# The mass gap in (1+1)-dim QCD


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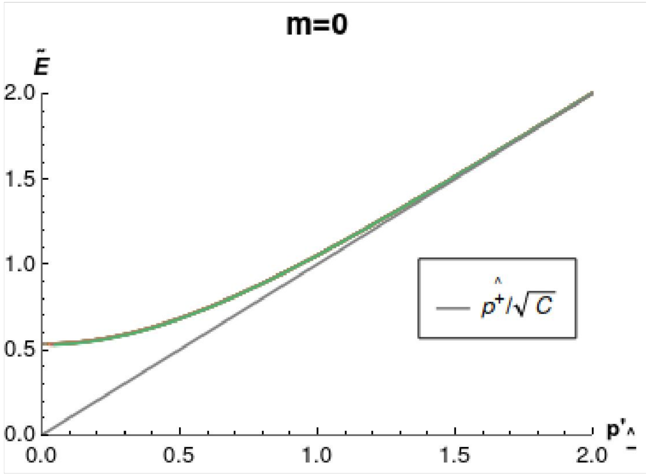
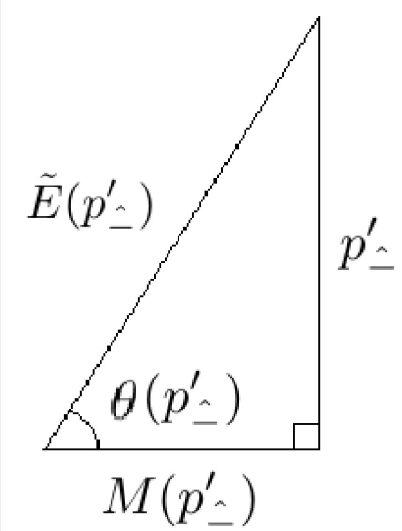


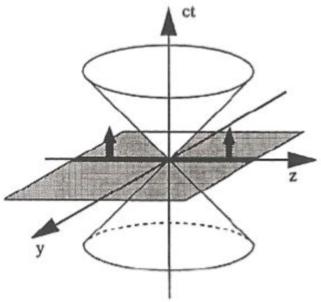




## July 22, 2021, PSQ@EIC

# Short List of References


- G.'tHooft, NPB75,461(74) - LFD
- Y.Frishman, et al., PRD15(75) - Interpol Gauges IFD&LFD
- I.Bars&M.Green, PRD17,537(78) - IFD(formulation)
- A.Zhitnitsky, PLB165,405(85) - LFD(chiral sym breaking)
- M.Li, et al., JPG13, 915(87) - IFD(rest frame)
- K.Hornbostel, Ph.D. Dissertation(88) - LFD(DLCQ)
- M.Burkardt, PRD53,933(96) - LFD(vacuum condensates)
- Y.Kalashnikov&A.Nefed'ev,Phys.-Usp.45,346('02) - IFD(rev)
- Y. Jia, et al., JHEP11, 151('17) - IFD(moving frame)
- Y. Jia, et al., PRD98, 054011('18) - IFD(quasi-PDFs)
- B.Ma&C.Ji, arXiv:2105.09388v1[hep-ph] - Link IFD&LFD

# Dirac's Proposition for Relativistic Dynamics



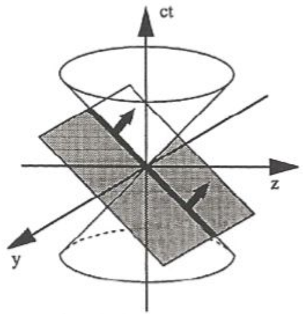
The instant form

*Equal t* 1949 *Equal τ*

$$p^0 \leftrightarrow p^- = p^0 - p^3$$

$$(p^1, p^2) \leftrightarrow \vec{p}_\perp$$

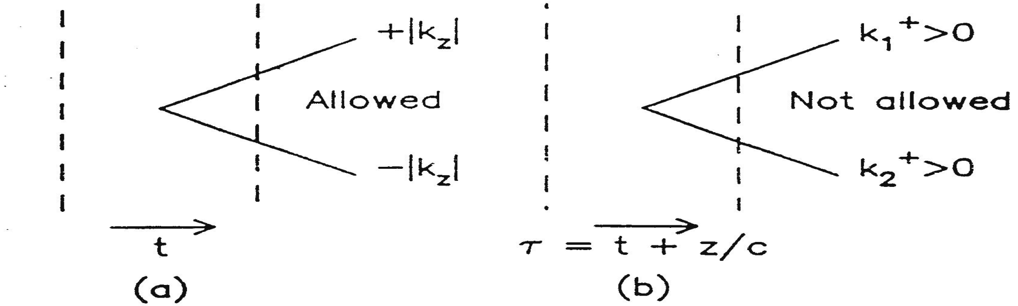
$$p^3 \leftrightarrow p^+ = p^0 + p^3$$



The front form

## Energy-Momentum Dispersion Relations

$$p^0 = \sqrt{\vec{p}^{\,2} + m^2} \qquad p^- = \frac{\vec{p}_\perp^{\,2} + m^2}{p^+}$$




Except zero-modes

$$k_1^+ = k_2^+ = 0$$

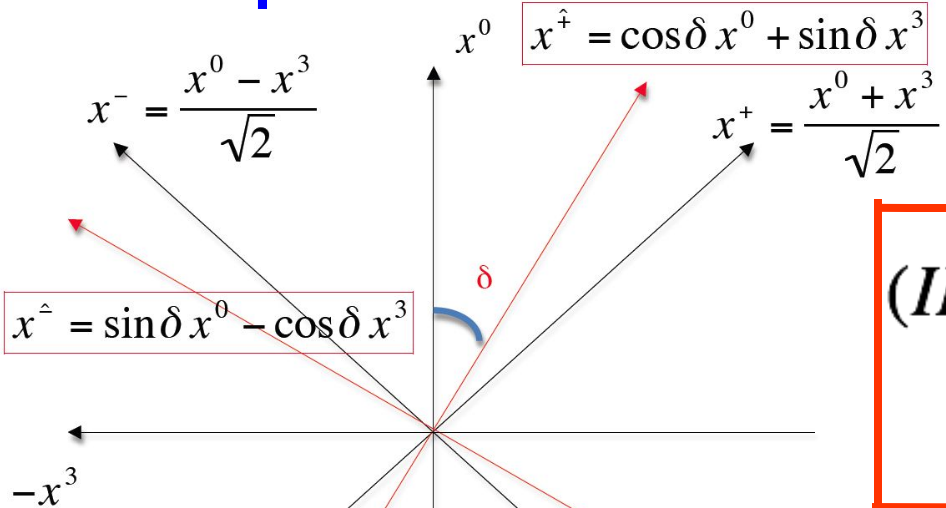
# IFD

Instant Form Dynamics

# LFD

Light-Front Dynamics

# Interpolation between IFD and LFD




$$(IFD) \quad 0 \le \delta \le \frac{\pi}{4} \quad (LFD)$$

$$1 \ge \quad C \equiv \cos(2\delta) \quad \ge 0$$

**K. Hornbostel, PRD45, 3781 (1992)** – RQFT

C.Ji and S.Rey, PRD53,5815(1996) – Chiral Anomaly

C.Ji and C. Mitchell, PRD64,085013 (2001) – Poincare Algebra

C.Ji and A. Suzuki, PRD87,065015 (2013) – Scattering Amps

C.Ji, Z. Li and A. Suzuki, PRD91, 065020 (2015) – EM Gauges

Z.Li, M. An and C.Ji, PRD92, 105014 (2015) – Spinors

C.Ji, Z.Li, B.Ma and A.Suzuki, PRD98, 036017(2018) – QED

B.Ma and C.Ji, arXiv:2105.09388v1[hep-ph],PRD in press – $QCD_{1+1}$

# Large $N_C$ QCD in 1+1 dim. ('tHooft Model)

$$\mathcal{L} = -\frac{1}{4} F^a_{\hat{\mu}\hat{\nu}} F^{\hat{\mu}\hat{\nu}a} + \bar{\psi}(i\gamma^{\hat{\mu}} D_{\hat{\mu}} - m)\psi$$

$$D_{\hat{\mu}} = \partial_{\hat{\mu}} - igA^a_{\hat{\mu}} t_a$$

$$F^a_{\hat{\mu}\hat{\nu}} = \partial_{\hat{\mu}} A^a_{\hat{\nu}} - \partial_{\hat{\nu}} A^a_{\hat{\mu}} + g f^{abc} A^b_{\hat{\mu}} A^c_{\hat{\nu}}$$

**'tHooft Coupling** $\lambda = \frac{g^2 (N_c - 1/N_c)}{4\pi}$ **and mass** $m$

$$g \to 0, N_C \to \infty; \lambda \to finite$$

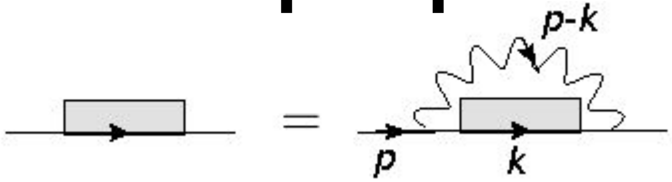
# Interpolating Axial Gauge

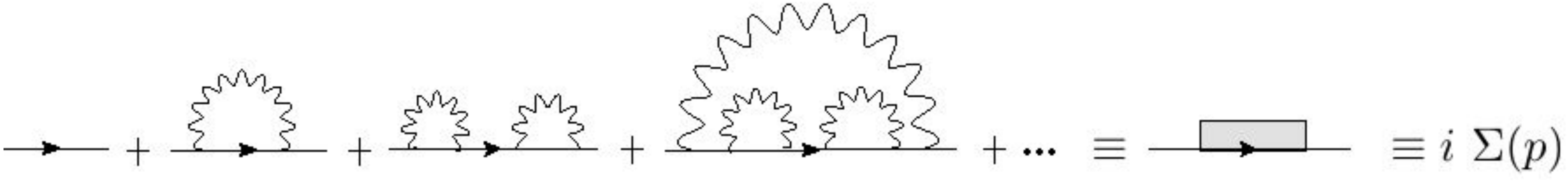
$$A^a_{\hat{-}} = 0$$

$$A^a_1 = 0 \qquad\qquad A^+_a = 0$$

$$\mathcal{L} = \frac{1}{2}(\partial_{\hat{-}} A^a_{\hat{+}})^2 + \bar{\psi}(i\gamma^{\hat{+}} D_{\hat{+}} + i\gamma^{\hat{-}}\partial_{\hat{-}} - m)\psi$$

## Mass Gap Equation




$$\Sigma(p_{\hat{-}}) = i\frac{\lambda}{2\pi}\int \frac{dk_{\hat{-}}\,dk_{\hat{+}}}{(p_{\hat{-}} - k_{\hat{-}})^2}\gamma^{\hat{+}}\frac{1}{\not{k} - m - \Sigma(k_{\hat{-}}) + i\epsilon}\gamma^{\hat{+}}$$

# Fermion Propagator

## Free Propagator

$$S_f(p) = \frac{1}{\not{p} - m + i\varepsilon}$$

## Interacting Propagator

$$S(p) = \frac{1}{\not{p} - m - \Sigma(p) + i\varepsilon} = \frac{F(p)}{\not{p} - M(p) + i\varepsilon}$$

$$\Sigma(p) = \Sigma_s(p) + \Sigma_v(p)\not{p}$$

$$F(p) = \left(1 - \Sigma_v(p)\right)^{-1}$$ "Wave function renormalization factor"

$$M(p) = \frac{m + \Sigma_s(p)}{1 - \Sigma_v(p)}$$ "Renormalized fermion mass function"

# Energy-Momentum Dispersion Relation

## Free particle

$$E = \sqrt{p_z^2 + m^2}$$

$$\theta_f = \tan^{-1}(p_z / m)$$

$$\beta = p_z / E$$
$$= \sin\theta_f$$
$$= \tanh\eta$$

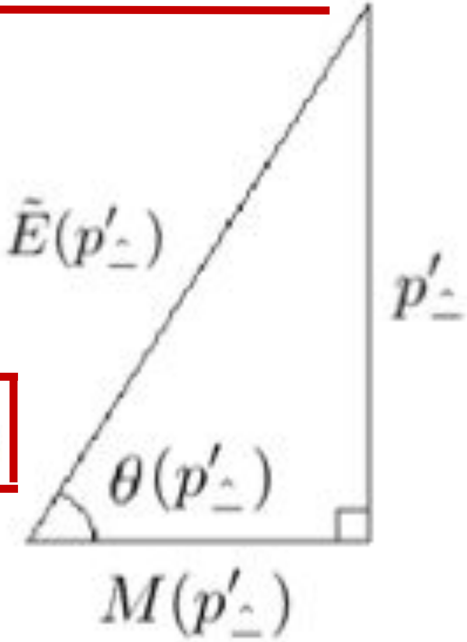
## Interacting particle

$$\frac{F(p'_{\hat{-}})E(p'_{\hat{-}})}{\sqrt{C}} = \sqrt{p'^2_{\hat{-}} + M(p'_{\hat{-}})^2} \equiv \tilde{E}(p'_{\hat{-}})$$

$$\theta(p'_{\hat{-}}) = \theta_f(p'_{\hat{-}}) + 2\zeta(p'_{\hat{-}})$$

$$\begin{pmatrix} b^i(p'_{\hat{-}}) \\ d^{+i}(p'_{\hat{-}}) \end{pmatrix} = \begin{pmatrix} \cos\zeta(p'_{\hat{-}}) & -\sin\zeta(p'_{\hat{-}}) \\ \sin\zeta(p'_{\hat{-}}) & \cos\zeta(p'_{\hat{-}}) \end{pmatrix} \begin{pmatrix} b^i_f(p'_{\hat{-}}) \\ d^{+i}_f(p'_{\hat{-}}) \end{pmatrix}$$

$b^i_f |0>=0, d^i_f |0>=0$ **vs.** $b^i |\Omega>=0, d^i |\Omega>=0$




**Interpolation**

$$(E, p_z) \Rightarrow (p^{\hat{+}} / \sqrt{C}, p_{\hat{-}} / \sqrt{C} \equiv p'_{\hat{-}})$$

# Mass Gap Equation in Scaled Variables

$$\bar{p}'_{\hat{-}} = \frac{\bar{p}_{\hat{-}}}{\sqrt{\mathbb{C}}}, \ \bar{E}' = \frac{\bar{E}}{\sqrt{\mathbb{C}}}, \bar{p}_{\hat{-}} = \frac{p_{\hat{-}}}{\sqrt{2\lambda}}, \ \bar{E} = \frac{E}{\sqrt{2\lambda}}, \bar{m} = \frac{m}{\sqrt{2\lambda}}$$

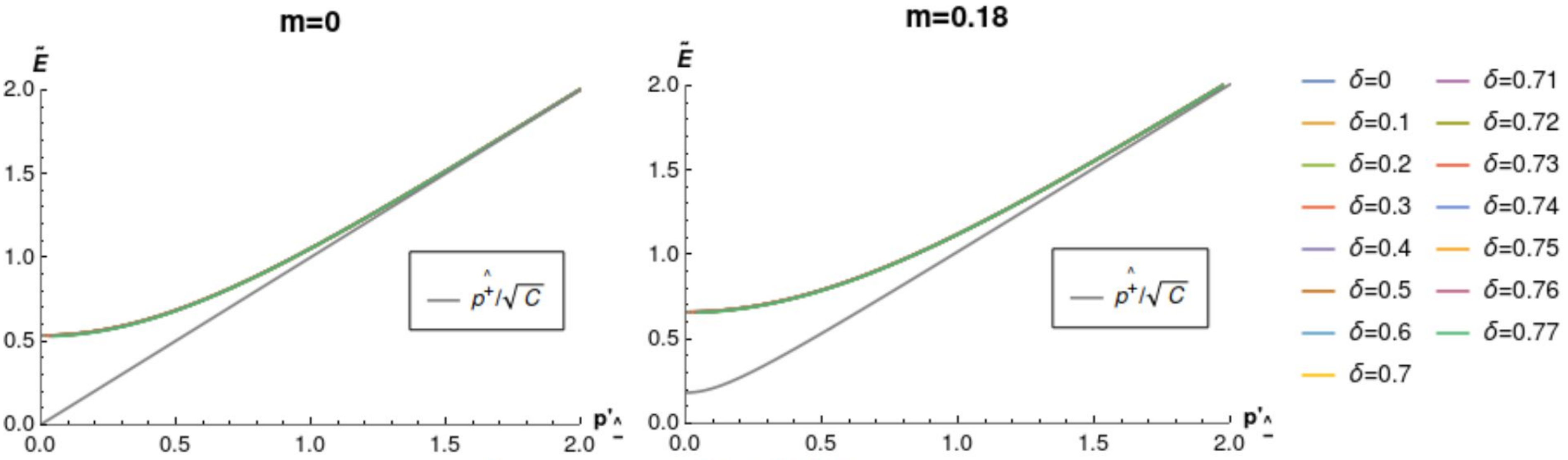
$$\bar{p}'_{\hat{-}} \cos\theta(\bar{p}'_{\hat{-}}) - \bar{m}\sin\theta(\bar{p}'_{\hat{-}}) = \frac{1}{4}\int \frac{d\bar{k}'_{\hat{-}}}{(\bar{p}'_{\hat{-}} - \bar{k}'_{\hat{-}})^2} \sin\left(\theta(\bar{p}'_{\hat{-}}) - \theta(\bar{k}'_{\hat{-}})\right)$$

$$\bar{E}'(\bar{p}'_{\hat{-}}) = \bar{p}'_{\hat{-}} \sin\theta(\bar{p}'_{\hat{-}}) + \bar{m}\cos\theta(\bar{p}'_{\hat{-}}) + \frac{1}{4}\int \frac{d\bar{k}'_{\hat{-}}}{(\bar{p}'_{\hat{-}} - \bar{k}'_{\hat{-}})^2} \cos\left(\theta(\bar{p}'_{\hat{-}}) - \theta(\bar{k}'_{\hat{-}})\right)$$

$$\frac{p_{\hat{-}}}{\mathbb{C}} \cos\theta(p_{\hat{-}}) - \frac{m}{\sqrt{\mathbb{C}}}\sin\theta(p_{\hat{-}}) = \frac{\lambda}{2}\int \frac{dk_{\hat{-}}}{(p_{\hat{-}} - k_{\hat{-}})^2} \sin\left(\theta(p_{\hat{-}}) - \theta(k_{\hat{-}})\right)$$

$$E(p_{\hat{-}}) = p_{\hat{-}} \sin\theta(p_{\hat{-}}) + \sqrt{\mathbb{C}} m\cos\theta(p_{\hat{-}}) + \frac{\mathbb{C}\lambda}{2}\int \frac{dk_{\hat{-}}}{(p_{\hat{-}} - k_{\hat{-}})^2} \cos\left(\theta(p_{\hat{-}}) - \theta(k_{\hat{-}})\right)$$

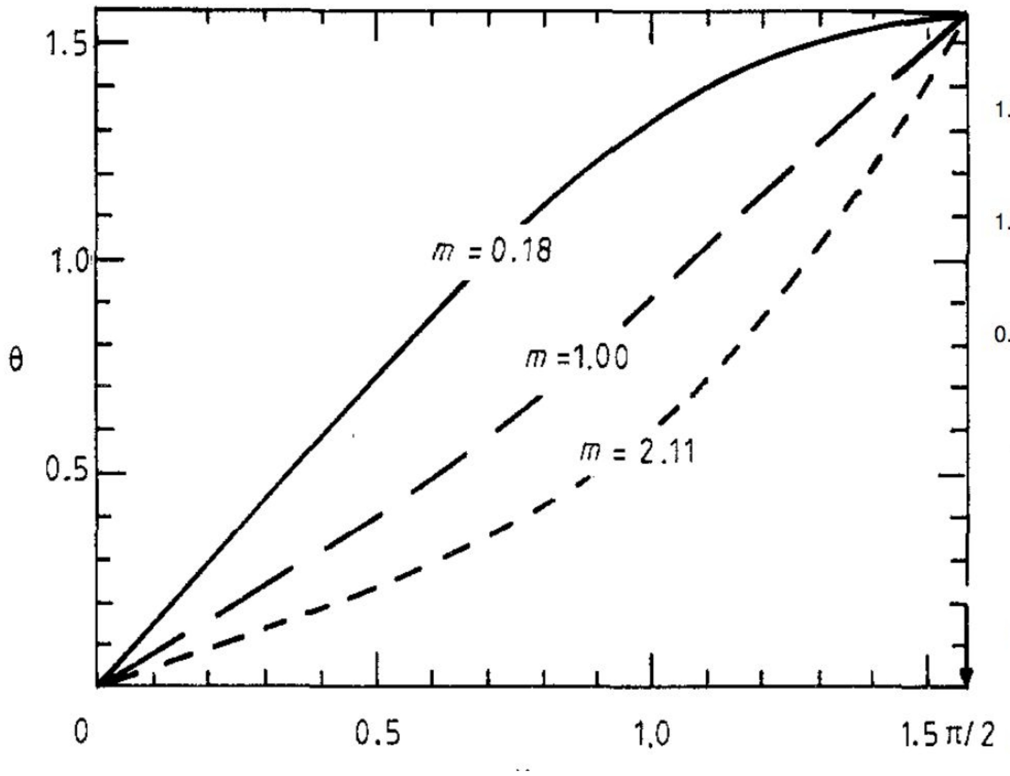
# Mass Gap Solutions

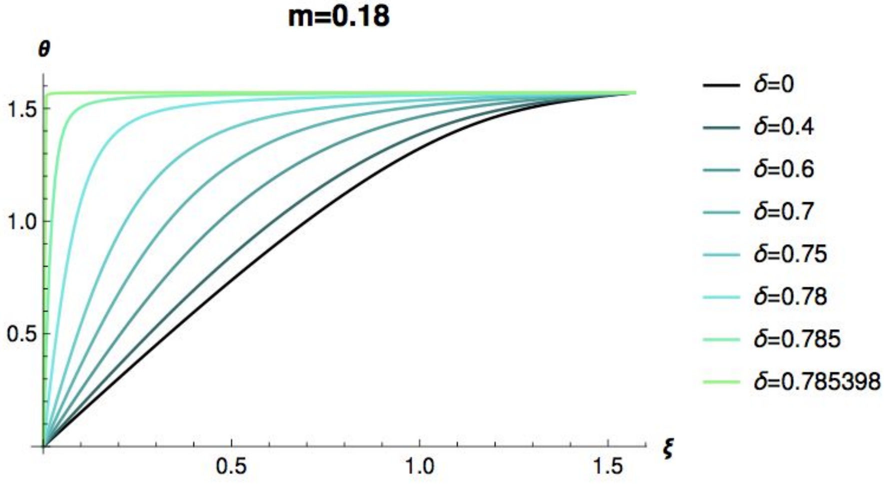



$$\tilde{E}(0) = \frac{F(0)E(0)}{\sqrt{\mathbb{C}}} = M(0)$$

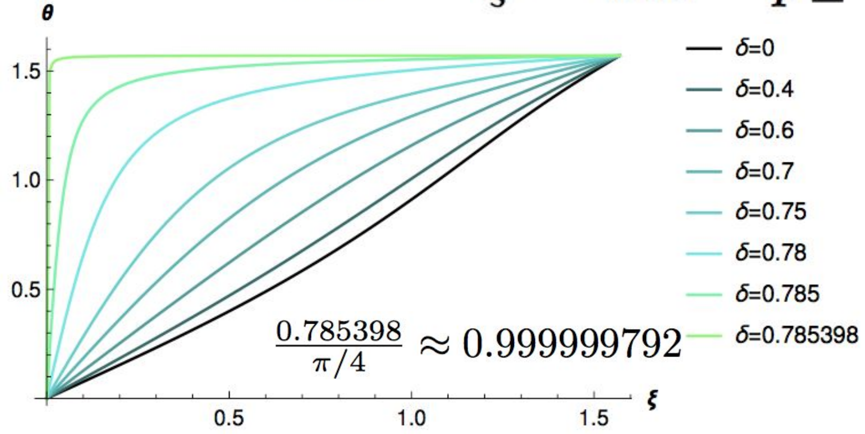
| $m$ | 0 | 0.045 | 0.18 | 0.749 | 1.00 | 2.11 | 4.23 |
|---|---|---|---|---|---|---|---|
| $M(0)$ | 0.532778 | 0.563644 | 0.659112 | 1.10105 | 1.31167 | 2.30969 | 4.34358 |
| $F(0)$ | −0.495173 | −0.584175 | −0.987673 | 4.11079 | 2.17976 | 1.22134 | 1.05526 |

$$m \lesssim 0.56$$



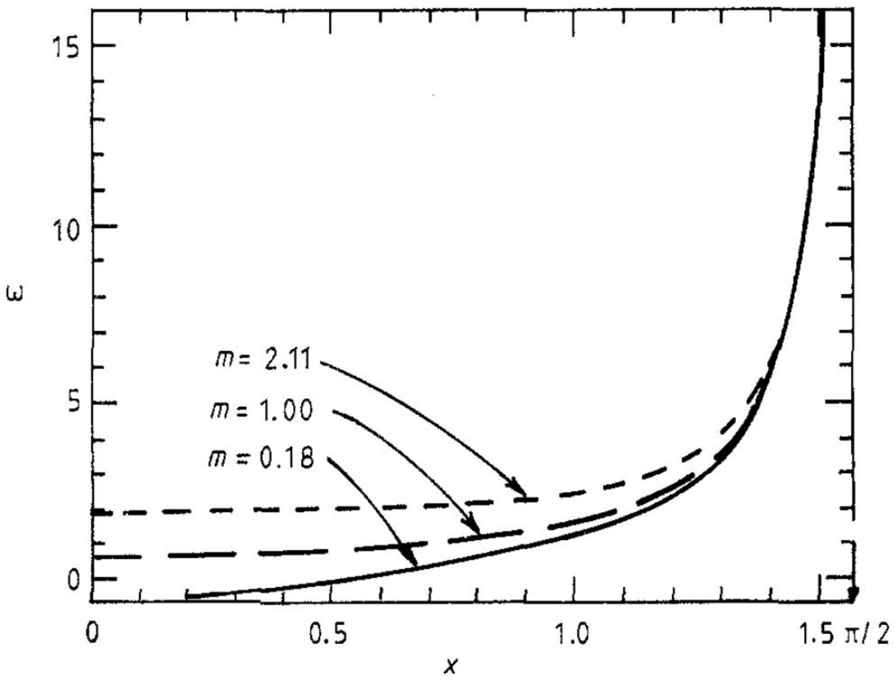



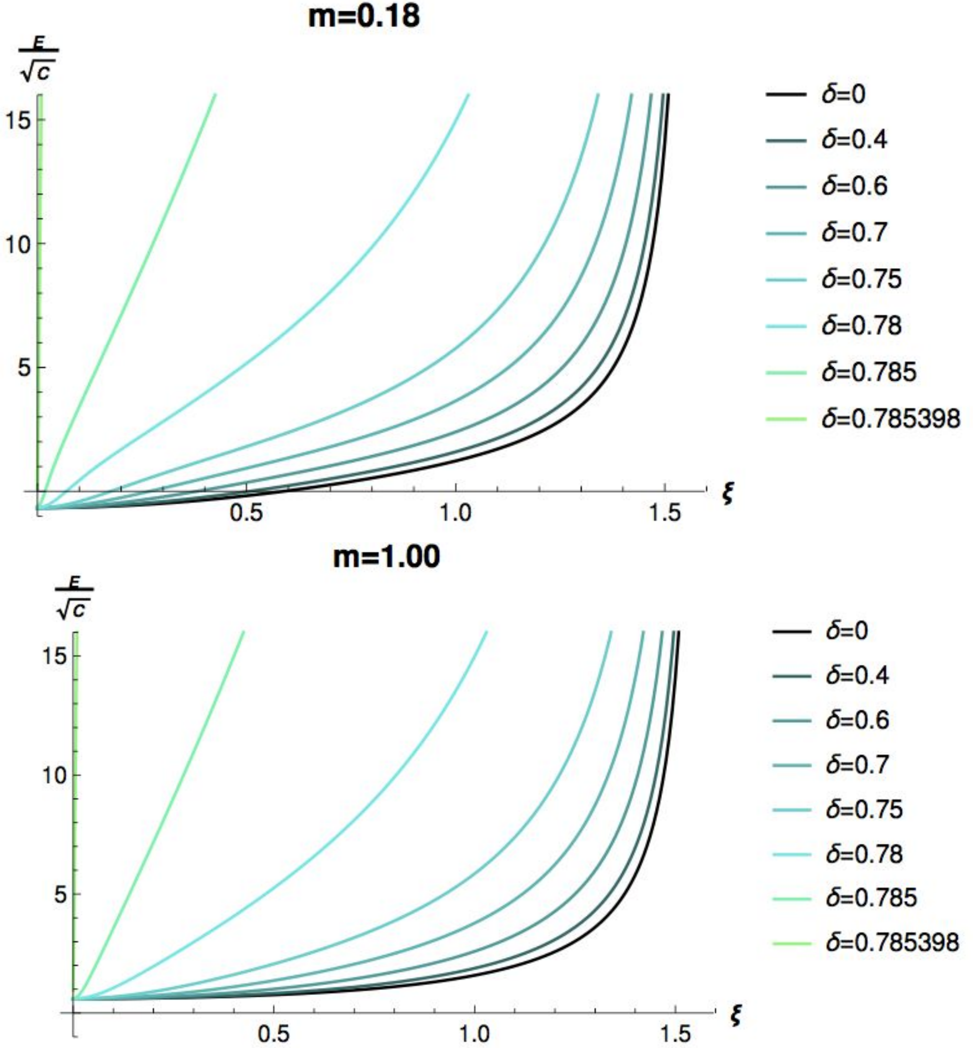

$$\xi = \tan^{-1} p_{\hat{\perp}}$$




$$\frac{0.785398}{\pi/4} \approx 0.999999792$$

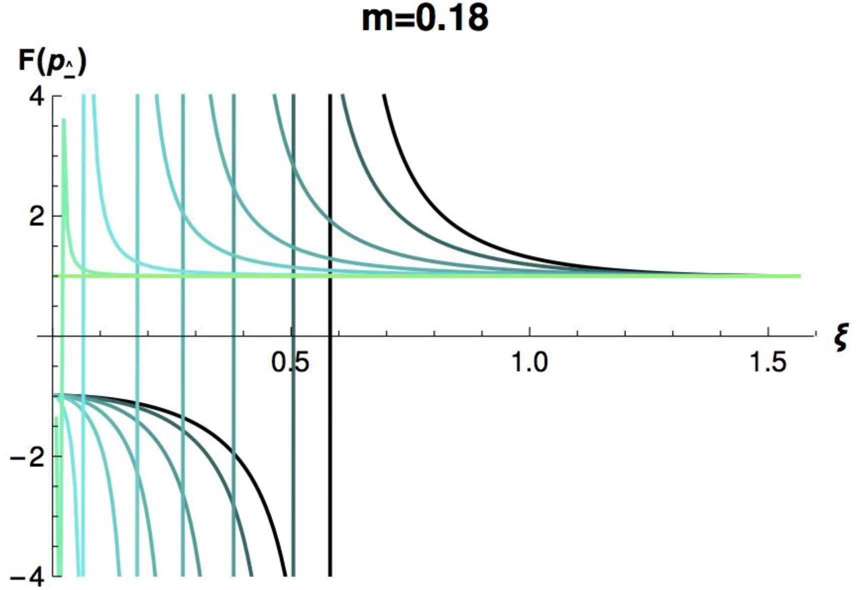
- M.Li, et al., JPG13, 915(87) - IFD(rest frame)

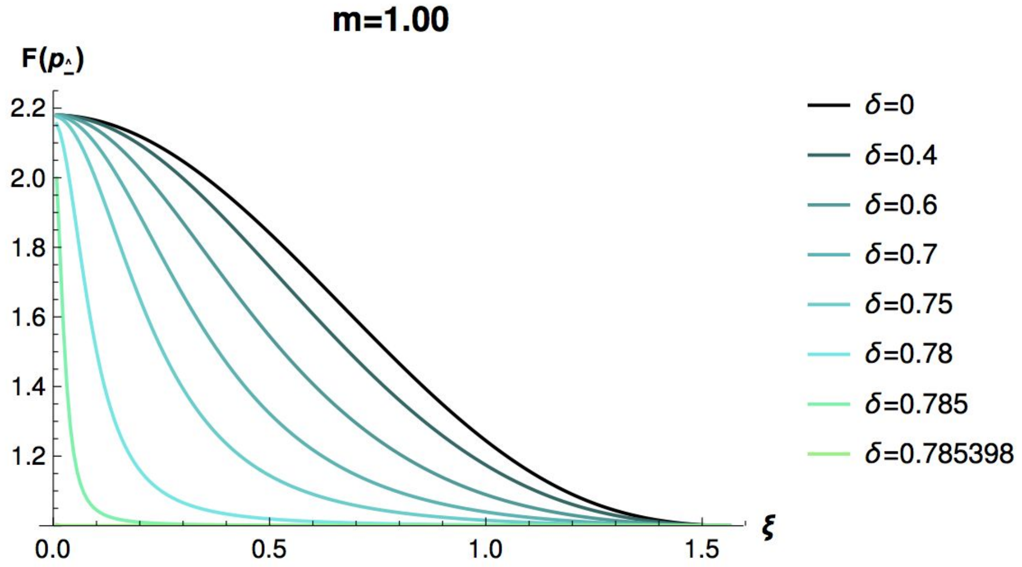


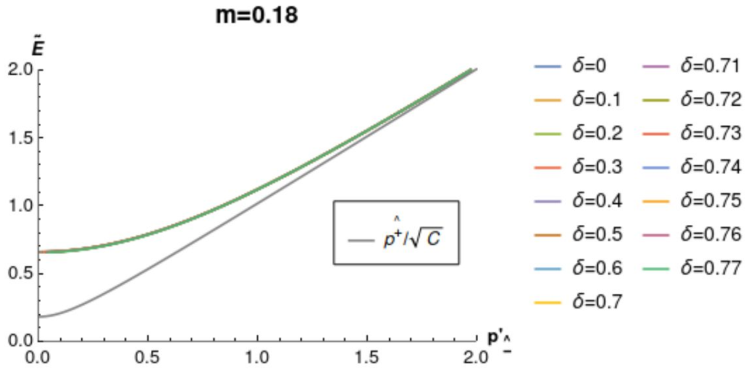




- M.Li, et al., JPG13, 915(87) - IFD(rest frame)

# Wave Function Renormalization Factors



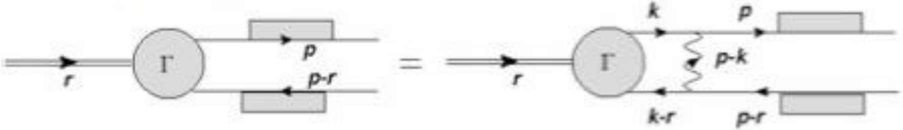




$$\frac{F(p'_{\hat{-}})E(p'_{\hat{-}})}{\sqrt{C}} = \sqrt{p'^2_{\hat{-}} + M(p'_{\hat{-}})^2} \equiv \tilde{E}(p'_{\hat{-}})$$

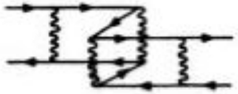
# BOUND-STATE EQUATION

$$\Gamma(r,p) = \frac{i\lambda}{2\pi}\int \frac{dk_{\hat{-}}\,dk_{\hat{+}}}{(p_{\hat{-}} - k_{\hat{-}})^2} S(p)\gamma^{\hat{+}}\Gamma(r,k)\gamma^{\hat{+}}S(p-r)$$




$$\left[-r_{\hat{+}} + \frac{-\mathbb{S}p_{\hat{-}} + E(p_{\hat{-}})}{\mathbb{C}} + \frac{\mathbb{S}(p_{\hat{-}} - r_{\hat{-}}) + E(p_{\hat{-}} - r_{\hat{-}})}{\mathbb{C}}\right]\hat{\phi}_{+}(r_{\hat{-}}, p_{\hat{-}})$$
$$= \lambda \fint \frac{dk_{\hat{-}}}{(p_{\hat{-}} - k_{\hat{-}})^2}\left[C(p_{\hat{-}}, k_{\hat{-}}, r_{\hat{-}})\hat{\phi}_{+}(r_{\hat{-}}, k_{\hat{-}}) - S(p_{\hat{-}}, k_{\hat{-}}, r_{\hat{-}})\hat{\phi}_{-}(r_{\hat{-}}, k_{\hat{-}})\right],$$

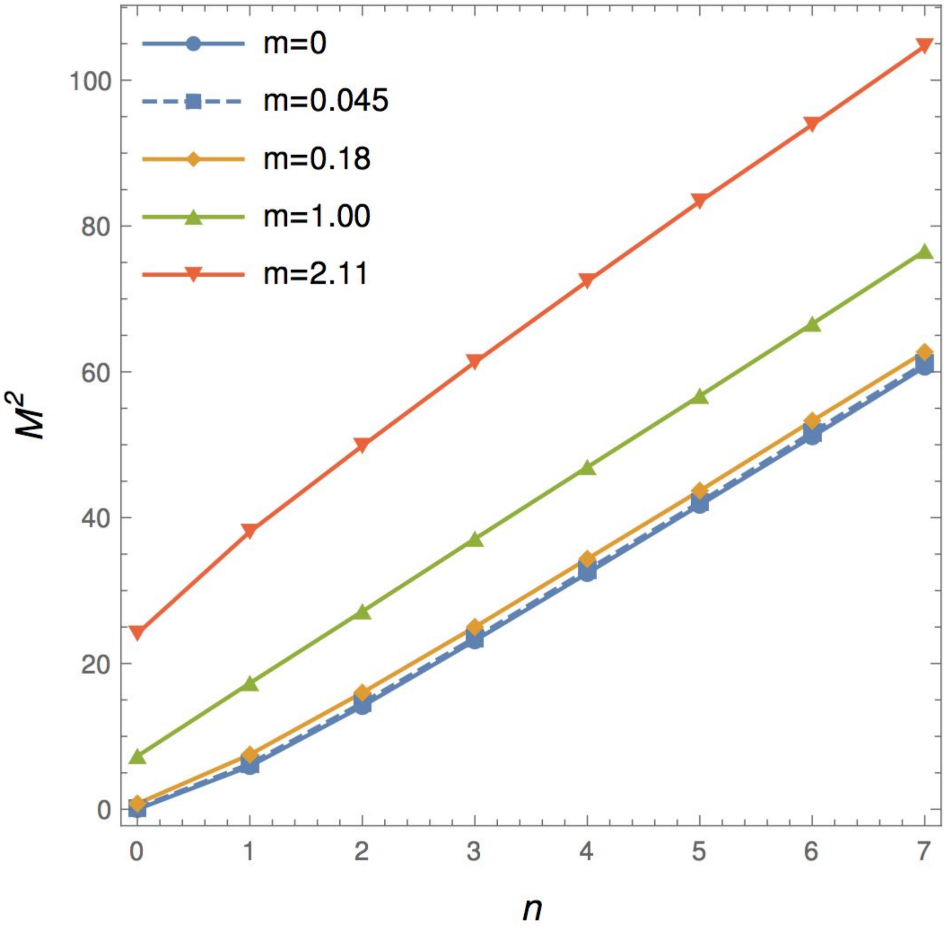
$$\left[r_{\hat{+}} + \frac{-\mathbb{S}(p_{\hat{-}} - r_{\hat{-}}) + E(p_{\hat{-}} - r_{\hat{-}})}{\mathbb{C}} + \frac{\mathbb{S}p_{\hat{-}} + E(p_{\hat{-}})}{\mathbb{C}}\right]\hat{\phi}_{-}(r_{\hat{-}}, p_{\hat{-}})$$
$$= \lambda \fint \frac{dk_{\hat{-}}}{(p_{\hat{-}} - k_{\hat{-}})^2}\left[C(p_{\hat{-}}, k_{\hat{-}}, r_{\hat{-}})\hat{\phi}_{-}(r_{\hat{-}}, k_{\hat{-}}) - S(p_{\hat{-}}, k_{\hat{-}}, r_{\hat{-}})\hat{\phi}_{+}(r_{\hat{-}}, k_{\hat{-}})\right].$$



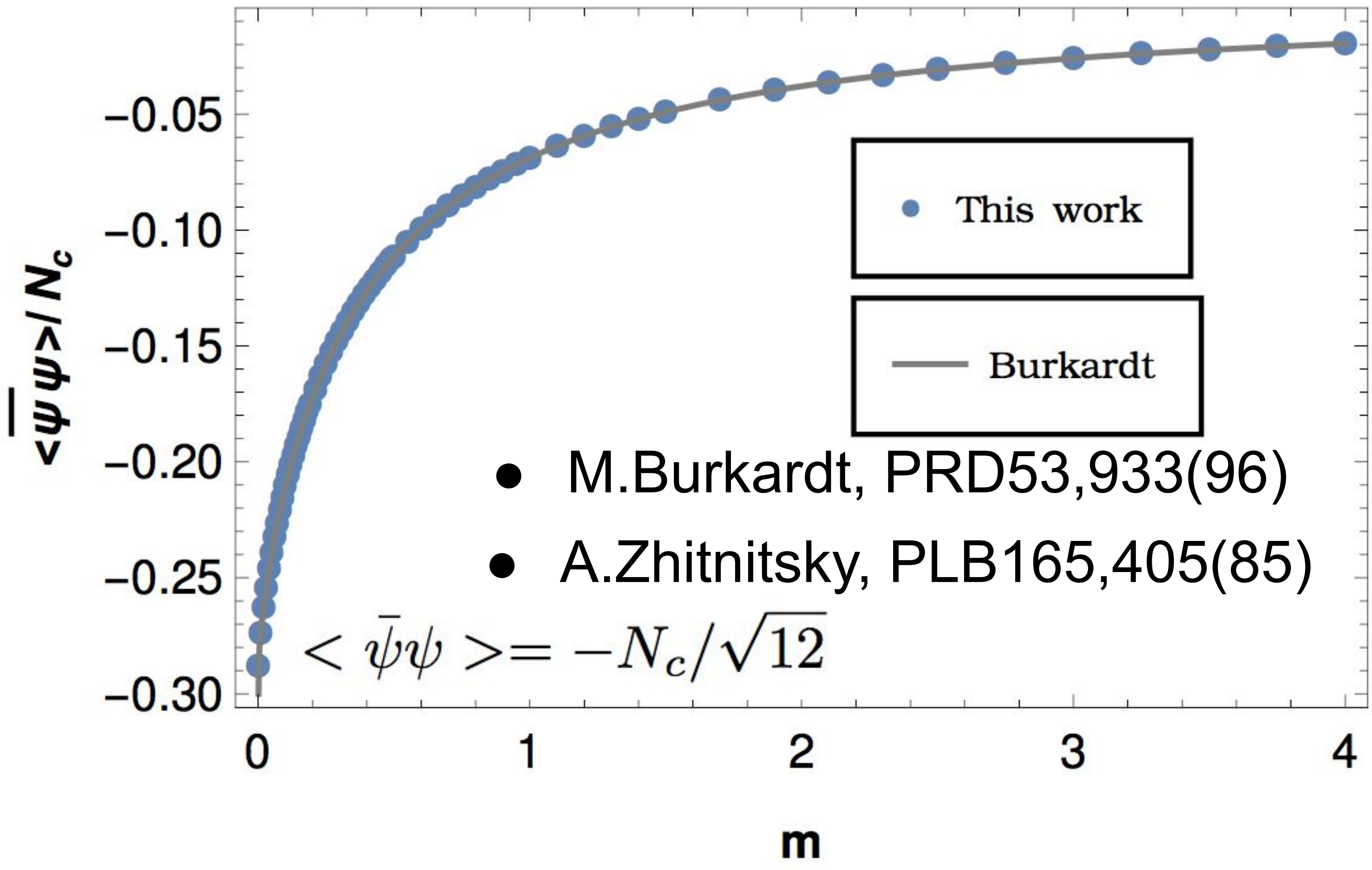
**LFD** $$\left[\mathcal{M}^2 - \frac{m^2 - 2\lambda}{x} - \frac{m^2 - 2\lambda}{1-x}\right]\phi(x) = -2\lambda \fint_0^1 \frac{dy}{(x-y)^2}\phi(y)$$

# Meson Spectroscopy

- G.'tHooft, NPB75, 461(74) - LFD

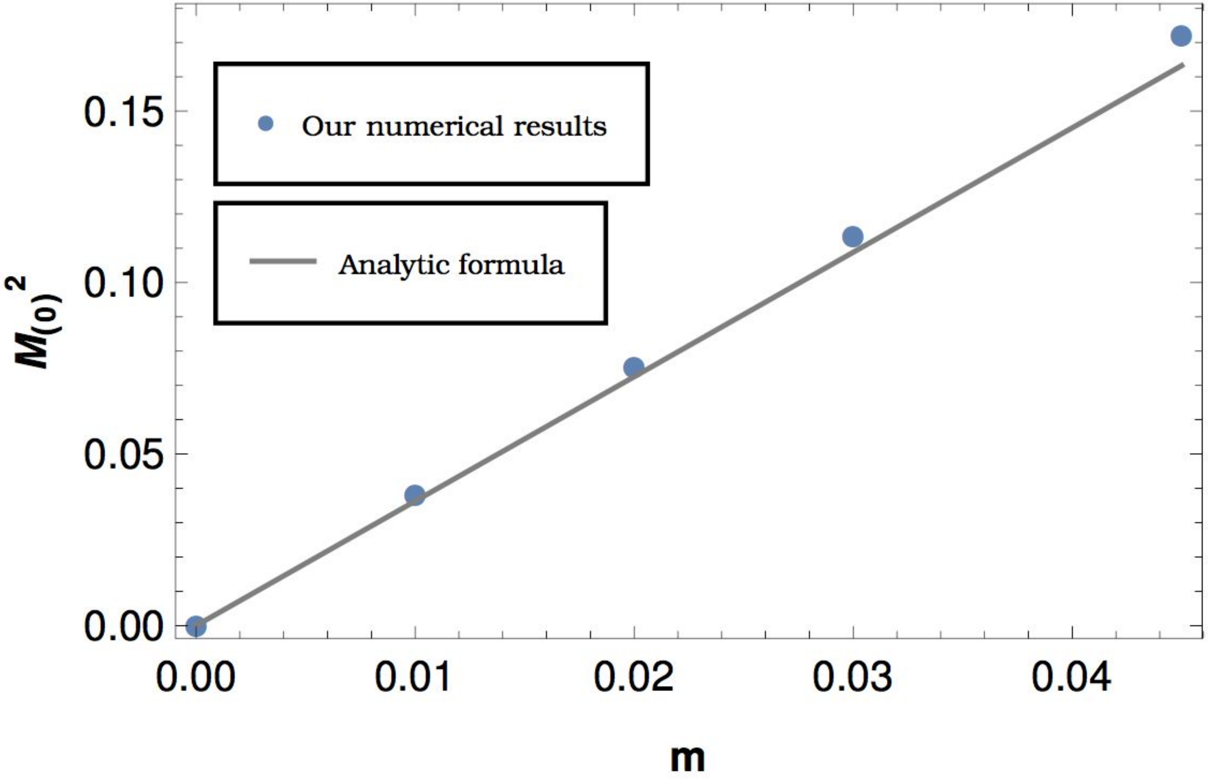



- M.Li, et al., JPG13, 915(87) - IFD (rest frame)
- Y. Jia, et al., JHEP11, 151('17) - IFD (moving frame)




- M.Burkardt, PRD53,933(96)
- A.Zhitnitsky, PLB165,405(85)

# Gell-Mann - Oaks - Renner Relation

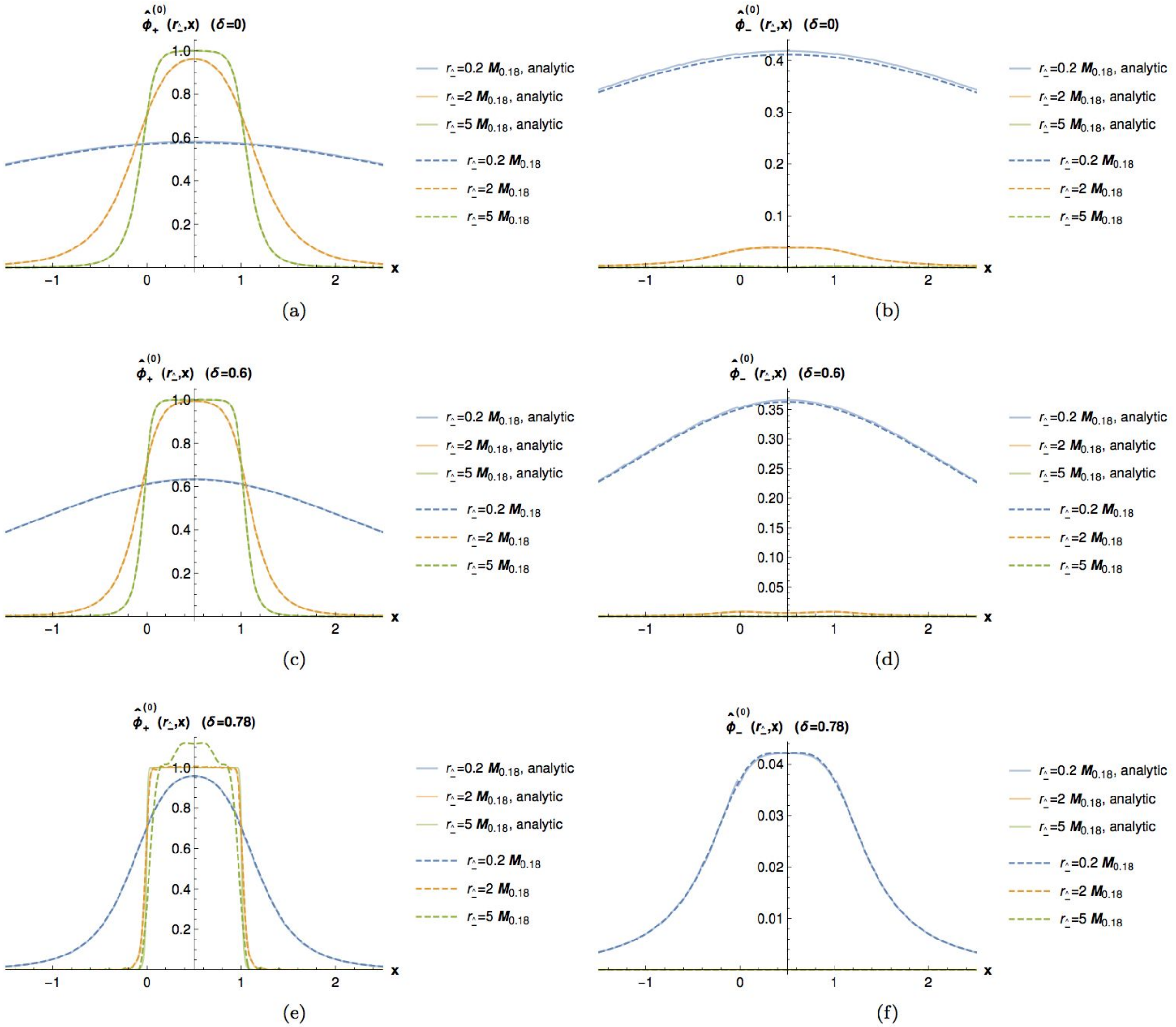



$$\mathcal{M}_\pi^2 = -\frac{4m < \bar{\psi}\psi >}{f_\pi^2} = \sqrt{\frac{8\pi^2 m^2 \lambda}{3}}$$

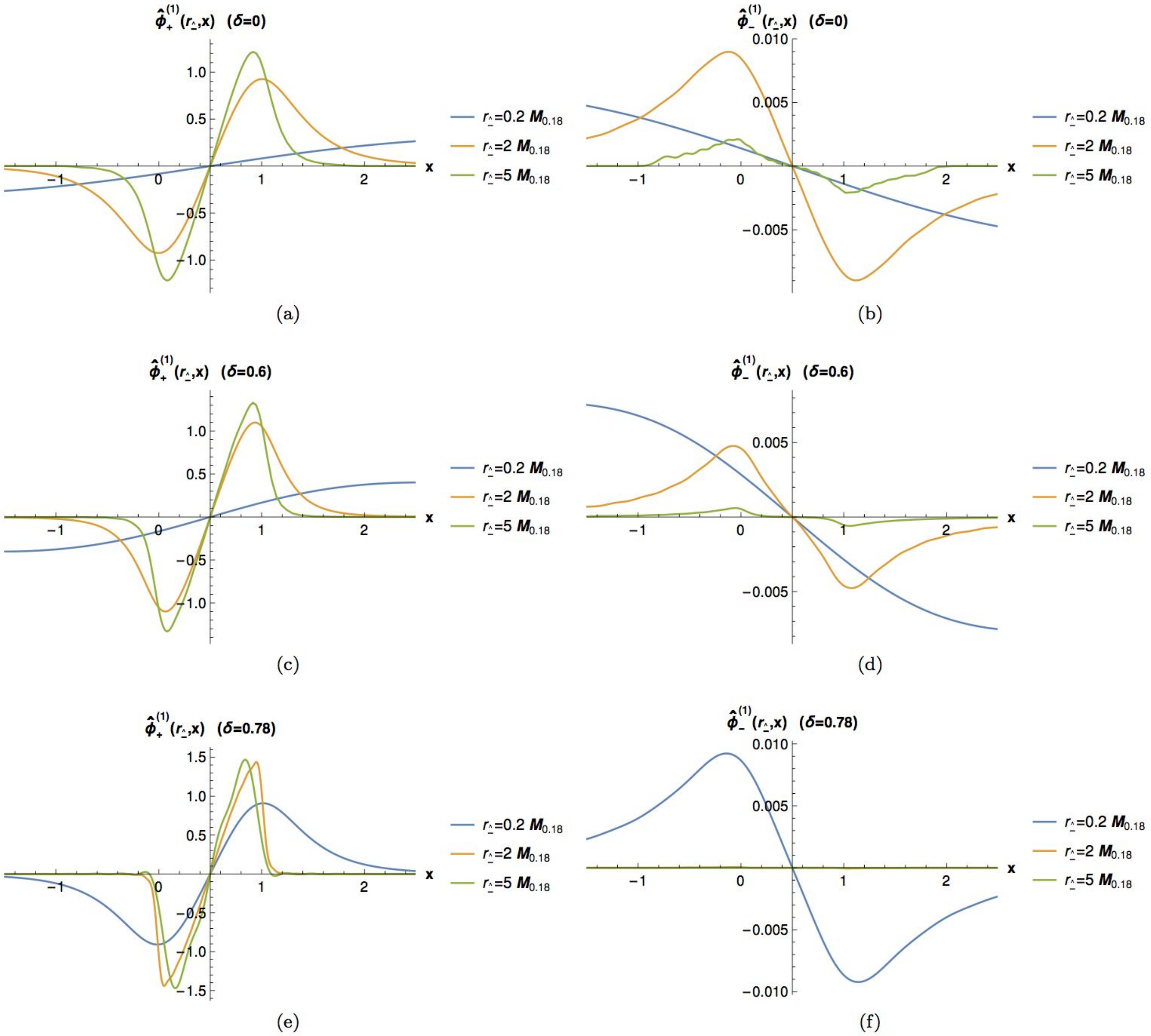
$$f_\pi = \sqrt{N_c/\pi}$$

# Meson Ground-state Wave-function for m=0 case

$$\hat{\phi}_{\pm}^{(0)}(r_{\hat{-}}, p_{\hat{-}}) = \frac{1}{2}\left(\cos\frac{\theta(r_{\hat{-}} - p_{\hat{-}}) - \theta(p_{\hat{-}})}{2} \pm \sin\frac{\theta(r_{\hat{-}} - p_{\hat{-}}) + \theta(p_{\hat{-}})}{2}\right)$$



# First Excited-state Meson Wave-functions for m=0 case



# Parton Distribution Functions (PDFs)

$$q_n(x) = \int_{-\infty}^{+\infty} \frac{d\xi^-}{4\pi} e^{-ixP^+\xi^-}$$

$$\times \langle P_n^-, P^+ | \bar{\psi}(\xi^-)\gamma^+ \mathcal{W}[\xi^-, 0]\psi(0) | P_n^-, P^+ \rangle_C,$$

$$\mathcal{W}[\xi^-, 0] = \mathcal{P}\left[\exp\left(-ig_s \int_0^{\xi^-} d\eta^- A^+(\eta^-)\right)\right]$$

**$A^+$=0 Gauge in LFD**

## Quasi-PDFs

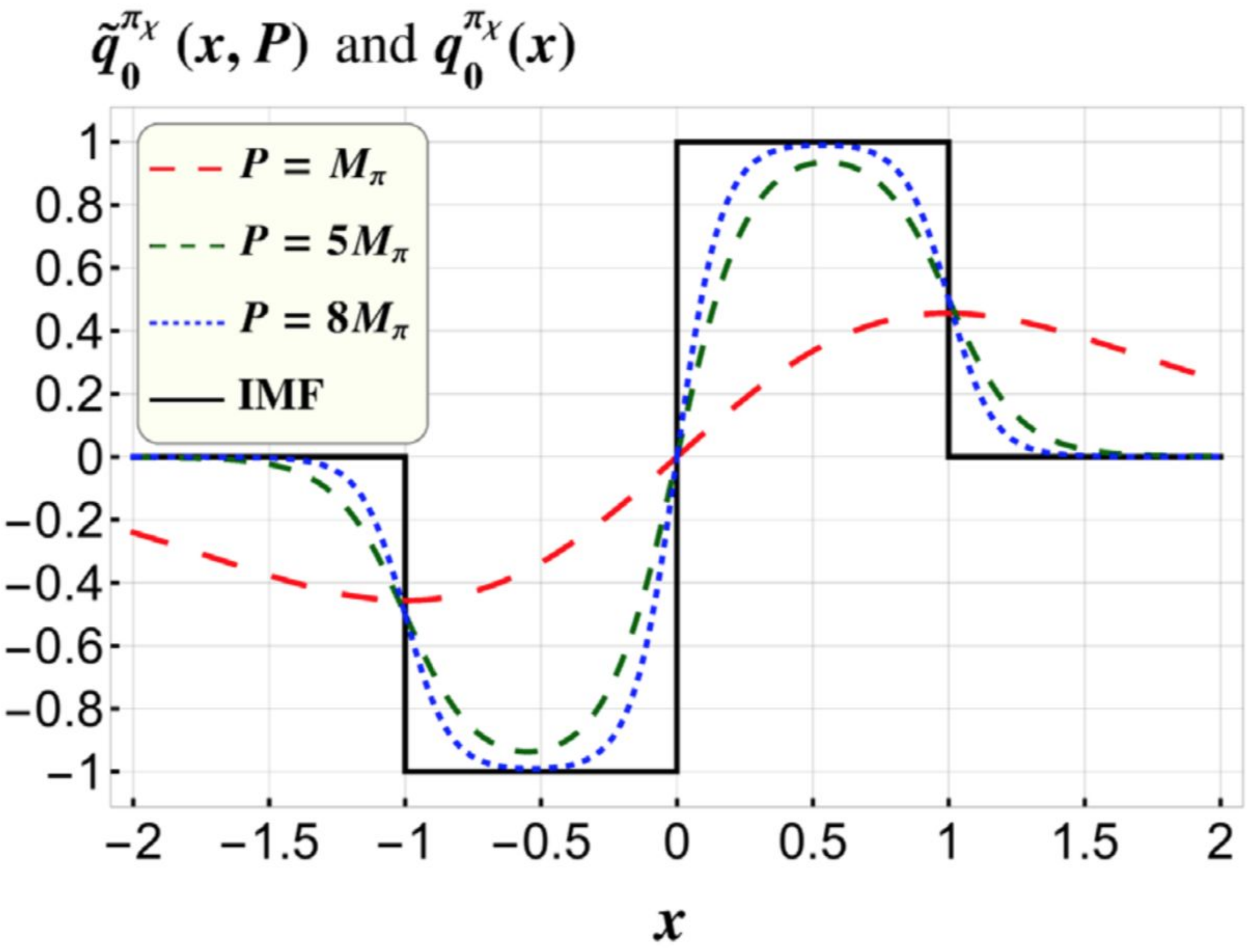
$$\tilde{q}_{(n)}(r_{\hat{-}}, x) = \int_{-\infty}^{+\infty} \frac{dx^{\hat{-}}}{4\pi} \, \mathrm{e}^{ix^{\hat{-}} r_{\hat{-}}}$$

$$\times < r^{\hat{+}}_{(n)}, r_{\hat{-}} \mid \bar{\psi}(x^{\hat{-}}) \, \gamma_{\hat{-}} \, \mathcal{W}[x^{\hat{-}}, 0] \, \psi(0) \mid r^{\hat{+}}_{(n)}, r_{\hat{-}} >_C,$$

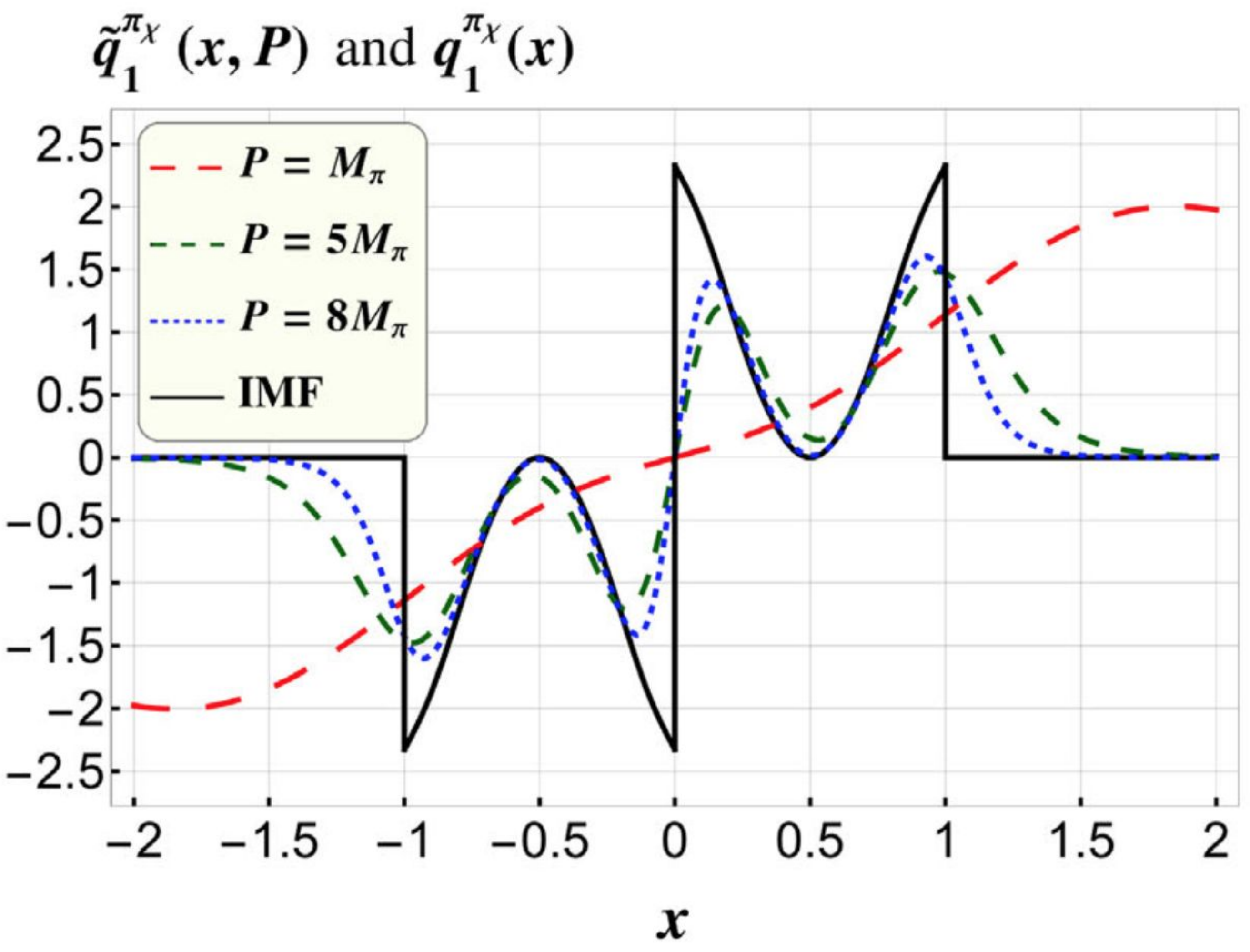
$$\mathcal{W}[x^{\hat{-}}, 0] = \mathcal{P}\left[\exp\left(-ig \int_0^{x^{\hat{-}}} dx'^{\hat{-}} A_{\hat{-}}(x'^{\hat{-}})\right)\right]$$

**Interpolating dynamics**

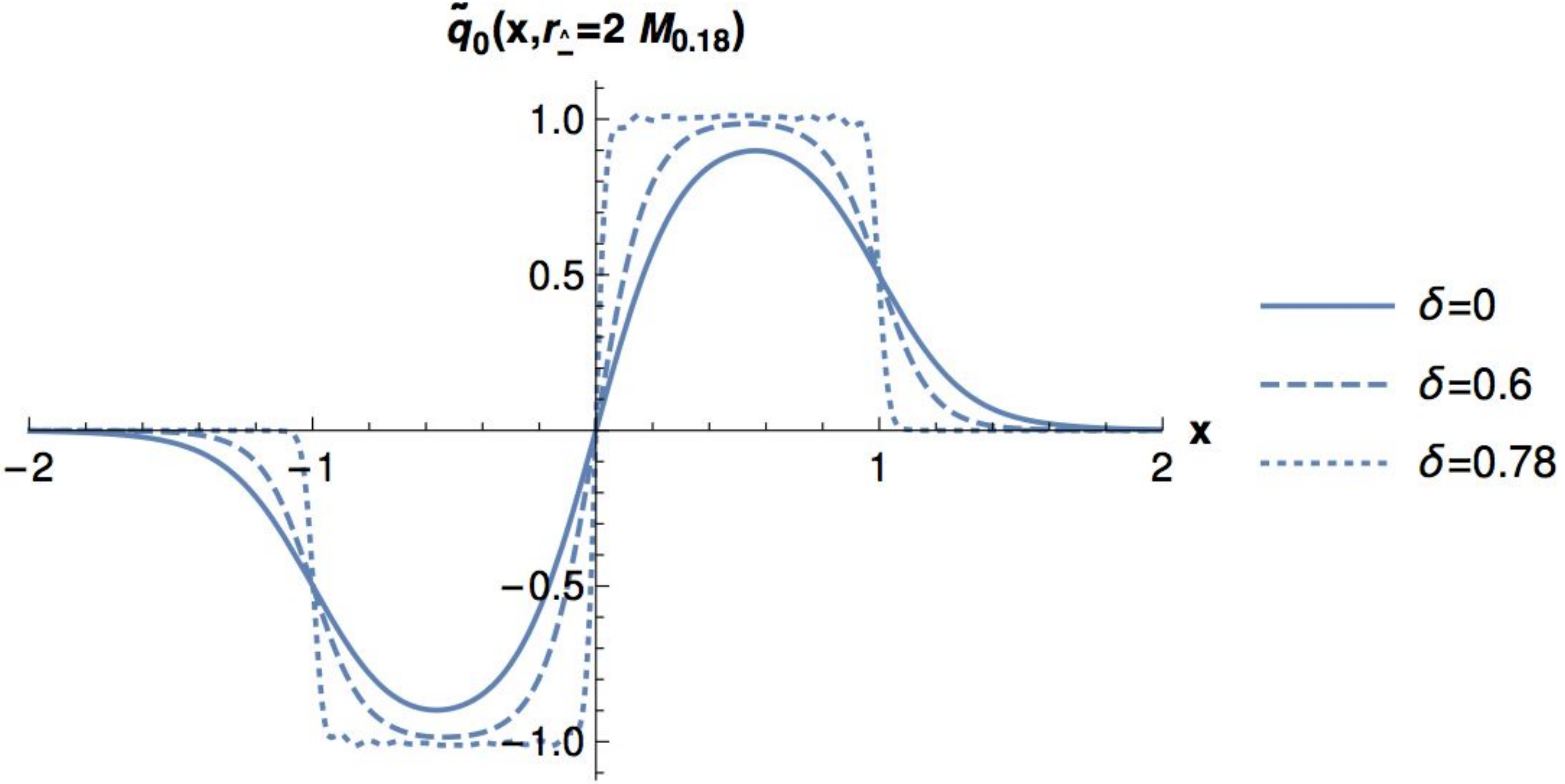
- Y. Jia, et al., PRD98, 054011('18)
  - IFD (quasi-PDFs)



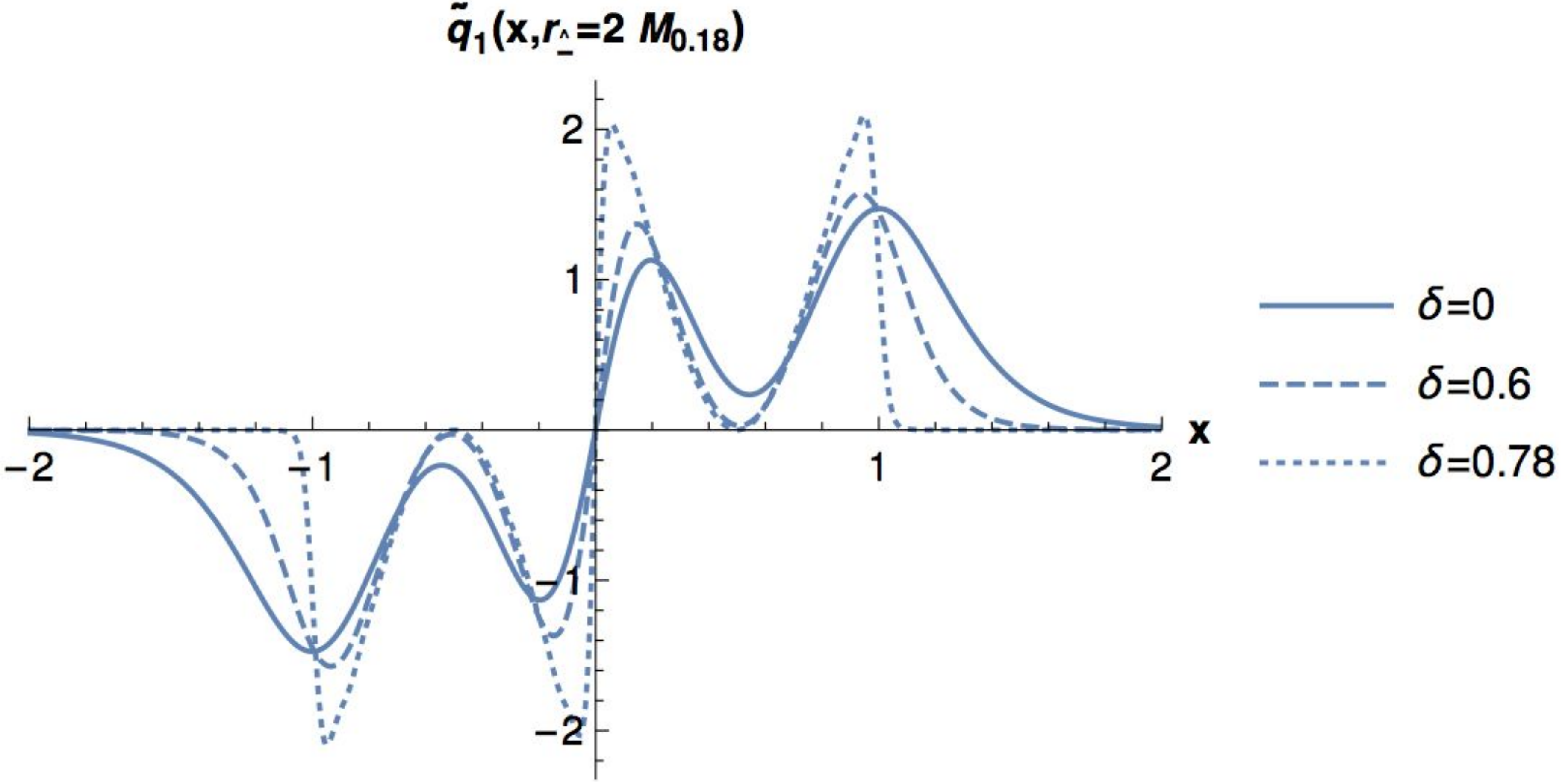
- Y. Jia, et al., PRD98, 054011('18)
  - IFD (quasi-PDFs)



- B.Ma&C.Ji, arXiv:2105.09388v1[hep-ph]



- B.Ma&C.Ji, arXiv:2105.09388v1[hep-ph]



# Extended Wick Rotation

$$p^0 \rightarrow \tilde{P}^0 = ip^0 \quad (\delta = 0)$$

$For \quad 0 < \delta < \pi / 4,$

$$p^{\hat{+}} / \sqrt{C} \rightarrow \tilde{P}^{\hat{+}} / \sqrt{C} = ip^{\hat{+}} / \sqrt{C}\,.$$

*Correspondence to Euclidean Space*

$$p'^{2}_{\underline{\wedge}} = p^{2}_{\underline{\wedge}} / C \leftrightarrow -\tilde{P}^2$$

# Conclusions and Outlook

- QCD(1+1) in large Nc “tHooft model’ is interpolated between IFD and LFD and solved for its mass gap to find interpolation angle independent energy function including the wavefunction renormalization.
- Chiral condensate is found independent of interpolation angle indicating the persistence of nontrivial vacuum even in LFD.

- Mass spectra of mesons bearing the feature of Regge trajectories are found and GOR relation for the pionic ground-state in the zero fermion mass limit.
- Applying to quasi-PDFs in the interpolating formulation, we note a possibility of utilizing not only the reference frame dependence but also the interpolation angle dependence to get an alternative effective approach to the LFD’s PDFs.