2(1+w^2z)^2(-(w+z)(1-w\,z)+\mathbf{r})^2(-(z-w)(1+w\,z)+r)^{-2}}$$

$$r^2(1-z)^2+2z^2(z+w^2)(1+w^2z)+\mathbf{r}\,(1-z)(1+z)\big(2w\,z-(w+z)(1+w\,z)\big) = \frac{2(-z)^2(1+w^2z)^2(-(w+z)(1-w\,z)+\mathbf{r})^2}{(-(z-w)(1+w\,z)+\mathbf{r})^2}$$

- ***Result: successful integration*** of symbol !
  - no roots of letters needed, much lower degree of arguments, no numerical letters !

A MATHEMATICAL RESULT

WITH PRACTICAL CONSEQUENCES

- In conclusion, despite the presence of non-rationalizable roots integrable in terms of standard multiple polylogarithms !

- **Algorithm** gives:

$$m_{32} = \epsilon^3 \left[4\,\mathrm{Li}_3\!\left(\frac{l_1 l_2 l_6 l_7 l_{10} l_{13}}{l_{14} l_{15} l_{16}}\right) - 2\,\mathrm{Li}_3\!\left(\frac{l_2^3 l_6 l_7^2}{l_{15} l_{16}}\right) + \ldots + 4\,\mathrm{Li}_2\!\left(\frac{l_6 l_{14} l_{16}}{l_7 l_9 l_{15}}\right)\ln(l_3) + \ldots \right]$$
$$+\epsilon^4 \left[-\mathrm{Li}_{2,2}\!\left(-\frac{l_1^2 l_3 l_{15}}{l_2^2 l_7 l_{14}}, \frac{l_2^2 l_7 l_{15}}{l_1 l_3 l_6 l_1 4}\right) + \ldots + \frac{701}{4}\,\mathrm{Li}_4\!\left(\frac{l_1 l_3^2 l_6^2 l_9 l_{14}}{l_2 l_7 l_{13} l_{15} l_{16}}\right) + \ldots \right] + O(\epsilon^5)$$

[Heller, AvM, Schabinger 2019]

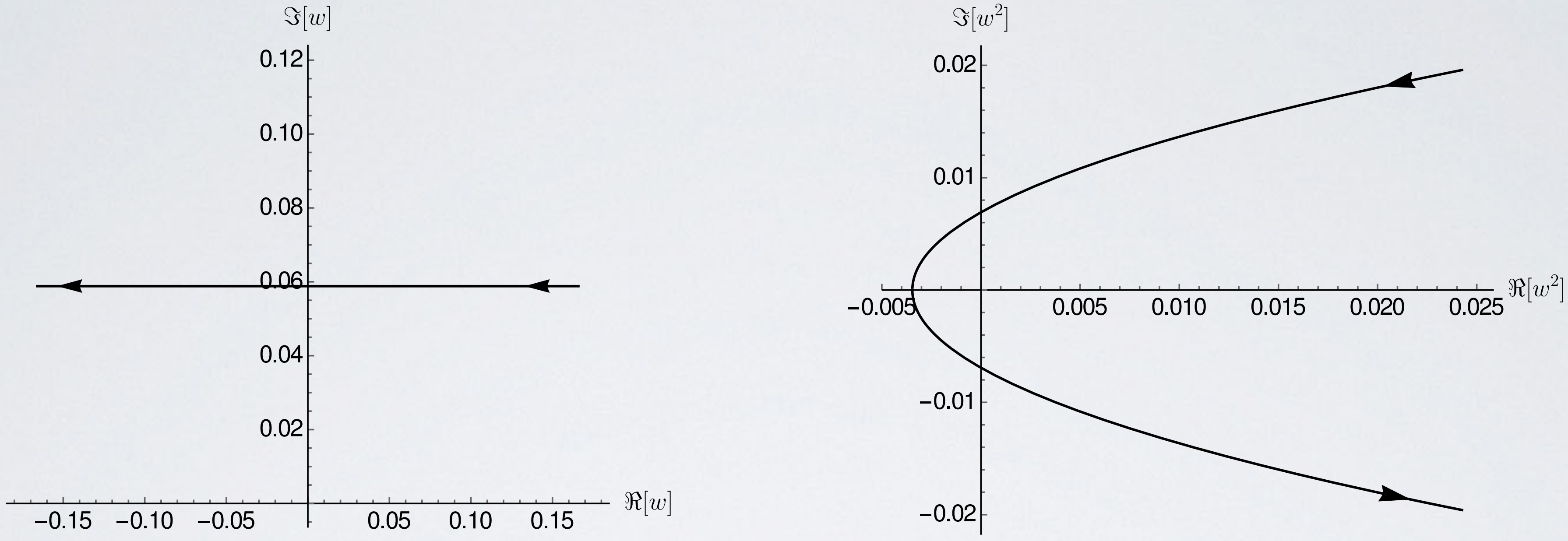
- ***Fast*** and ***robust*** numerical evaluations in Monte Carlo programs

- O(s) for all master integrals for generic point (double precision)

- Avoid pseudo-thresholds in functional basis

ANALYTIC CONTINUATION

Option 1:



$$\ln\left(w^2\right) \to \ln\left(w^2\right) + 2\pi i$$

$$\ln\left(w^2\right) = 2\ln\left(w\right) \to 2\ln(w) = 2\ln(-w) + 2\pi i = \ln\left(w^2\right) + 2\pi i$$

$$\mathrm{Li}_2\left(1-w^2\right) \to \mathrm{Li}_2\left(1-w^2\right) - 2\pi i \ln\left(1-w^2\right),$$

monodromies of multiple poylogs: with coproduct *[Goncharov ’01, Duhr ’11]*

Option 2: fix boundaries in each region separately

Option 3: solve diff. eqs. by expansion, fit precise numerics to transport analyt. constants. *[Lee, Smirnov, Smirnov ‘18, Moriello ’19]*

RESTRICTIONS ON FUNCTIONAL BASIS

- Wish to ***avoid i0 prescriptions***, cancelations at pseudo-thresholds
- ***Select functions*** based on absence of cuts (read off from symbol)
- ***Rich structure of pseudo-thresholds*** in physical region
- Already integrals with rationalizable letters require sub-domains:
  $0 < s < m^2, \quad m^2 < s < 2m^2, \quad 2m^2 < s < 4m^2, \quad s > 4m^2$
  (note: only $s = 0,\ m^2,\ 4m^2$ physical thresholds)

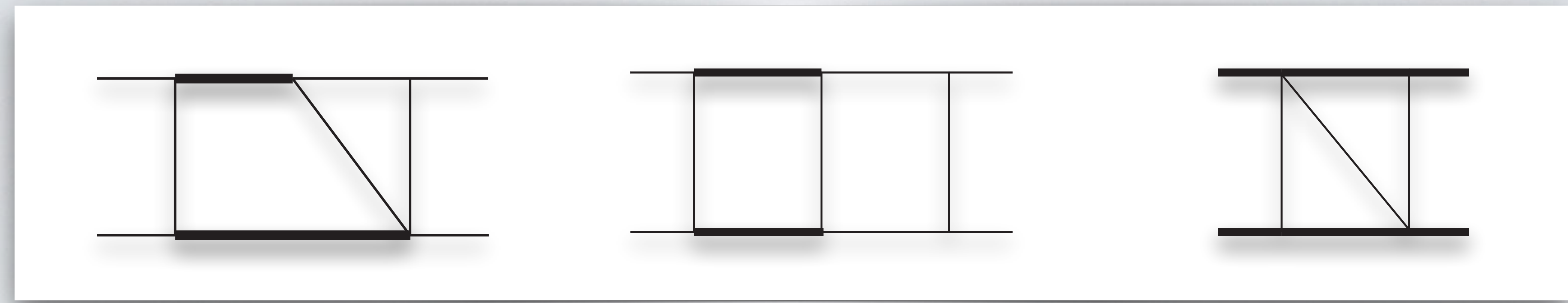
NUMERICAL PERFORMANCE

- Our representation allows ***fast and robust*** usage in Monte-Carlo
- E.g. at $(s, t, m^2) = (17, -7, 6241/1681)$:

$$m_{32} \approx \epsilon^3\left(0.066537984962080530758\ldots - 27.508245870011457529\ldots\ i\right)$$
$$+\epsilon^4\left(51.615607433806381131\ldots - 149.06326619542437190\ldots\ i\right) + \mathcal{O}\left(\epsilon^5\right),$$

all master integrals: O(second) for double precision with GiNaC's polylogs *[Vollinga, Weinzierl '04]*

SCOPE OF METHOD



- We also considered DY integrals and planar Bhabha integrals in ***direct integration*** approach *[Brown '08, Panzer '14]*
- Found ***variable changes*** to prove multiple polylog solution possible to ***all orders in*** ϵ
- Obtained explicit results, but not as nice as differential equations
- Note: DY and Bhabha K3s not isomorphic *[Besier, Festi, Harrison, Naskrecki '19]*
- Constructed alphabets with up to 5 simultaneous roots for HH/VV production

multivariateapart.nb

Univariate partial fractions separate terms with different poles:

In[1]:= $\texttt{Apart}\left[\frac{1}{x\,(1+x)},\ x\right]$

Out[1]= $\frac{1}{x}-\frac{1}{1+x}$

Let's consider a multivariate example:

In[2]:= $\texttt{multi} = \frac{2\,y-x}{y\,(x+y)\,(y-x)};$

Naive iteration introduces spurious poles (here 1/x) for multivariate case:

In[3]:= `Apart[multi, y]`

Out[3]= $\frac{1}{x\,y}+\frac{1}{2\,x\,(-x+y)}-\frac{3}{2\,x\,(x+y)}$

Solution: multivariate partial fractions using methods from polynomial ideal theory:

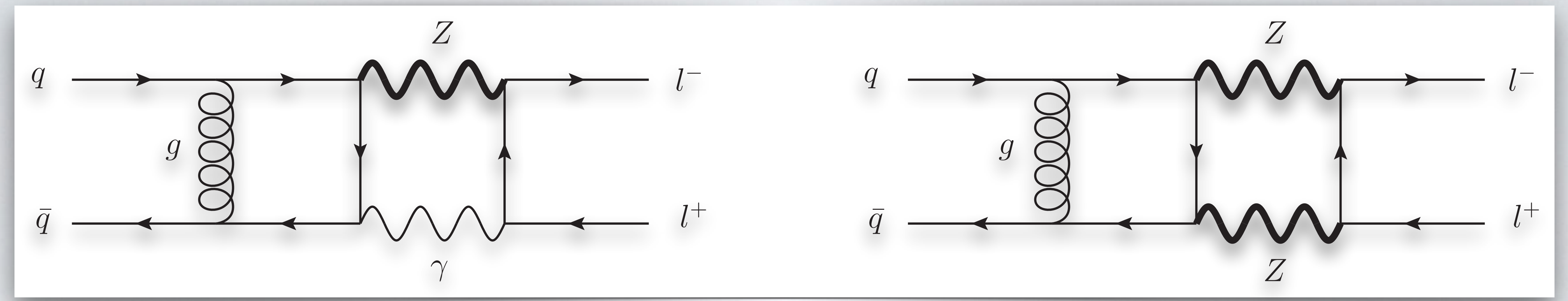
In[4]:= ``<< MultivariateApart` ``

```
MultivariateApart -- Multivariate partial fractions. By Matthias Heller and Andreas von Manteuffel.
```

In[5]:= `MultivariateApart[multi]`

Out[5]= $-\frac{1}{2\,(x-y)\,y}+\frac{3}{2\,y\,(x+y)}$

RESULTS FOR AMPLITUDES



- Calculated $O(\alpha_s), O(\alpha), O(\alpha_s\alpha)$ corrections to sufficient order in ϵ
- Amplitudes ***finite*** after UV renormalizations and IR subtractions
- Confirm known ***QED-QCD*** result *[Kilgore, Sturm '11]*
- γ^5 scheme dependent results, but ***finite remainders coincide*** !

NOTATION

- Perturbative expansion:

$$\overset{(-)}{\mathcal{A}}_{\mathrm{DY}} = 4\pi\alpha\left(\overset{(-)}{\mathcal{A}}{}^{(0,0)}_{\mathrm{DY}} + \overset{(-)}{\mathcal{A}}{}^{(0,1)}_{\mathrm{DY}}\left(\frac{\alpha_s}{4\pi}\right) + \overset{(-)}{\mathcal{A}}{}^{(1,0)}_{\mathrm{DY}}\left(\frac{\alpha}{4\pi}\right) + \overset{(-)}{\mathcal{A}}{}^{(1,1)}_{\mathrm{DY}}\left(\frac{\alpha}{4\pi}\right)\left(\frac{\alpha_s}{4\pi}\right) + \cdots\right)$$

- From form factors to helicity amplitudes:

$$\begin{aligned}
\mathcal{H}^{(m,n)}_{+-+-} &= -2(s+t)\left(\mathbf{C}^{(m,n),\,\mathrm{fin}}_{\mathrm{VV}} + \mathbf{C}^{(m,n),\,\mathrm{fin}}_{\mathrm{AA}} + \mathbf{C}^{(m,n),\,\mathrm{fin}}_{\mathrm{VA}} + \mathbf{C}^{(m,n),\,\mathrm{fin}}_{\mathrm{AV}}\right), \\
\mathcal{H}^{(m,n)}_{-+-+} &= -2(s+t)\left(\mathbf{C}^{(m,n),\,\mathrm{fin}}_{\mathrm{VV}} + \mathbf{C}^{(m,n),\,\mathrm{fin}}_{\mathrm{AA}} - \mathbf{C}^{(m,n),\,\mathrm{fin}}_{\mathrm{VA}} - \mathbf{C}^{(m,n),\,\mathrm{fin}}_{\mathrm{AV}}\right), \\
\mathcal{H}^{(m,n)}_{+--+} &= -2t\left(\mathbf{C}^{(m,n),\,\mathrm{fin}}_{\mathrm{VV}} - \mathbf{C}^{(m,n),\,\mathrm{fin}}_{\mathrm{AA}} - \mathbf{C}^{(m,n),\,\mathrm{fin}}_{\mathrm{VA}} + \mathbf{C}^{(m,n),\,\mathrm{fin}}_{\mathrm{AV}}\right), \\
\mathcal{H}^{(m,n)}_{-++-} &= -2t\left(\mathbf{C}^{(m,n),\,\mathrm{fin}}_{\mathrm{VV}} - \mathbf{C}^{(m,n),\,\mathrm{fin}}_{\mathrm{AA}} + \mathbf{C}^{(m,n),\,\mathrm{fin}}_{\mathrm{VA}} - \mathbf{C}^{(m,n),\,\mathrm{fin}}_{\mathrm{AV}}\right).
\end{aligned}$$

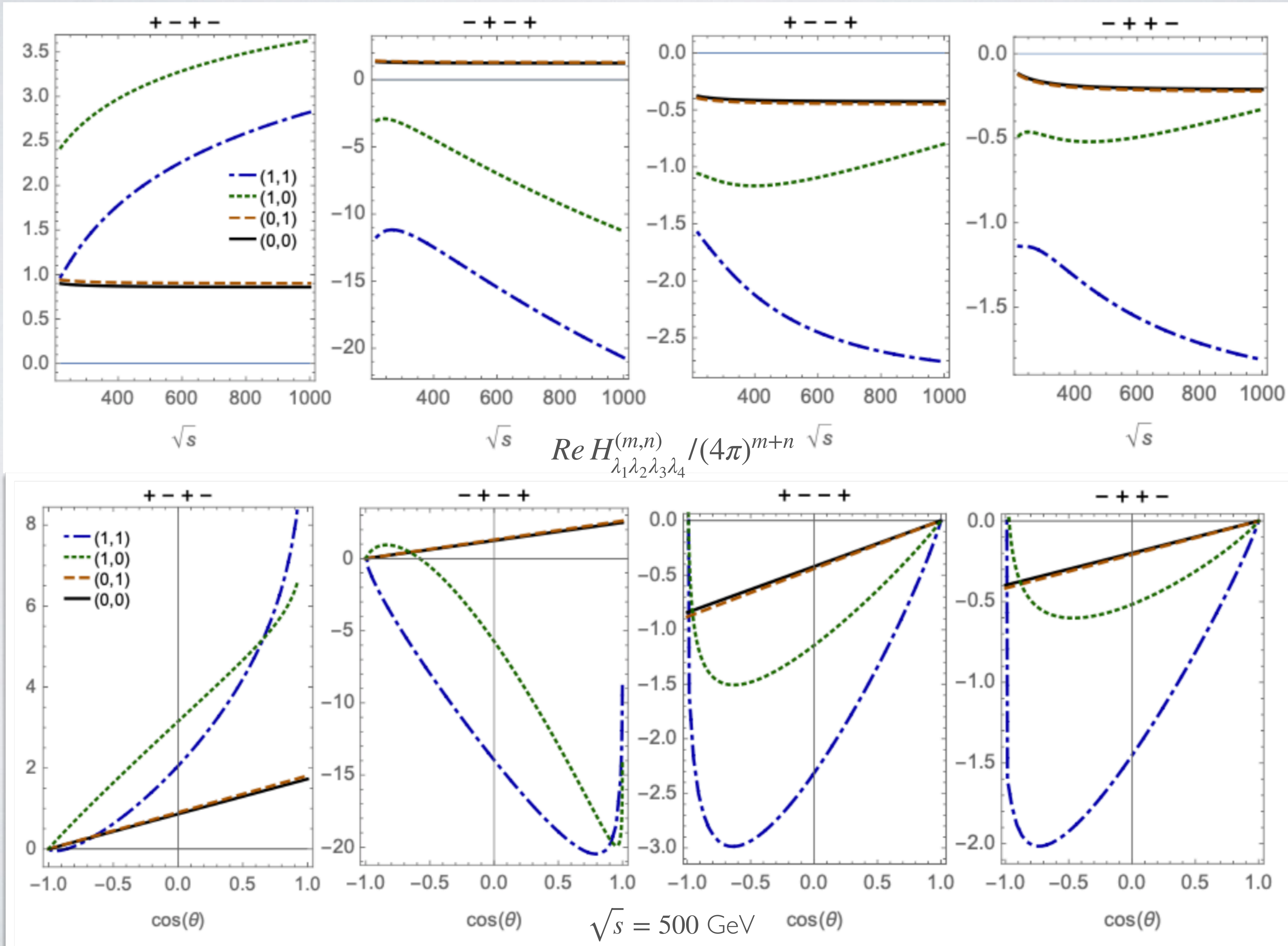
- NLO QCD:

$$\mathcal{H}^{(0,1)}_{\lambda_1\lambda_2\lambda_3\lambda_4}/(4\pi) = \left(\frac{\pi}{3} - i\right)\mathcal{H}^{(0,0)}_{\lambda_1\lambda_2\lambda_3\lambda_4}$$

(note: $\pi/3 \approx 1.05$)

HELICITY AMPLITUDES

[Heller, AvM, Schabinger, Spiesberger 2020]



CONCLUSIONS

- ***Drell-Yan process*** important for SM precision physics and BSM searches
  - Want control at highest energies
  - Here: mixed QCD-EW corrections to dilepton production (“off-shell DY")
- ***Two-loop Feynman integrals***
  - ϵ dln basis and root-valued letters
  - possible to solve in terms of standard multiple polylogarithms
  - new method to construct algebraic letters
- ***Two-loop amplitudes***
  - Analytical calculation in two γ_5 schemes: HVBM and Kreimer’s scheme
  - Finite remainders agree
  - MC-friendly compact results, ready-to-go for cross section calculation