Gravitational Particle Production Using Adiabatic Invariants

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Outline

- Review mechanism of gravitational particle production (GPP)
- Elaborate on fast-slow decomposition of the Bogoliubov integral
- Introduce adiabatic invariant formalism to compute slow components
- Obtain numerical and analytical results
- Develop time model to approximate the Boltzmann equation
- Obtain Fermi's golden rule for equivalent scattering process

Review of GPP and fast-slow decomposition

Particle production is due to time dependence of ω_k

Consider a real scalar field χ. The dispersion relation of its Fourier modes χ_k is

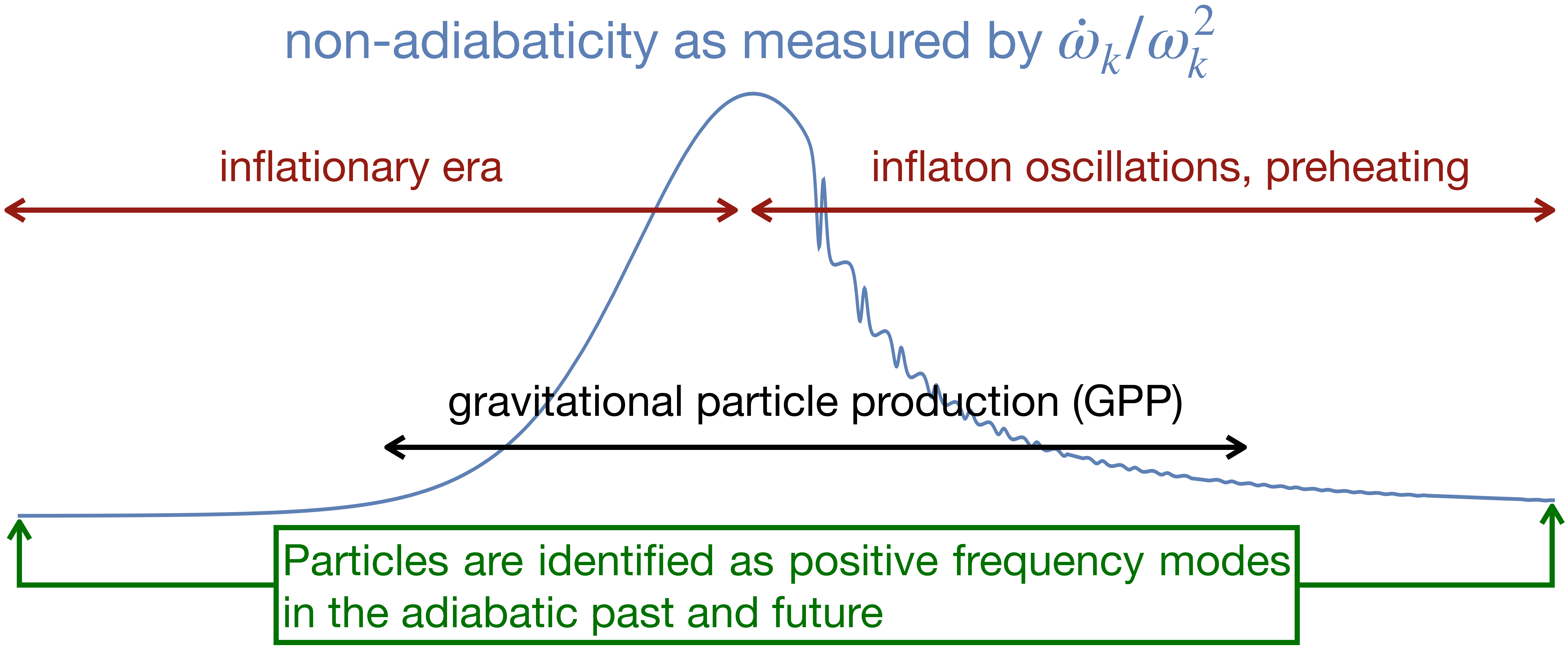
$$\omega_k = \sqrt{k^2 + a^2 m_\chi^2 + \frac{1}{6}(1 - 6\xi) a^2 R}$$

FLRW background

non-minimal coupling

The expansion of the universe causes ω_k to be time dependent. As a result, **the vacuum cannot be defined in a time invariant way**. This mismatch at different times leads to particle production.

Particle production peaks around the end of inflation



GPP comes down to evaluating an oscillatory integral

Write χ_k modes with time dependent **Bogoliubov coefficients** that mix positive and negative frequency modes. The mode equation $\chi_k'' + \omega_k^2 \chi_k = 0$ is then solved by $\alpha_k \simeq 1$ and the **oscillatory integral**

$$\beta_k(t_f) \simeq \int_{t_i}^{t_f} dt \frac{\dot{\omega}_k(t)}{2\omega_k(t)} e^{-2i \int_{t_i}^{t} ds \frac{\omega_k(s)}{a(s)}}$$

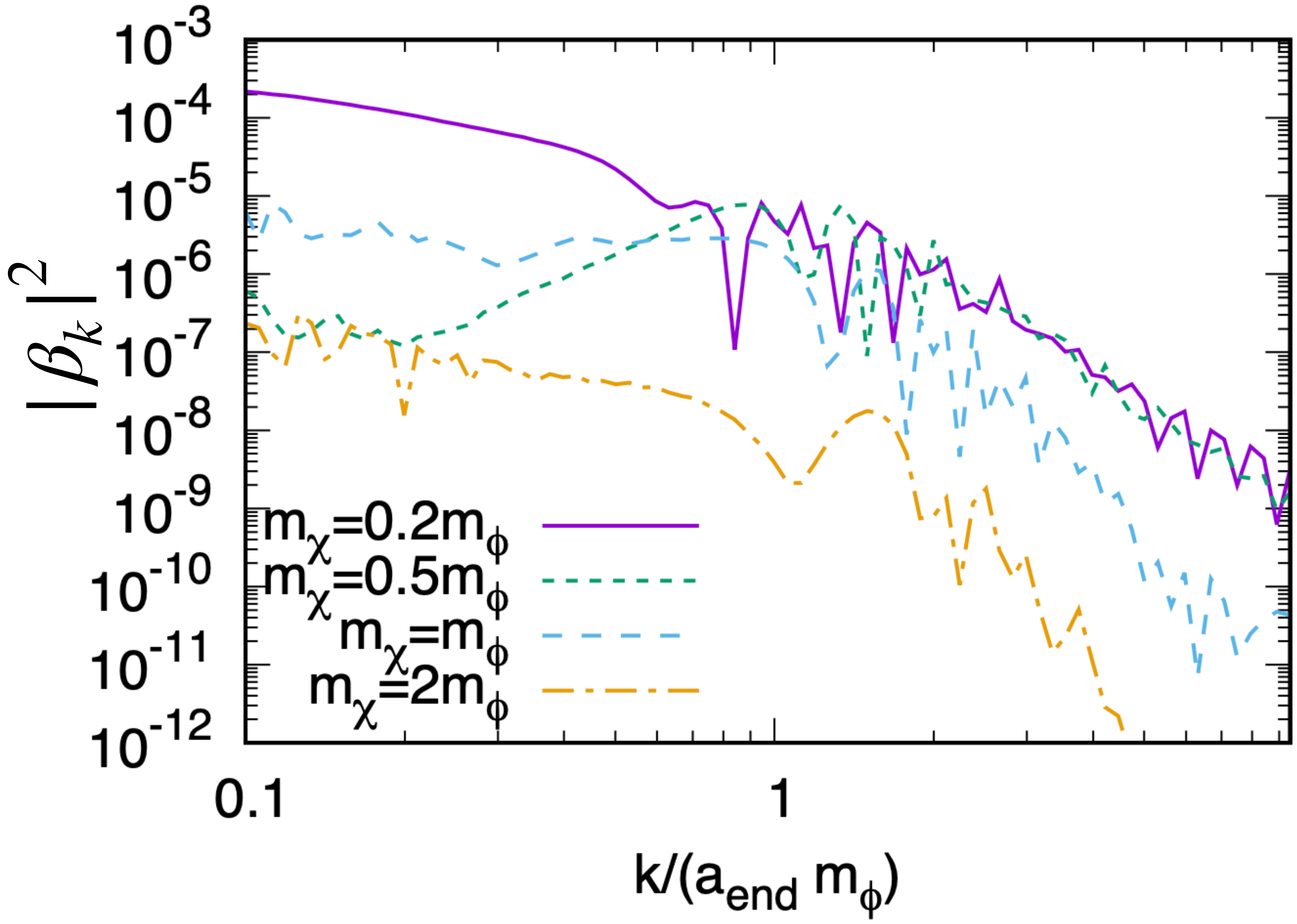
Use this to compute the **number density** of produced particles:

$$a^3(t) n_\chi(t) = \int \frac{d^3k}{(2\pi)^3} |\beta_k(t)|^2$$

Numerical results from previous work

Ema, Nakayama, & Tang [1804.07471] studied scalar particle production in a hill top model. Numerically, they found $n_\chi \propto H_{\text{end}}^3$ for $m_\chi > H_{\text{end}}$

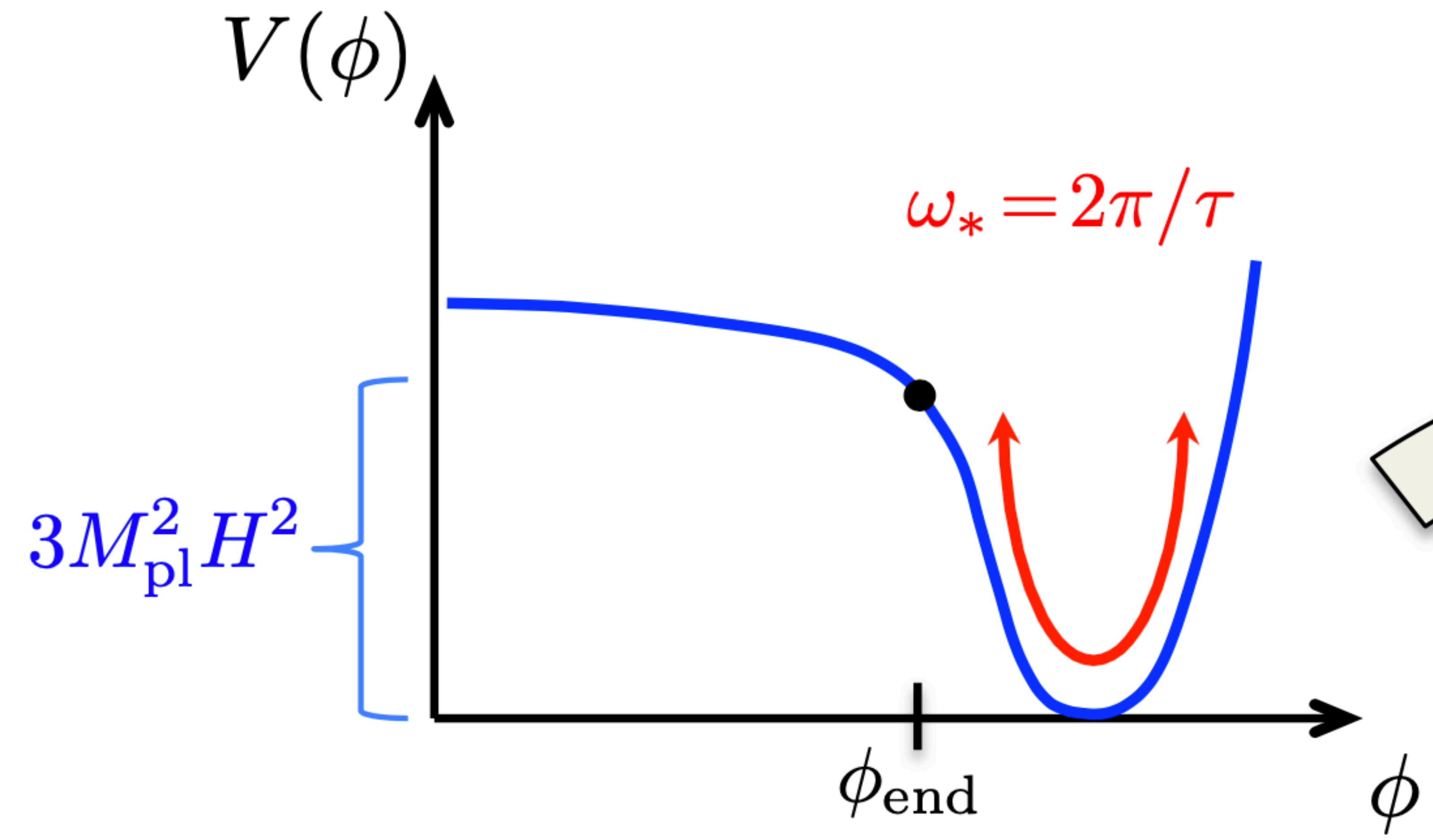
The inflaton mass was super-Hubble in this model, i.e. $H_{\text{end}} \ll m_\phi$. This is the regime of interest for our work.



An analytic approach to particle production

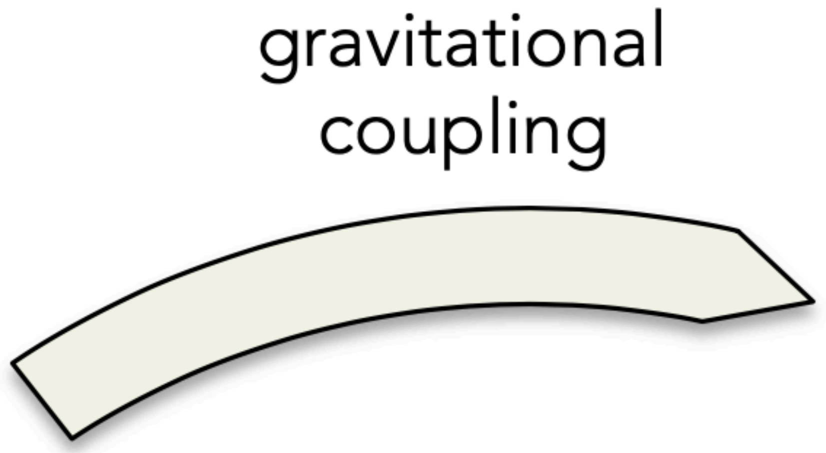
Chung, Kolb, & Long [1812.00211] developed an analytic formalism for GPP by treating ratios H/ω_* and H/m_χ as small, where $\omega_* \simeq m_\phi$ is oscillatory frequency of the inflaton motion.

Inflation Sector

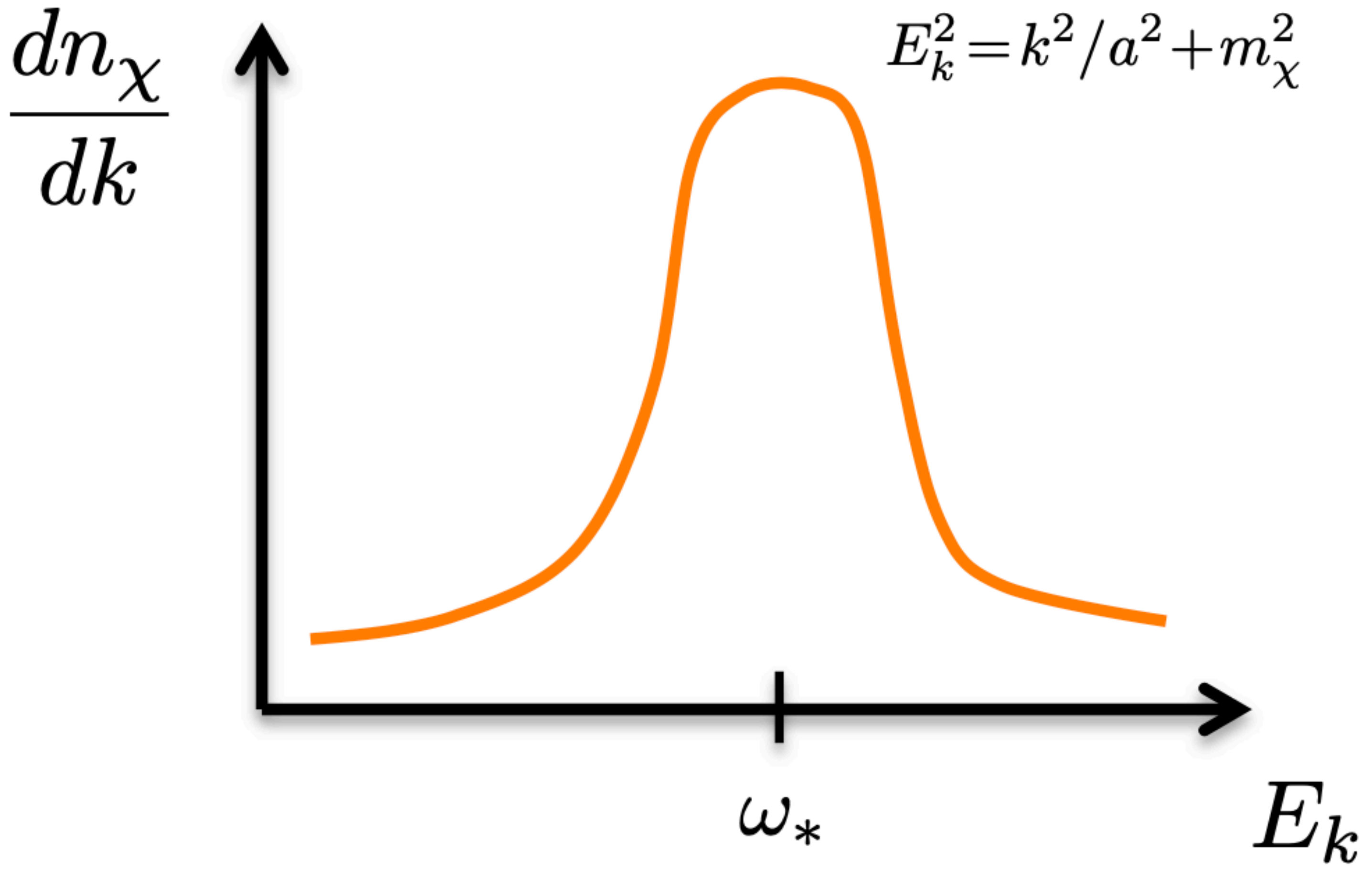


$\omega_* \gg H$

λ-expansion



Heavy Spectator Sector



$m_\chi \gg H$

ε-expansion

Approximations due to $H/m_\phi \ll 1$ and $H/m_\chi \ll 1$

$$\beta_k(t_f) = \frac{1}{2}\int_{\cancelto{t_\mathrm{end}}{t_i}}^{t_f} dt \left(m_\chi^2 \cancelto{H_\mathrm{fast}}{H(t)} + \frac{1}{6}(1-6\xi)\left(\frac{1}{2}\cancelto{\dot{R}_\mathrm{fast}}{\dot{R}(t)} + \cancelto{0}{H(t)R(t)}\right)\right) \cancelto{E_k^2(t)}{\frac{a^2(t)}{\omega_k^2(t)}} e^{-2i\int_{t_i}^{t_f} \cancelto{E_k(s)}{\frac{\omega_k(s)}{a(s)}} ds}$$

$$\frac{\omega_k}{a} = \sqrt{\frac{k^2}{\cancelto{a_\mathrm{slow}^2 + O(H^2/m_\phi^2)}{a^2}} + m_\chi^2 + \cancelto{0}{\frac{1}{6}(1-6\xi)R}} \quad \rightarrow \quad E_k = \sqrt{\frac{k^2}{a_\mathrm{slow}^2} + m_\chi^2}$$

$$\beta_k(t_f) \rightarrow \frac{1}{2}\int_{t_\mathrm{end}}^{t_f} dt \left(H_\mathrm{fast} + \frac{1}{12}(1-6\xi)\frac{\dot{R}_\mathrm{fast}}{m_\chi^2}\right)\frac{m_\chi^2}{E_k^2} e^{-2i\int_{t_i}^{t} E_k ds}$$

Computing the fast and slow components is the purpose of our work

Adiabatic invariant formalism

Adiabatic invariants can describe slow time behavior

- Time dependent Hamiltonian: $\mathscr{H}(\phi, \pi, a) = \frac{\pi^2}{2a^3} + a^3 V(\phi) = E(t)$
- Scale of $E(t)$ time dependence: $H = \frac{\dot{a}}{a}$
- Scale of inflaton oscillations: $m_\phi = \sqrt{V''(\phi_{\min})}$
- Adiabatic invariant: $Q = \oint_C d\phi_C \, \Pi(\phi_C, E(t), a(t))$
- Time average of Q is approximately constant to order H^2/m_ϕ^2
- Fast-slow decomposition exists if $H \ll m_\phi$

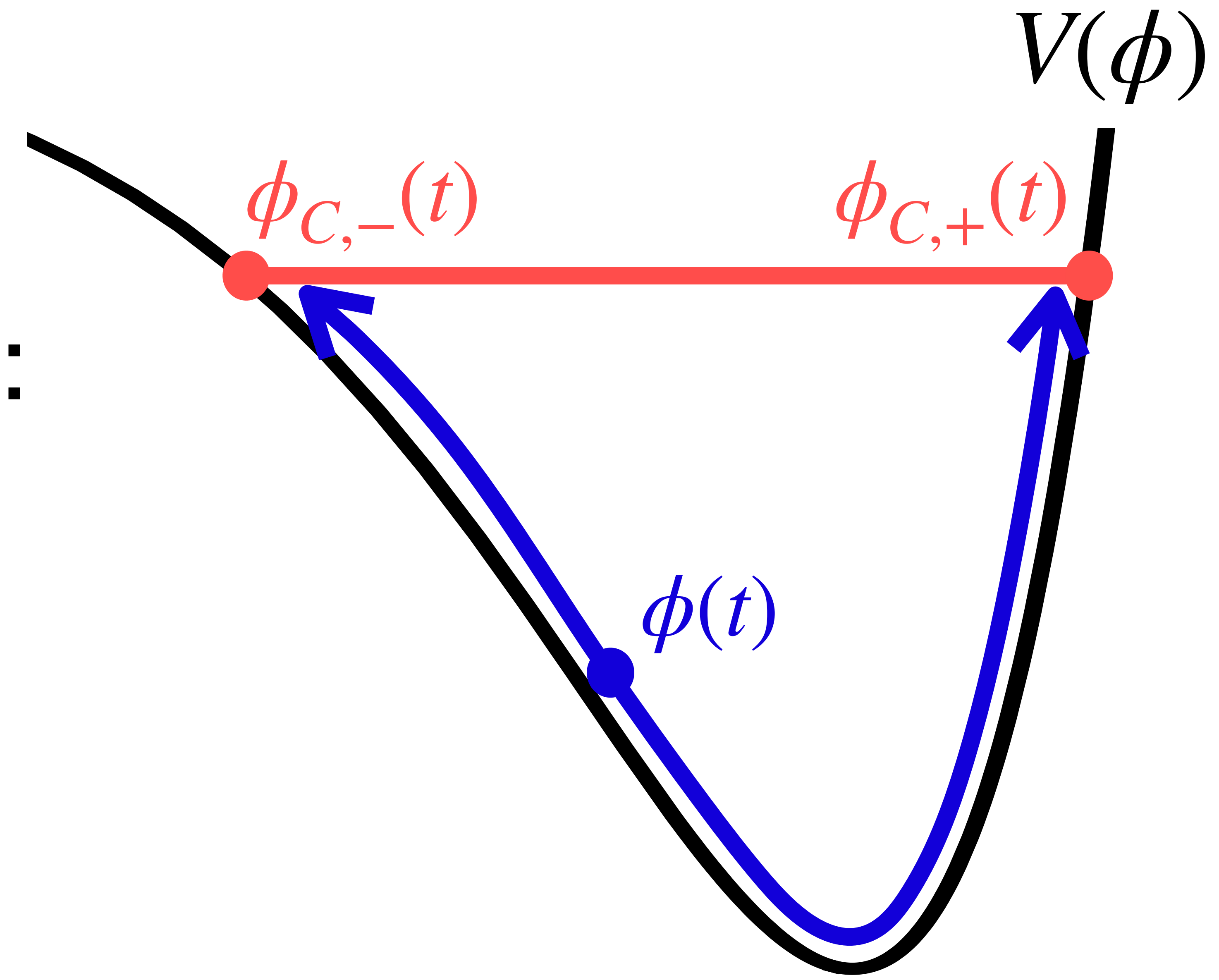
Use adiabatic invariant to compute slow and fast

Subtract “slow” from total to get “fast,” i.e. $H_{\text{fast}} = H - H_{\text{slow}}$

The adiabatic invariant Q determines turning points of inflaton motion as

$$Q = 2a_{\text{slow}}^3(t)\int_{\phi_{C,-}(t)}^{\phi_{C,+}(t)} d\phi\sqrt{2V(\phi_{C,\pm}(t)) - 2V(\phi)}$$

where $V(\phi_{C,+}) = V(\phi_{C,-})$ is related to the total energy.



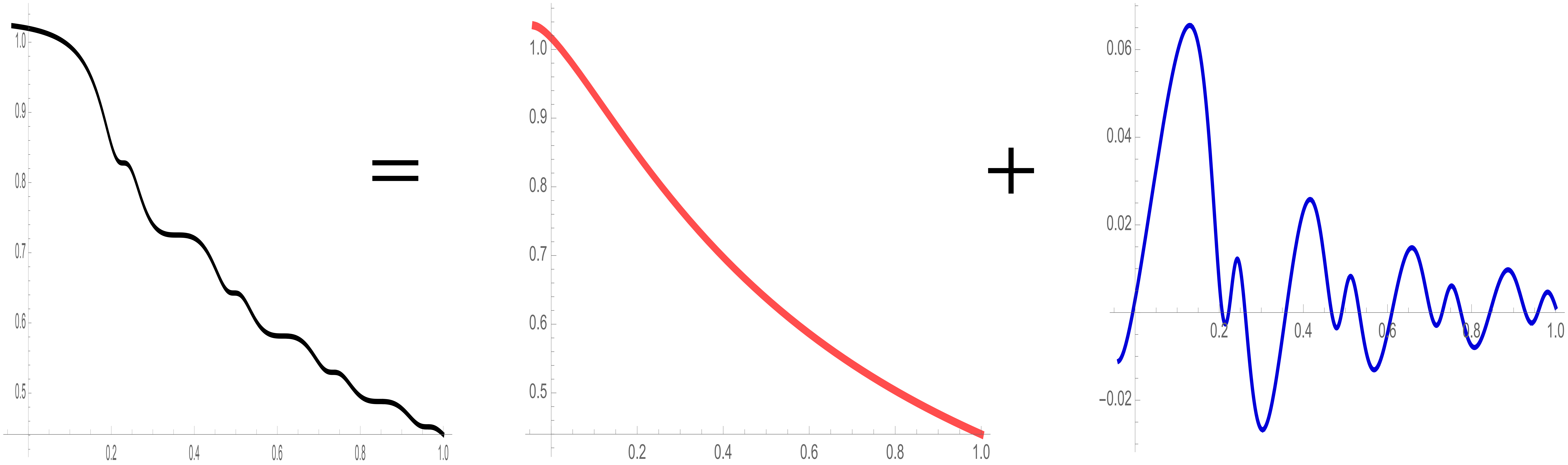
Time dependence determined by slow Friedman equation:

$$\frac{\dot{a}_{\text{slow}}^2(t)}{a_{\text{slow}}^2(t)} = H_{\text{slow}}^2(t) = \frac{V(\phi_{C,\pm}(a_{\text{slow}}(t)))}{3M_P^2}$$

Fast-slow decomposition of H based on $\omega_* \gg H_{\mathrm{end}}$

Use slow components and EOM solution to obtain $H_{\mathrm{fast}} = H_{\mathrm{EOM}} - H_{\mathrm{slow}}$.

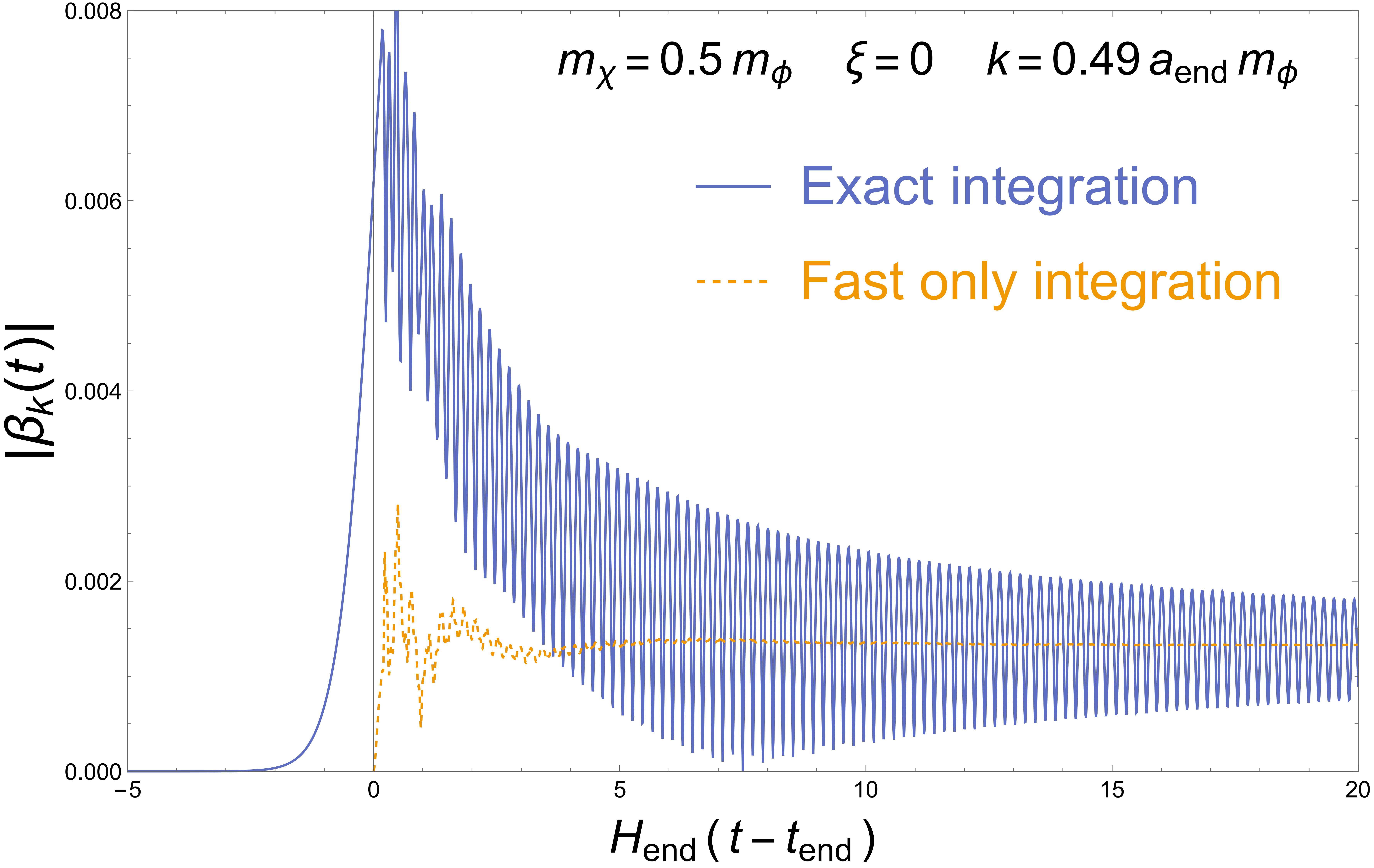
$$H(t) = H_{\mathrm{slow}}(t) + H_{\mathrm{fast}}(t)$$



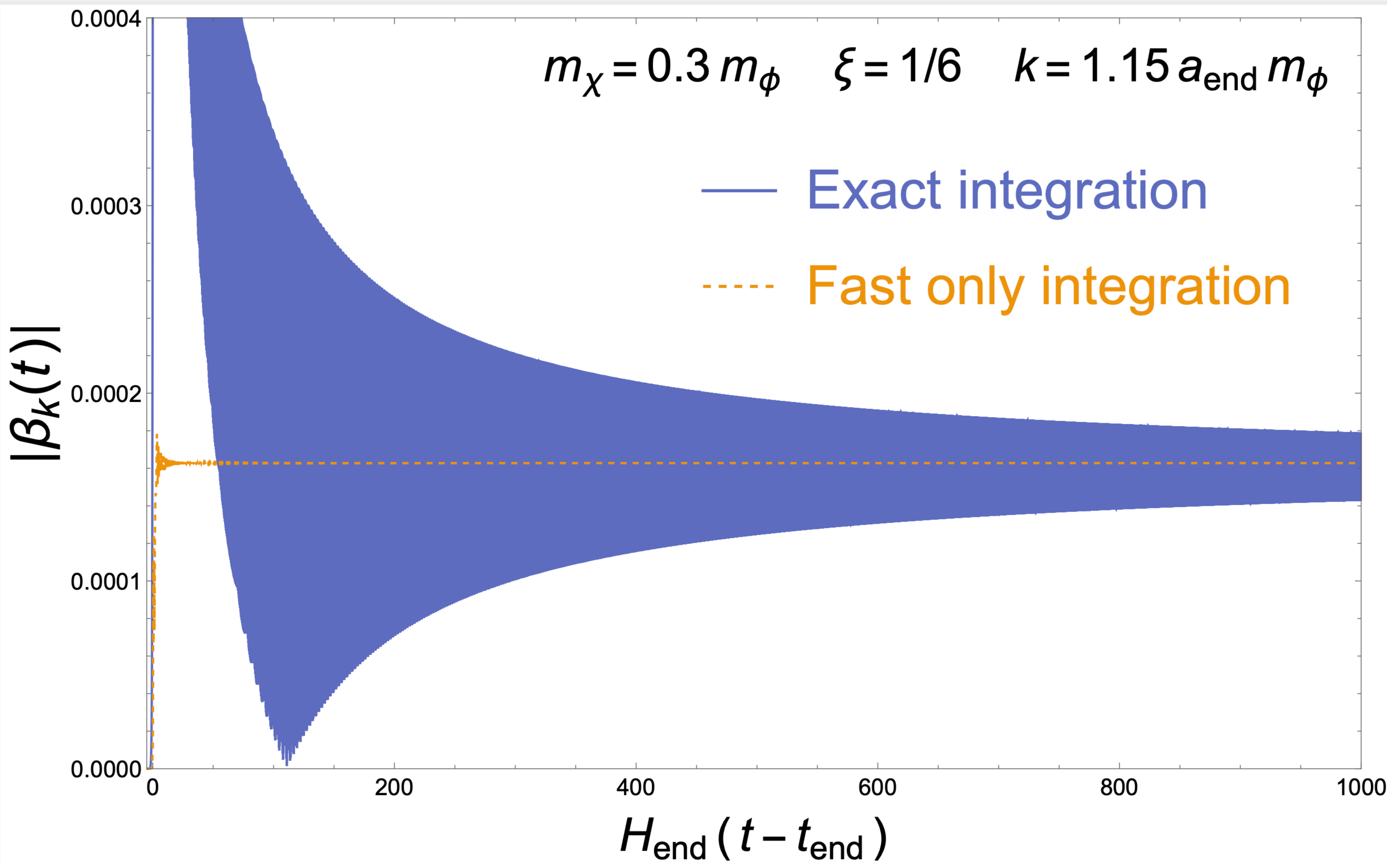
Fast-only integration avoids slow convergence

Due to the properties of oscillatory integrals, the slow components of $\beta_k^{(\mathrm{exact})}$ are exponentially suppressed but slow to converge, as illustrated by the figure.

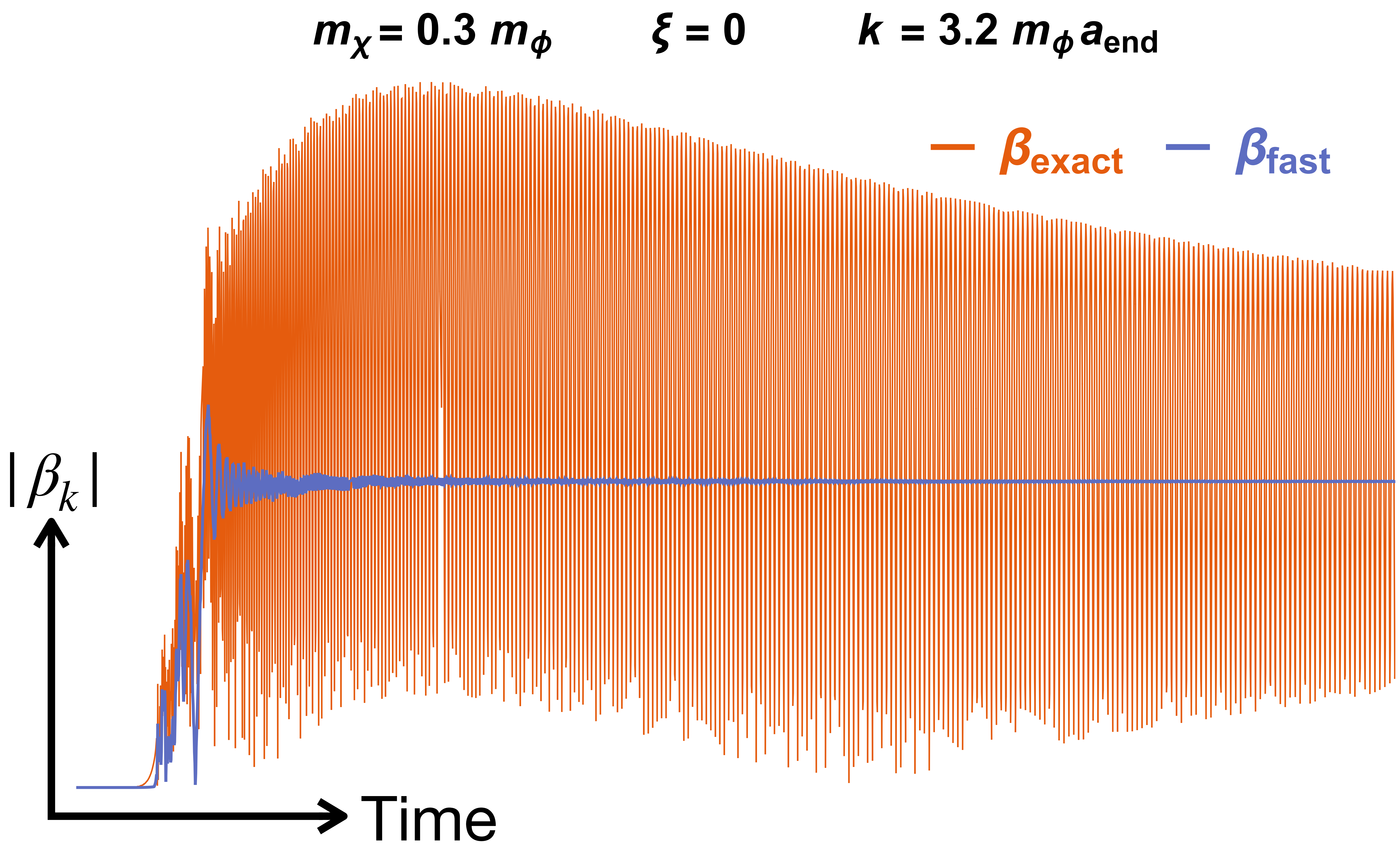
Without these negligible contributions, $\beta_k^{(\mathrm{fast})}$ converges on an O(1000) shorter time scale.



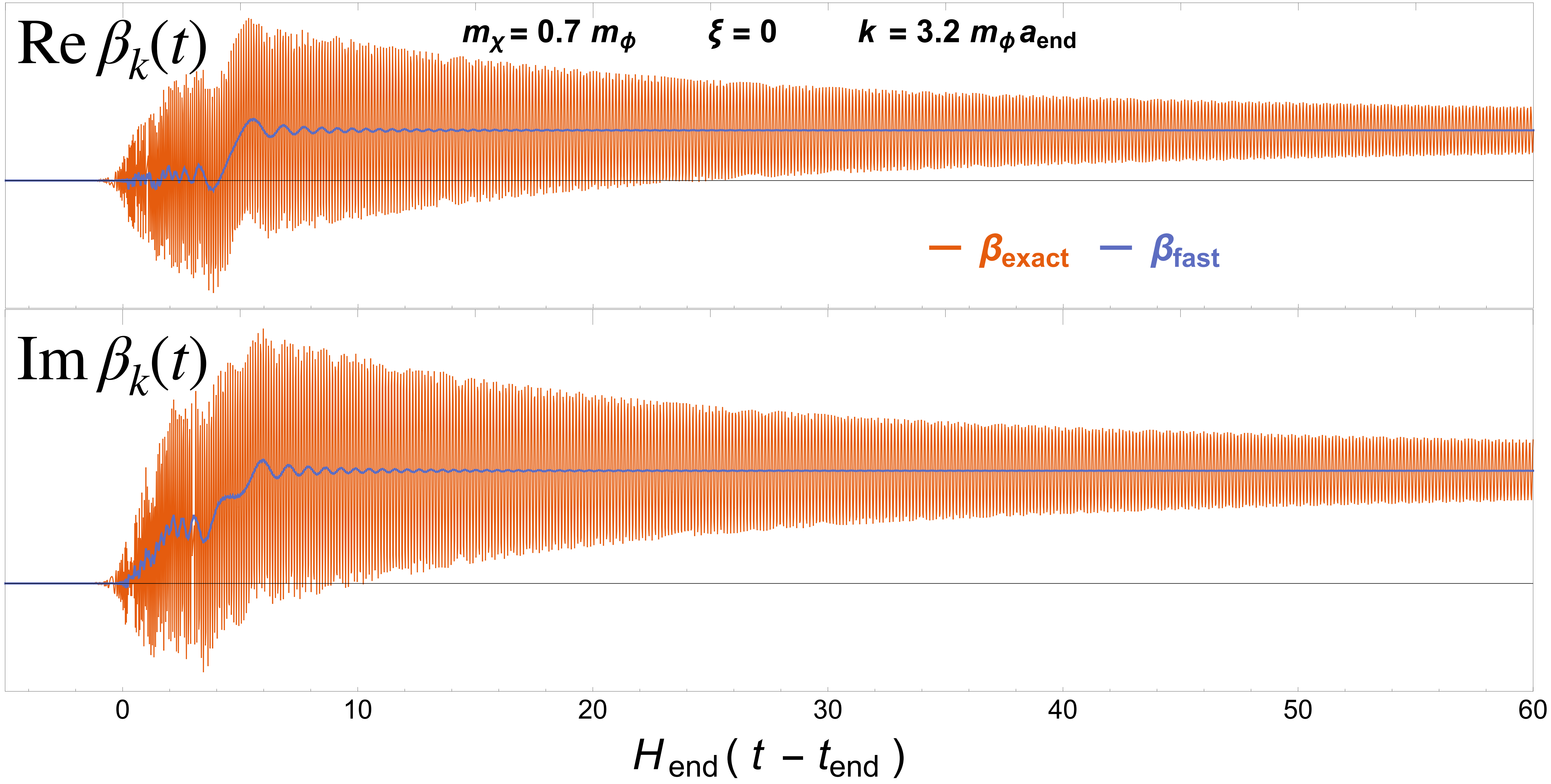
Fast-only integration avoids slow convergence



Faster convergence after slow time subtraction



Integration converges faster using adiabatic invariant formalism



Approximation of the Boltzmann equation using time model

Approximation of the Boltzmann equation

Given the time scale separation of β_k, Boltzmann equation is approximated as

$$\frac{d}{dt}\left(a^3(t)n_\chi(t)\right) = \int \frac{d^3k}{(2\pi)^3}\frac{\partial}{\partial t}\left|\beta_k(t)\right|^2 \approx \int \frac{d^3k}{(2\pi)^3}\frac{1}{\Delta t(t)}\left|\tilde{b}_k(t)\right|^2$$

$$\beta_k(t) = \int_{t_{\mathrm{end}}}^{t} ds B_k(s) e^{-2i\int_{t_{\mathrm{end}}}^{s} E_k(s')ds'} \qquad \tilde{b}_k(t) \equiv \int_0^{\Delta t(t)} ds B_k(t+s) e^{-2iE_k(t)s}$$

and where Δt is a coarse graining time that encompasses many oscillations. Chosen to minimize error of this approximation. Turns out to be

$$\Delta t = \sqrt{\frac{2\pi}{H_{\mathrm{slow}} m_\phi}} \longleftarrow \text{Intermediate time scale}$$

Use $\phi_{C,\pm}$ to derive time model of inflaton dynamics

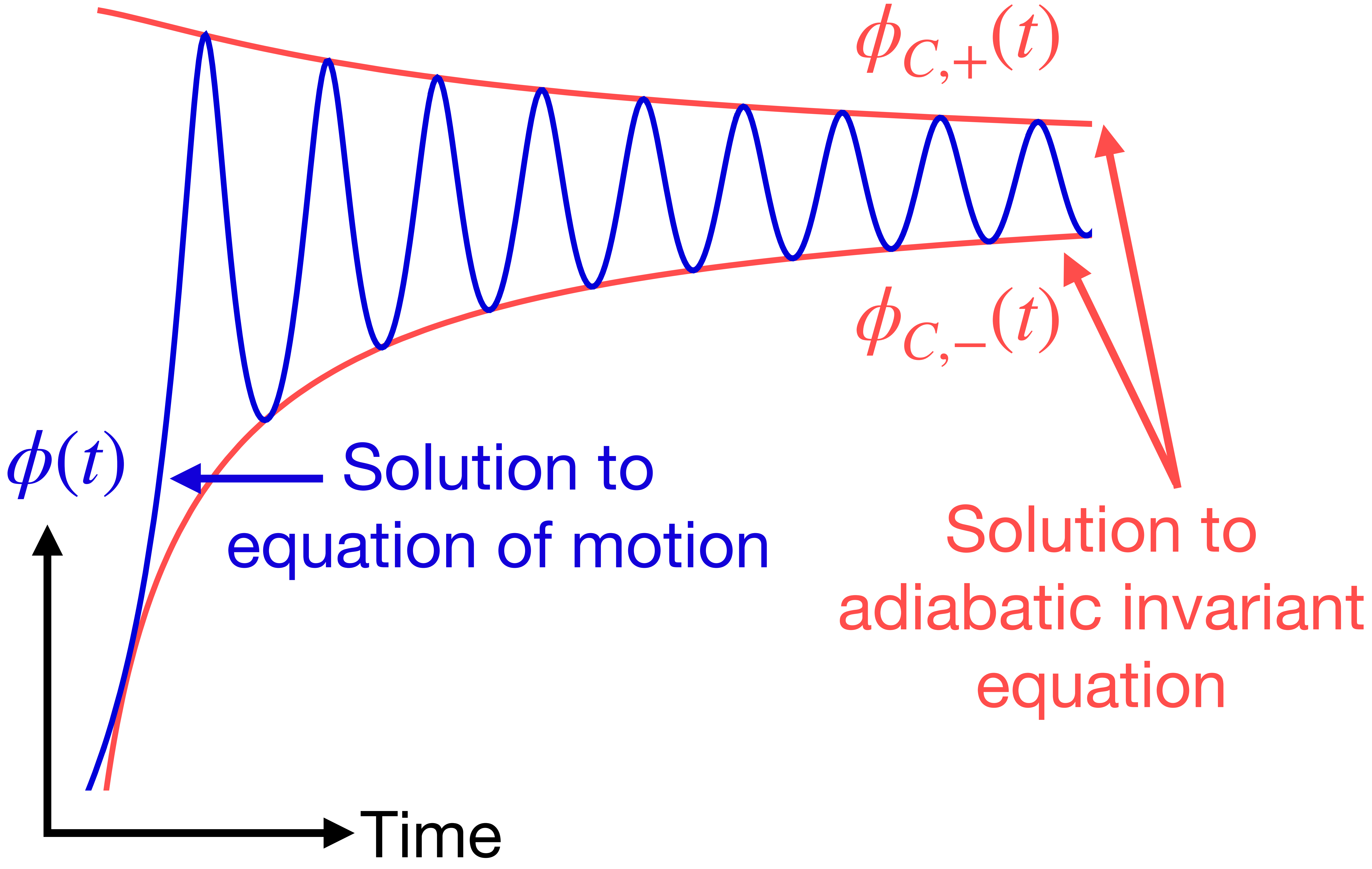
Since $\phi_{C,\pm}$ are bounds on $\phi(t)$, there is an oscillatory phase $\Xi_\phi(t)$ such that

$$\phi(t) = \bar{\phi}(t) + \Delta\phi(t)\cos\Xi_\phi(t)$$

where

$$\bar{\phi} = \frac{\phi_{C,+} + \phi_{C,-}}{2}$$

$$\Delta\phi = \frac{\phi_{C,+} - \phi_{C,-}}{2}$$



Separation of time scales allows simple solution of Ξ_ϕ

Over a time scale of Δt, slow quantities are approximately constant. Separate into slow time v and fast time s:

$$\partial_s \phi(v,s) \approx -\Delta\phi(v) \frac{\partial \Xi_\phi(v,s)}{\partial s} \sin \Xi_\phi(v,s)$$

$$\frac{1}{2}\left[\partial_s \phi(v,s)\right]^2 \approx V(\phi_C(v)) - V\left(\phi(v,s)\right)$$

This leads to quadrature integral for Ξ_ϕ, with the results

$$s = \int_{\Xi_\phi(v,0)}^{\Xi_\phi(v,s)} \frac{\Delta\phi(v) \sin \Xi \, d\Xi}{\sqrt{2V(\phi_C(v)) - 2V(\bar{\phi}(v) + \Delta\phi(v) \cos \Xi)}}$$

Expand potential to evaluate

Consider potential with leading asymmetric cubic term:

$$V(\phi) = m_\phi^2 M_P^2 \left(\frac{1}{2} \left(\frac{\phi - \phi_{\min}}{M_P} \right)^2 + \alpha_3 \left(\frac{\phi - \phi_{\min}}{M_P} \right)^3 + \ldots \right)$$

$$m_\phi^2 = V''(\phi_{\min}) \qquad \alpha_3 \equiv M_P V'''(\phi_{\min})/(6m_\phi^2)$$

Solution to quadrature for Ξ_ϕ ultimately gives

$$[\partial_s \phi(v,s)]^2 \approx V(\phi_C(v))\big(1 - \cos 2\tau(v,s)\big) \left(1 + 4\alpha_3 \sqrt{\frac{2V(\phi_C(v))}{m_\phi^2 M_P^2}} \cos \tau(v,s) \right)$$

$$\tau(v,s) = \sqrt{\frac{2V_m(v)}{[\Delta\phi(v)]^2}}\, s$$

Use this to evaluate $\tilde{b}_k$

Production rate as estimated by time model

$$\frac{\partial}{\partial t} f_\chi(k,t) \approx \frac{1}{\Delta t(t)} \left| \tilde{b}_k(t) \right|^2$$

$$\tilde{b}_k(v) = \frac{1}{2} \int_0^{\Delta t(v)} ds \left(H_{\text{fast}}(v+s) + \frac{1}{12}(1-6\xi) \frac{\dot{R}_{\text{fast}}(v+s)}{m_\chi^2} \right) \frac{m_\chi^2}{E_k^2(v)} e^{-2iE_k(v)s}$$

$$\approx F(E_k(v)) + F^*(-E_k(v))$$

$$F(E_k) = \frac{A_2}{2} \frac{e^{i\left(2\omega_* - 2E_k\right)\Delta t} - 1}{i(2\omega_* - 2E_k)} + \frac{A_3}{2} \frac{e^{i\left(3\omega_* - 2E_k\right)\Delta t} - 1}{i(3\omega_* - 2E_k)} + \frac{A_1}{2} \frac{e^{i\left(\omega_* - 2E_k\right)\Delta t} - 1}{i(\omega_* - 2E_k)}$$

$$\omega_*(t) = \sqrt{\frac{2V(\phi_{C,\pm}(t))}{[\Delta\phi(t)]^2}}$$

A_1, A_2, A_3 evaluated with time model

Amplitudes computed from time model

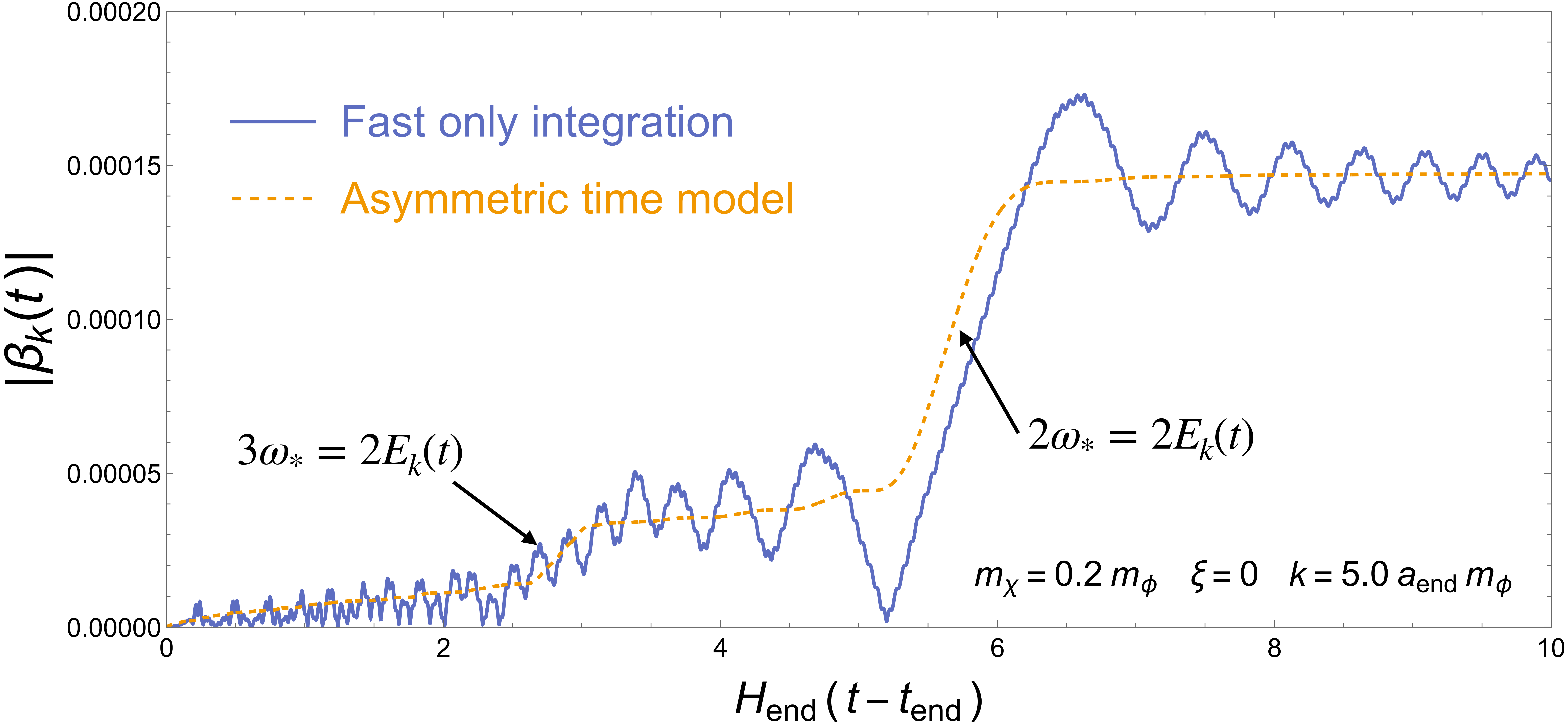
$$A_2(k,t) = \frac{3H^2_{\rm slow}(t)}{4i\omega_*(t)}\frac{(1-6\xi)\omega_*^2(t)+\frac{1}{2}m_\chi^2}{E_k^2(t)}$$

$$A_3(k,t) = +\,2\alpha_3\sqrt{\frac{2V(\phi_C(t))}{[\Delta\phi(t)]^2}}\frac{3H^2_{\rm slow}(t)}{4i\omega_*(t)}\frac{\frac{3}{2}(1-6\xi)\omega_*^2(t)+\frac{1}{3}m_\chi^2}{E_k^2(t)}$$

$$A_1(k,t) = -\,2\alpha_3\sqrt{\frac{2V(\phi_C(t))}{[\Delta\phi(t)]^2}}\frac{3H^2_{\rm slow}(t)}{4i\omega_*(t)}\frac{\frac{1}{2}(1-6\xi)\omega_*^2(t)+m_\chi^2}{E_k^2(t)}$$

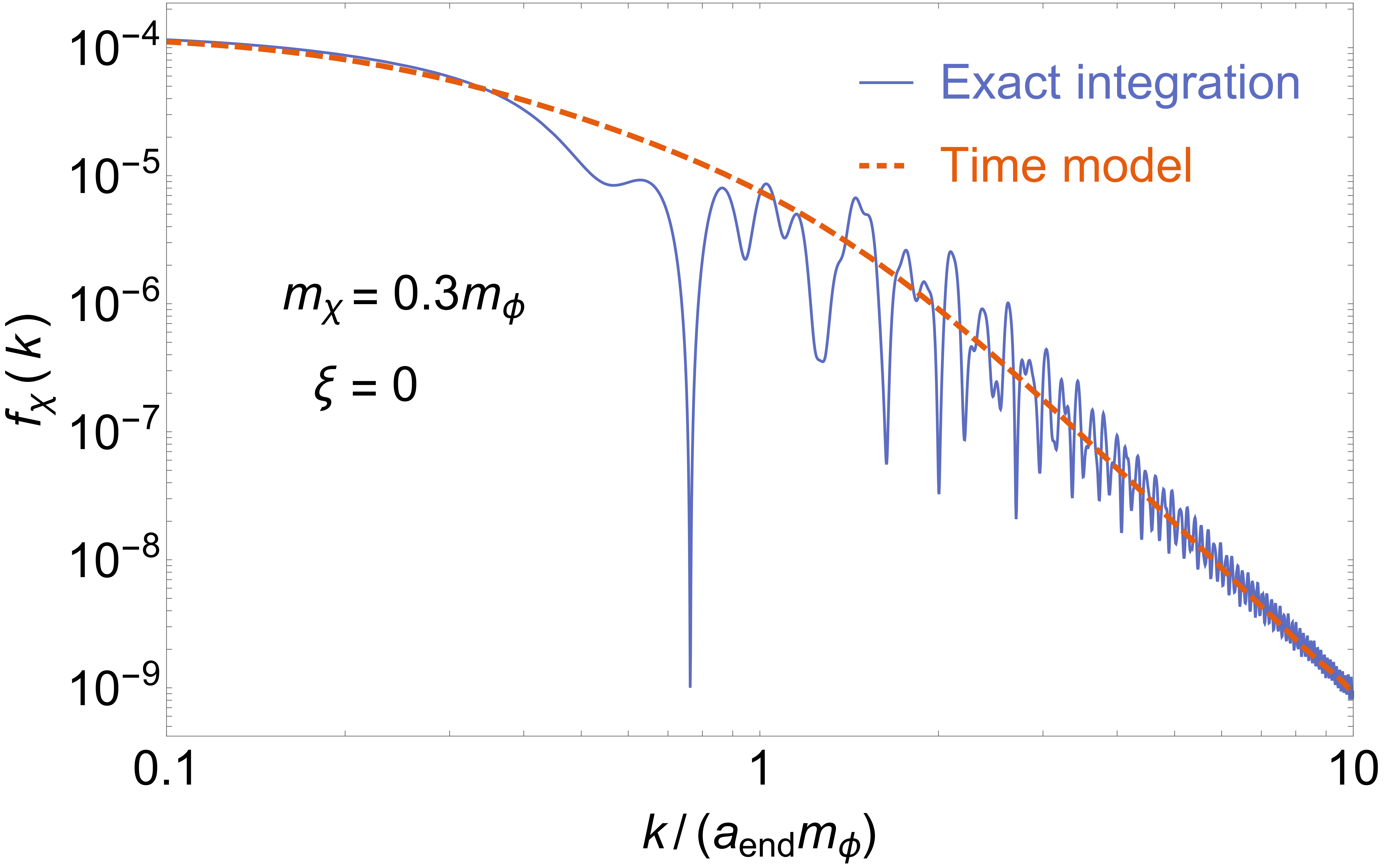
$$\omega_*(t) = \sqrt{\frac{2V(\phi_{C,\pm}(t))}{[\Delta\phi(t)]^2}}$$

Fast only integration undergoes resonances



Spectrum predictions of the time model

The time model matches the average behavior of the spectrum $f_\chi(k) = |\beta_k|^2$, which scales as $k^{-9/2}$ at large k. The oscillations of f_χ will be studied in future work.



Obtaining Fermi's golden rule

At large times, $\Delta t \to \infty$ and the production rate limits to Fermi's golden rule:

$$\frac{\partial}{\partial t} f_\chi(k,t) \to 2\pi \left| \frac{A_2(t)}{2} \right|^2 \delta(2\omega_*(t) - 2E_k(t))$$

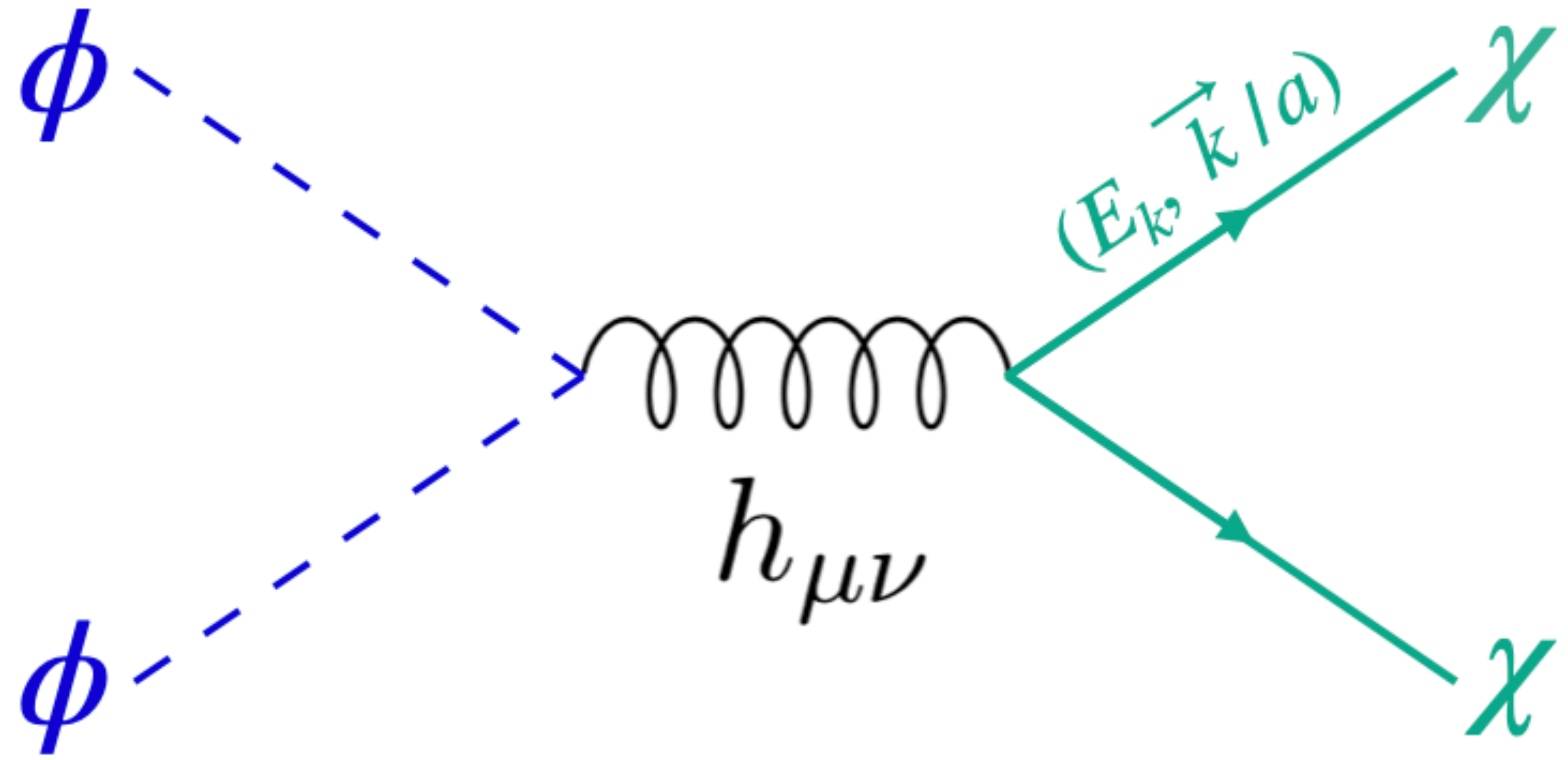
The amplitude $A_2/2$ matches tree level results of graviton-mediated $\phi\phi \to \chi\chi$ annihilation as computed by 1708.05138 and 2102.06214. Note $\omega_* \simeq m_\phi$

Resonance occurs when $2m_\phi$ = $2E_k$

Energy of χ

$$\beta_k = \int B_k e^{-2i\int E_k dt} dt \quad \sim$$

Oscillates at frequency $2m_\phi$



Obtaining Fermi's golden rule

Consider only the dominant $2\omega_* = 2E_k$ resonance:

$$\frac{d}{dt}\left(a^3 n_\chi\right) = \int \frac{d^3k}{(2\pi)^3}\frac{\partial}{\partial t} f_\chi(k,t) \approx \int \frac{d^3k}{(2\pi)^3}\left|\frac{A_2(t)}{2}\right|^2 \Delta t(t)\mathrm{sinc}^2\left[\Delta t(t)\left(\omega_*(t) - E_k(t)\right)\right]$$

At large resonance times, $\Delta t \to \infty$ and $\frac{\partial}{\partial t} f_\chi(k,t) \to 2\pi \left|\frac{A_2(t)}{2}\right|^2 \delta(2\omega_*(t) - 2E_k(t))$

which is Fermi's golden rule with $A_2/2$ as the amplitude, $2\omega_*$ as the initial energy, and $2E_k$ as the final energy. Amplitude matches tree level results of graviton-mediated annihilation found by 1708.05138 and 2102.06214. As $\omega_* \simeq m_\phi$ at large times, delta function enforces same energy conservation of $\phi\phi \to \chi\chi$ process.

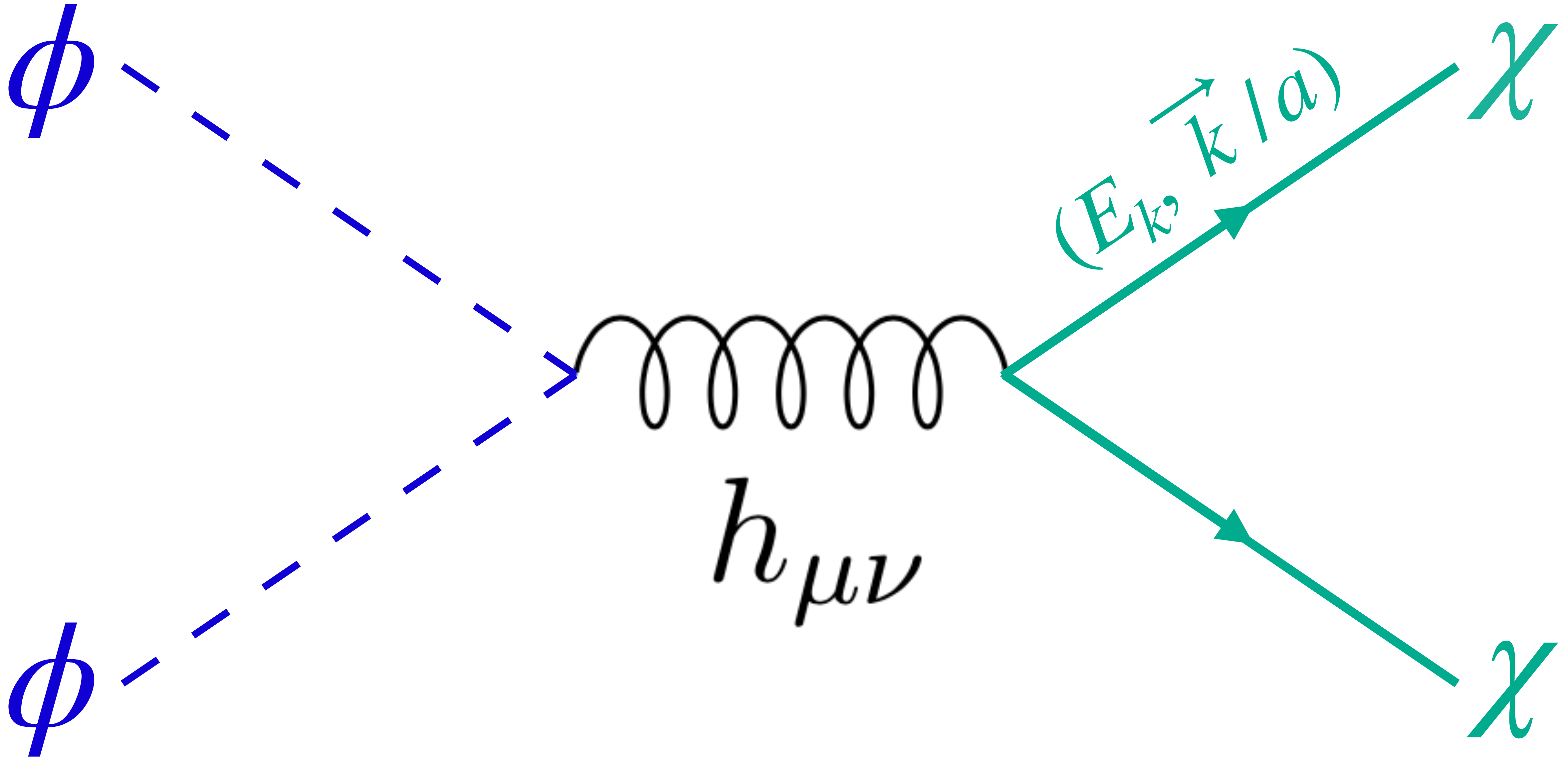
Main resonance analogous to $\phi\phi \to \chi\chi$ annihilation process

Resonance occurs when $2m_\phi = 2E_k$

Energy of χ

$$\beta_k = \int B_k e^{-2i\int E_k dt} dt \quad \sim$$

Oscillates at frequency $2m_\phi$



Tang & Wu 1708.05138

Mambrini & Olive 2102.06214

Summary

- Computed necessary components to implement fast-slow decomposition of β_k
- Obtained an integration technique that converges on O(1000) shorter time scale
- Created a time model that takes advantage of separation of time scales to estimate number density using only adiabatic invariant quantities
- Lead to Fermi's golden rule for the graviton-mediated $\phi\phi \to \chi\chi$ process that creates a correspondence between the Bogoliubov and scattering methods of computing gravitational particle production
- The oscillatory nature of spectrum vs k remains unexplained. Ongoing research by our group suggests this is due to quantum interference between different resonances of the Bogoliubov integral.

Thank you for listening!

Thank you to my collaborator and advisor Daniel Chung
Thank you to Brookhaven Forum for this speaking opportunity
A particular thanks to Nicole and Peter for logistical support

Questions?