Photomultiplier tube charge and time measurement basics

ratna bose

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7/23/2021

BNL physics department summer lectures.

Library of materials from this and other lectures
https://www.phy.bnl.gov/~diwan/#photo

Also see: https://arxiv.org/abs/1909.05373 by MVD to understand the calculation in these notes.

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Outline

- *In this lecture we are going to learn the basics of how to analyze detector functions.*

  1. *We will learn the basics of how a photomultiplier tube works. This is a very important sensor. The electrical signal out of a PMT has been analyzed in a separate lectures. This will focus on the statistics.*

  2. *We will derive the expected charge spectra for photo-multiplier tubes. The techniques can be applied to other detectors as well.*

  3. *We will learn about the statistics of gain through multiplication.*

  4. *We will also learn about time measurement and its resolution from such devices.*

  5. *And we will learn about statistics of a counting process.*

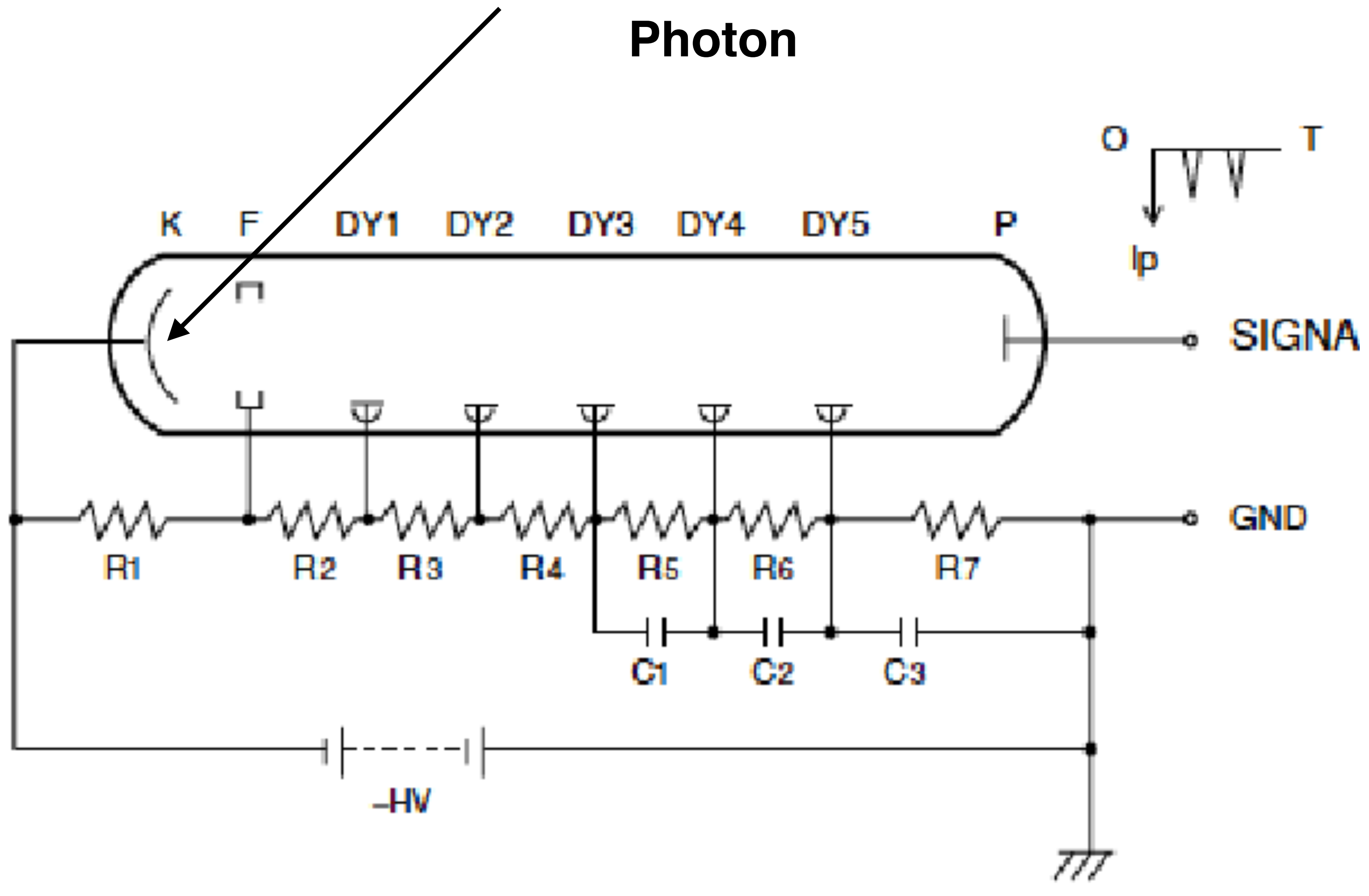
  6. *To derive these we will need some powerful mathematical tools. We will get an introduction to these tools.*

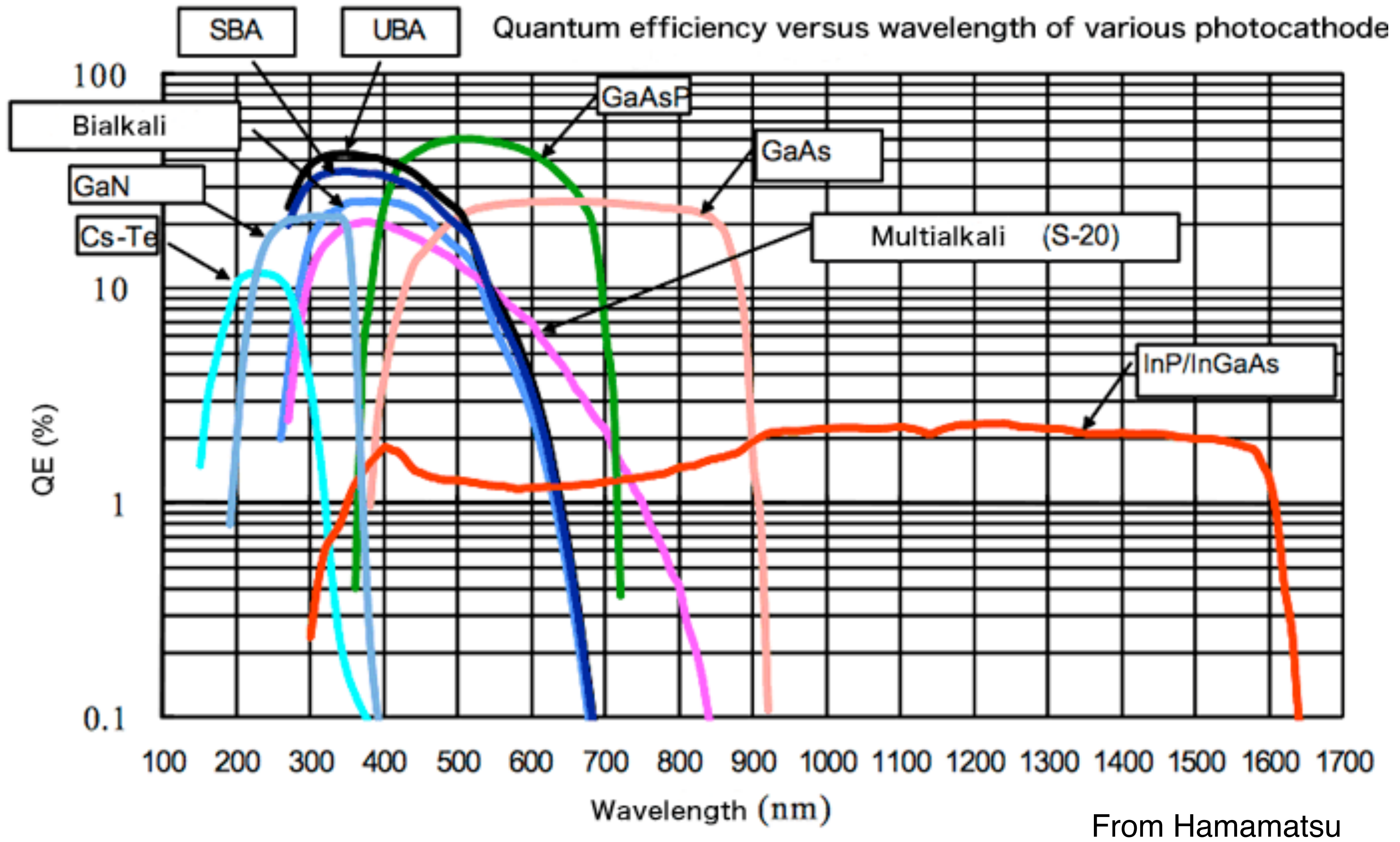
- *The assumption in these slides is that we have integrated the charge coming from a PMT. The only sources of noise is the PMT itself.*

- *Reference: See the guide from Hamamatsu, Photomultiplier tubes: Basics and Applications (2007) 3rd edition. There are many papers in journals such as Nuclear Instruments and Methods.*

This is a lot, but I will provide all materials. About 20-30% of this is original and rest can be found in textbooks.

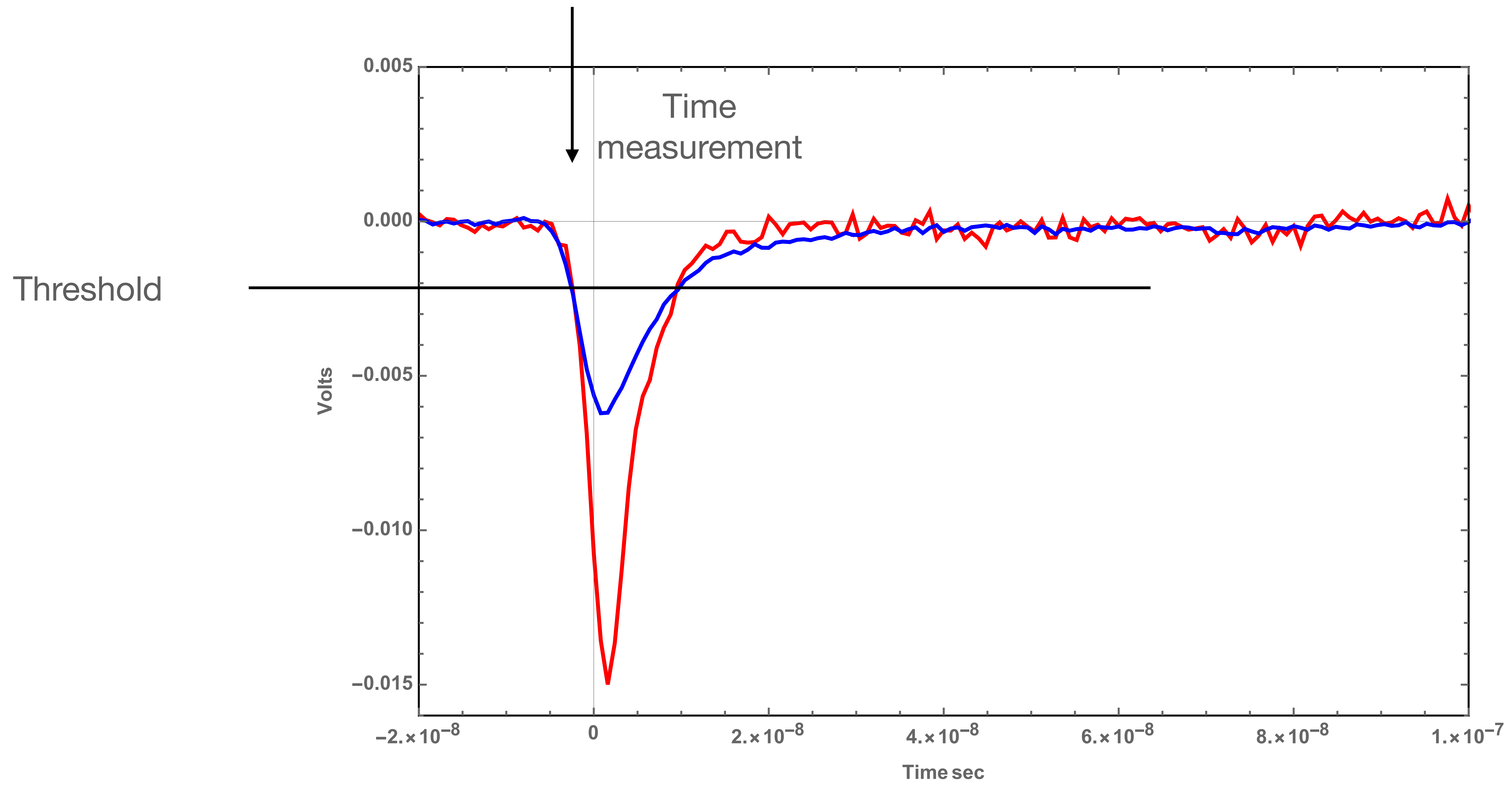
Photo-Multiplier Tube





From Hamamatsu

- The photoelectric effect(Hertz 1887!) causes metals to eject electrons in response to light.
- Einstein's Nobel prize was for explaining the photoelectric effect with quantum mechanics !
- Electric fields accelerate and multiply the primary electron in several stages. Each stage has multiplication of ~4-5.
- Typical Gain = A* $V^n \sim 10^6$- 10^7 where V is the typical voltage ~ few 1000 V.
- Time resolution < 10 ns. From varying transit time.
- Transit time can be <1 microsec
- PMT first stage is sensitive to small magnetic fields.
- Many clever geometries.



$$Q \approx \frac{-0.007\ Volt \times 5 \times 10^{-9}\ sec}{50\ Ohm} = 0.7\ pC$$

4 million electrons

These are a couple of typical single pe pulses. Red is with a short cable, and Blue is with a 40 meter RG316 cable. A cable will smooth out a pulse and reduce its amplitude. What happens to the charge ?

We are going to integrate pulses like these to get the total charge.

The time is obtained by looking for the pulse to go above threshold. Typically this is set to see a single photo-electron. We will deal with time resolution and counting later in this lecture.

R has two grandparents. Each is going to give R some money for her birthday. \$10, \$20, \$30, or \$40 with probability of 0.1, 0.2, 0.3, 0.4. What is the probability that R gets a total of \$50 ?

How do we make a method that provides an answer quickly ? It is called a characteristic function. It keeps track of numbers. Gauss described this as a clothes-line to find the number that you need.

$$\phi(s)_R = 0.1e^{10s} + 0.2e^{20s} + 0.3e^{30s} + 0.4e^{40s}$$

To get the total you multiply the char. functions.

$$\phi(s)_{R1+R2} = (0.1e^{10s} + 0.2e^{20s} + 0.3e^{30s} + 0.4e^{40s})^2$$

$$\phi(s)_{R1+R2} = 0.01e^{20s} + 0.04e^{30s} + 0.1e^{40s} + 0.2e^{50s} + 0.25e^{60s} + 0.24e^{70s} + 0.16e^{80s}$$

Notice that you can just read off the probability that R gets \$50

The characteristic function always exists and encodes all information about a probability density.

Some definitions

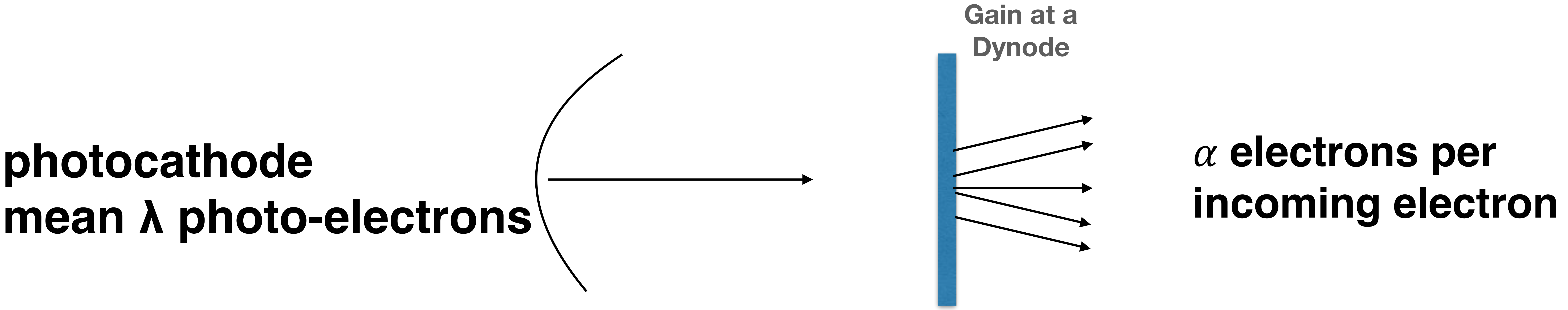
- X — Continuous random variable with probability density $P_X(x)$
- $\varphi_X(k) \equiv E[e^{ikx}] = \int_{-\infty}^{+\infty} P_X(x) e^{ikx} dx$ is the characteristic function.
- x is a realization of X over the its domain. $E[]$ is the expectation value its argument.
- Some special points
  - The characteristic function tags the probability at x with a unique identifier e^{ikx}
  - If all the probability is only at some value $x = 5$, for example then $\varphi_X(k) = e^{ik5}$
  - $\varphi_X(k=0) = 1$ since it is simply the total probability
  - $\langle x \rangle = -i\dfrac{\partial \varphi}{\partial k}(k=0)$; higher derivatives lead to high moments.
- Imagine X, Y are two random variables and $z = f(x,y)$ then the characteristics function for Z is
- $\varphi_Z(k) = \int\int e^{ikf(xy)} P_X(x) Q_Y(y) dx dy$ => for $z = x + y$ then $\varphi_Z(k) = \varphi_X(k)\varphi_Y(k)$

Basics of photomultiplier charge

Mean of λ photo electrons come from the photo-cathode to the first dynode.

Each electron generates α electrons at the first dynode.

Each subsequent stage produces gain of few electrons per incoming electrons leading to gains of ~$10^{6\text{-}7}$ after ~10 stages



We are going to assume that the stages after the first stage do not contribute to the variance of the gain. This is a good assumption as long as the gain at the first stage is reasonable > 3. We will deal with this later.

If mean of λ photons convert in a photo-sensor with a mean gain of α electrons per photon what is the distribution of the output number of electrons ?

Basically, an average of λ packets arrive each with an average of α items in each packet. What is the mean and variance of the total number of items ? How do we calculate this…

Poisson probability mass function and its Char. function

- **Incoming photoelectrons are Poisson distributed with mean λ; K is the number of electrons then**
- $P_K(k) = \dfrac{\lambda^k e^{-\lambda}}{k!}$

This has a characteristic function $\varphi_K(s) = \sum_{k=0}^{\infty} e^{isk} \dfrac{\lambda^k e^{-k}}{k!} = e^{\lambda(e^{is}-1)}$

This looks crazy, but remember that it is just a sum with each probability labeled.

- **For the first stage the gain is also Poisson distributed with mean α; L is the number of secondary electrons**

And so we have two characteristics functions: $\varphi_K(s), \varphi_L(s)$

Also recall: mean for Poisson with parameter λ: $E[k] = \lambda$

variance for Poisson with parameter λ: $E[k^2] - (E[k])^2 = \lambda$

Debatable points

- ***Why should the number of ejected electrons from the photo-cathode be Poisson distributed ?***
  - ***Recall that only a small fraction of total photons from a scintillation event are caught by the PMT, and then with an efficiency of 20-30% converted to electrons. And so this is a Poisson/ Binary process. The result is a Poisson distribution for the photo-electrons.***
- ***Why should the emission from the first dynode be Poisson distributed ?***
  - ***If the dynode is uniform then only a small fraction of total excitations from a penetrating electron results in an emitted electron. However, there could be many geometrical reasons why the dynode response is not uniform. For example, an electron could miss a dynode entirely.***
  - ***We are also going to do the calculation assuming a normally distributed gain. This is good enough for most cases when the gain from the first dynode is high.***
- ***And so both of these are reasonable assumptions for well performing detectors in normal situations.***

Simple calculation of mean and variance

- λ and α are the Poisson parameters for the incident photoelectrons and the first stage gain. Total charge random variable will be called Q
- Obviously, $\langle Q \rangle = \lambda \times \alpha$

$Q = \sum_{i=1}^{K} L_i$ where K is the random number of incident photoelectrons and L_i is the random number of secondaries for each photoelectrons.

$$\frac{Var[Q]}{\langle Q \rangle^2} = \frac{Var[K]}{\langle K \rangle^2} + \frac{1}{K}\frac{Var[L]}{\langle L \rangle^2} = \frac{1}{\lambda} + \frac{1}{\lambda \times \alpha}$$

$$Var[Q] = \lambda\alpha(\alpha + 1)$$

suppose L is actually Normally (Gaussian) distributed with parameters α (mean) and σ (standard deviation)

$$\frac{Var[Q]}{\langle Q \rangle^2} = \frac{1}{\lambda} + \frac{1}{\lambda}\frac{\sigma^2}{\alpha^2}$$

$$Var[Q] = \lambda(\sigma^2 + \alpha^2)$$

Now calculate the PDF for the total charge

- **Total charge Z (or Q) is a discrete random variable**

$$Z = \sum_{i=1}^{K} L_i$$

This is a sum of K random numbers. K itself is a random number.

There is often a naive mistake made here. The total charge is not the product of two random numbers (K, L)

We want the PDF for the random variable Z.

If $K = k$ were a fixed number then the C.F. for Z is simply $\varphi_L(s)^k$ (recall that the for sums of random numbers the characteristic functions are multiplied.)

Since K is a random number we have to take the expectation value of $\varphi_L(s)^k$ with respect to K. (law of total expectation).

$$\varphi_Z(s) = \sum_{k=0}^{\infty} \varphi_L(s)^k P_K(k)$$

C.F. expressions for Poisson and Normal gain

- **For Poisson gain with parameter α:** $\varphi_L(s) = e^{\alpha(e^{is}-1)}$
- **For Normally distributed gain $N(\mu, \sigma)$:** $\varphi_L(s) = e^{is\mu} \times e^{\frac{-s^2\sigma^2}{2}}$

- **Using the above we get the characteristic function for the total charge Z**

For Poisson gain:

$$\varphi_Z(s) = \sum_{k=0}^{\infty} (e^{\alpha(e^{is}-1)})^k \frac{\lambda^k e^{-\lambda}}{k!} = \sum_{k=0}^{\infty} e^{k\alpha(e^{is}-1)} \frac{\lambda^k e^{-\lambda}}{k!}$$

For Normal Gain:

$$\varphi_Z(s) = \sum_{k=0}^{\infty} (e^{is\mu} e^{\frac{-s^2\sigma^2}{2}})^k \frac{\lambda^k e^{-\lambda}}{k!} = \sum_{k=0}^{\infty} e^{isk\mu} e^{\frac{-s^2k\sigma^2}{2}} \frac{\lambda^k e^{-\lambda}}{k!}$$

Probability densities for Poisson and Normal gain for signal only

We have not added the noise and dark rate to the formula yet. And so this is just a middle step. To do this we just take the inverse transform of the previous C.F.

- **For Poisson gain (This is called the compound Poisson or Jumping Poisson, etc.)**

$$P_Z(n) = e^{-\lambda} + \sum_{k=1}^{\infty} e^{-k\alpha} \times \frac{e^{-\lambda}\lambda^k}{k!} \qquad \text{for n} = 0$$

$$= \sum_{k=1}^{\infty} \frac{e^{-k\alpha}(k\alpha)^n}{n!} \times \frac{e^{-\lambda}\lambda^k}{k!} \qquad \text{for n} > 0$$

- **For Normal gain the formula needs some care because charge z is now a continuous variable (can be negative). But there is no physical way for charge to be negative here. We will deal with this shortly.**

$$P_Z(z) = e^{-\lambda} + \qquad \text{for z} = 0$$

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 k}} e^{-\frac{(z-k\mu)^2}{2\sigma^2 k}} \times \frac{e^{-\lambda}\lambda^k}{k!} \qquad \text{for z} \geq 0$$

Pedestal, noise and background

- We have to now include noise and background dark rate in the description of the charge.

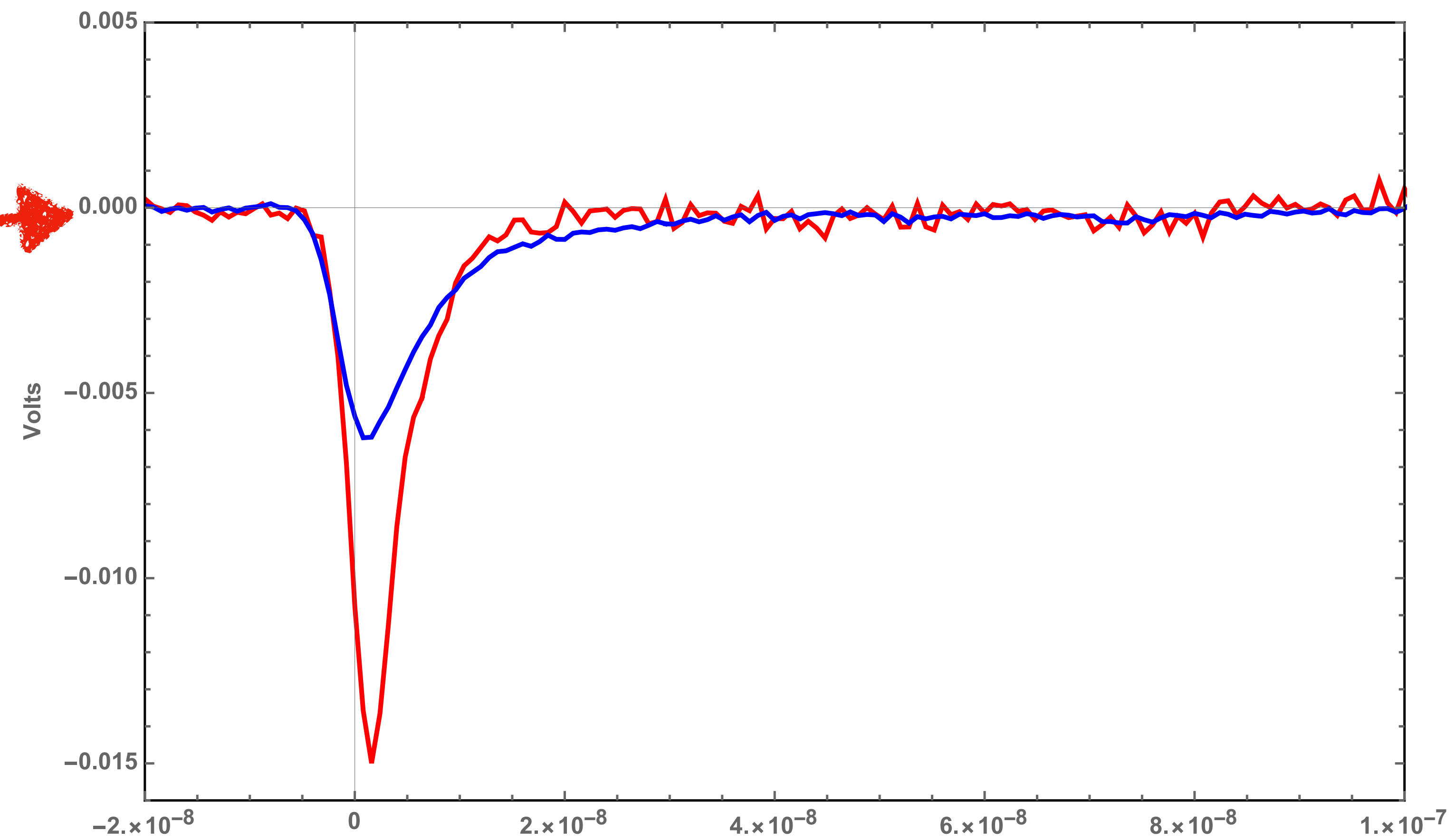
- The fluctuation of the baseline due to electronic noise and pickup noise will introduce a Gaussian width with some pedestal shift.

$$P_Q(x) = \frac{1}{\sqrt{2\pi\sigma_0^2}} \times e^{-\frac{(x-q_0)^2}{2\sigma_0^2}}$$ **this has the CF** $\varphi_Q(s) = e^{isq_0} \times e^{\frac{-s^2\sigma_0^2}{2}}$

Thermal emission of real electrons from metals inside the PMT is also a contributor. This is called dark rate and the charge is exponentially distributed, but there is a chance that there is no emission at all.

$$P_D(x) = (1-w)\delta(x) + w\theta(x)c_0e^{-c_0x}$$ **this has the CF** $\varphi_D(x) = (1-w) + w\frac{1}{1-is/c_0}$

The baseline fluctuates producing noise. When this is summed it will contribute a Gaussian distributed charge. The baseline could be shifted and has a fluctuation.



This is a typical pulse to be integrated to produce the total signal charge

Total response

To get the complete response we have to get the PDF for $Y = B + Z$ where

B is the background and Z is the signal. $B = D + Q$ as we calculated for the background.

and so

$$\varphi_Y(s) = \varphi_Z(s) \bullet \varphi_D(s) \bullet \varphi_Q(s)$$

$$\varphi_Y(s) = \left(\sum_{k=0}^{\infty} (e^{is\mu k} \bullet e^{-\frac{s^2\sigma^2 k}{2}}) \bullet \frac{e^{-\lambda}\lambda^k}{k!} \right) \times \left((1-w) + w\frac{1}{1-is/c_0} \right) \times e^{isq_0} e^{-\frac{1}{2}\sigma_0^2 s^2}$$

This is the full and complete expression for the PMT response assuming the gain is normally distributed. We will break this up in 6 pieces and analyze it for special conditions.

$$\varphi_Y(s) = \left(\sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} (e^{is(\mu k + q_0)} \bullet e^{-\frac{s^2(\sigma^2 k + \sigma_0^2)}{2}}) \right) \times \left((1-w) + w\frac{1}{1-is/c_0} \right) \times e^{isq_0} e^{-\frac{1}{2}\sigma_0^2 s^2}$$

We break this up in three cases for p.e. count: $k = 0$, $k = 1$, and $k > 1$

And additional two cases for $(1 - w)$, without dark rate

and (w), with dark rate addition.

Gaussian-modified-exponential

A normally distributed random number with an addition of an exponential random number is called an exponential-Gaussian or Gaussian-exponential.

The characteristic function is

$$\varphi_{EG}(s) = \frac{e^{isq_0} e^{-\frac{1}{2}\sigma_0^2 s^2}}{(1 - is/c_0)}$$

The PDF that corresponds to this is

$$P_{EG}(x) = \frac{c_0}{2} e^{\frac{c_0^2\sigma_0^2}{2}} e^{-c_0(x-q_0)} Erfc\left[\frac{1}{\sqrt{2}}\left(c_0\sigma_0 - \frac{x-q_0}{\sigma_0}\right)\right]$$

Recall that c_0 is the constant for dark rate, σ_0 is the std. dev. of the pedestal and q_0 is the pedestal.

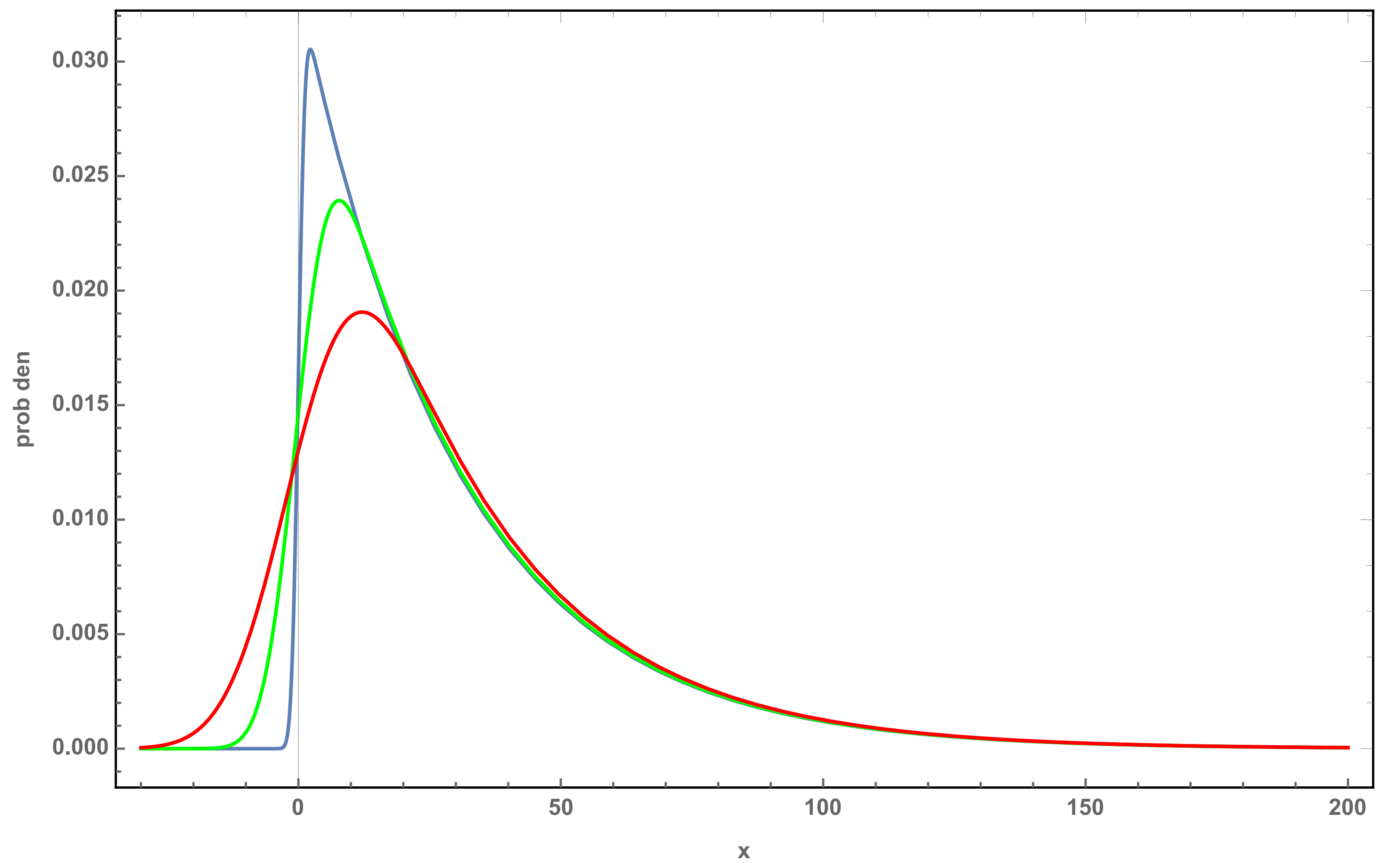
$Erfc[x]$ is the complement of the error fuction.

$$Erfc[x] = 1 - Erf[x] = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2}\, dt$$

When $c_0\sigma_0$ is small the *Erfc* acts like a step function. Some care is needed in calculation in case of negative or very large arguments.

The Mean of the PDF is $(q_0 + 1/c_0)$

The Variance is $(\sigma_0^2 + 1/c_0^2)$



Couple of definitions

Let's define some PDFs to get a compact expression

The Normal PDF

$$N(x:\mu,\sigma^2)=\frac{1}{\sqrt{2\pi\sigma^2}}\ \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

The Exponential modified Normal PDF

$$E_N(x:\mu,\sigma^2,\lambda)=\frac{\lambda}{2}e^{\frac{\lambda^2\sigma^2}{2}}e^{-\lambda(x-\mu)}Erfc\left[\frac{1}{\sqrt{2}}\left(\lambda\sigma-\frac{x-\mu}{\sigma}\right)\right]$$

μ: mean of the Gaussian

σ^2: variance of the Gaussian

λ: exponential decay parameter

All terms broken out for the characteristic function. Check that when s=0 the sum adds to 1

	Terms	$(1-w)\times$	$w\times$
no signal	$e^{-\lambda}\times$	$e^{isq_0}e^{-\frac{s^2\sigma_0^2}{2}}$	$e^{isq_0}e^{-\frac{s^2\sigma_0^2}{2}}\times\frac{1}{1-is/c_0}$
single pe	$\lambda e^{-\lambda}\times$	$e^{is(\mu+q_0)}e^{-\frac{s^2(\sigma^2+\sigma_0^2)}{2}}$	$e^{is(\mu+q_0)}e^{-\frac{s^2(\sigma^2+\sigma_0^2)}{2}}\times\frac{1}{1-is/c_0}$
many pe	$\sum_{k=2}^{\infty}\frac{e^{-\lambda}\lambda^k}{k!}\times$	$e^{is(\mu k+q_0)}e^{-\frac{s^2(\sigma^2 k+\sigma_0^2)}{2}}$	$e^{is(\mu k+q_0)}e^{-\frac{s^2(\sigma^2 k+\sigma_0^2)}{2}}\frac{1}{1-is/c_0}$
		no dark current	with dark current

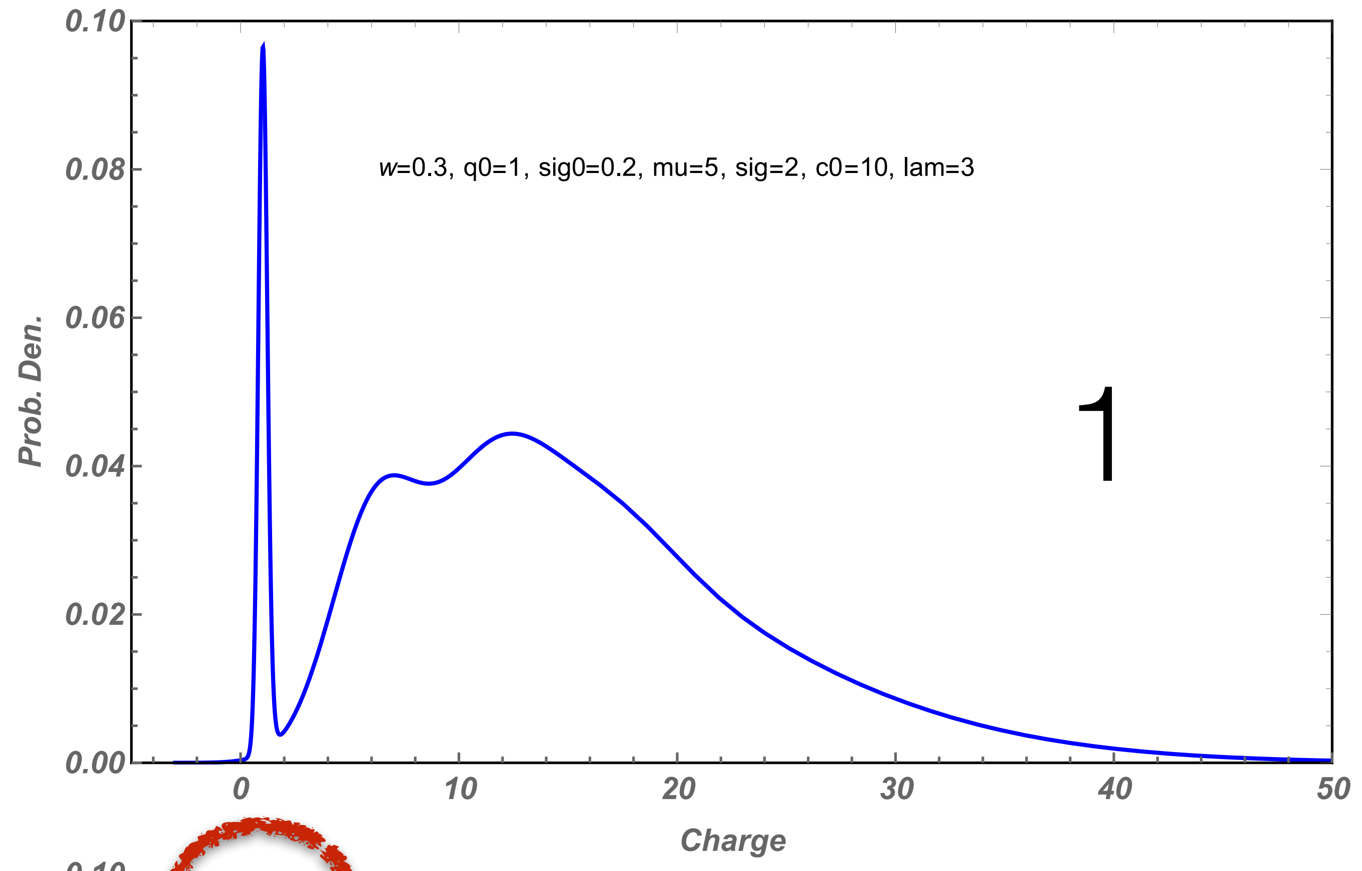
all these are to be added together. When transformed to PDF, each term will convert to a normal PDF or an exponential-normal PDF.

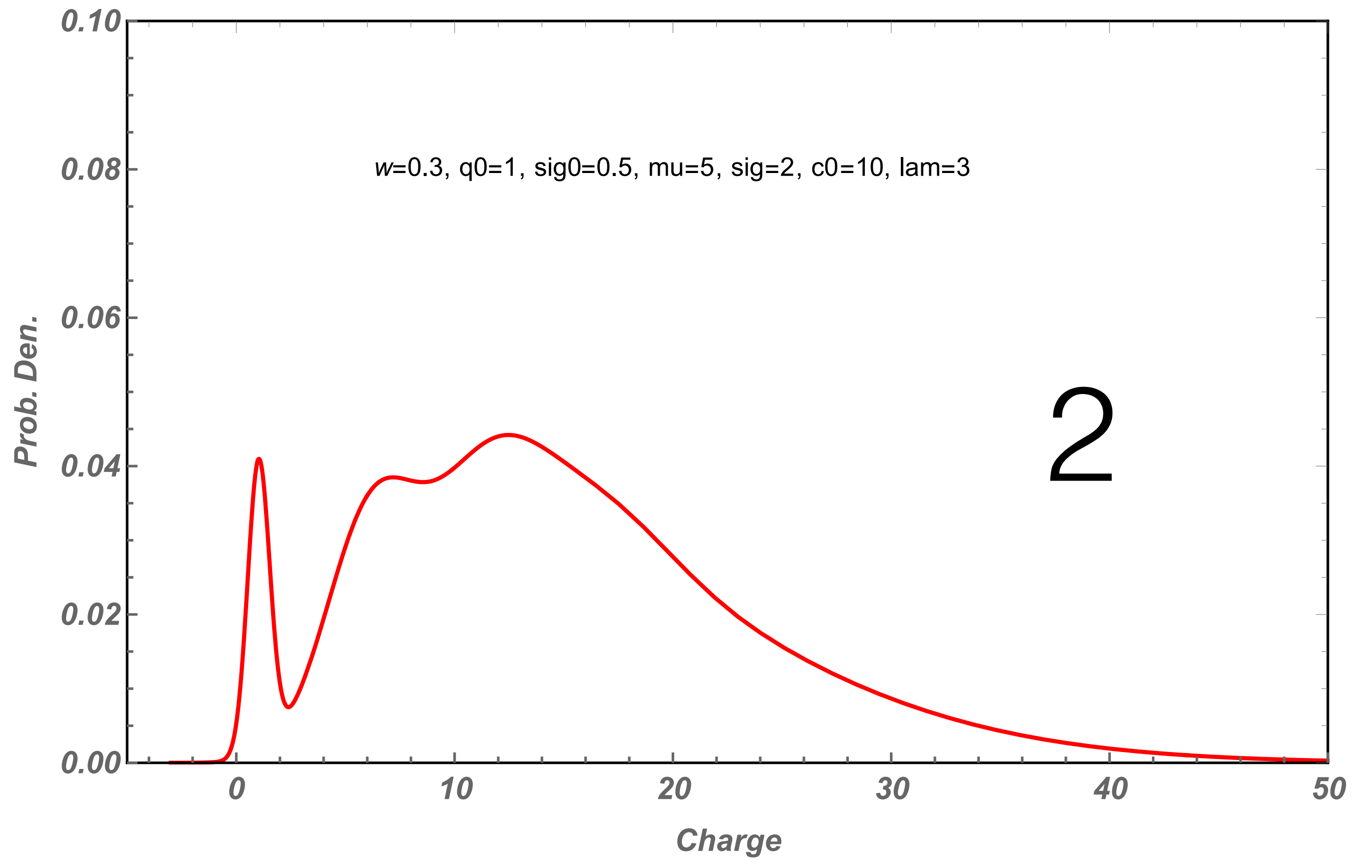
All terms broken out for the PDF in compact notation

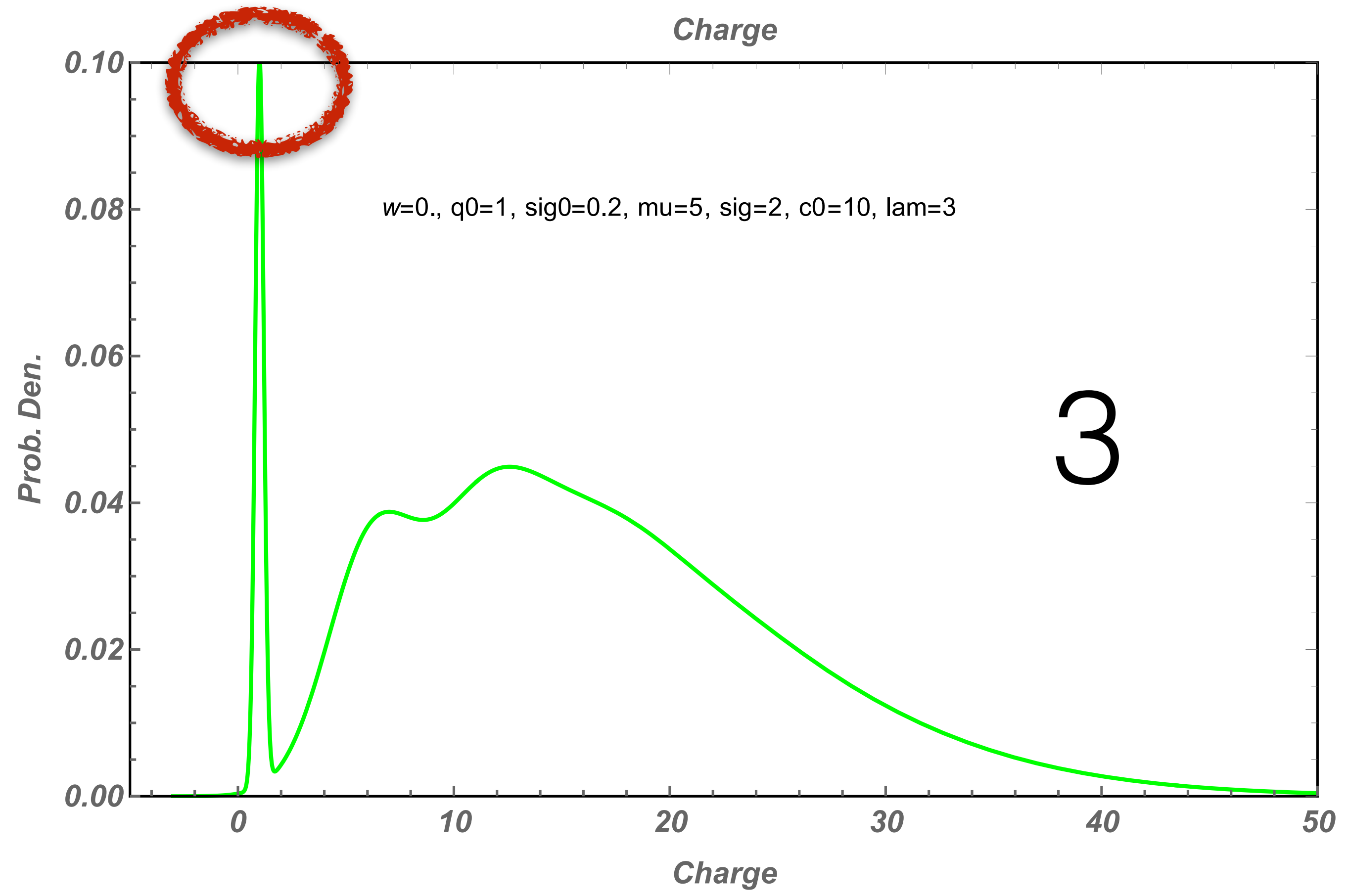
	$Terms$	$(1-w)\times$	$w\times$
no signal	$e^{-\lambda}\times$	$N(x:q_0,\sigma_0^2)$	$E_N(x:q_0,\sigma_0^2,c_0)$
single pe	$\lambda e^{-\lambda}\times$	$N(x:\mu+q_0,\sigma^2+\sigma_0^2)$	$E_N(x:\mu+q_0,\sigma^2+\sigma_0^2,c_0)$
many pe	$\sum_{k=2}^{\infty}\frac{e^{-\lambda}\lambda^k}{k!}\times$	$N(x:\mu k+q_0,\sigma^2 k+\sigma_0^2)$	$E_N(x:\mu k+q_0,\sigma^2 k+\sigma_0^2,c_0)$
		no dark current	with dark current

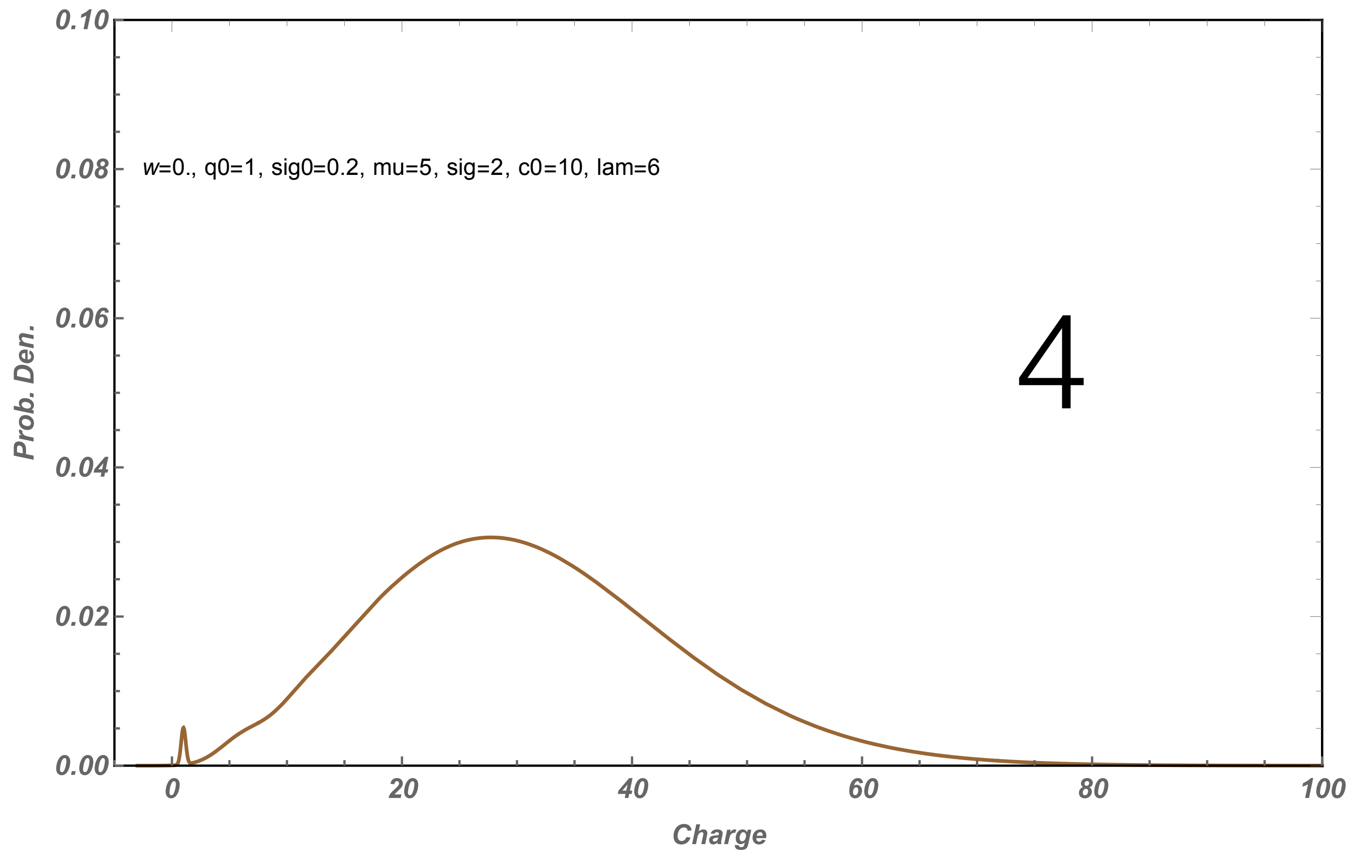
all these are to be added together to get the full PDF.
The sum is applied across the row.

Plot some examples







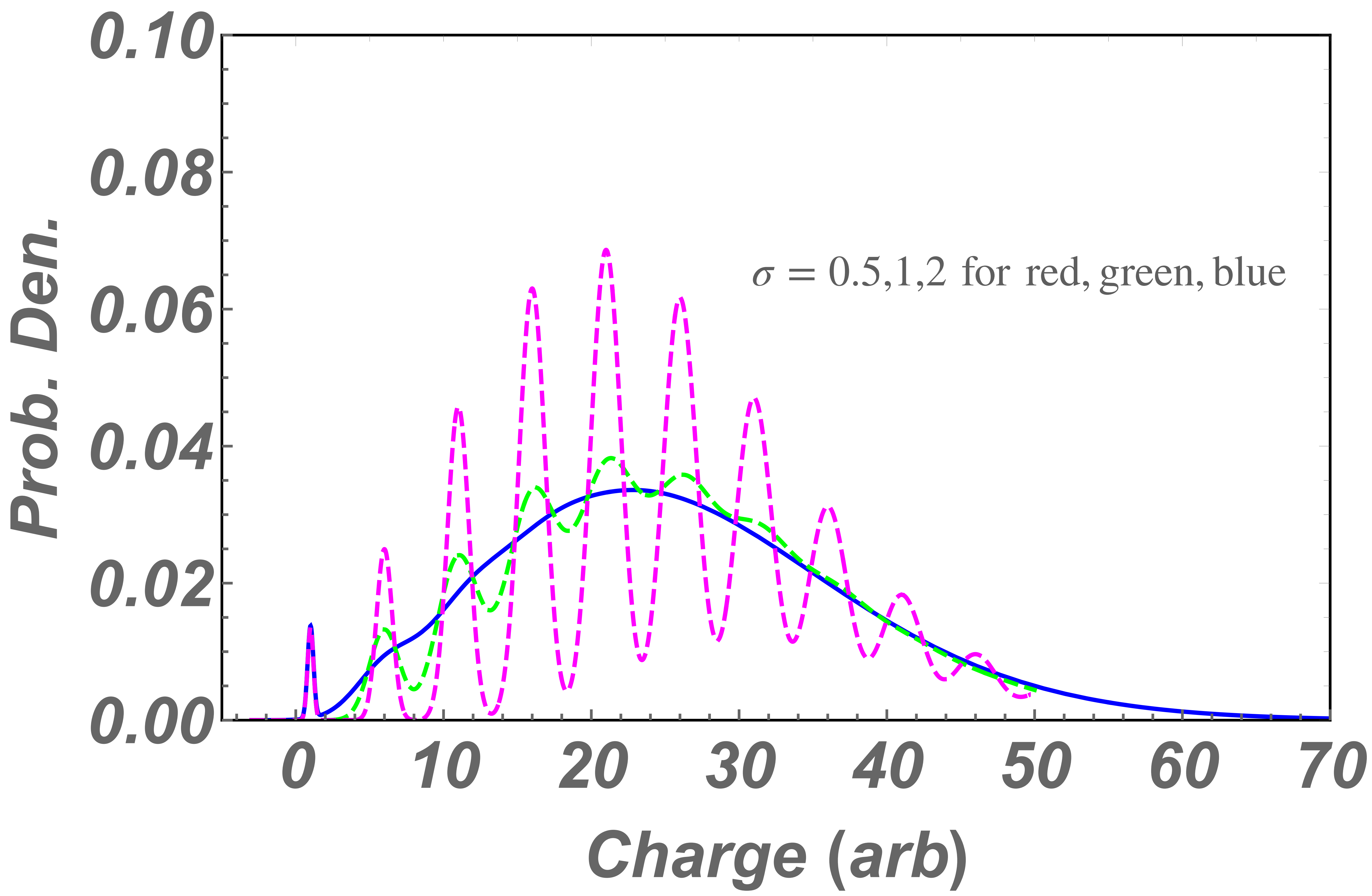


plot	λ	w	q_0	σ_0	μ	σ	c_0
1	3	0.3	1	0.2	5	2	10
2	3	0.3	1	0.5	5	2	10
3	3	0	1	0.2	5	2	10
4	6	0	1	0.2	5	2	10

with dark current, narrow/wide pedestal

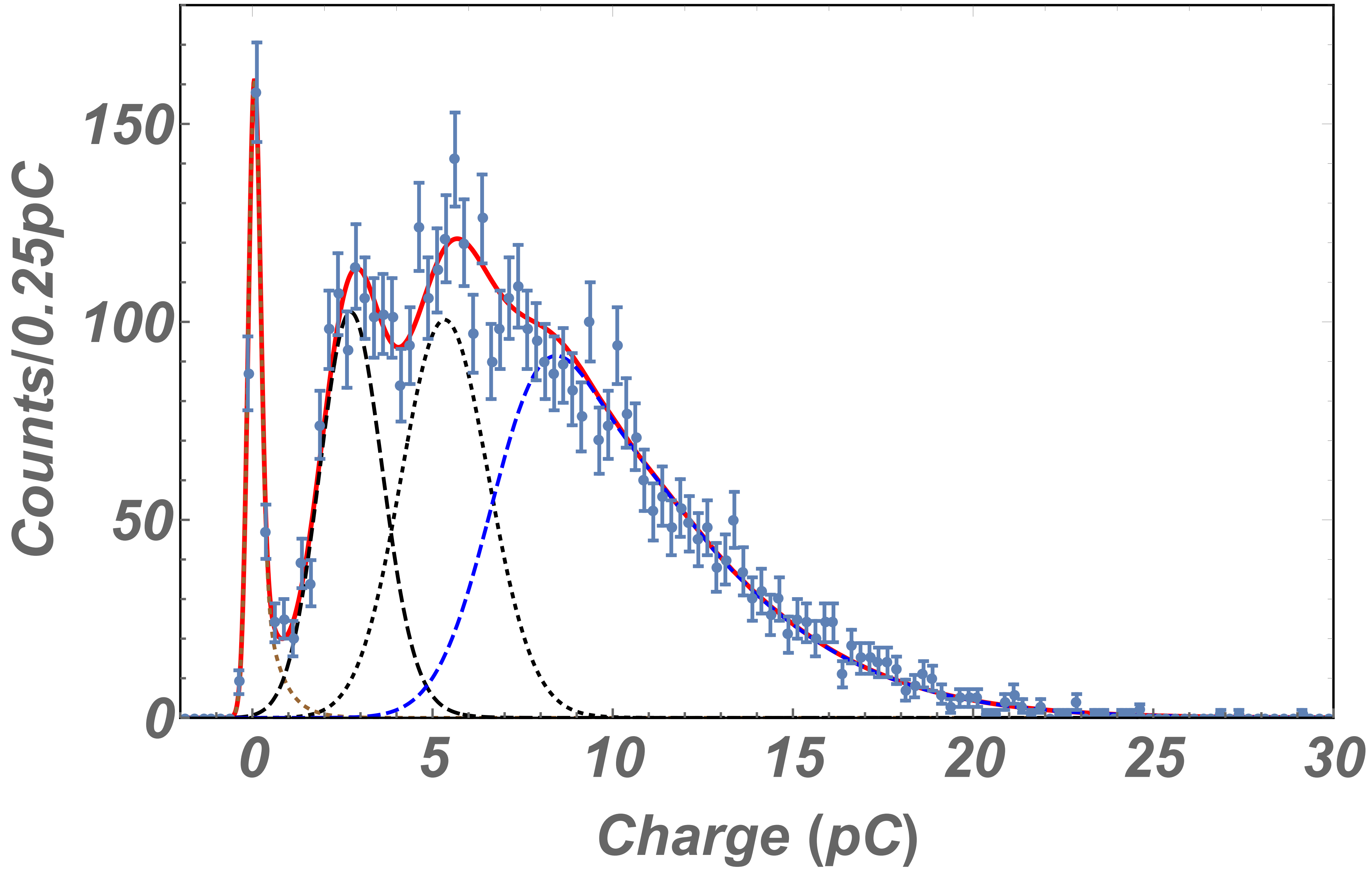
no dark current, less/more p.e.

Example where the gain has a narrow variance; and individual photoelectrons can be separated and counted.



The probability of dark pulses is set to zero,w=0. The values of other parameters are set to beλ=5(mean photo-electrons),q0=1(baseline shift),σ0=0.2(baseline fluctuation),μ=5(mean gain),σ=0.5, 1, 2(gain fluctuation) for the magenta-dashed, green-dashed, and blue-solid curves, respectively

A detailed fit to data.



The data contains 5000 pulses. The 7 parameter fit resulted in a reduced χ2/DOF=1.08 for 77 DOF. The parameters extracted were the baseline shift q0=0.044±0.031pC, the baseline fluctuation σ0=0.169±0.018pC, the dark rate probability w=0.30±0.12, the dark rate exponential parameter c0=2.1±2.1pC−1, the mean gain μ=2.59±0.06pC,the gain fluctuation σ=0.826±0.057pC and mean number of photo-electrons λ=2.69±0.10. The figure shows the best fit curve (red) as well as individual components of the spectrum: the charge spectrum for no photo-electron emission (brown dashed) shows a small dark rate component as a tail on the positive side; the single photo-electron spectrum (black dashed), a two photo-electron spectrum (black dotted), and greater than two photo-electrons (blue dashed) are shown separately

- ***I have provided some data from a HPK R5912-mod 10 stage PMT. It has very low dark rate. An LED was flashed thru a fiber at the PMT.***
- ***Data was taken with a scope and so the pedestal noise is very low also.***
- ***There has been no selection of data. 5000 pulses were integrated in a fixed time interval and the LED pulse charge plotted with no cuts.***
- ***Homework: fit the 6 spectra I have provided.***
- ***https://www.phy.bnl.gov/~diwan/software/pmt-spec-code-and-data.tar***

How to calculate the gain distribution from a single photo-electron ?

Or the statistics of electron multiplication.

$$P(1) = 1 \xrightarrow{P_K(k)} P_1(k) \xrightarrow{P_K(k)} P_2(k) \xrightarrow{P_K(k)} P_3(k) \Longrightarrow \xrightarrow{P_K(k)} P_{Nstage}(k)$$

- **The solution is a couple of iteration equations. These can be applied to any gain PDF.**

1. $P_{stg}(0) = P_1(P_{stg-1}(0)) = e^{-\lambda}e^{\lambda \times P_{stg-1}(0)}$ **.... Assumed Poisson stage gain for each stage**

2. $$P_{stg}(n) = \frac{\lambda}{n}\sum_{i=0}^{n-1}(n-i)P_{stg}(i) \times P_{stg-1}(n-i)$$

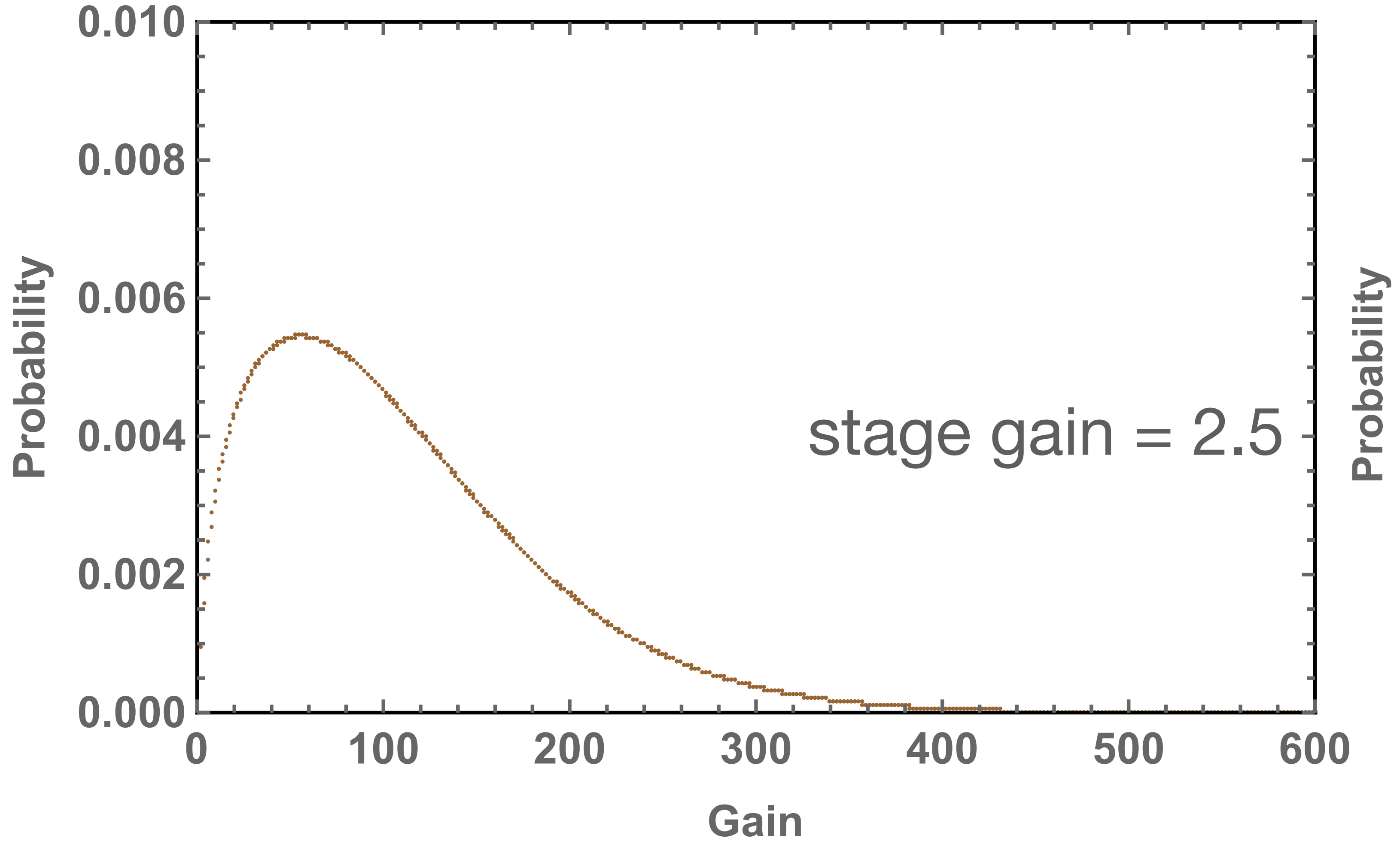
- **To perform the calculation start with the evaluation at 0 and proceed to n=1, 2, 3, for each stage; error is introduced because for each stage we must cut off n at a finite value. It can be kept small.**

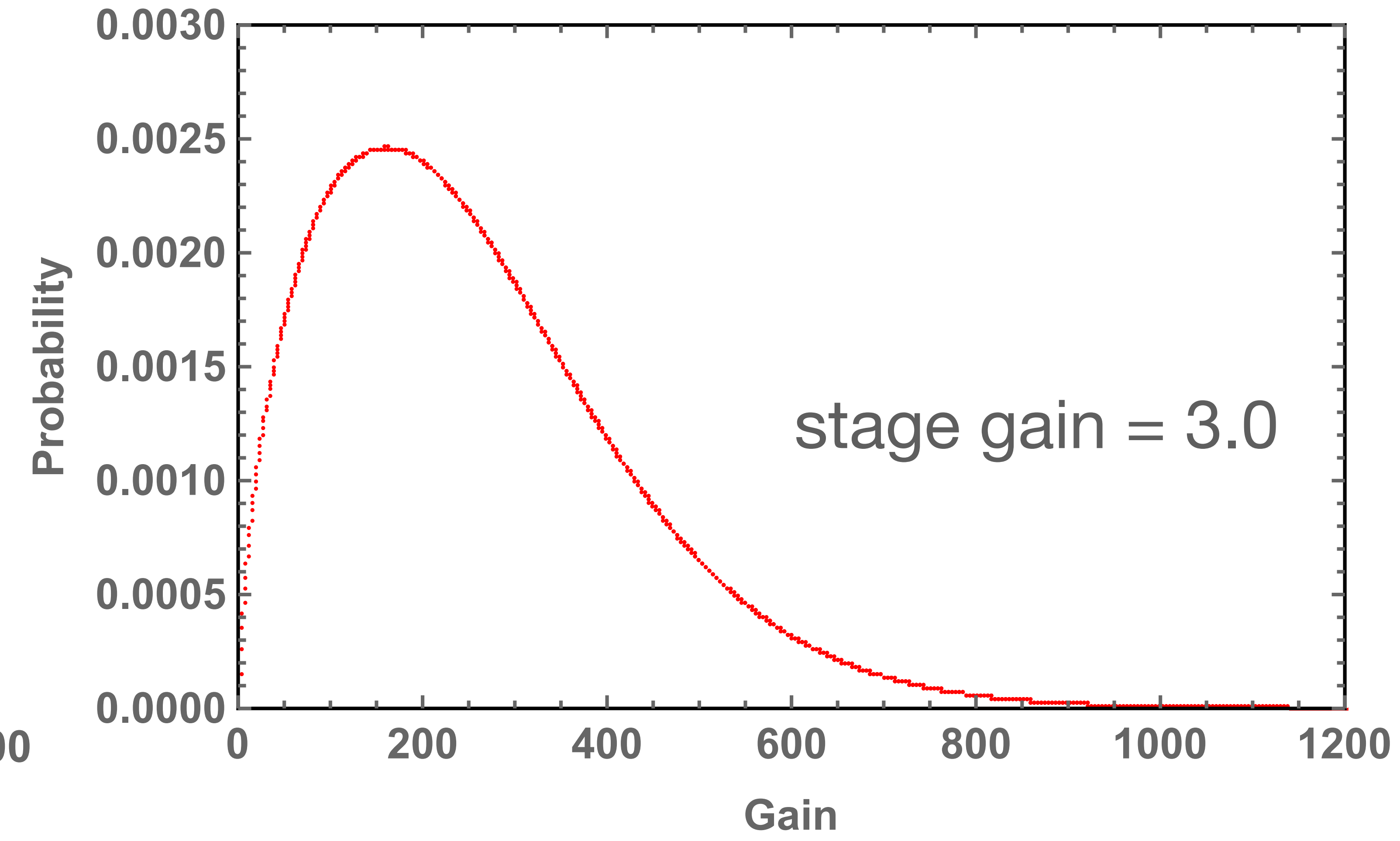
- **There is a lot of literature on this, even recently, but there is also a lot of confusion over nomenclature.**

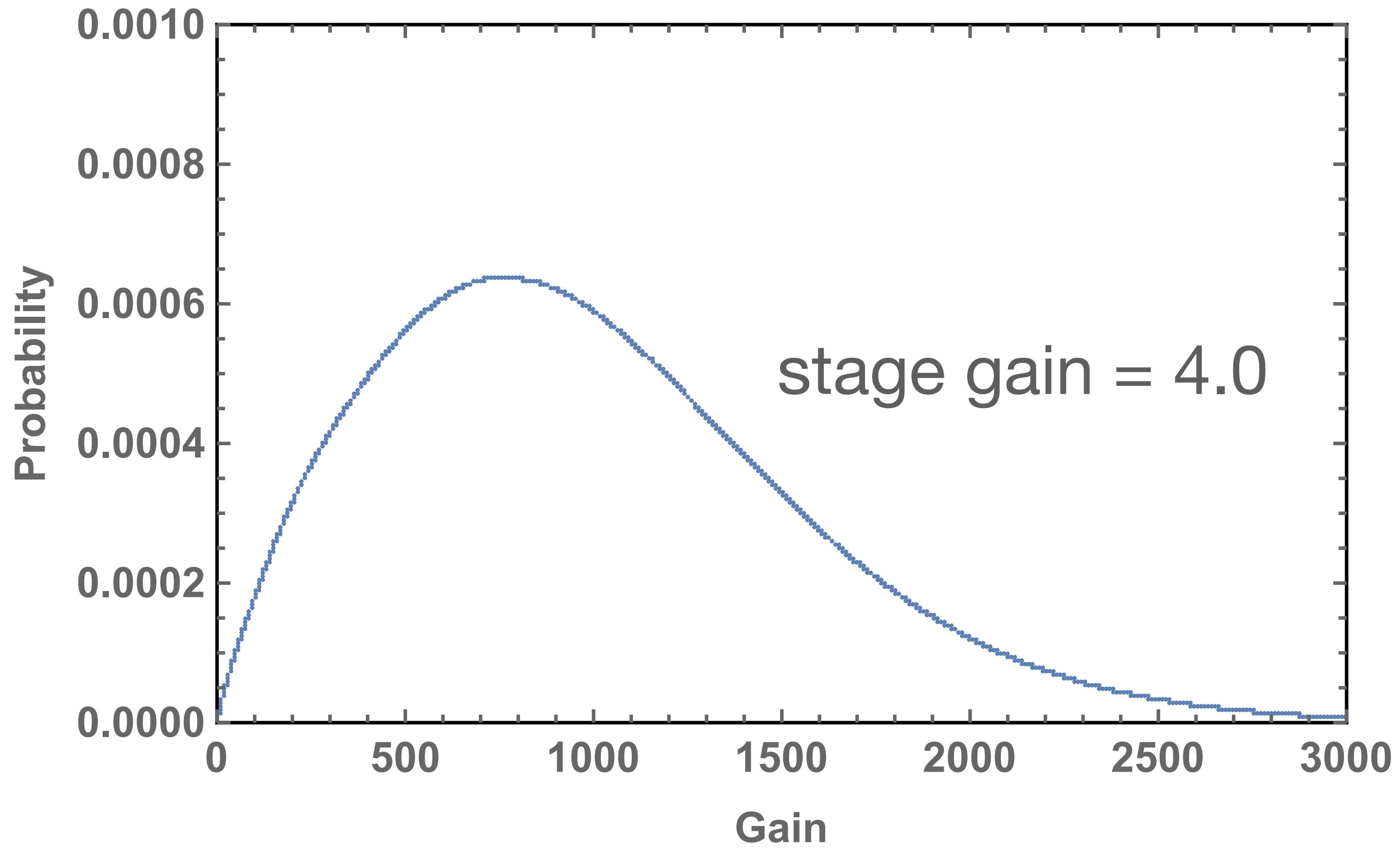
- **Ref: Lombard and Martin, Review of Scientific Instruments, Vol 32, Feb 1961**

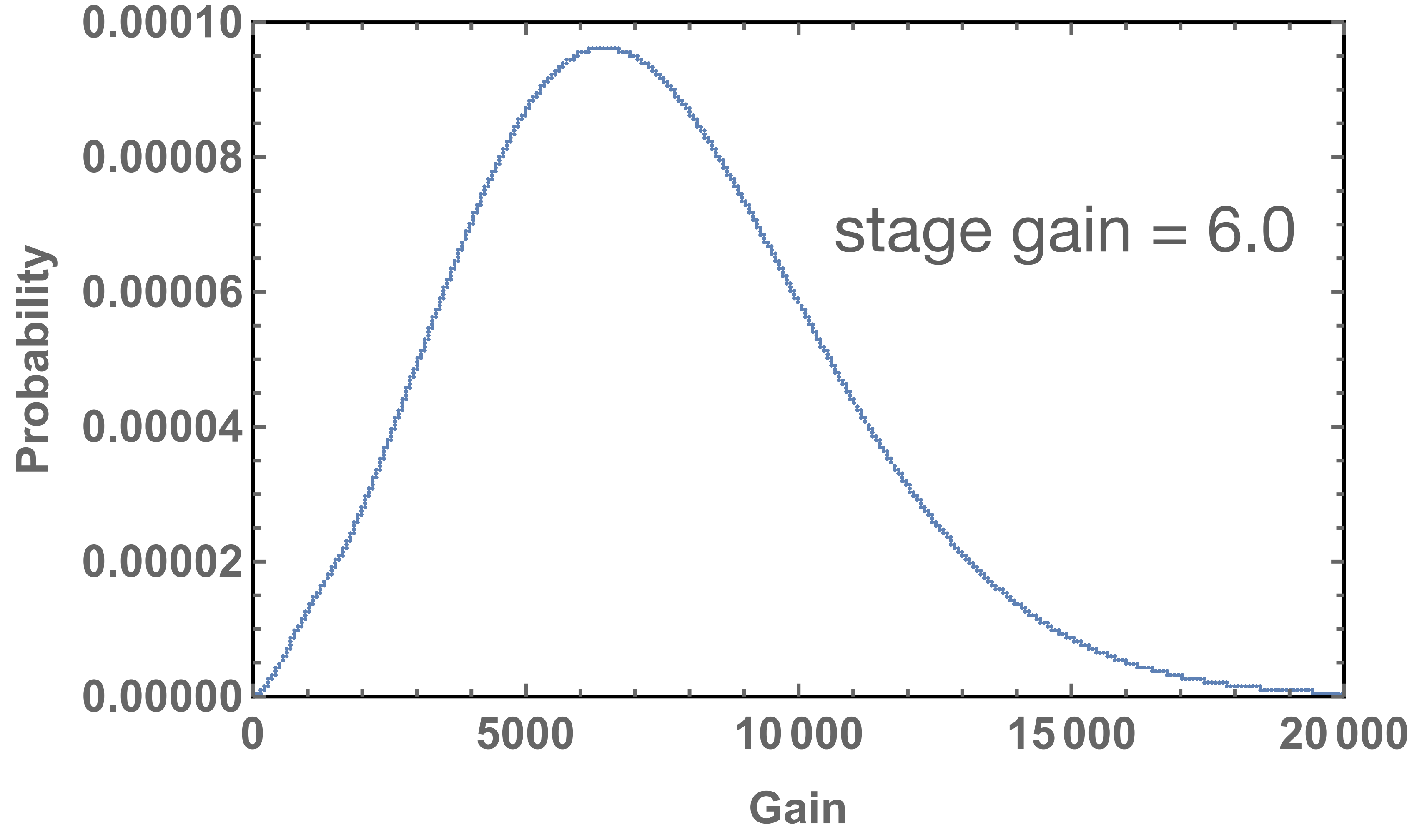
Gain calculation for single pe using Poisson gain at each stage

Only up to stage 5.









- For modest stage gains there is a substantial possibility of getting a zero (or no signal) $>e^{-\lambda}$
- To get close to a Normal PDF, need to get a first stage gain of > 4

Time probability distribution and resolution in scintillation counters.

- ***Scintillation counters often are coupled to photo-multipliers.***
- ***Scintillation photons have an exponential probability density distribution in time.***
- ***A collection of n scintillation photons, each independent, will form a time series. Such a series will form a signal from a photo-multiplier tube.***
- ***For our purposes here, we will assume that the detector has fast time response, and low noise. When we measure the time of the pulse, it is essentially the time of the first photo-electron in the time series of n photo-electrons***
- ***Time resolution from such a device will have 3 components: 1) fluctuation in the time of the photon, 2) fluctuation from transit time spread, 3) fluctuation from the gain process in the PMT.***
- ***For the time PDF, We will combine (2) and (3) into a single Normal distribution.***

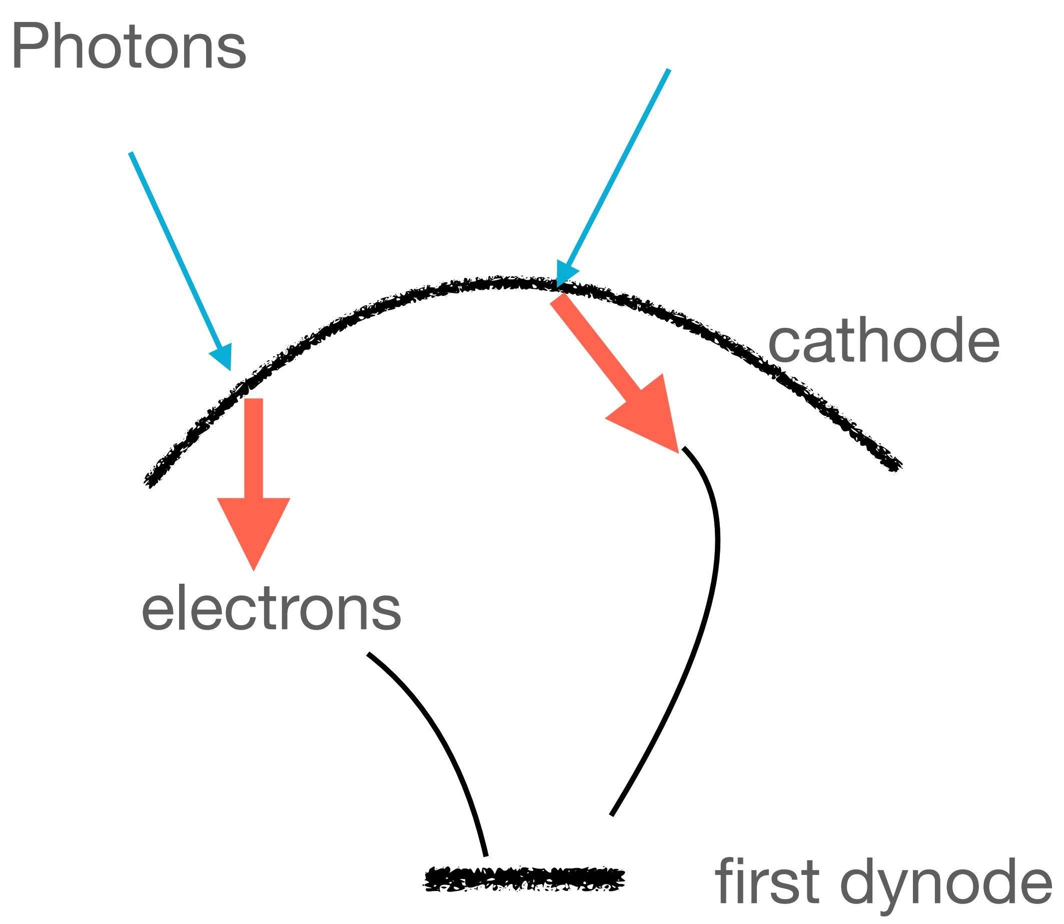


Photo-electrons from the cathode will arrive at different times at the first dynode where they are multiplied

Time resolution formula.

Proof can be found in my detailed notes.

- If $P_X(t)$ is the PDF of events in time, then $F_X(t) = \int_0^t P_X(x)dx$ is the cumulative probability that $(X \leq t)$
- If there are n events then the PDF for the minimum is given by

  $$P_T(t) = nP_X(t)[1 - F_X(t)]^{n-1}$$

  for an exponential PDF for X, $P_T = \frac{n}{\tau} e^{-nt/\tau}$

  The time resolution improves as $\sigma = \tau / n$

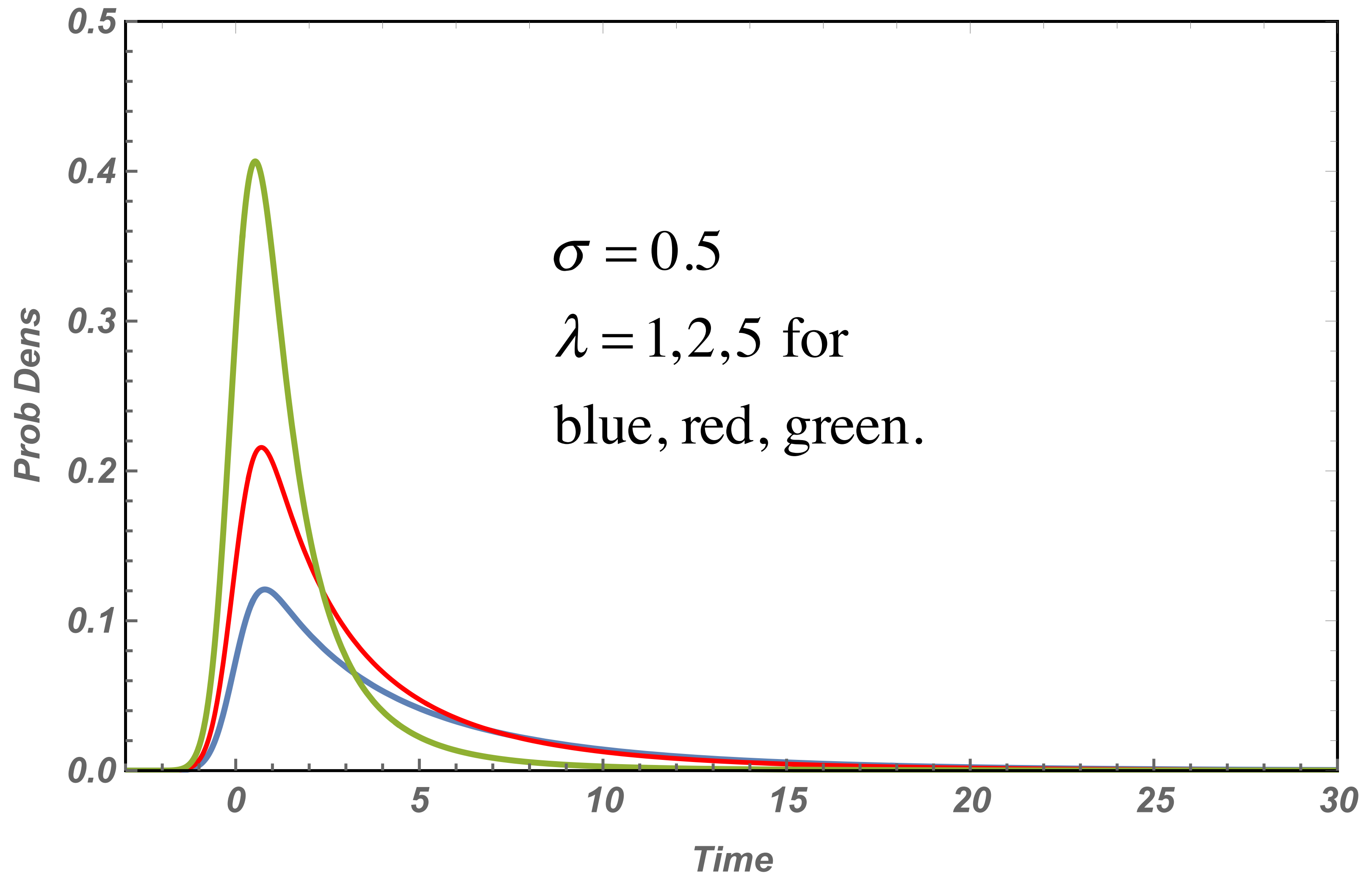
  We have to convolute this with Gaussian resolution due to noise and transit time spread.

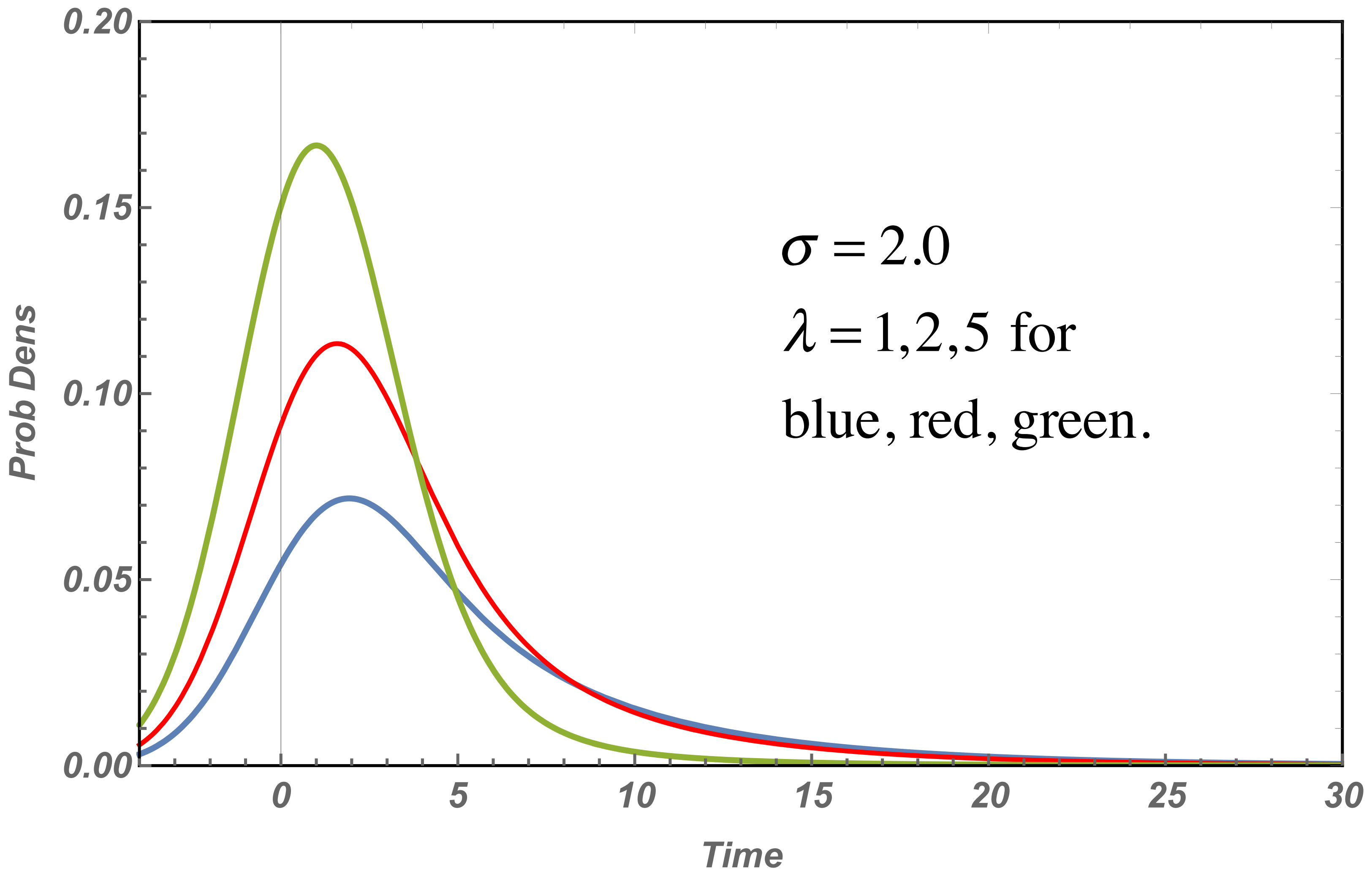
  We also have to include the Poisson probability for n events.

The answer for the distribution of first photon time including the effects of noise is now given by

$$\text{Probability(t or no-t)} = \left[\begin{array}{ll} e^{-\lambda} & \text{for no definite event time.} \\ \sum_{n=1}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} \times E_N(t:0,\sigma^2,n/\tau) & \text{when there is a time} \end{array} \right.$$

We plot this for some examples. We set $\tau=6$; $\sigma=0.5$, 2 and $\lambda=1, 2, 5$



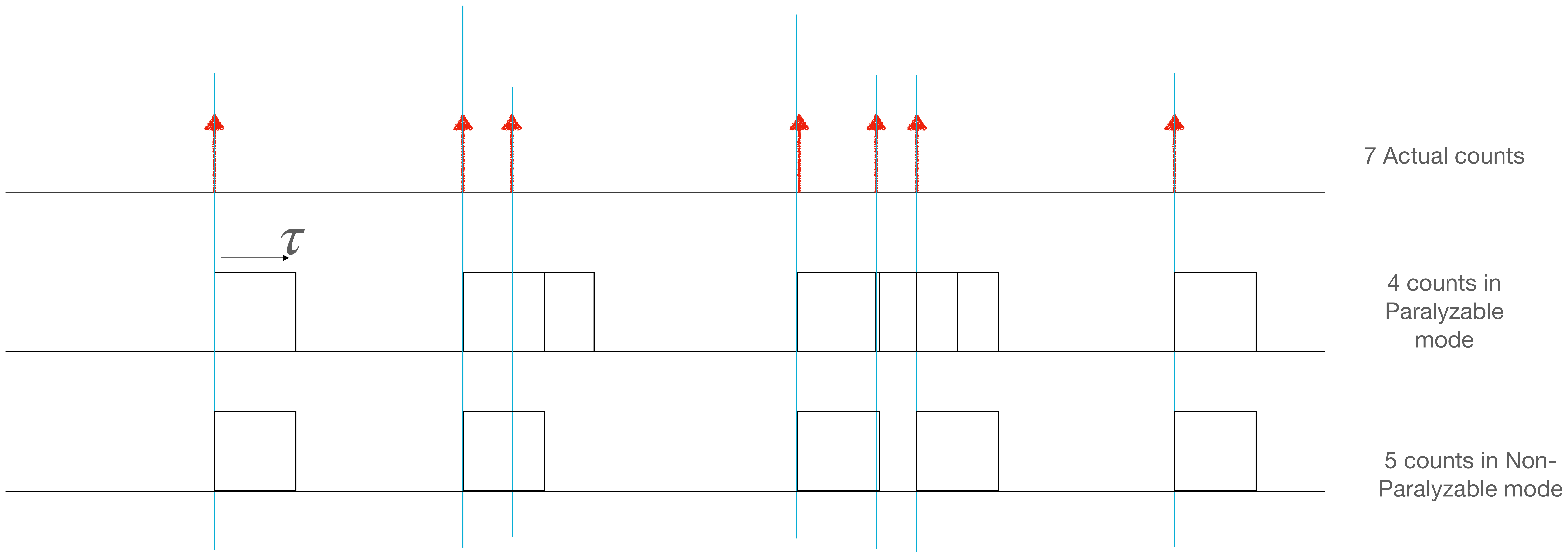


Notice how there is a tail for low number of photons. We leave it to the reader to calculate the mean and variance of these.

Random processes and Pulse Counting in typical experiments

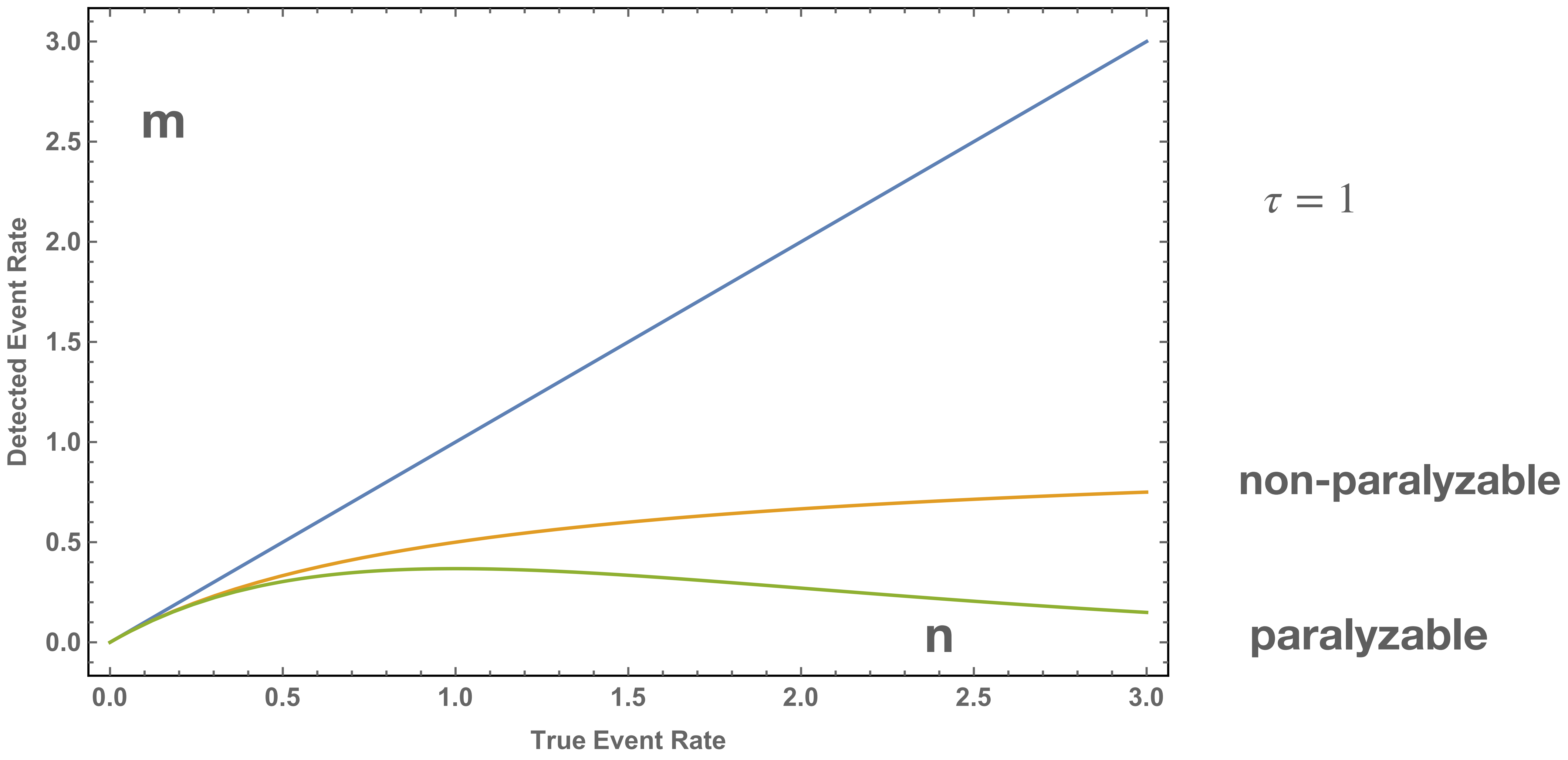
Using a thresholding algorithm

- In typical experiments, such as accelerator or underground radiation counters, we count events after some processing.
- A continuous measurement in time may contain a signal as pulses, and some random noise. A simple pulse finding method will count a signal pulse when the measurement makes a transition from low to high across a given threshold.
- The statistics of such a thresholding filter are non-trivial, but calculable under some assumptions.
- We will do a few calculations for reference. This is actually a subject of interest in both the statistics and electrical engineering communities.



- ***First we will do some simple stuff covered in textbooks (see Glen F. Knoll, for example). Feller (1948) Courant Lectures has a complete description. Feller originally identified these are Type 1, and Type 2 processes.***
- ***Let's imagine that there are n real events in a unit of time.***
- ***The pulse that is produced and used for counting has a holding time (width) of τ .... And m pulses are actually counted.***
- ***The relation between $m < n$ depends on the nature of the pulses: if the filter produces pulses of arbitrary length (thus not counting events when the events are closer than the resolving time) then it is called paralyzable. If the filter produced pulses of fixed length no matter what then it is called Non-paralyzable.***
- ***The real situation can be more complex and could be based on the shape of the event pulse. But these are useful extremes to analyze.***

Non-paralyzable (Type 1) and Paralyzable (type 2) dead time loss



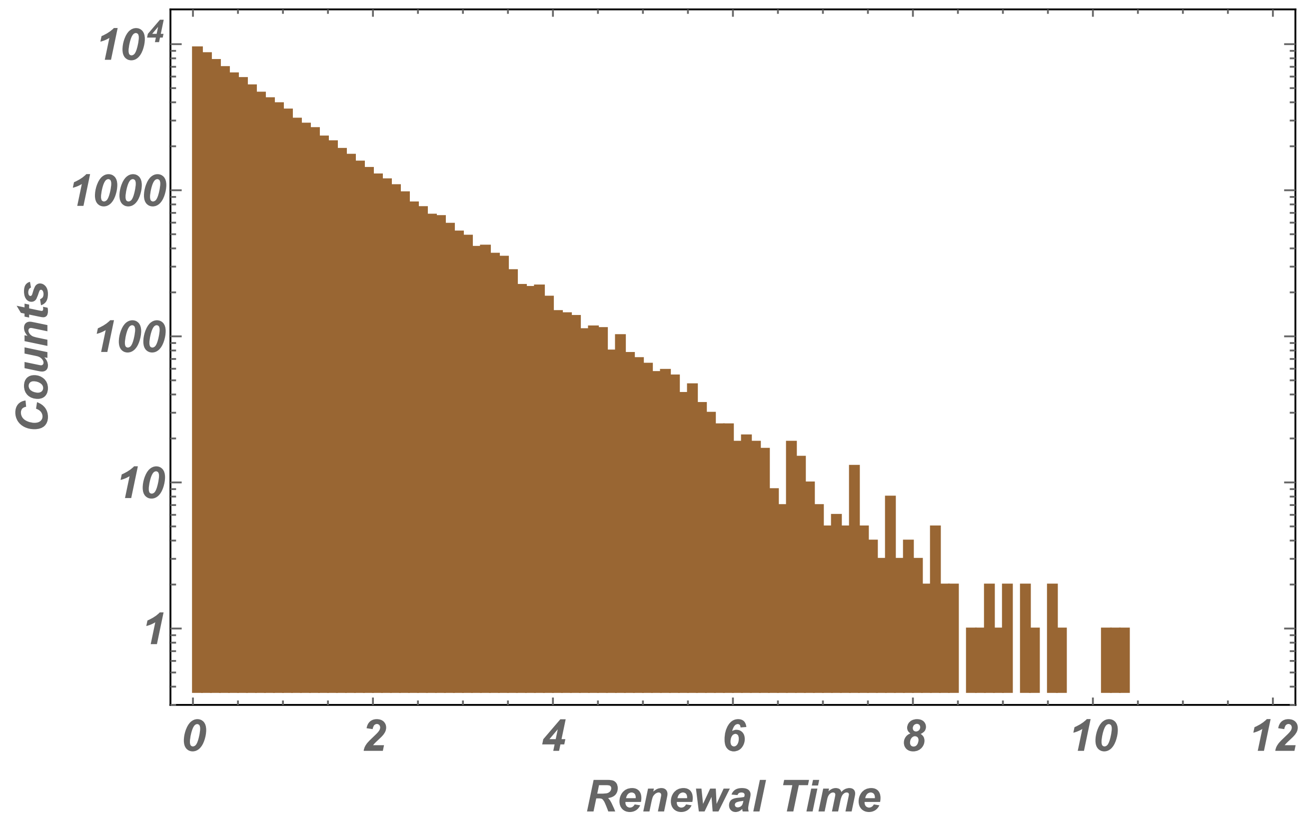
- For a randomly distributed event rate of r what is the probability density function for the time intervals between events ? $P_{gap}(t) = re^{-rt}$
- Non-paralyzable dead time formula: $m = \dfrac{n}{(1 + n\tau)}$
- Paralyzable dead time formula: $m = ne^{-nt}$

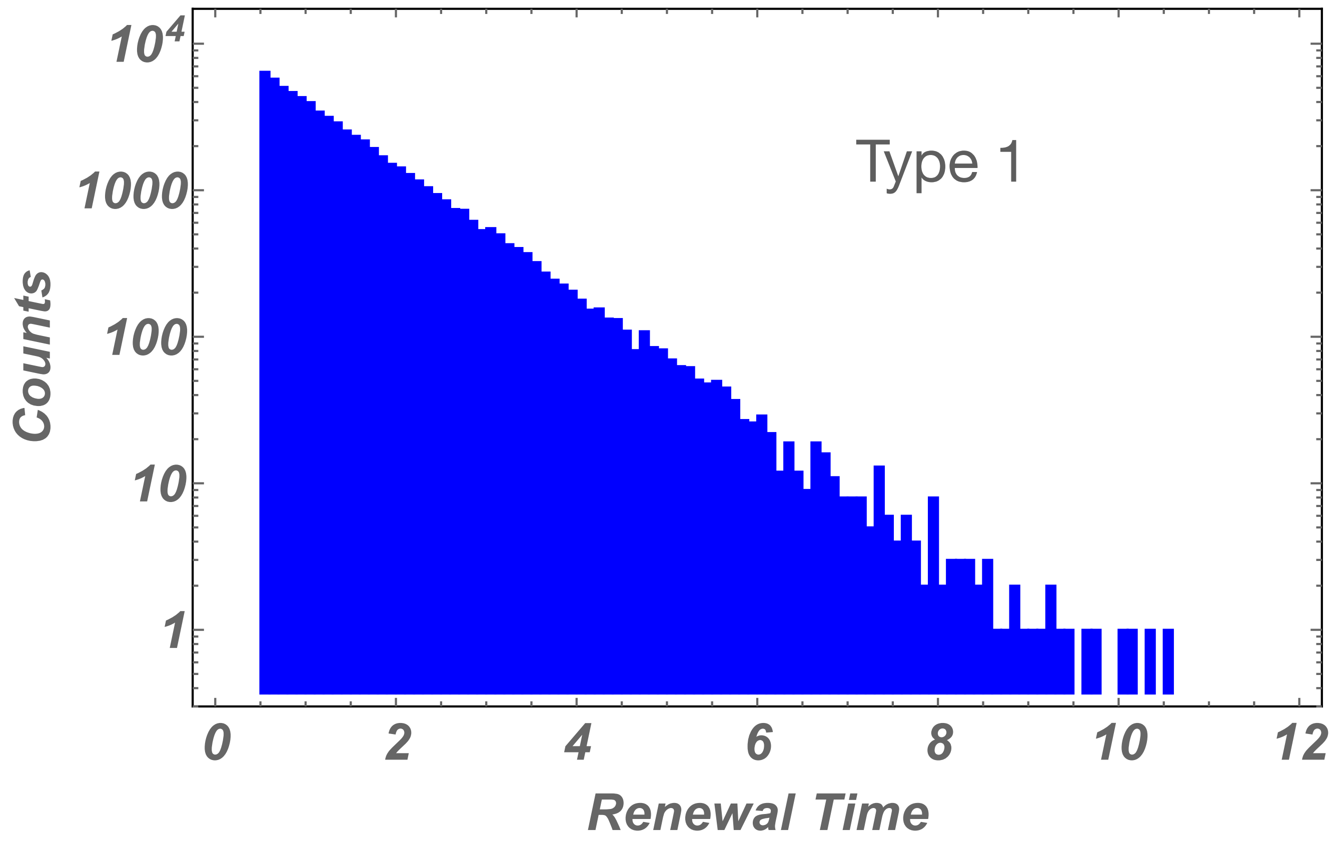
More generally: what can we monitor to make sure the detector and pulse counting system are working as intended ?

- Clearly we cannot monitor the ratio m/n since we do not know n
- We could make a plot of the rate m and make sure it stays constant if that is what we expect in a data taking run.
- We could plot the width of the pulses, the minimum will be τ and then it will drop with some PDF. This PDF is known, what is it?
- We could plot the time between the edges of the pulses ? This would be called the renewal time distribution. This is actually the best monitor for each channel and also for the trigger. It should be exponential except for small times where it will deviate because of dead time.
- What if the rate n is not random in the sense of a Poisson process ? What if some of the rate is correlated. e.g, the pulses are more probable if they are close to each other.

Simulation of renewal times and pulse widths

1 Hz,   holding time = 0.5 sec





- **You can find the proof of how to analytically calculate the renewal time PDF from Feller.**
- **You can also take a lecture on *Renewal theory* from MIT opencourses.**
- **Type 2 renewal times get pushed out longer because of the multiple overlapping pulses.**
- **Plot on right: The minimum pulse width is 0.5, When there is another pulse within 0.5 sec, this extends to 1 sec, a third pulse has to be present within the second 0.5 sec internal. Therefore there will be lower probability >1 sec.**

Conclusion.

- ***We derived the full expression for the charge spectrum from a typical photo-multiplier.***
- ***The expression has parameters for the pedestal, width of the pedestal, the dark current, and the signal. Expression assumes that gain has a Gaussian PDF.***
- ***The method for deriving the expression is very general, and can be applied to any detector system with appropriate changes.***
- ***We also understood how to obtain the PDF for gain using similar techniques.***
- ***We examined time resolution as well as the statistics of counting from detectors.***
- ***I wrote a paper last year with more detail. https://arxiv.org/abs/1909.05373***
- ***Also look at https://www.phy.bnl.gov/~diwan/#photo for much more details.***