Density perturbations in axions: Classical vs quantum field

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- QCD axions or axion-like particles are highly motivated candidates for CDM.

- QCD axions have very high phase-space degeneracy:

$$\mathcal{N} = n(t)\frac{(2\pi)^3}{\left(\frac{4\pi}{3}\,\delta p(t)^3\right)} \sim 10^{61}\left(\frac{6\times 10^{-6}\ \mathrm{eV}}{m}\right)^{\frac{8}{3}}$$

(Sikivie and Yang; 2009)

$\mathcal{N}$ is time-independent because $n(t) \propto a(t)^{-3}$ and $\delta p(t) \propto a(t)^{-1}$.

Generally, in cosmology, highly degenerate scalar fields are assumed to follow the classical field equations.

- Thermalization:

  Classical fields → Maxwell-Boltzmann distribution (UV catastrophe)

  Bosonic quantum fields → Bose-Einstein distribution

  There must be a time scale after which the quantum evolution of a highly degenerate scalar field differs from its classical evolution.

- Evolution of occupation numbers in quantum field theory differs from its classical counterpart.

Is the difference between classical and quantum field treatments apparent for density or velocity field?

Classical vs Quantum field treatment

- Non-relativistic limit: *(we are interested in cold axions)*

$$\phi(\vec{x},t) = \frac{1}{\sqrt{2m}}\left[\psi(\vec{x},t)\, e^{-imt} + \psi^\dagger(\vec{x},t)\, e^{imt}\right]$$

- Klein-Gordon equation reduces to a Schrödinger-like equation:

$$i\partial_t\psi = -\frac{1}{2m}\,\nabla^2\psi + V(\psi)\psi$$

- Classical: Quantum field $\psi(\vec{x},t)$ ⟹ Wave-function $\Psi(\vec{x},t)$

- Quantum: $\psi(\vec{x},t)$ is treated as an operator.

$$[\psi(\vec{x},t),\psi(\vec{y},t)] = 0, \quad \left[\psi(\vec{x},t),\psi^\dagger(\vec{y},t)\right] = \delta^3(\vec{x}-\vec{y})$$

Classical Field Treatment

Wavefunction: $\Psi(\vec{x},t) = A(\vec{x},t)e^{i\beta(\vec{x},t)}$

Number density: $n(\vec{x},t) = \mathrm{A}^2(\vec{x},t)$

Velocity: $\vec{v}(\vec{x},t) = \frac{1}{m}\vec{\nabla}\beta(\vec{x},t)$

- Continuity equation: $\partial_t n + \vec{\nabla}\cdot(n\vec{v}) = 0$
- Euler-like equation: $\partial_t\vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{m}\vec{\nabla}V - \vec{\nabla}q$

Evolution is uniquely specified by $\boldsymbol{n(\vec{x},0)}$ and $\boldsymbol{\vec{v}(\vec{x},0)}$.

Potential: $V(\vec{x},t) = \frac{\lambda}{8m^2}n(\vec{x},t)$

Quantum pressure: $q(\vec{x},t) = -\frac{1}{2m^2}\frac{\nabla^2\sqrt{n}}{\sqrt{n}}$

- Equations for a fluid of classical particles except for the quantum pressure term.

(SSC, Enomoto, Han, Sikivie, Todarello; 2018)

Quantum Field Treatment

- Number density: $\hat{n}(\vec{x},t) = \psi^\dagger\psi$
- Current density: $\hat{\vec{j}}(\vec{x},t) = \frac{1}{2im}\left(\psi^\dagger\,\vec{\nabla}\psi - \vec{\nabla}\psi^\dagger\,\psi\right)$
- Evolution:

$$\partial_t\hat{n} + \vec{\nabla}.\hat{\vec{j}} = 0$$

$$\partial_t\hat{\vec{j}} + \psi^\dagger\,\vec{\nabla}\hat{V}\,\psi = \frac{1}{4m^2}(\psi^\dagger\,\vec{\nabla}\nabla^2\psi - \nabla^2\psi^\dagger\,\vec{\nabla}\psi + h.c.)$$

Evolution is NOT uniquely specified by $\langle\hat{n}(\vec{x},0)\rangle$ and $\langle\hat{\vec{j}}\,(\vec{x},0)\rangle$.

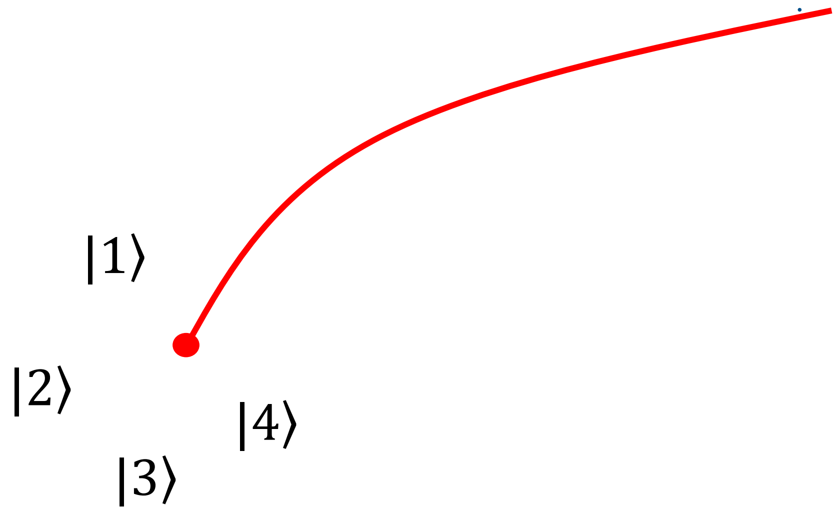
In quantum field treatment, the evolution is uniquely determined only by initial quantum state of the system.

SSC: arXiv 2105:09749, Phys. Rev. D 2021

- Initial density and velocity field: $n(\vec{x},0)$ and $\vec{v}(\vec{x},0)$.

- There exist many quantum states consistent with the initial conditions.

$$\langle \mathrm{in}|\hat{n}_Q(\vec{x},0)|\mathrm{in}\rangle = n(\vec{x},0),$$
$$\langle \mathrm{in}|\hat{\vec{j}}_Q(\vec{x},0)|\mathrm{in}\rangle = \vec{j}(\vec{x},0).$$



Do all these quantum states yield the same evolution of density?

In general, they are not identical!

Number density in quantum field treatment

- We expand the quantum field in terms of momentum eigenstates.

$$u^{\vec{k}}(\vec{x}) = \frac{1}{\sqrt{V}} e^{-i\vec{k}\cdot\vec{x}}$$

- Any number eigenstate results into homogeneous number density

$$|\text{in}\rangle = |N_1(p_1), N_2(p_2), N_3(p_3), \ldots\rangle$$

$$\langle N_1, \ldots, N_M | \psi^\dagger(\vec{x}, t_0)\psi(\vec{x}, t_0) | N_1, \ldots, N_M \rangle = \frac{N}{V}$$

- This is consistent with the uncertainty principle.

SSC: arXiv 2105:09749, Phys. Rev. D 2021

- How to represent inhomogeneous number density?

$$|B\rangle = c_1|N_1, N_2\rangle + c_2|N_1 - 1, N_2 + 1\rangle$$

$$\langle B|\psi^\dagger(\vec{x}, t_0)\psi(\vec{x}, t_0)|B\rangle = \frac{N}{V}\left[1 + 2|c_1||c_2|\sqrt{\frac{N_1}{N}\frac{(N_2+1)}{N}}\cos(\vec{k}_{21}\cdot\vec{x} + \delta_{21})\right]$$

$$\vec{k}_{21} = (\vec{k}_2 - \vec{k}_1)$$

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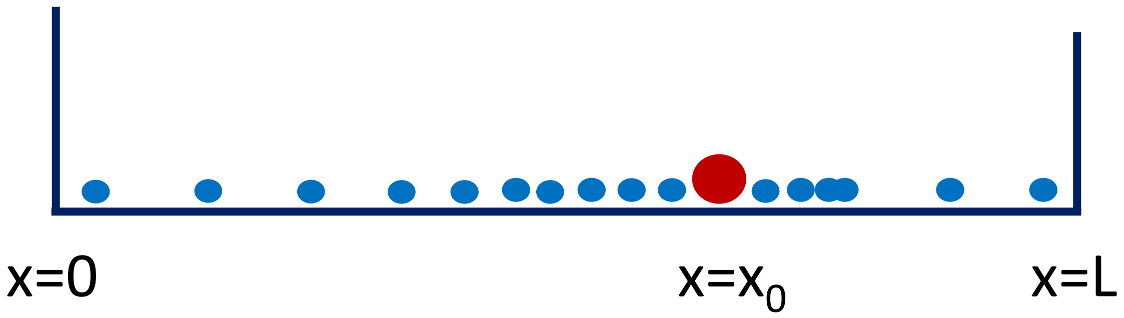
A toy model in 1D

N axions, each with mass *m*, moving under a static point mass *M*.



- Initial condition: $n(x,0) = \frac{N}{L}$ and $j(x,0) = 0$
- Quantum states:

$$|f_1(n_1), f_2(n_2), \ldots\rangle \text{ where } f_i = \frac{N_i}{N} \text{ and } p_i = \frac{2\pi n_i}{L}$$

$$\sum_i N_i = N \quad \text{and} \quad \sum_i N_i p_i = 0$$

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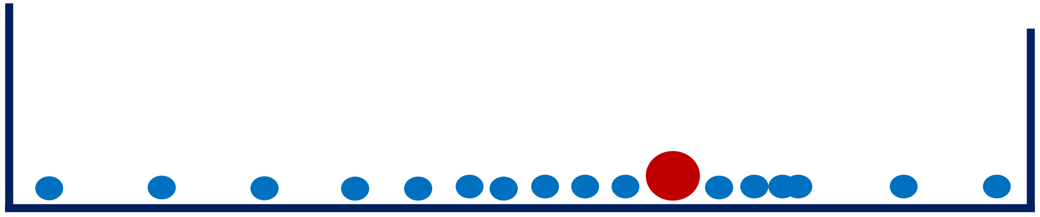
$$n(x,t) = \frac{N}{L}[1 + \delta(x,t)]$$



- Classical result:

$$\delta_{\rm cl}(x,t) = 2\frac{GMmt}{L}\tilde{F}(p=0,x,t)$$

- Quantum result: $|f_1(n_1), f_2(n_2), \ldots\rangle$ where $f_i = \frac{N_i}{N}$ and $p_i = \frac{2\pi n_i}{L}$

$$\delta_Q(x,t) = 2\left(\frac{GMmt}{L}\right)\sum_i f_i \tilde{F}(p_i,x,t)$$

They are certainly not identical. When are they indistinguishable?

$$\Omega_k^p t = \left(\frac{p^2}{2m} - \frac{k^2}{2m}\right)t \quad \ll 1$$

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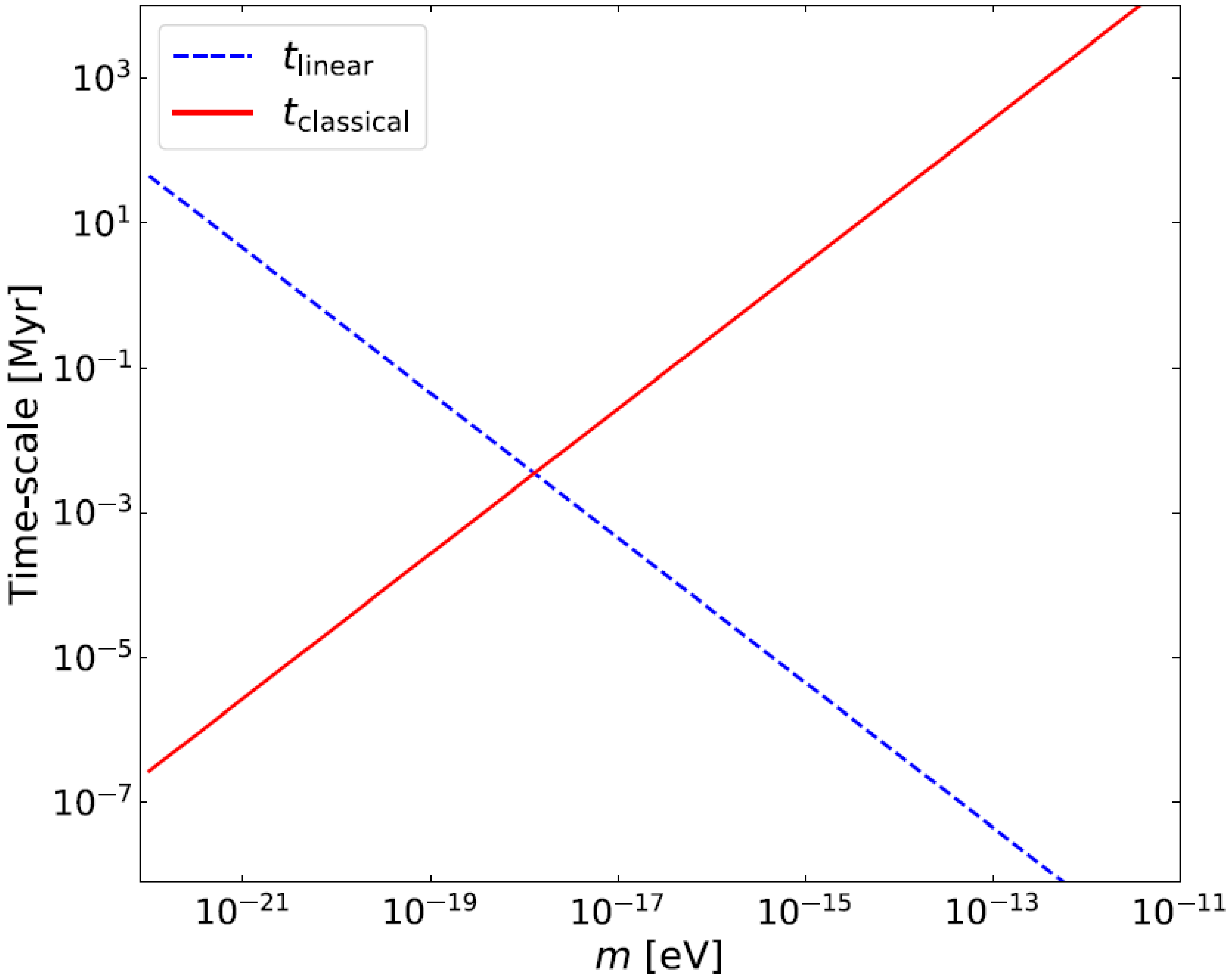
Relevant time-scales:

- Validity of linear perturbation:

$$\frac{GMmt_{\rm linear}}{L} = 10^{-5}$$

$$t_{\rm linear} = 0.45 \text{ Myr}\left(\frac{10^{-20}\text{ eV}}{m}\right)\left(\frac{M_\odot}{M}\right)\left(\frac{L}{1\text{ pc}}\right)$$



- Validity of classical approximation:

$$\Omega_k^p t_{\rm classical} = (n_p^2 - n_k^2)\frac{2\pi^2 t_{\rm classical}}{mL^2} = 0.1(n_p^2 - n_k^2)$$

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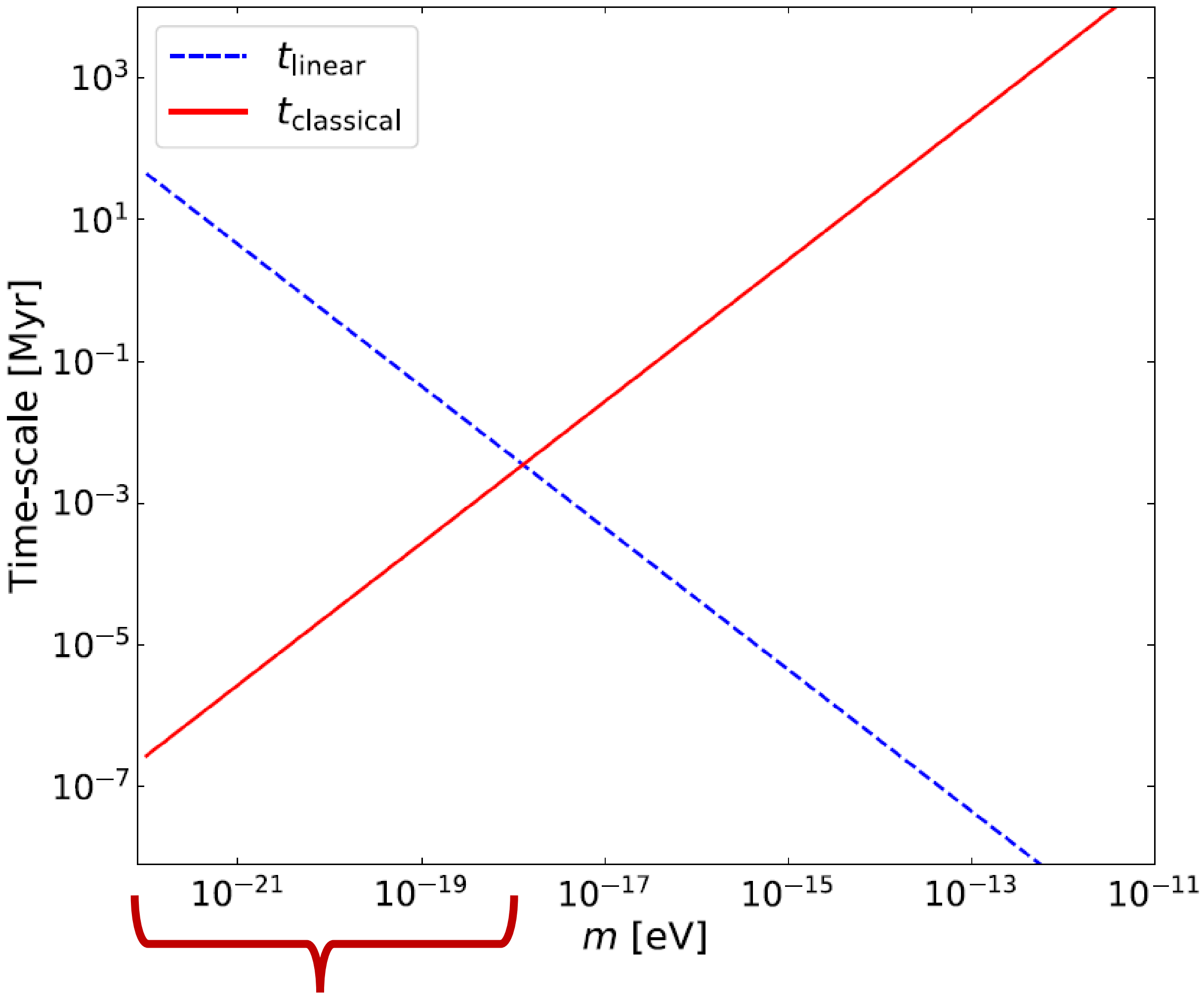

Classical approximation is invalid even within the linear regime.

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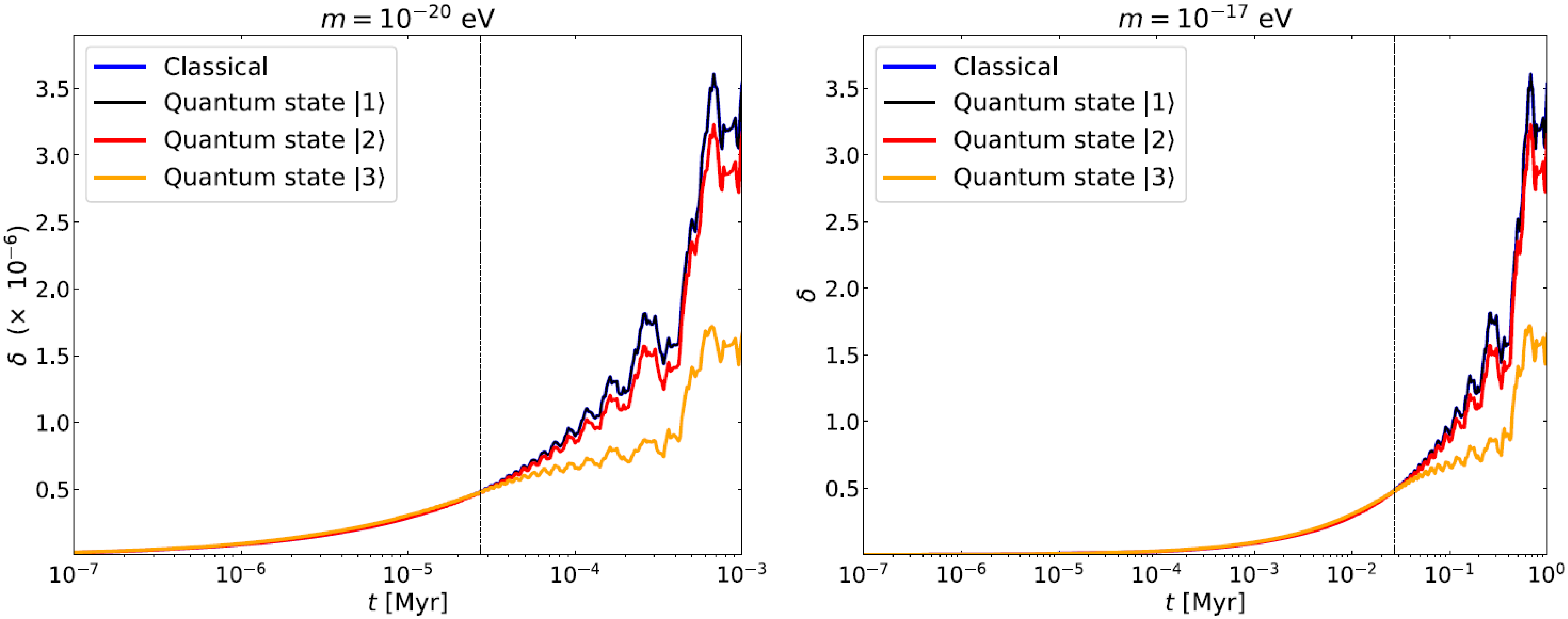

$$|f_1(n_1), f_2(n_2), \ldots\rangle$$

where $f_i = \frac{N_i}{N}$ and $p_i = \frac{2\pi n_i}{L}$

$$|1\rangle = |1.0(0), 0, 0, \ldots\rangle,$$
$$|2\rangle = |0.9(0), 0.05(-2), 0.05(2), 0, 0, \ldots\rangle,$$
$$|3\rangle = |0.5(0), 0.3(2), 0.15(-3), 0.15(-1), 0, 0, \ldots\rangle.$$

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Summary

- Evolution of axions in classical field theory are is uniquely specified by initial density and velocity field.
- In quantum theory, there are many quantum states consistent with the initial condition.
- Evolution corresponding to all these quantum states are not necessarily identical to the unique classical evolution.
- We outlined the formalism to calculate density perturbation using quantum field theory.
- With a toy model, we show that for certain mass of the axions, the classical approximation may break down within the linear regime.

Is classical theory of structure formation valid for all mass of cosmological axions?