

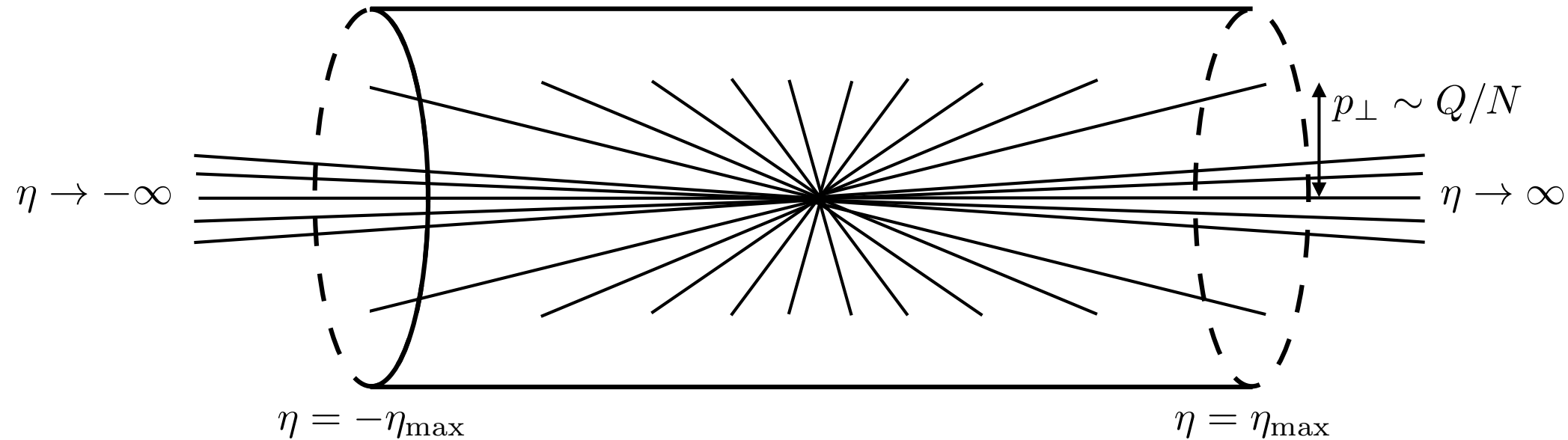
A Large- N Expansion for Minimum Bias

with Tom Melia, 2107.04041

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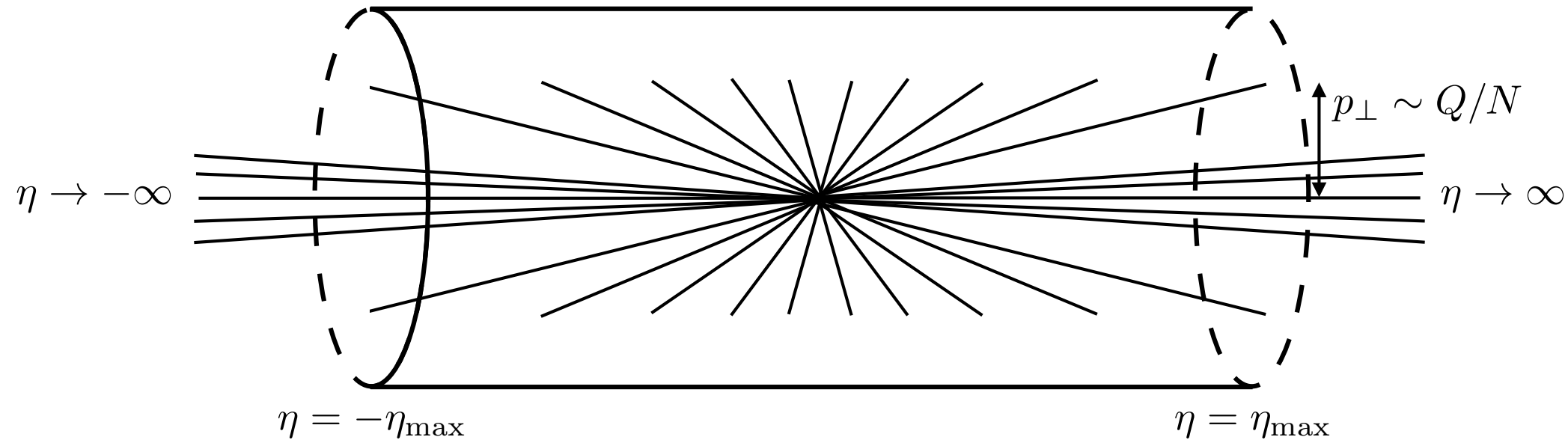
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Minimum Bias Power Counting



- 1) $|\eta| \ll \eta_{\max}$ (detected particle region is large)
- 2) $p_{\perp} \gg m_{\pi}$ (hadron masses are irrelevant)
- 3) $E_{|\eta| > \eta_{\max}} \sim Q$ (large energy lost down beams)
- 4) $N \gg 1$ (large number of detected particles)
- 5) $\langle p_{\perp} \rangle \sim \sqrt{\langle p_{\perp}^2 \rangle}$ (particle p_T is representative)

Minimum Bias Symmetries



- 1) $O(2)$ about beam
- 2) $\eta \rightarrow -\eta$ reflection of beams
- 3) $\eta \rightarrow \eta + \Delta\eta$ translation for $\eta_{\max} \rightarrow \infty$
- 4) S_N permutation symmetry of particles
- 5) “Ergodicity”: any two detected particles can be interchanged by symmetries

Expanding the Cross Section

$$\begin{aligned}
 \sigma &\sim \int d\Pi_{N+N_{B_1}+N_{B_2}} |\mathcal{M}(1, \dots, N, N+1, \dots, N+N_{B_1}, N+N_{B_1}+1, \dots, N+N_{B_1}+N_{B_2})|^2 \\
 &\sim \int_0^Q dk^+ \int_0^Q dk^- \int \prod_{i=1}^N \left[p_{\perp i} dp_{\perp i} d\eta_i \frac{d\phi_i}{2\pi} \right] f(k^+ k^-) |\mathcal{M}(1, 2, \dots, N)|^2 \\
 &\quad \times \delta \left(k^- - \sum_{i=1}^N p_{\perp i} e^{\eta_i} \right) \delta \left(k^+ - \sum_{i=1}^N p_{\perp i} e^{-\eta_i} \right) \delta^{(2)} \left(\sum_{i=1}^N \vec{p}_{\perp i} \right)
 \end{aligned}$$

Exclusively expressed in terms of detected particles

$f(k^+ k^-)$ is like a parton/nucleon distribution function

Transverse momentum is conserved in the large- N limit because of “random walk” of particles in event

Not differential in N nor Q ; will only be able to make statements at fixed N and Q or in the large- N limit

Expanding the Matrix Element

$$|\mathcal{M}(1, 2, \dots, N)|^2 = 1 + \mathcal{O}(Q^{-2})$$

As a physical probability density, assume analytic, finite, and non-negative on all of phase space

Expand in linearly-independent irreducible polynomial representations of symmetries of minimum bias

Momentum conservation:

$$0 = \left(\sum_{i=1}^N \vec{p}_{\perp i} \right)^2 = \sum_{i=1}^N p_{\perp i}^2 + \sum_{i \neq j} p_{\perp i} p_{\perp j} \cos(\phi_i - \phi_j)$$

$$k^+ k^- = \left(\sum_{i=1}^N p_{\perp i} e^{-\eta_i} \right) \left(\sum_{j=1}^N p_{\perp j} e^{\eta_j} \right) = \sum_{i=1}^N p_{\perp i}^2 + \sum_{i \neq j} p_{\perp i} p_{\perp j} \cosh(\eta_i - \eta_j)$$

Only $\sum_{i=1}^N p_{\perp i}^2$ is independent at order Q^{-2}

Expanding the Matrix Element

$$|\mathcal{M}(1, 2, \dots, N)|^2 = 1 + \frac{c_1^{(2)}}{Q^2} \sum_{i=1}^N p_{\perp i}^2 + \mathcal{O}(Q^{-4})$$

Ergodicity and central limit theorem means that matrix element reduces to a constant on phase space as $N \rightarrow \infty$

$$\lim_{N \rightarrow \infty} |\mathcal{M}(1, 2, \dots, N)|^2 \rightarrow 1 + \frac{c_1^{(2)}}{Q^2} N \langle p_{\perp}^2 \rangle + \dots$$

N dependence of coefficients is just constrained by finiteness (for now)

$$\lim_{N \rightarrow \infty} \frac{c_1^{(2)}}{Q^2} N \langle p_{\perp}^2 \rangle \sim \frac{c_1^{(2)}}{N} < \infty$$

Application 1: p_T Spectrum

Take $N \rightarrow \infty$ limit so squared matrix element is just a constant

Use permutation symmetry to fix to p_T of particle 1:

$$p(p_\perp) \sim \lim_{N \rightarrow \infty} \int_0^Q dk^+ \int_0^Q dk^- \int \prod_{i=1}^N \left[p_{\perp i} dp_{\perp i} d\eta_i \frac{d\phi_i}{2\pi} \right] f(k^+ k^-) \delta(p_\perp - p_{\perp 1}) \\ \times \delta \left(k^- - \sum_{i=1}^N p_{\perp i} e^{\eta_i} \right) \delta \left(k^+ - \sum_{i=1}^N p_{\perp i} e^{-\eta_i} \right) \delta^{(2)} \left(\sum_{i=1}^N \vec{p}_{\perp i} \right)$$

Application 1: $p_\perp$ Spectrum

Take $N \rightarrow \infty$ limit so squared matrix element is just a constant

Integrate over particles 2 through N :

$$p(p_\perp) \sim \lim_{N \rightarrow \infty} \int_0^Q dk^+ \int_0^Q dk^- f(k^+ k^-) \\ \times \int_{-\infty}^{\infty} d\eta_1 p_\perp [(k^+ - p_\perp e^{-\eta_1})(k^- - p_\perp e^{\eta_1}) - p_\perp^2]^N$$

Application 1: p_τ Spectrum

Take $N \rightarrow \infty$ limit so squared matrix element is just a constant

Take $N \rightarrow \infty$ to exponentiate integrand:

$$p(p_\perp) \sim \int_0^Q dk^+ \int_0^Q dk^- f(k^+ k^-) \int_{-\infty}^{\infty} d\eta_1 p_\perp e^{-\frac{k^+ e^\eta + k^- e^{-\eta}}{k^+ k^-} N p_\perp}$$

Application 1: p_T Spectrum

Take $N \rightarrow \infty$ limit so squared matrix element is just a constant

Take $N \rightarrow \infty$ to exponentiate integrand:

$$p(p_\perp) \sim \int_0^Q dk^+ \int_0^Q dk^- f(k^+ k^-) \int_{-\infty}^{\infty} d\eta_1 p_\perp e^{-\frac{k^+ e^\eta + k^- e^{-\eta}}{k^+ k^-} N p_\perp}$$

Insert appropriate factors to normalize distribution:

$$\begin{aligned} p(p_\perp) &= \frac{1}{Q^2} \int_0^Q dk^+ \int_0^Q dk^- f(k^+ k^-) \frac{4N^2 p_\perp}{k^+ k^-} K_0 \left(\frac{2N p_\perp}{\sqrt{k^+ k^-}} \right) \\ &= \frac{4N^2 p_\perp}{Q^2} \int_0^1 dx \log \frac{1}{x} f(x) \frac{1}{x} K_0 \left(\frac{2N p_\perp}{\sqrt{x} Q} \right) \end{aligned}$$

$$\int_0^1 dx \log \frac{1}{x} f(x) = 1$$

Application 1: p_τ Spectrum

$$p(p_\perp) = \frac{4N^2 p_\perp}{Q^2} \int_0^1 dx \log \frac{1}{x} f(x) \frac{1}{x} K_0 \left(\frac{2N p_\perp}{\sqrt{x} Q} \right)$$

Problem 1: Explicit function of N ; want a result valid for any large N

Solution: Replace it with the mean p_τ

$$\begin{aligned} \langle p_\perp \rangle &= \int_0^\infty dp_\perp p_\perp p(p_\perp) = \frac{\pi}{4} \frac{Q}{N} \int_0^1 dx \log \frac{1}{x} f(x) \sqrt{x} \\ &\equiv \frac{\pi}{4} \frac{Q}{N} \langle \sqrt{x} \rangle \end{aligned}$$

Eliminate explicit N dependence:

$$p(p_\perp) = \frac{\pi^2 \langle \sqrt{x} \rangle^2}{4 \langle p_\perp \rangle^2} p_\perp \int_0^1 dx \log \frac{1}{x} f(x) \frac{1}{x} K_0 \left(\frac{\pi \langle \sqrt{x} \rangle}{2 \langle p_\perp \rangle \sqrt{x}} p_\perp \right)$$

Application 1: p_T Spectrum

$$p(p_{\perp}) = \frac{\pi^2 \langle \sqrt{x} \rangle^2}{4 \langle p_{\perp} \rangle^2} p_{\perp} \int_0^1 dx \log \frac{1}{x} f(x) \frac{1}{x} K_0 \left(\frac{\pi \langle \sqrt{x} \rangle}{2 \langle p_{\perp} \rangle \sqrt{x}} p_{\perp} \right)$$

Problem 2: Need an explicit form of $f(x)$ to evaluate the distribution

Solution: $f(x)$ is a physical distribution, so should be analytic. IR divergences in pQCD or Landau pole suggest that $f(x)$ should peak at low total visible energy.

$$f(k^+ k^-) \sim \frac{n}{\gamma_E + \log n} e^{-n \frac{k^+ k^-}{Q^2}}$$
$$f(x) \sim \frac{n}{\gamma_E + \log n} e^{-nx}$$

We assume $n \gg 1$

Application 1: $p_\perp$ Spectrum

$$p(p_\perp) = \frac{\pi^2 \langle \sqrt{x} \rangle^2}{4 \langle p_\perp \rangle^2} p_\perp \int_0^1 dx \log \frac{1}{x} f(x) \frac{1}{x} K_0 \left(\frac{\pi \langle \sqrt{x} \rangle}{2 \langle p_\perp \rangle \sqrt{x}} p_\perp \right)$$

Can integrate over x with saddle-point approximation:

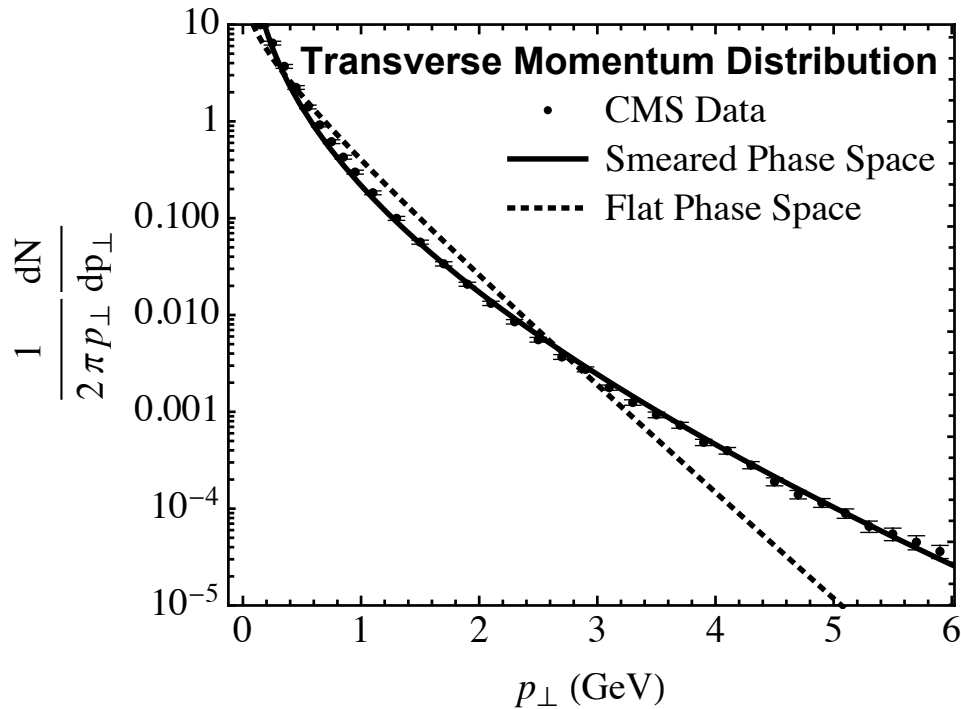
$$\lim_{z \rightarrow \infty} K_0(z) \rightarrow \sqrt{\frac{\pi}{2z}} e^{-z}$$

$$p(p_\perp) \sim \sqrt{p_\perp} \int_0^1 dx \log \frac{1}{x} \frac{e^{-nx - \frac{\pi^{3/2} p_\perp}{4 \langle p_\perp \rangle \sqrt{x} n}}}{x^{3/4}} \sim e^{-\frac{3\pi}{4} \frac{p_\perp^{2/3}}{\langle p_\perp \rangle^{2/3}}}$$

Suggestive of fractional-power dispersion relation? [arXiv:1403.3365](https://arxiv.org/abs/1403.3365)

Points way forward for developing honest EFT for minimum bias?

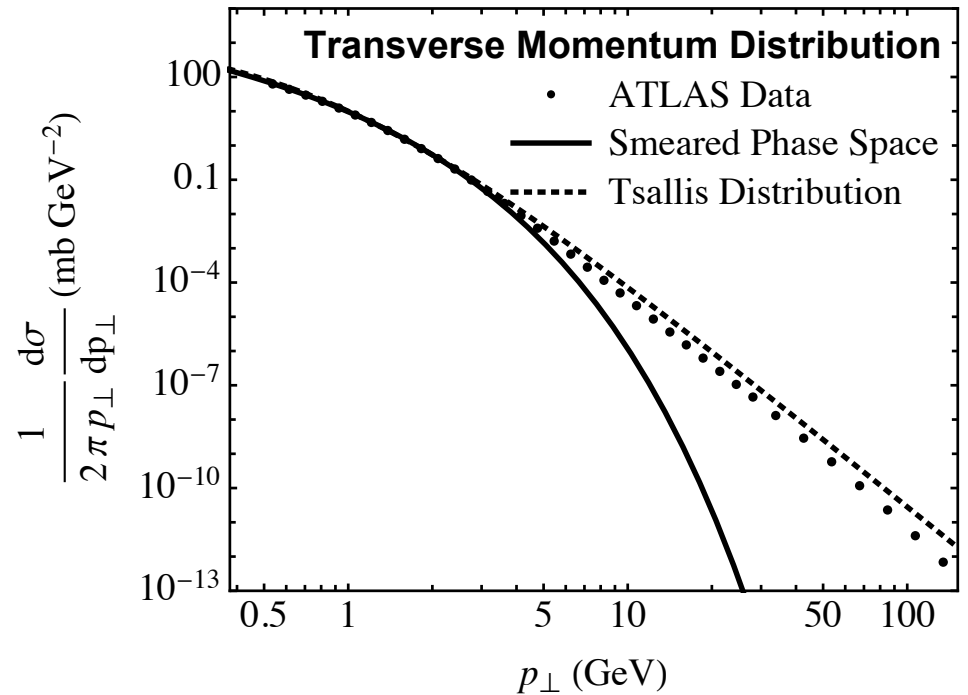
Application 1: p_T Spectrum



Prediction compared to 7 TeV
CMS data [arXiv:1005.3299](https://arxiv.org/abs/1005.3299)

$$\langle p_\perp \rangle = 0.65 \text{ GeV}$$

Qualitatively different shape than
flat phase space assumption



Prediction compared to 2.76 TeV
ATLAS data [arXiv:1504.04337](https://arxiv.org/abs/1504.04337)

$$\langle p_\perp \rangle = 0.5 \text{ GeV}$$

cf. Tsallis distribution

$$\frac{d\sigma}{dp_\perp} \sim p_\perp \left(1 + \frac{p_\perp}{nT} \right)^{-n}$$

Application 2: $\Delta\phi$ Correlations

Need a non-trivial squared matrix element to have non-trivial $\Delta\phi$ correlations

Expand pairwise particle azimuthal correlations in Fourier expansion

$$\begin{aligned} |\mathcal{M}|^2 &\supset 1 + \sum_{n=1}^{\infty} g_n(k^+k^-, N) \sum_{i \neq j}^N \frac{(\vec{p}_{\perp i} \cdot \vec{p}_{\perp j})^n}{Q^{2n}} \\ &\supset 1 + \sum_{n=1}^{\infty} g_n(k^+k^-, N) \sum_{i \neq j}^N \frac{p_{\perp i}^n p_{\perp j}^n}{Q^{2n}} \cos(n(\phi_i - \phi_j)) \end{aligned}$$

Focus on scaling with number of particles N :

$$|\mathcal{M}|^2 \sim 1 + \sum_{n=1}^{\infty} \frac{g_n(k^+k^-, N)}{N^{2n}} \sum_{i \neq j}^N \cos(n(\phi_i - \phi_j))$$

Application 2: $\Delta\phi$ Correlations

Pairwise azimuthal correlations imposed by momentum conservation trivialize as $N \rightarrow \infty$

- 1) Momentum conservation involves all N particles
- 2) ~Transverse random walk conserves momentum anyway

$$\lim_{N \rightarrow \infty} \int d\Pi_N \delta^{(2)} \left(\sum_{i=1}^N \vec{p}_{\perp i} \right) \rightarrow \int \prod_{i=1}^N \frac{d\phi_i}{2\pi}$$

Large- N limit for non-trivial azimuthal correlations:

$$\begin{aligned} \sigma &\sim \int_0^Q dk^+ \int_0^Q dk^- f(k^+ k^-) \int \prod_{i=1}^N \frac{d\phi_i}{2\pi} \left(1 + \sum_{n=1}^{\infty} \frac{g_n(k^+ k^-, N)}{N^{2n}} \sum_{j \neq k} \cos(n(\phi_j - \phi_k)) \right) \\ &\sim \int \prod_{i=1}^N \frac{d\phi_i}{2\pi} \left(1 + \sum_{n=1}^{\infty} \frac{d_n(N)}{N^{2n}} \sum_{j \neq k} \cos(n(\phi_j - \phi_k)) \right) \end{aligned}$$

Application 2: $\Delta\phi$ Correlations

Constraints on Fourier coefficients from positivity

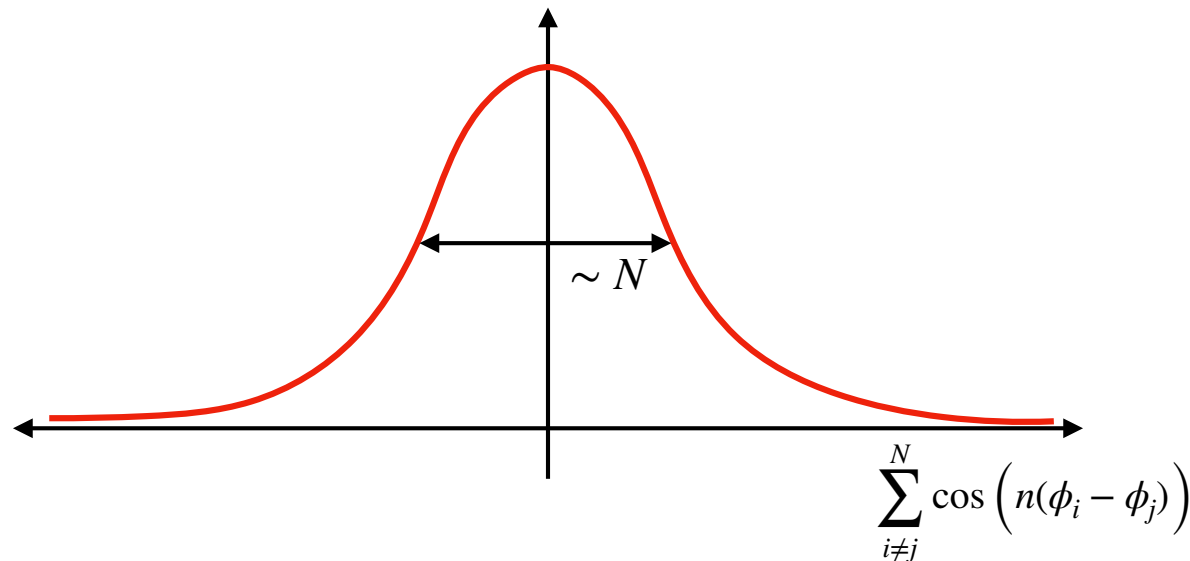
Central Limit Theorem in large- N limit

Mean:

$$\int_0^{2\pi} \prod_{i=1}^N \frac{d\phi_i}{2\pi} \sum_{j \neq k} \cos(n(\phi_j - \phi_k)) = 0$$

Variance:

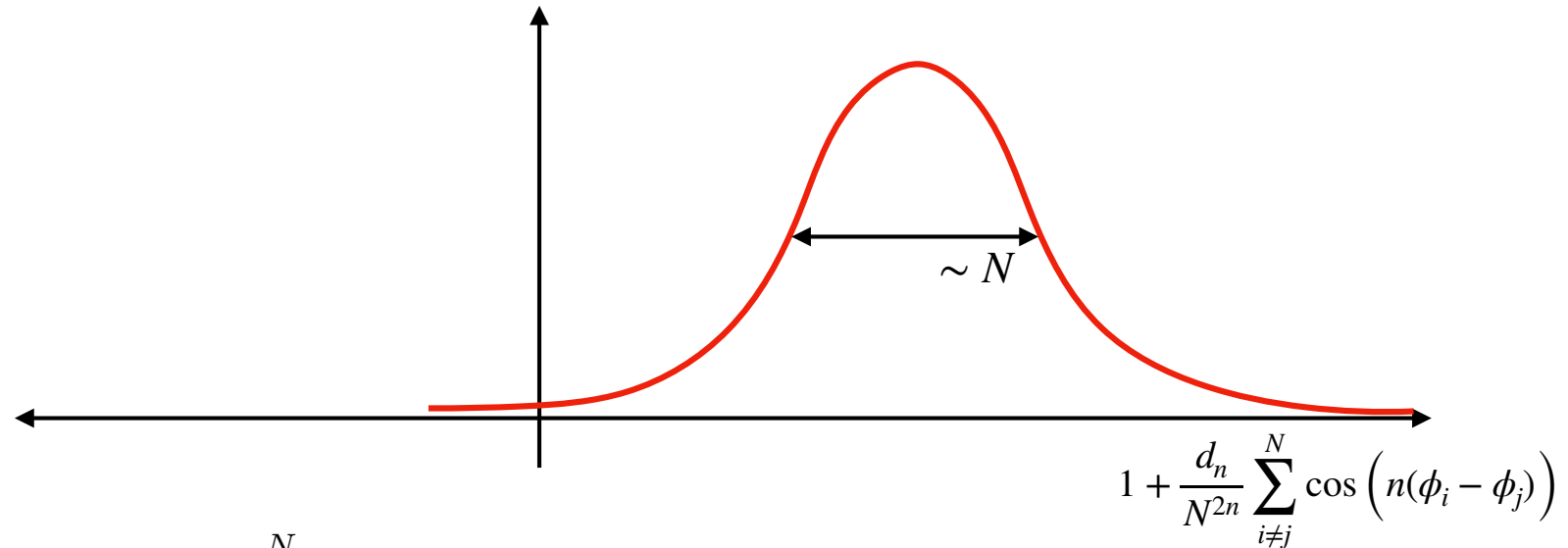
$$\sigma^2 \equiv \int_0^{2\pi} \prod_{i=1}^N \frac{d\phi_i}{2\pi} \left[\sum_{j \neq k} \cos(n(\phi_j - \phi_k)) \right]^2 = N^2 \int_0^{2\pi} \prod_{i=1}^N \frac{d\phi_i}{2\pi} \cos^2(n(\phi_1 - \phi_2)) = \frac{N^2}{2}$$



Application 2: $\Delta\phi$ Correlations

Constraints on Fourier coefficients from positivity

Squared Matrix Element must be positive!



$$1 \gtrsim \frac{d_n(N)}{N^{2n}} \sum_{i \neq j}^N \cos(n(\phi_i - \phi_j)) \sim \frac{d_n(N)}{N^{2n}} \sigma \sim \frac{d_n(N)}{N^{2n-1}}$$

$$\lim_{N \rightarrow \infty} d_n(N) \lesssim N^{2n-1}$$

Application 2: $\Delta\phi$ Correlations

$$\lim_{N \rightarrow \infty} d_n(N) \lesssim N^{2n-1}$$

Consequences for $\Delta\phi$ distribution:

$$p(\Delta\phi) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{d_n(N)}{N^{2n}} \cos(n \Delta\phi)$$

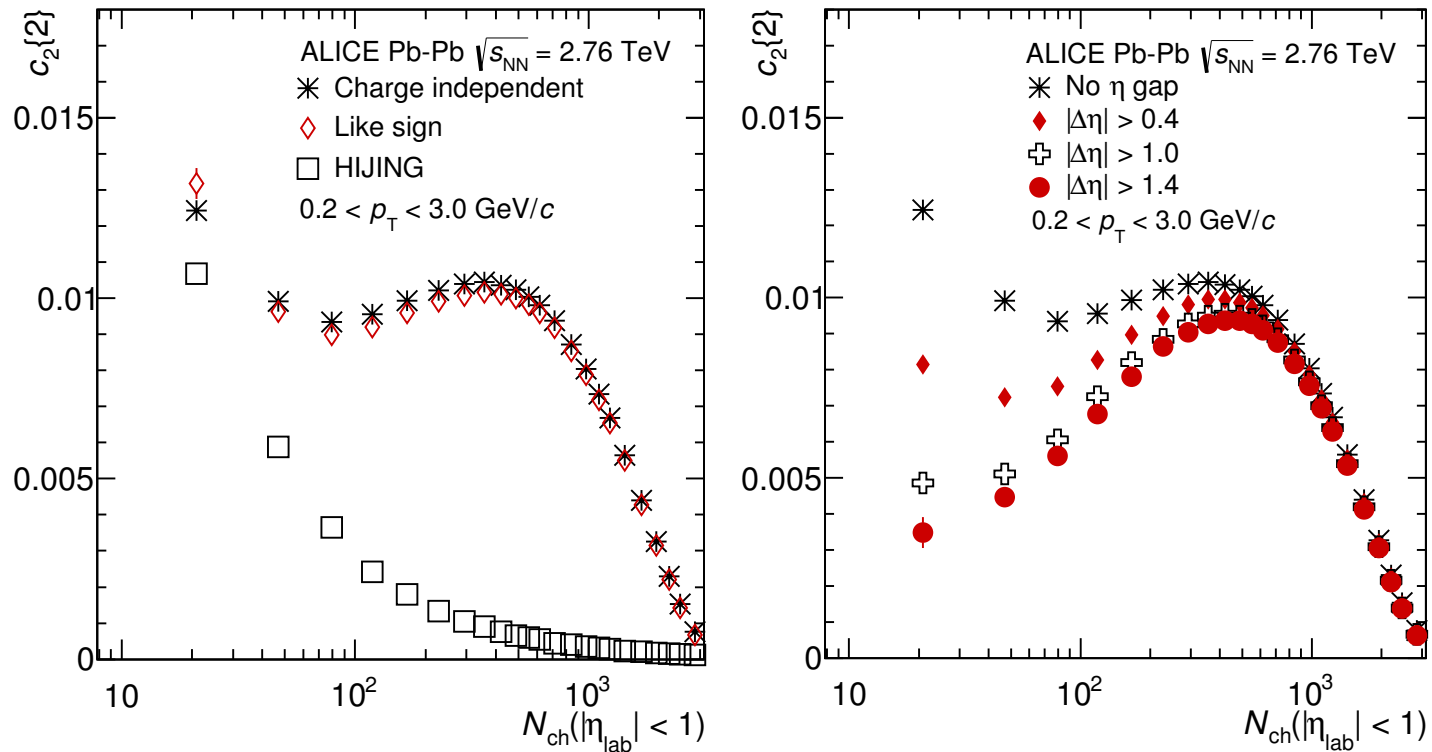
$$\lim_{N \rightarrow \infty} p(\Delta\phi) = \frac{1}{2\pi}$$

Ellipticity measure:

$$c_2\{2\} = \frac{1}{N} + \langle \cos(2\Delta\phi) \rangle = \frac{1}{N} + \int_0^{2\pi} d\Delta\phi p(\Delta\phi) \cos(2\Delta\phi) = \frac{1}{N} + \frac{d_2(N)}{N^4}$$

Ellipticity vanishes at large- N /low centrality because of positivity of the squared matrix element

Application 2: $\Delta\phi$ Correlations



ALICE data of ellipticity as a function of number of observed charged particles at central rapidities N_{ch} [arXiv:1406.2474](https://arxiv.org/abs/1406.2474)

Vanishes at large- N ; independent of $\Delta\eta$ spread of particle pairs

Azimuthal correlations also trend to 0 at small centralities

Conclusions

Identified several other scaling results that follow from minimum bias power counting: η distribution, CMS “ridge”, hadronic correlations in e^+e^- colliders

What other results follow from simple scaling arguments?

How does this expansion match to pQCD?

How does this expansion match to hydrodynamics?

Can the energy dependence of the “Wilson coefficients” in the matrix element expansion be predicted?

Is there a perturbative EFT for minimum bias?