High energy evolution and S-channel Unitarity: an interesting problem that bothers no one.

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BNL, May 6, 2021

A.K., E. Levin, M. Lublinsky; JHEP 08 (2016) 031; A.K., E. Levin, Ming Li, M. Lublinsky, JHEP 09 (2020) 199, JHEP 10 (2020) 185

The over arching motivation

High energy evolution: hadronic scattering amplitudes change with growth of total energy.

We know how to describe it in some limits: BFKL (dilute on dilute); JIMLWK/KLWMIJ (dilute on dense).

We would like to have an approach that works for asymptotically large energy, where both scattering objects become dense. We do not have it.

JIMWLK etc. must break down at very high energy. Try to understand what goes wrong. If we do understand, then perhaps we can fix the bug and can have a workable evolution beyond present approaches.

The Question

One standard way to encode the evolution of amplitudes is via a "Hamiltonian" - JIMWLK, KLWMIJ, BK, Braun...

"Boost" the wave function of one of the objects (Projectile) and calculate the change in the S-matrix for scattering on the other object (Target).

Examine and deduce "Hamiltonian" which should be unversally applicable to evolve scattering amplitudes for any QCD scattering process.

Hamiltonian should be consistent with basic quantum mechanical properties of the projectile and target state: positivity of probabilities.

WELL, IS IT?

The eikonal S - matrix

Every projectile gluon keeps its transverse position but acquires a color "phase"

$$|x, a\rangle \rightarrow S^{ab}(x)|x, b\rangle$$

$$S^{ab}(x) = \mathcal{P} \exp\left\{ i \int dx^- \; T^a \, A_T^{+a}(x, x^-) \right\}^{ab} \; .$$

The scattering amplitude of $|P\rangle$ on T:

$$\mathcal{S} \; = \langle \mathrm{IN}|\mathrm{OUT}\rangle =< \langle P|\hat{S}|P\rangle >_T$$

$$=< \int d\rho \; W^P[\rho] \;\; \exp\left\{ i \int \; d^2x \, \rho_P^a(x) \; A_T^{+a}(x) \right\} >_T$$

$W^P[\rho]$ is the "probability distribution" of the projectile color charge density.

A^+ is the color field of the target, which is also distributed according with weight $W_T[A^+]$.

The “Hamiltonian” evolution.

Boost the projectile - more gluons become resolvable by the target .

The S-matrix changes, since $W^P[\rho]$ changes.

The change is due to “materialization” of soft modes (growth of coherence time of soft fluctuations).

Find the “soft gluon” part of the hadronic wave function, then calculate the change in $S-matrix$.

The “Vacuum” of the soft gluons in the presence of “valence” charge density ρ:

$$|P_{\rm soft}\rangle \equiv P_{\rm soft}[a^\dagger_{soft};\rho(x)]|0_{soft}\rangle$$

Then

$$\mathcal{S}_{Y+\Delta Y} = \langle {\rm IN}|{\rm OUT}\rangle = \langle\langle P_{\rm valence}|\langle P_{\rm soft}|\hat{S}|P_{\rm soft}\rangle|P_{\rm valence}\rangle\rangle_T$$

The phase space of $|P_{\rm soft}\rangle$ is proportional to ΔY, hence differential evolution in Y.

Thus we can write

$$\langle P_{\rm soft}|\hat{S}|P_{\rm soft}\rangle = [1 - \mathcal{H}[\rho, \delta/\delta\rho]\ \Delta Y + ...]\hat{S}_{valence}$$

Or

$$\mathcal{S}_{Y+\Delta Y} = [1 - \mathcal{H}[\rho, \delta/\delta\rho]\ \Delta Y]\mathcal{S}_Y$$

$\mathcal{H}$ generates a Hamiltonian evolution for any observable which is calculated as average over the probability distribution W:

$$\frac{d}{dY} W^P[\rho] = -\mathcal{H}[\rho, \delta/\delta\rho]\ W^P[\rho]$$

This is the JIMWLK framework. Also the "Reggeon Field theory" defined by $\mathcal{H}$.

Hamiltonian Assembly Instructions.

1. Calculate the soft gluon wave function of the Projectile at fixed valence color charge density $P_{soft}[a^{a\dagger}(x), \rho^a(x)]|0\rangle$.

2. Eikonally propagate $|P_{soft}\rangle$ through the target fields:

$$|IN\rangle = P_{soft}[a^\dagger(x), \rho(x)]|0\rangle \to |OUT\rangle = P_{soft}[S(x)^{ab}a^{b\dagger}(x), S(x)^{ab}\rho^a(x)]|0\rangle$$

3. Calculate the soft gluon part of the overlap: $\langle IN|OUT\rangle$

4. Expand to first order in ΔY and extract $\mathcal{H}$.

5. If we can do it for arbitrarily dense projectile and arbitrarily dense target, we have solved QCD. We can fill out the travel request form and buy a ticket to Stockholm.

No can do in general, but can in simple limits.

The JIWMLK limit.

JIMWLK Hamiltonian: projectile is dense $\alpha_s \rho_P(x) \sim 1$,

but the target is dilute $A_T^{+a}(x) \sim g$.

Or alternatively (KLWMIJ limit): projectile is dilute, $\rho_P(x) \sim 1$, but the target is dense $A_T^{+}(x) \sim 1/g$.

The eigenfunction P_{soft} is found to leading perturbative order

The eikonal S-matrix on the target field is not expanded in α_s.

This has been extended to NLO, but for our purposes this is not important.

The JIMWLK Hamiltonian.

We end up with a 2+1 dimensional Euclidean quantum field theory, with 2 transverse spatial dimensions, and rapidity as time.

$$H^{JIMWLK} = \frac{\alpha_s}{2\pi^2} \int d^2z\, Q_i^a(z)\, Q_i^a(z)$$

the Hermitian amplitudes $Q_i^a(z)$ are "single inclusive gluon emission amplitude"

$$Q_i^a(z) = \int d^2x \frac{(x-z)_i}{(x-z)^2}[S^{ab}(z) - S^{ab}(x)]J_R^b(x)\,.$$

the generators of color rotation J_R

$$J_R^a(x) = -\mathrm{tr}\left\{S(x)\,T^a \frac{\delta}{\delta S^\dagger(x)}\right\}$$

Dense- Dilute Duality (DDD).

Could evolve the Target instead

$$\mathcal{S} =< \int d\rho\ W^P[\rho]\ \ \exp\left\{i\int\ d^2x\,\rho^a_P(x)\ A^{+a}_T(x)\right\} >_T$$

$$= \int dA^+_T d\rho\, W^P[\rho]\ \ \exp\left\{i\int\ d^2x\,\rho^a_P(x)\ A^{+a}_T(x)\right\} W^T[A^+]$$

$$\frac{d}{dY}\mathcal{S} = -\int dA^+ d\rho\ \mathcal{H}[\rho,\delta/\delta\rho] W^P[\rho]\exp\left\{i\int\ d^2x\,\rho^a_P(x)\ A^{+a}_T(x)\right\} W^T[A^+$$

$$= -\int dA^+ d\rho\ W^P[\rho]\exp\left\{i\int\ d^2x\rho^a_P(x)\ A^{+a}_T(x)\right\}\mathcal{H}[-i\delta/\delta A^+, iA^+] W^T[A^+$$

Dense-Dilute Duality: $S = e^{iT^aA^{+a}(x)} \leftrightarrow R \equiv e^{T^a\frac{\delta}{\delta\rho^a(x)}}$

If W^P evolves with $\mathcal{H}[R]$, then W^T evolves with $\mathcal{H}[S]$.

RECAP (almost)

Parametrize:

$$W^P[\rho] = \delta(\rho) W_P[R]$$

Integrate (formally) over the target degrees of freedom:

$$\begin{aligned}\mathcal{S} &= \int d\rho dA^+_T \delta(\rho) W_P[R] e^{i\int_z g^2 \rho^a(z) A^{+a}_T(z)} \tilde{W}_T[A^+_T] \\ &= \int d\rho \delta(\rho) W_P[R] W_T[S]\end{aligned}$$

$W_T[S]$ - functional Fourier transform of $\tilde{W}_T[A^+_T]$ depends on the projectile density DoF

$$R_x = e^{t^a \frac{\delta}{\delta \rho^a_x}} \equiv U(x); \qquad S_x = e^{ig^2 t^a \alpha^a_x} \equiv \bar{U}(x); \qquad \alpha^a \approx \frac{1}{\nabla^2} \rho^a$$

t-channel unitarity.

Evolution:

$$\mathcal{S}(Y) = \int d\bar{U}\delta(\bar{U}-1)W_P[U]e^{-\mathcal{H}[U,\bar{U}]Y}W_T[\bar{U}]$$

$\mathcal{H}$ can act either on $W_P[U]$ or on $W_T[\bar{U}]$.

Target and projectile should evolve the same way.

The hamiltonian should thus be invariant under the Dense Dilute Duality $U \leftrightarrow \bar{U}$.

$$\mathcal{H}[U,\bar{U}] = \mathcal{H}[\bar{U},U]$$

Self duality is the same as t-channel unitarity.

How does W look like?

Can W be arbitrary? No - it is related to a probability distribution.

Every factor of $U(x)$ in $W_P[U]$- a gluon in the **QCD wave function** of the projectile.

Every factor $\bar{U}(x)$ in $W_T[\bar{U}]$ - a gluon in **QCD wave function** of the target.

Suppose QCD WF of the projectile before scattering

$$|\Psi_i\rangle = \sum_{n;\mathbf{x}_i;a_i} C_{a_1,a_2\ldots a_n}|\mathbf{x}_1, a_1; \ldots; \mathbf{x}_n, a_n\rangle$$

and after scattering

$$|\Psi_f\rangle = \sum_{n;\mathbf{x}_i;b_i} C_{b_1,b_2\ldots b_n}|\mathbf{x}_1, b_1; \ldots; \mathbf{x}_n, b_n\rangle$$

S-channel unitarity - projectile.

Then

$$W_P = \sum_{n,\{a,b;x\}} F^n(\{a,b;x\}) \prod_{i=1}^{n} [U^{a_i b_i}(x_i)]$$

$$F^n(\{a,b;x\}) = C_{a_1,a_2...a_n}(x_1...x_n) C^*_{b_1,b_2...b_n}(x_1...x_n)$$

For $\{a_i\} = \{b_i\}$ the function F has the meaning of the normalized probability density,

$$F^n(\{a,a;x\}) \geq 0$$

$$\sum_{n,\{a\}} \int_{\{x\}} F^n(\{a,a;x\}) = 1;$$

S-channel unitarity - target.

Same for the target:

$$W_T = \sum_{n,\{c,d;y\}} \bar{F}^n(\{c,d;x\}) \prod_{i=1}^{n} [\bar{U}^{c_i,d_i}(y_i)]$$

with the constraints

$$\bar{F}^n(\{b,b;y\}) \geq 0$$

and normalization

$$\sum_{n,\{b\}} \int_{\{y\}} \bar{F}^n(\{b,b;y\}) = 1;$$

Unitarity of evolution?

Suppose we start our evolution with "good" W and evolve it with JIMWLK (or the like). Does it stay "good" or does it go "bad" at higher energy?

This is not a trivial question.
We derive evolution by calculating WF of the soft gluons of the projectile. The evolution of W_P better be unitary by construction.

But what about W_T?

The upshot: target evolution is **NOT** unitary in JIMWLK, or in any other approximate $\mathcal{H}$ that we know. It gets worse the higher in energy you go.

We know how to fix this in a toy 0+1 dimensional model.

UNITARIY IN REGGEON FIELD THEORY.

A.K., E. Levin, M. Lublinsky, 2016

Reggeon field theory $\equiv$ JIMWLK (like) Hamiltonian restricted to act on dipoles.

$$\mathcal{S} = \int d\rho\delta(\rho)W_P[U]W_T[\bar{U}]$$

The "POMERONS" (a.k.a. "Dipoles")

$$P(x,y) = 1 - \frac{1}{N_c}tr[U_x U_y^\dagger]; \qquad \bar{P}(x,y) = 1 - \frac{1}{N_c}tr[\bar{U}_x \bar{U}_y^\dagger]$$

"Dipole Model" $\equiv$: Large N_c $\equiv$ "Pomeron Calculus"

$$\mathcal{S} = \int d\bar{P}\delta(\bar{P})W_P[P]W_T[\bar{P}]$$

S-channel unitarity conditions

In dipole model:

$$W_P = \sum_{n,\{x,\bar{x}\}} F^n(\{x,\bar{x}\}) \prod_{i=1}^{n} [1 - P(x_i,\bar{x}_i)]$$

$$W_T = \sum_{n,\{x,\bar{x}\}} \bar{F}^n(\{x,\bar{x}\}) \prod_{i=1}^{n} [1 - \bar{P}(x_i,\bar{x}_i)]$$

Probability densities:

$$0 < F^n, \bar{F}^n < 1 : \quad \sum_n \int_{\{x,\bar{x}\}} F^n(\{x,\bar{x}\}) = 1; \qquad \sum_n \int_{\{x,\bar{x}\}} \bar{F}^n(\{x,\bar{x}\}) = 1$$

Unitary evolution must preserve this property of F^n, $\bar{F}^n$ at all energies!

The Hamiltonian(s).

Evolution of the S-matrix:

$$\mathcal{S} = \int d\bar{P}\delta(\bar{P})\, W_P[P]\, e^{-H_{RFT}[P,\bar{P}]Y}\, W_T[\bar{P}]$$

BK Hamiltonian (JIMWLK)

$$H_{BK} = -\frac{\bar{\alpha}_s}{2\pi}\int K(x,y|z)\, P^\dagger(x,y)$$

$$\times\left[P(x,z) + P(z,y) - P(x,y) - P(x,z)P(z,y)\right]$$

with $K(x,y|z) = \frac{(x-y)^2}{(x-z)^2(y-z)^2}$

BK Hamiltonian is applicable for dense-dilute scattering and is not self dual.

The Hamiltonian(s).

The Braun Hamiltonian

$$H_B = -\frac{N_c^2}{2\pi\bar{\alpha}_s}\int \bar{P}(x,y)\nabla_x^2\nabla_y^2\left[K(x,y|z)[P(x,z)+P(z,y)-P(x,y)\right.$$

$$\left.-P(x,z)P(z,y)] - P(x,y)\nabla_x^2\nabla_y^2\left[K(x,y|z)\bar{P}(x,z)\bar{P}(z,y)\right]$$

Suggested by Misha Braun for description of Nucleus-Nucleus (dense-dense) scattering. Is DDD invariant a.k.a. Self dual.

To calculate need to know the algebra of P and $\bar{P}$. Both, BK and Braun setups use "dilute limit" commutators:

$$[P^\dagger, P] = -1; \qquad P^\dagger(x,y) = \frac{N_c^2}{4\pi^4\bar{\alpha}_s^2}\nabla_x^2\nabla_y^2\bar{P}(x,y)$$

Solutions are very strange: bifurcate above Y_C and collapses onto the solution of BK or its dual. Bondarenko and Motyka, 2007

Escape into toy world.

Zero transverse dimensions: all dipole sit at the same transverse position.

m dipoles scatter on n dipoles:

$$\langle m|\bar{n}\rangle = \int d\bar{P}\delta(\bar{P})(1-P)^m(1-\bar{P})^{\bar{n}}$$

To calculate: commute every $\bar{P}$ to the left through all the P's and annihilate the δ-function. This amplitude is evolved in energy according to

$$\langle m|\bar{n}\rangle_Y = \int d\bar{P}\delta(\bar{P})(1-P)^m e^{-HY}(1-\bar{P})^{\bar{n}} \equiv \langle m|e^{-HY}|\bar{n}\rangle$$

Act with H to the right - evolve the target wave function; act with H to the left - evolve the projectile wave function.

Question: Does the evolution preserve unitarity of both wave functions?

Trouble in the toy world.

$$H_{BK} = -\frac{1}{\gamma}\left[\bar{P}P - \bar{P}P^2\right]$$

Need to commute $\bar{P}$'s through P's: what is the algebra?
In dilute limit:

$$[\bar{P}, P] = \gamma; \qquad \gamma \sim \alpha_s^2 > 0$$

$$\langle m|e^{-\Delta H} \approx \langle m|(1-\Delta H) = (1-\Delta m)\langle m| + \Delta m\langle m+1|$$

$$e^{-\Delta H}|\bar{n}\rangle = (1+\Delta\bar{n})|\bar{n}\rangle - \Delta\bar{n}[1+\gamma(\bar{n}-1)]|\bar{n}-1\rangle + \Delta\gamma\bar{n}(\bar{n}-1)|\bar{n}-2\rangle$$

Projectile evolves unitarily, but target evolution generates negative probabilities! Also the number of dipoles decreases with energy?!

Maybe Braun?

BK is not self dual - maybe that's the problem?

Maybe Braun Hamiltonian?

$$H_B = -\frac{1}{\gamma}\left[\bar{P}P - \bar{P}P^2 - \bar{P}^2 P\right]$$

$$e^{-\Delta H_B}|\bar{n}\rangle = (1-\Delta\bar{n})|\bar{n}\rangle + \Delta\bar{n}|\bar{n}+1\rangle - \Delta\gamma\bar{n}(\bar{n}-1)|\bar{n}-1\rangle + \Delta\gamma\bar{n}(\bar{n}-1)|\bar{n}-2\rangle$$

Unitarity is violated by an amount γ, but for both, projectile and target.

New and improved algebra.

Maybe the commutators are to blame? Relax the "dilute limit algebra": Commuting $\bar{P}$ through P is like scattering a target dipole on a projecitle dipole. A dipole does not disappear after it scatters, so they can scatter on other dipoles as well

$$(1-P)(1-\bar{P}) = [1-\gamma](1-\bar{P})(1-P)$$

$1-\gamma$ - a dipole-dipole s-matrix factor.

$$e^{-\Delta H_{BK}}|\bar{n}\rangle = \left[1 - \frac{\Delta}{\gamma}\left[1-(1-\gamma)^{\bar{n}}\right](1-\gamma)^{\bar{n}}\right]|\bar{n}\rangle$$
$$+\frac{\Delta}{\gamma}\left[1-(1-\gamma)^{\bar{n}}\right](1-\gamma)^{\bar{n}}|\bar{n}+1\rangle$$

$$\langle m|e^{-\Delta H_{BK}} = \langle m| - \frac{\Delta}{\gamma}[1-(1-\gamma)^m]\langle m+1| + \frac{\Delta}{\gamma}[1-(1-\gamma)^m]\langle m+2|$$

No luck - now the projectile evolution violates unitarity!

Braun Hamiltonian only makes things even worse...

Panacea?

Can we construct a unitary, DDD invariant Toy Hamiltonian, which reduces to BK in the appropriate limit?

The answer is YES!

$$H_{UTM} = -\frac{1}{\gamma}\bar{P}P$$

$$e^{-\Delta H_{UTM}}|\bar{n}\rangle = \left[1 - \frac{\Delta}{\gamma}[1-(1-\gamma)^{\bar{n}}]\right]|\bar{n}\rangle + \frac{\Delta}{\gamma}[1-(1-\gamma)^{\bar{n}}]|\bar{n}+1\rangle$$

This is *t*-channel unitary.

This is *s*-channel unitary.

The evolution at small n is the same as projectile in BK.

At large n the growth rate saturates, and is not proportional to n anymore - just what the doctor ordered!

JIMWLK in 3D?

Technically much more involved.

A.K., E. Levin, Ming Li, M. Lublinsky, JHEP 09 (2020) 199, if not for Ming, we would never have been able to get through the calculation!

Algebra of the RFT observables:

$$U(\mathbf{x}) = e^{T^a \frac{\delta}{\delta \rho^a(\mathbf{x})}} \; ; \qquad \bar{U}(\mathbf{x}) = e^{igT^a \int_y \phi(\mathbf{x}-\mathbf{y})\rho^a(\mathbf{y})}$$

$$\phi(\mathbf{x}-\mathbf{y}) = \frac{g}{2\pi} \ln \frac{|\mathbf{x}-\mathbf{y}|}{L}$$

Schematically, suppressing color, transverse coordinates and polarization:

$$\langle m|\bar{n}\rangle = \int d\bar{U}\, U^m \bar{U}^n$$

However cannot write $\mathcal{H}_{JIMWLK}$ only in terms of U and $\bar{U}$ - have to work harder to calculate evolution of states.

JIMWLK is not in great shape.

Nevertheless we were able to explicitly verify the following:

1. For a dilute projectile state $\langle m|$ unitarity is preserved, f_a are positive (a - color, transverse position and polarization) :

$$\langle m|e^{-\mathcal{H}_{JIMWLK}\Delta} \approx (1 - f_a\Delta)\langle m_a| + f_a\Delta\langle(m+1)_a|$$

2. For a dense target state $|\bar{n}\rangle$ initarity is violated

$$e^{-\mathcal{H}_{JIMWLK}\Delta}|\bar{n}\rangle \approx (1 + g_a\Delta)|\bar{n}_a\rangle - g_a\Delta|(\bar{n} - 1)_a\rangle$$

g_a are not of fixed sign - some are negative. Also the number of gluons is apparently decreasing throughout the evolution, which is clearly unphysical!

Can we cure it *a la* toy model?

A.K., E. Levin, Ming Li, M. Lublinsky, JHEP 10 (2020) 185

Can we come up with a Hamiltonian that:

1. Reduces to JIMWLK in dense-dilute limit;
2. Is self dual, a.k.a. t-channel unitary;
3. Is "s"-channel unitary.

We can do two out of three, but do not know about the third one.

"Diamond - like action".

Have to define $\bar{V}_L^{\beta\gamma}(\mathbf{x})$; $\bar{V}_R^{\delta\alpha}(\mathbf{x})$; $V_L^{\beta\gamma}(\mathbf{x})$; $V_R^{\delta\alpha}(\mathbf{x})$ kinda like $\bar{U}(x)$ and $U(x)$, but not quite.

$$
\begin{aligned}
H_{RFT} =& \frac{1}{\pi g^2} \int d^2\mathbf{x}\, \bar{V}_L^{\beta\gamma}(\mathbf{x}) \bar{V}_R^{\delta\alpha}(\mathbf{x}) \partial^2 [V_L^{\alpha\beta}(\mathbf{x}) V_R^{\gamma\delta}(\mathbf{x})] \\
=& \frac{1}{\pi g^2} \int d^2\mathbf{x}\, \partial^2 [\bar{V}_L^{\beta\gamma}(\mathbf{x}) \bar{V}_R^{\delta\alpha}(\mathbf{x})] V_L^{\alpha\beta}(\mathbf{x}) V_R^{\gamma\delta}(\mathbf{x})
\end{aligned}
$$

"Daimond" - like action Y. Hatta, E. Iancu, L. McLerran, A. Stasto, D.N. Triantafyllopoulos, Nucl.Phys.A 764 (2006) 423 (and its cousins), although not quite.

S-channel unitarity? We don't know...