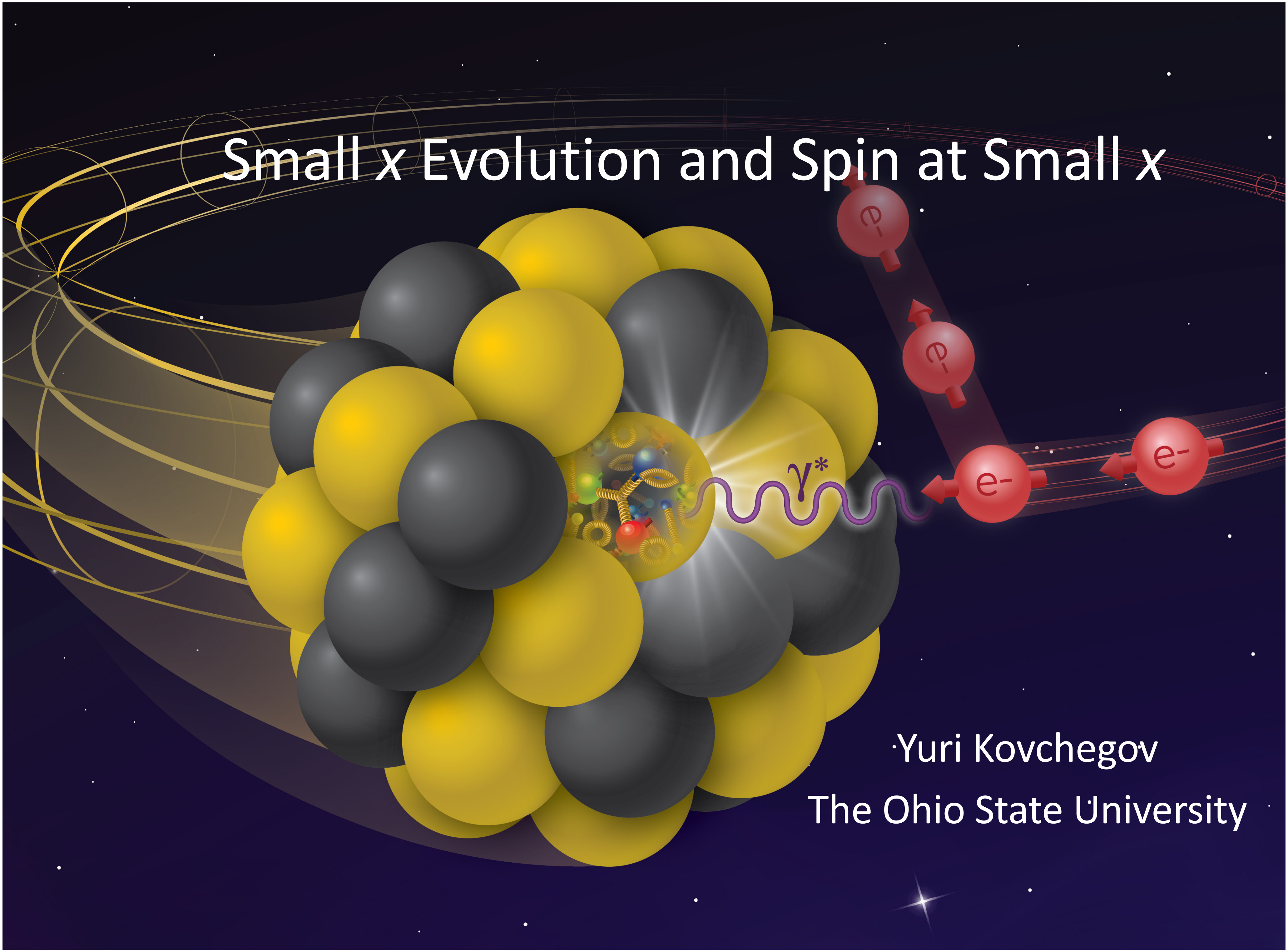
Small x Evolution and Spin at Small x



Yuri Kovchegov

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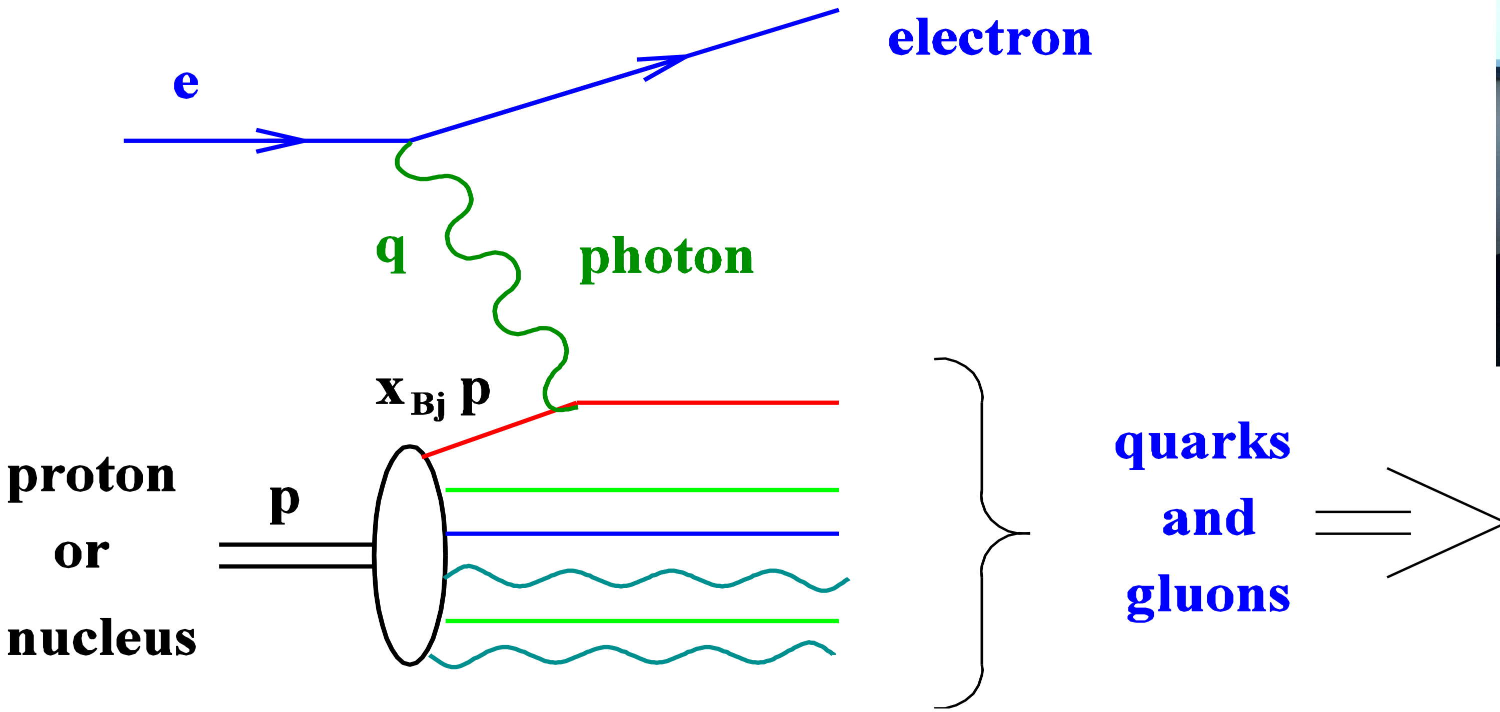
Outline

- Classical small-x physics:
  - DIS in the dipole picture, Glauber-Gribov-Mueller formula
  - Black disk limit, parton saturation, saturation scale
  - McLerran-Venugopalan model, saturation scale for a nucleus
- Nonlinear small-x evolution:
  - Non-linear BK and JIMWLK evolution equations
  - Solution of BK and JIMWLK equations, unitarity, energy dependence of the saturation scale
  - Map of high-energy QCD
- Proton spin at small x:
  - A new small-x evolution equation for helicity
  - Its solution and some phenomenology
- Conclusions

Quasi-Classical Approximation: Multiple Rescatterings

Kinematics of DIS



- Photon carries 4-momentum q_μ , its virtuality is

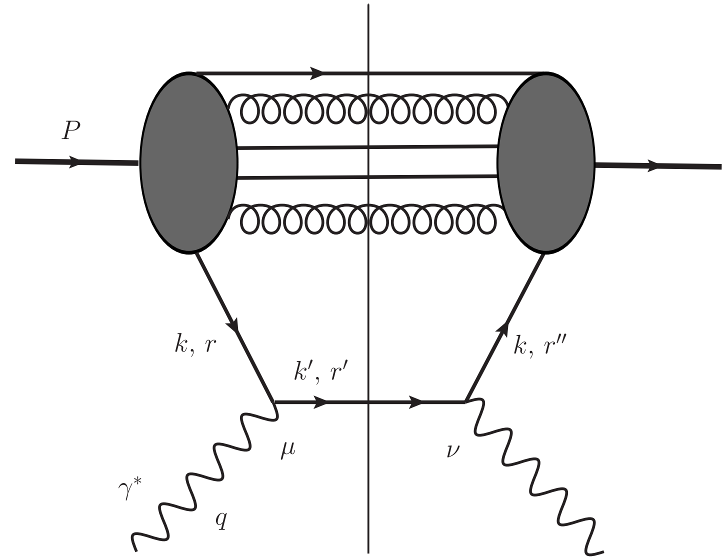
$$Q^2 = -q_\mu \, q^\mu$$

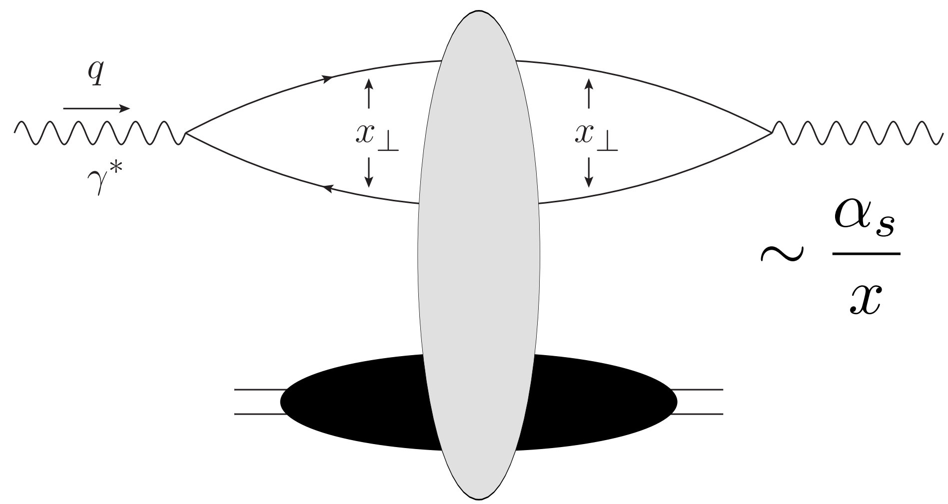
- Photon hits a quark in the proton carrying momentum $x_{Bj} p$ with p being the proton's momentum. Parameter x_{Bj} is the Bjorken *x* variable.

Dipole picture of DIS

- At small x, the dominant contribution to DIS structure functions does not come from the handbag diagram.

- Instead, the dominant terms comes from the dipole picture of DIS, where the virtual photon splits into a quark-antiquark pair, which then interacts with the target.





$$\sim \frac{\alpha_s}{x}$$

Dipole picture of DIS

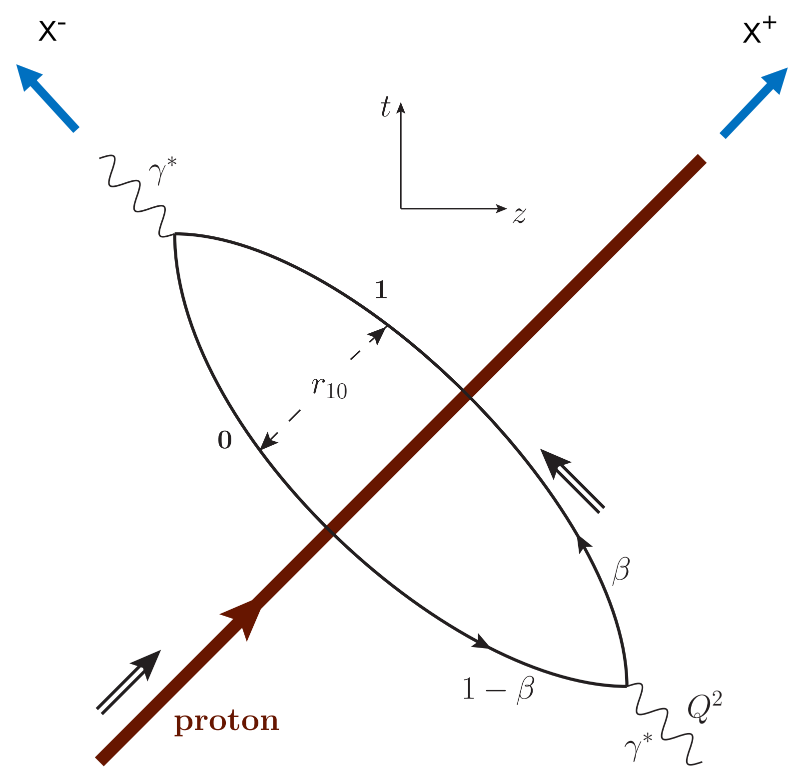
$$W^{\mu\nu} = \frac{1}{4\pi M_p} \int d^4x\, e^{iq\cdot x} \langle P | j^\mu(x)\, j^\nu(0) | P \rangle$$

$$q^\mu = \left(\frac{Q^2}{2q^-}, q^-, 0_\perp \right)$$

Large q^- → large x^- separation

Same is true for PDFs: small x means large x^- spread

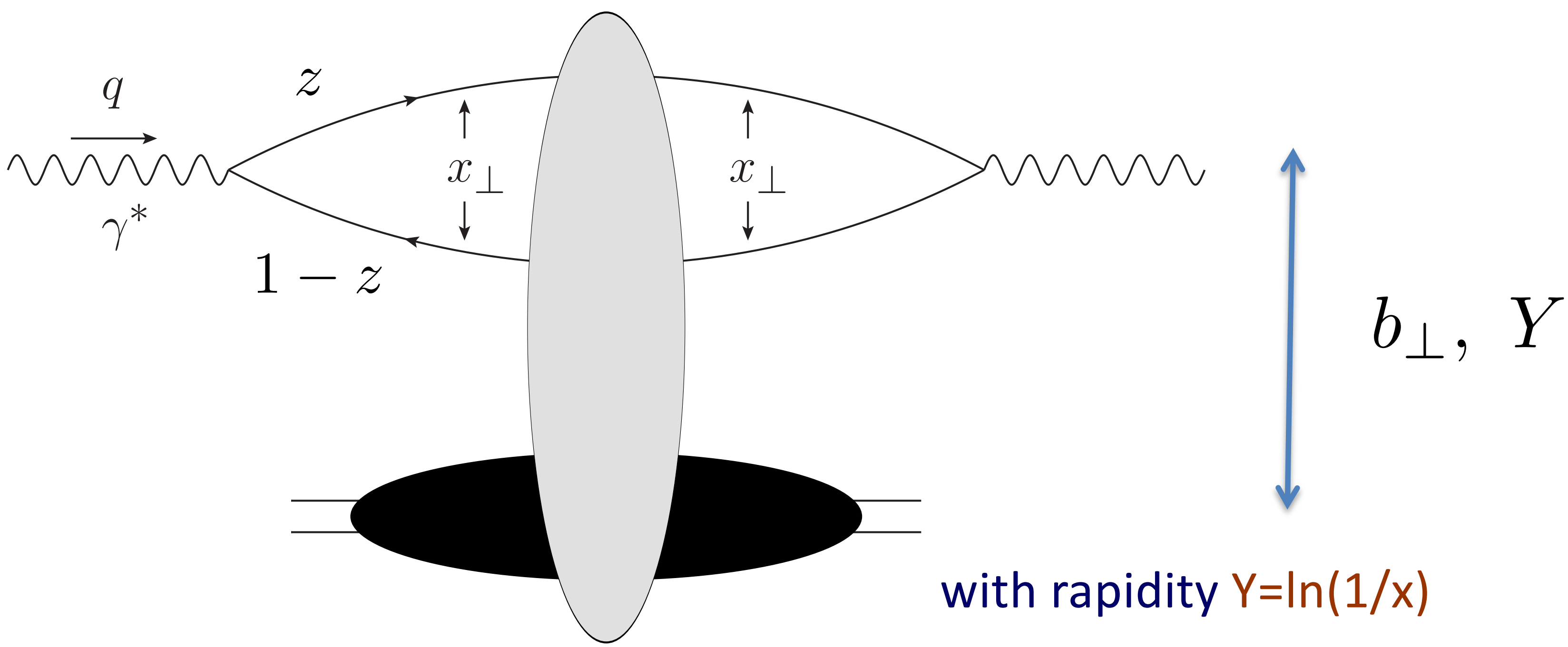
$$q(x, Q^2) = \frac{1}{4\pi} \int_{-\infty}^{\infty} dx^-\, e^{ixP^+x^-} \langle P | \bar{q}(x^-) \gamma^+ \mathcal{U}\, q(0) | P \rangle$$



Dipole Amplitude

- The total DIS cross section is expressed in terms of the (Im part of the) forward quark dipole amplitude N:

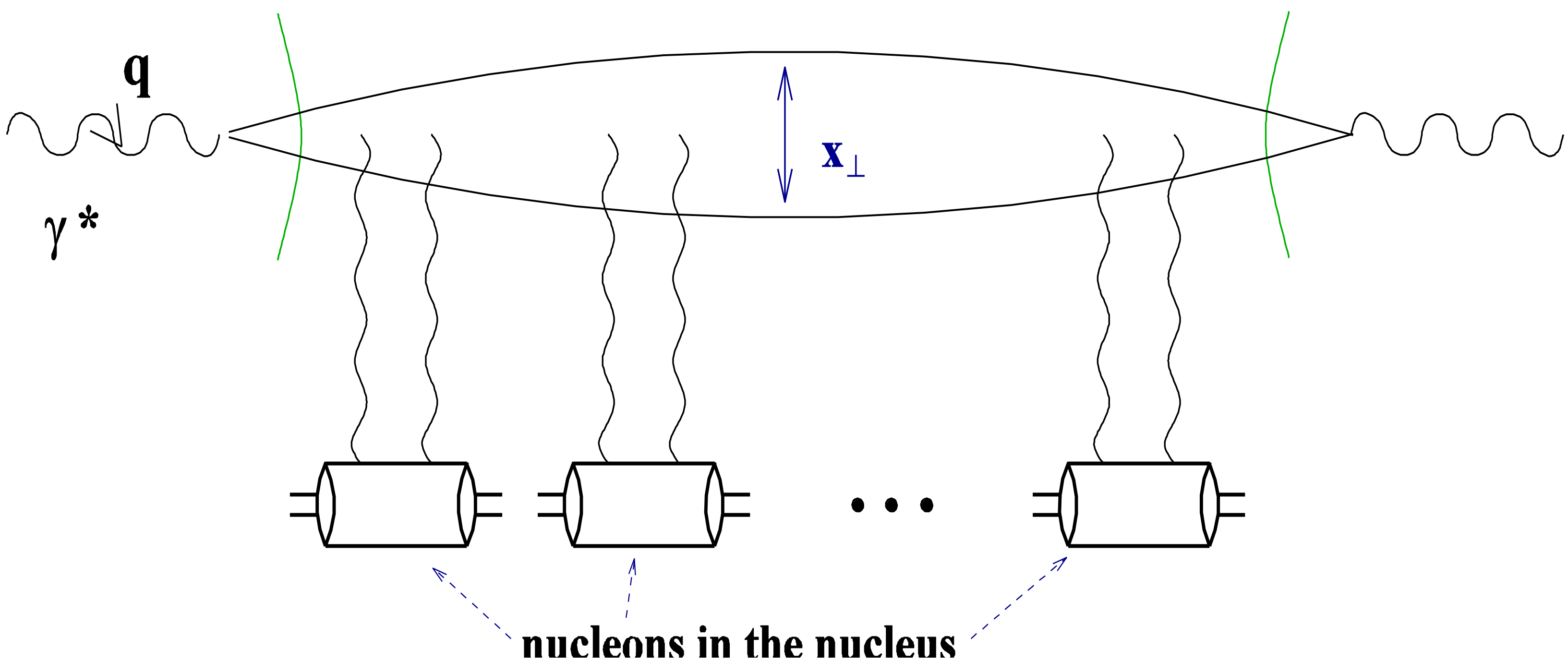
$$\sigma_{tot}^{\gamma^* A} = \int \frac{d^2 x_\perp}{2\,\pi}\, d^2 b_\perp \int\limits_0^1 \frac{dz}{z\,(1-z)}\, |\Psi^{\gamma^* \to q\bar{q}}(\vec{x}_\perp, z)|^2 \; N(\vec{x}_\perp, \vec{b}_\perp, Y)$$



with rapidity Y=ln(1/x)

DIS in the Classical Approximation

The DIS process in the rest frame of the target nucleus is shown below.



$$\sigma_{tot}^{\gamma * A}(x_{Bj}, Q^2) = |\Psi^{\gamma * \to q\bar{q}}|^2 \otimes N(x_\perp, Y = \ln 1/x_{Bj})$$

with rapidity Y=ln(1/x)

Dipole Amplitude

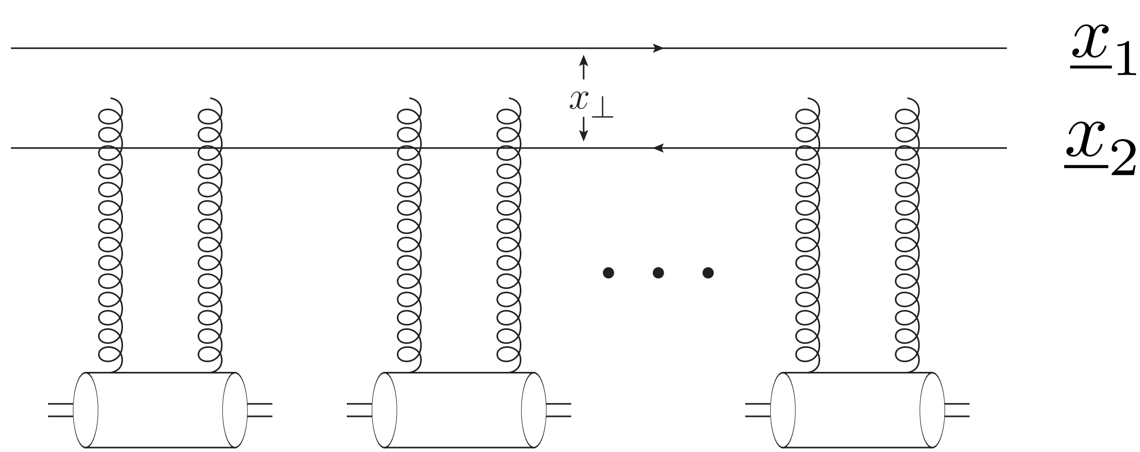
- The quark dipole amplitude is defined by

$$N(\underline{x}_1, \underline{x}_2) = 1 - \frac{1}{N_c} \left\langle \mathrm{tr}\left[V(\underline{x}_1)\, V^\dagger(\underline{x}_2) \right] \right\rangle$$

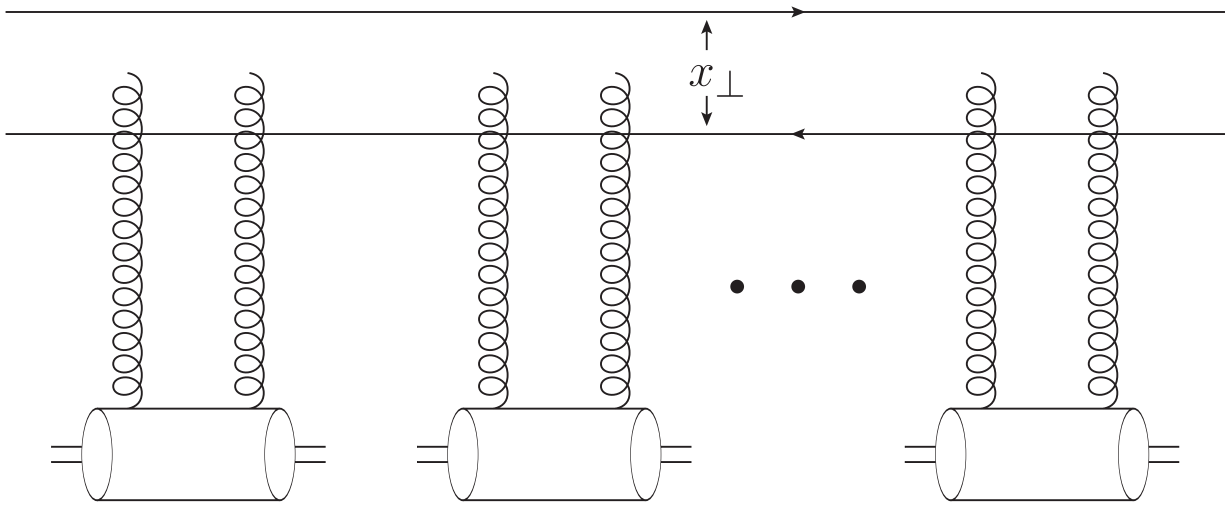
- Here we use the Wilson lines along the light-cone direction

$$V_{\underline{x}} \equiv \mathcal{P} \exp \left[ig \int\limits_{-\infty}^{\infty} dx^+ \, A^-(x^+, 0^-, \underline{x}) \right]$$

- In the classical Glauber-Mueller/McLerran-Venugopalan approach the dipole amplitude resums multiple rescatterings:

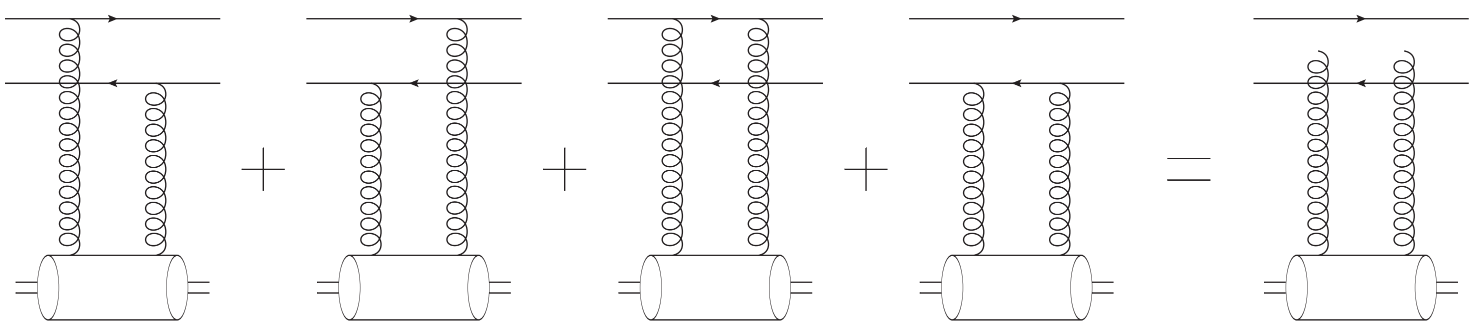


Quasi-classical dipole amplitude



A.H. Mueller, ‘90

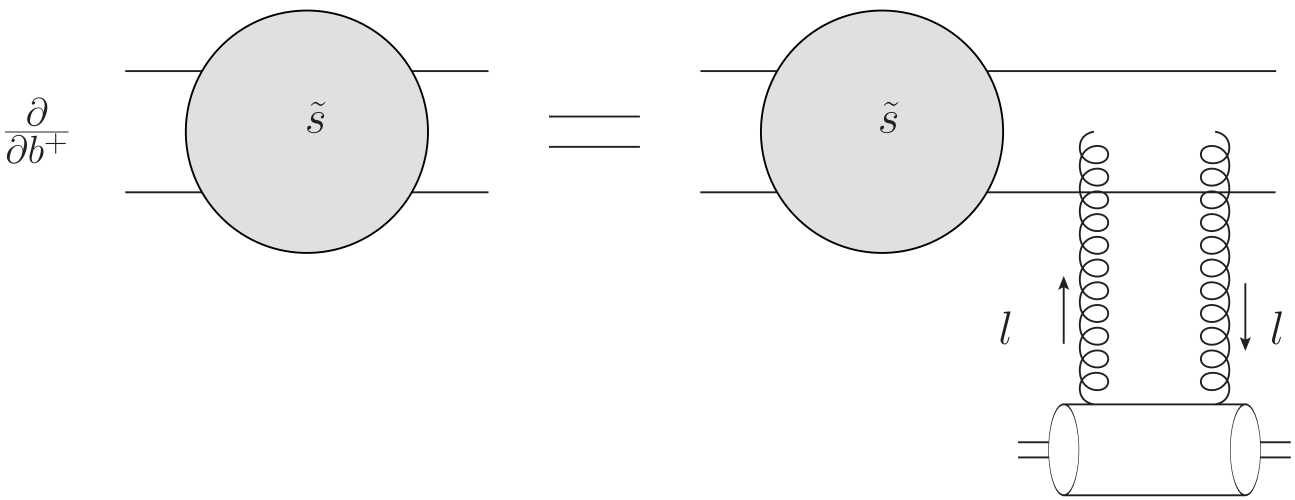
Lowest-order interaction with each nucleon – two gluon exchange – the same resummation parameter as in the MV model: $\alpha_s^2\, A^{1/3}$



Quasi-classical dipole amplitude

- To resum multiple rescatterings, note that the nucleons are independent of each other and rescatterings on the nucleons are also independent.

- One then writes an equation (Mueller '90)



Each scattering!

$$N(x_\perp, Y) = 1 - \exp\left[-\frac{x_\perp^2\, Q_s^2}{4}\,\ln\frac{1}{x_\perp\,\Lambda}\right]$$

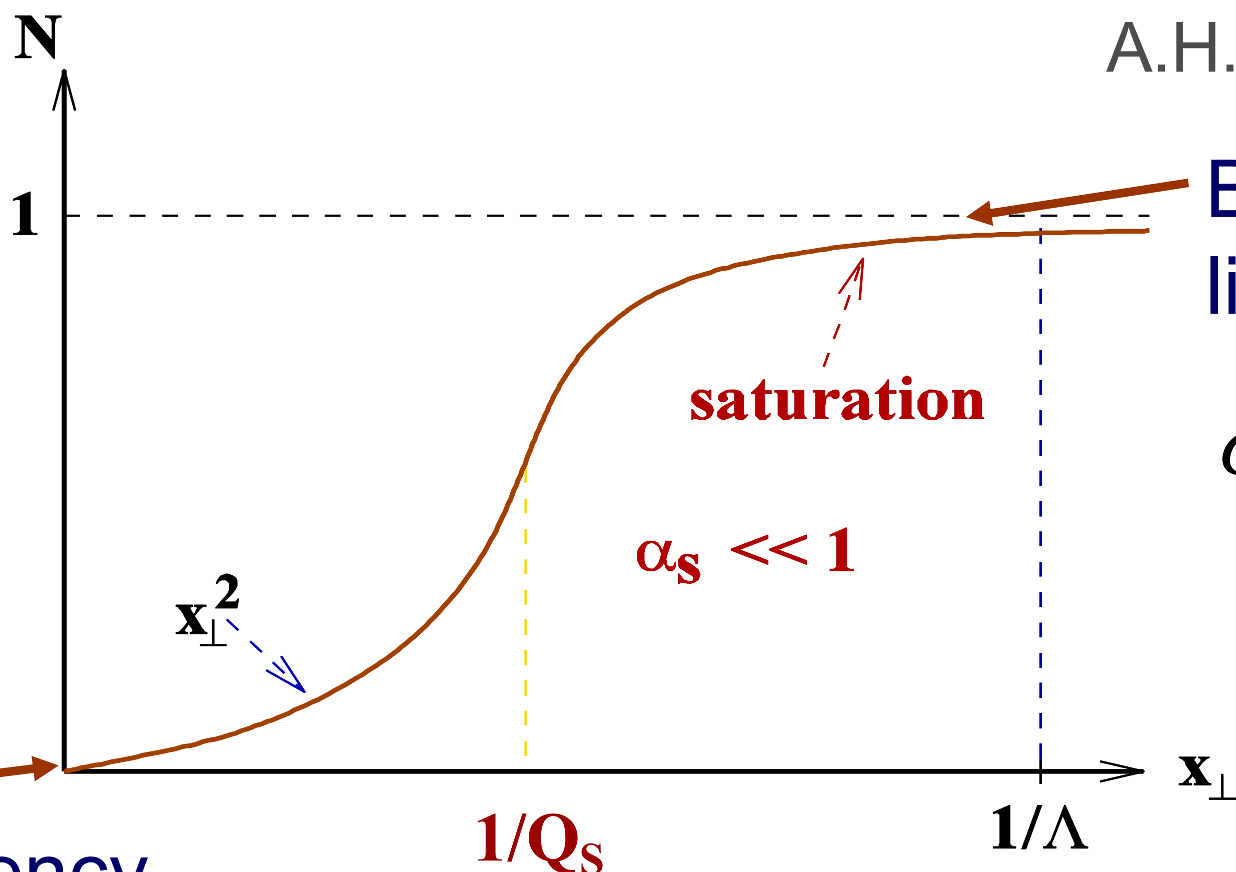
DIS in the Classical Approximation

The dipole-nucleus amplitude in the classical approximation is

$$\sigma^{q\bar{q}A} = 2 \int d^2 b\, N(x_\perp, b_\perp, Y)$$

$$N(x_\perp, Y) = 1 - \exp\left[-\frac{x_\perp^2\, Q_s^2}{4}\, \ln \frac{1}{x_\perp\, \Lambda}\right]$$

A.H. Mueller, '90



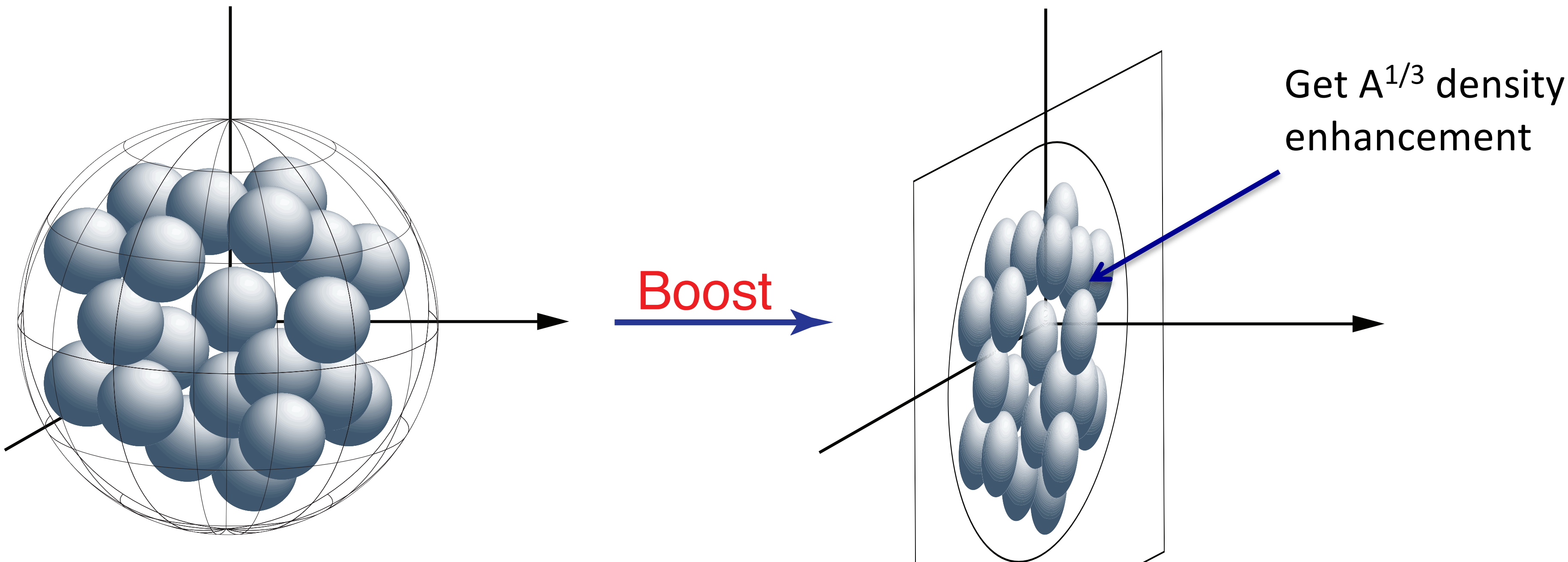
Black disk limit,

$$\sigma_{tot} < 2\pi R^2$$

Color transparency

$$Q_S^2 \sim A^{1/3}$$

McLerran-Venugopalan Model



$$M = \rho V = \rho \frac{4}{3} \pi R^3 \sim A \quad \Rightarrow \quad R \sim A^{1/3}$$

$$Q_s^2 \sim A^{1/3}$$

- Large gluon density gives a large momentum scale Q_s (the saturation scale): Q_s^2 ~ # gluons per unit transverse area ~ $A^{1/3}$ (nuclear oomph).
- For $Q_s >> \Lambda_{QCD}$, get a theory at weak coupling $\alpha_s(Q_s^2) \ll 1$ and the leading gluon field is <u>classical</u>.

Monte Carlo implementation

- Sartre (T. Toll, T. Ullrich, 2012) has the multiple rescatterings in it.

$$\sigma_{tot}^{\gamma^* A} = \int \frac{d^2 x_\perp}{2\,\pi}\, d^2 b_\perp \int\limits_0^1 \frac{dz}{z\,(1-z)}\ |\Psi^{\gamma^* \to q\bar{q}}(\vec{x}_\perp, z)|^2\ N(\vec{x}_\perp, \vec{b}_\perp, Y)$$

$$N(x_\perp, Y) = 1 - \exp\left[-\frac{x_\perp^2\, Q_s^2}{4}\ \ln\frac{1}{x_\perp\,\Lambda}\right]$$

Glauber-Gribov-Mueller formula

- It has saturation and realistic A-dependence for nuclear targets.

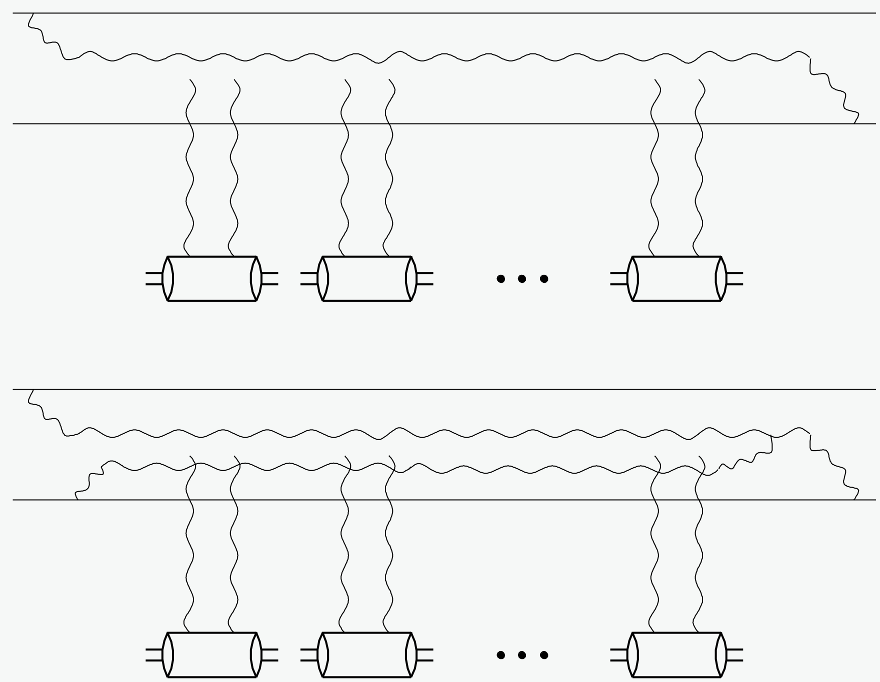
- But: quasi-classical formula above has no energy/Bjorken *x* dependence.

Small-x Evolution Equations

Quantum Evolution

- The energy dependence comes in through the long-lived s-channel gluon corrections (higher Fock states):

$$\alpha_s \ln s \sim \alpha_s Y \sim 1$$

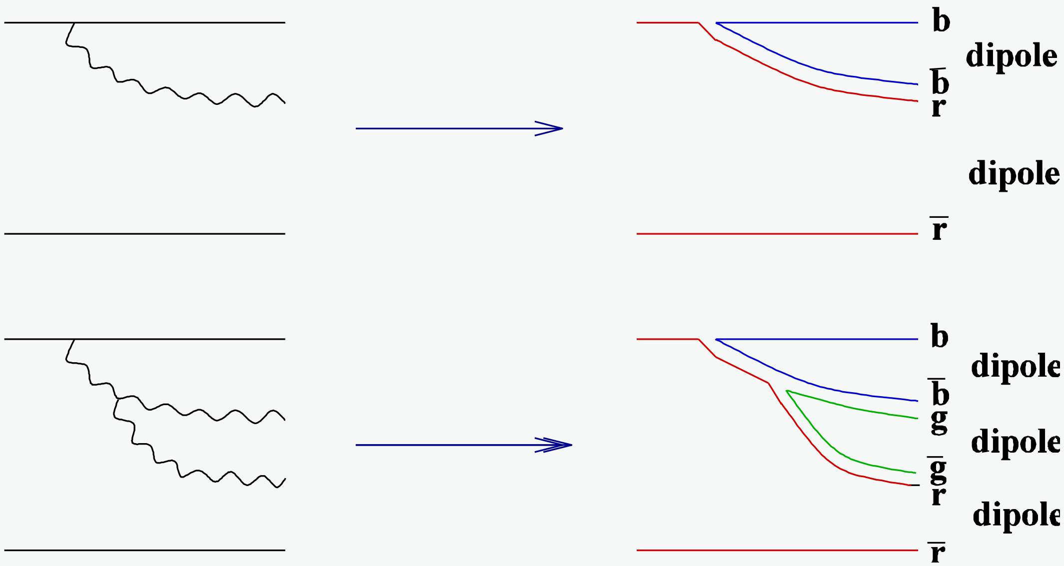


These extra gluons bring in powers of $\alpha_S \ln s$, such that when $\alpha_S << 1$ and $\ln s >> 1$ this parameter is $\alpha_S \ln s \sim 1$ (leading logarithmic approximation, LLA).

Mueller's Dipole Model

To include the quantum evolution in a dipole amplitude one can use the approach developed by A. H. Mueller in '93-'94.

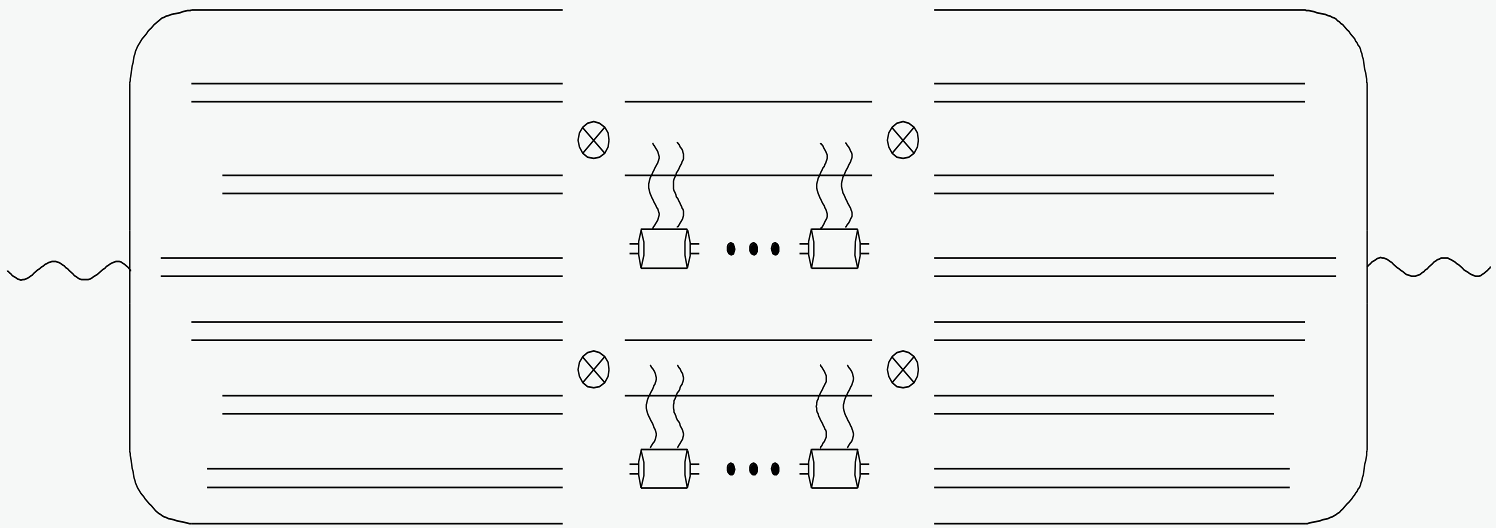
Emission of a small-x gluon taken in the large-N_C limit would split the original color dipole in two:



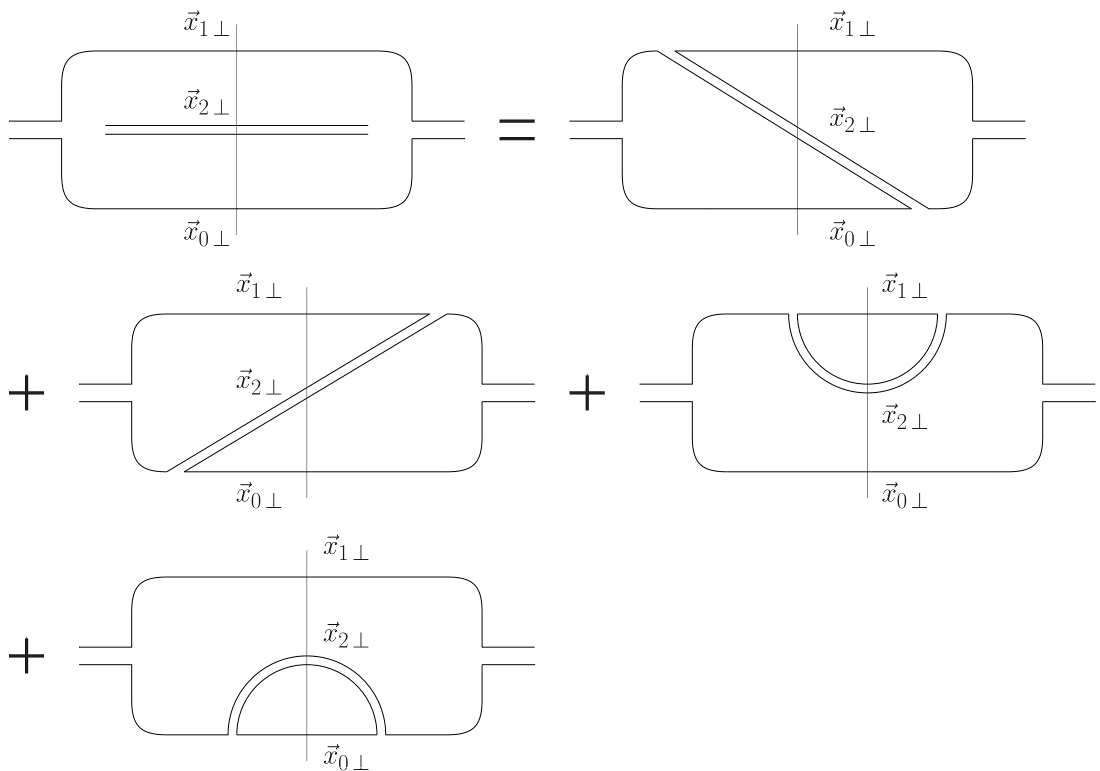
$$3 \otimes \bar{3} = 1 \oplus 8 \quad \Rightarrow \quad N_c \otimes \bar{N}_c = 1 \oplus (N_c^2 - 1) \approx N_c^2 - 1$$

Re-summing gluon cascade

- At large N_c the gluon cascade turns into a dipole cascade. We need to resum the dipole cascade, with each dipole interacting with the target independently.



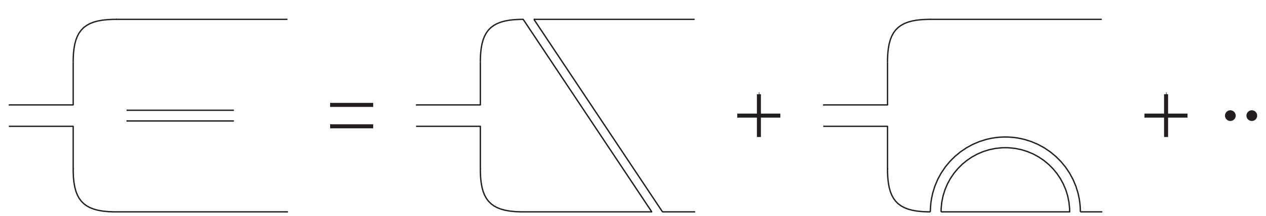
Notation (Large-N_C)



Real emissions in the amplitude squared

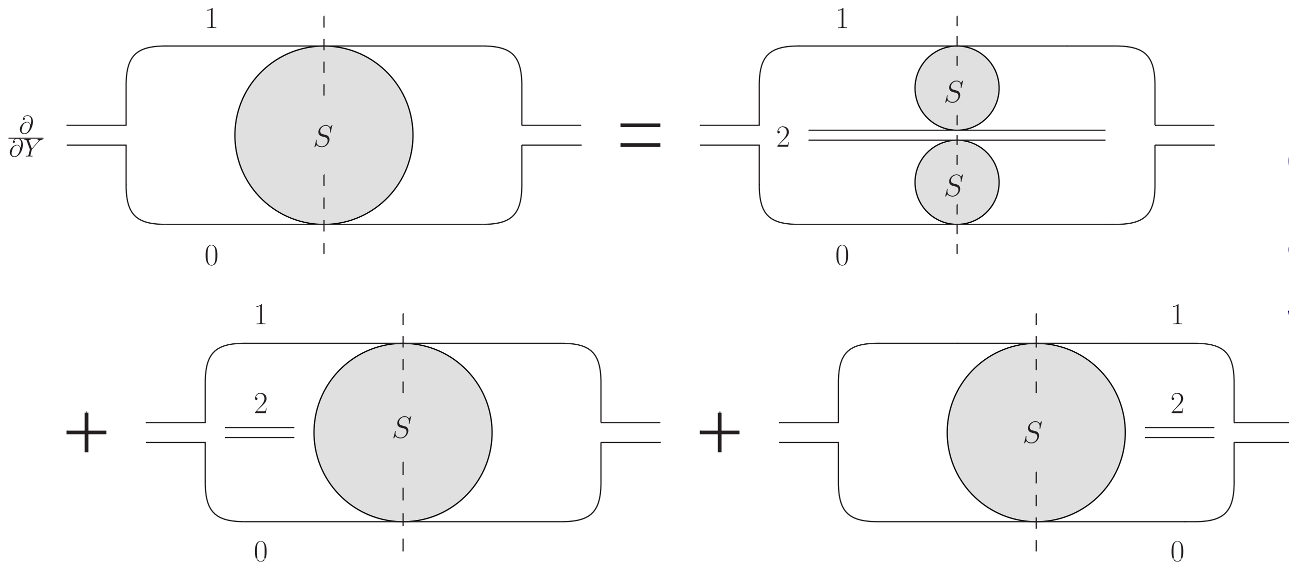
(dashed line – all Glauber-Mueller exchanges at light-cone time =0)

Virtual corrections in the amplitude (wave function)



Nonlinear Evolution

To sum up the gluon cascade at large-N_C we write the following equation for the dipole S-matrix:



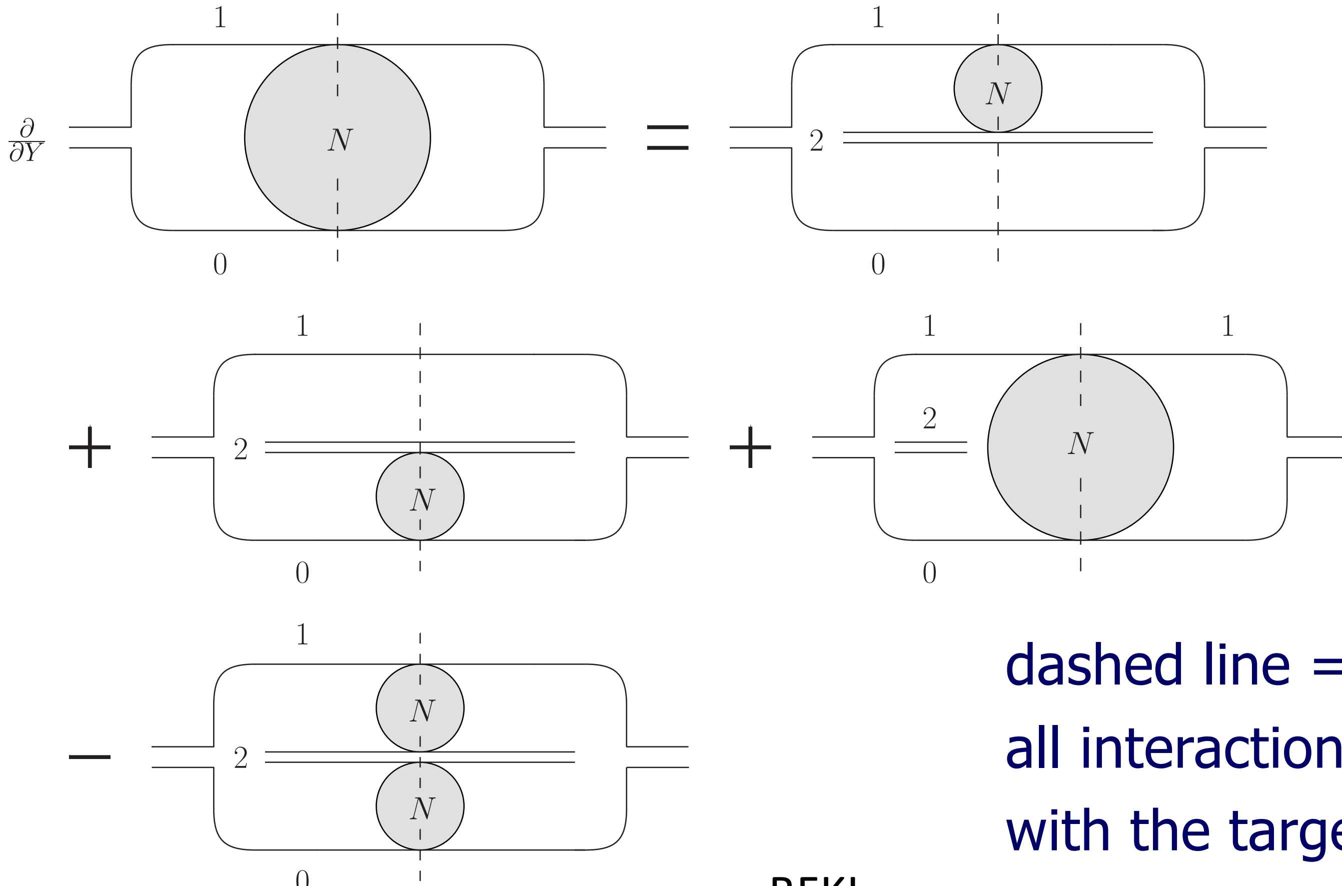
dashed line = all interactions with the target

$$\partial_Y S_{\mathbf{x}_0,\mathbf{x}_1}(Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2 x_2 \frac{x_{01}^2}{x_{02}^2\, x_{21}^2} \left[S_{\mathbf{x}_0,\mathbf{x}_2}(Y)\, S_{\mathbf{x}_2,\mathbf{x}_1}(Y) - S_{\mathbf{x}_0,\mathbf{x}_1}(Y)\right]$$

Remembering that S= 1-N we can rewrite this equation in terms of the dipole scattering amplitude N.

Nonlinear evolution at large N_c

As N=1-S we write



dashed line = all interactions with the target

BFKL

$$\partial_Y N_{\mathbf{x}_0,\mathbf{x}_1}(Y) = \frac{\alpha_s N_c}{2\pi^2} \int d^2x_2 \frac{x_{01}^2}{x_{02}^2 x_{21}^2} \left[N_{\mathbf{x}_0,\mathbf{x}_2}(Y) + N_{\mathbf{x}_2,\mathbf{x}_1}(Y) - N_{\mathbf{x}_0,\mathbf{x}_1}(Y) - N_{\mathbf{x}_0,\mathbf{x}_2}(Y)\, N_{\mathbf{x}_2,\mathbf{x}_1}(Y)\right]$$

Balitsky ‘96, Yu.K. ‘99

Resummation parameter

- BK equation resums powers of

$$\alpha_s \, N_c \, Y$$

- The Gribov-Glauber-Mueller/McLerran-Venugopalan initial conditions for it resum powers of

$$\alpha_s^2 \, A^{1/3}$$

$$\langle \hat{O}_\alpha \rangle_Y = \int \mathcal{D}\alpha \; \hat{O}_\alpha \, W_Y[\alpha]$$

JIMWLK Equation

- The JIMWLK evolution equation (1997-2002):

$$\partial_Y W_Y[\alpha] = \alpha_s \left\{ \frac{1}{2} \int d^2x_\perp \, d^2y_\perp \, \frac{\delta^2}{\delta\alpha^a(x^-, \vec{x}_\perp) \, \delta\alpha^b(y^-, \vec{y}_\perp)} \left[\eta^{ab}_{\vec{x}_\perp \vec{y}_\perp} W_Y[\alpha] \right] \right.$$
$$\left. - \int d^2x_\perp \, \frac{\delta}{\delta\alpha^a(x^-, \vec{x}_\perp)} \left[\nu^a_{\vec{x}_\perp} W_Y[\alpha] \right] \right\}$$

with

$$\eta^{ab}_{\vec{x}_{1\perp} \vec{x}_{0\perp}} = \frac{4}{g^2 \, \pi^2} \int d^2x_2 \, \frac{\vec{x}_{21} \cdot \vec{x}_{20}}{x_{21}^2 \, x_{20}^2} \left[\mathbf{1} - U_{\vec{x}_{1\perp}} U^\dagger_{\vec{x}_{2\perp}} - U_{\vec{x}_{2\perp}} U^\dagger_{\vec{x}_{0\perp}} + U_{\vec{x}_{1\perp}} U^\dagger_{\vec{x}_{0\perp}} \right]^{ab}$$

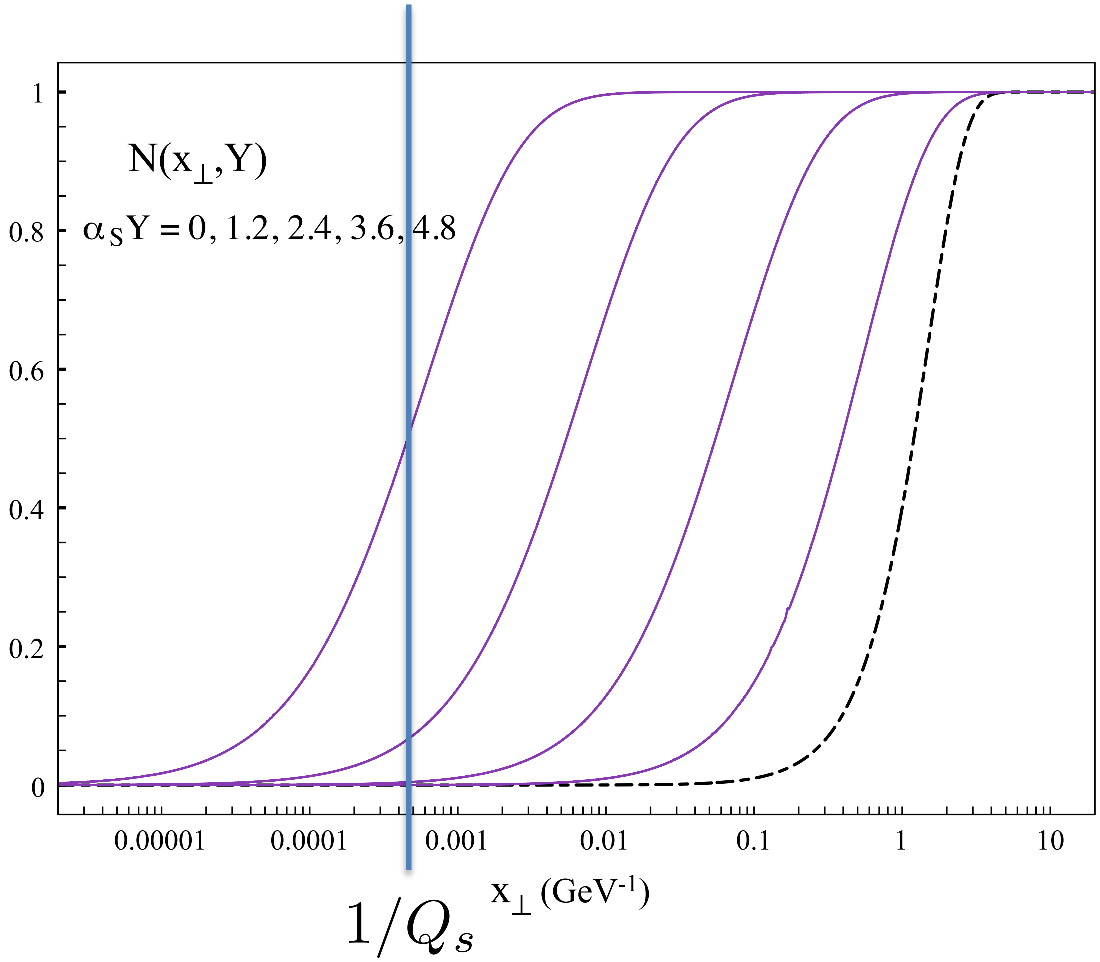
$$\nu^a_{\vec{x}_{1\perp}} = \frac{i}{g \, \pi^2} \int \frac{d^2x_2}{x_{21}^2} \, \mathrm{Tr} \left[T^a U_{\vec{x}_{1\perp}} U^\dagger_{\vec{x}_{2\perp}} \right]$$

- Here U is the adjoint Wilson line on a light cone,

$$U_{\vec{x}_\perp} = \mathrm{P} \exp \left\{ i \, g \int\limits_{-\infty}^{\infty} dx^- \, \mathcal{A}^+(x^+ = 0, x^-, \vec{x}_\perp) \right\}$$

Solution of the Nonlinear Equation

Solution of BK equation

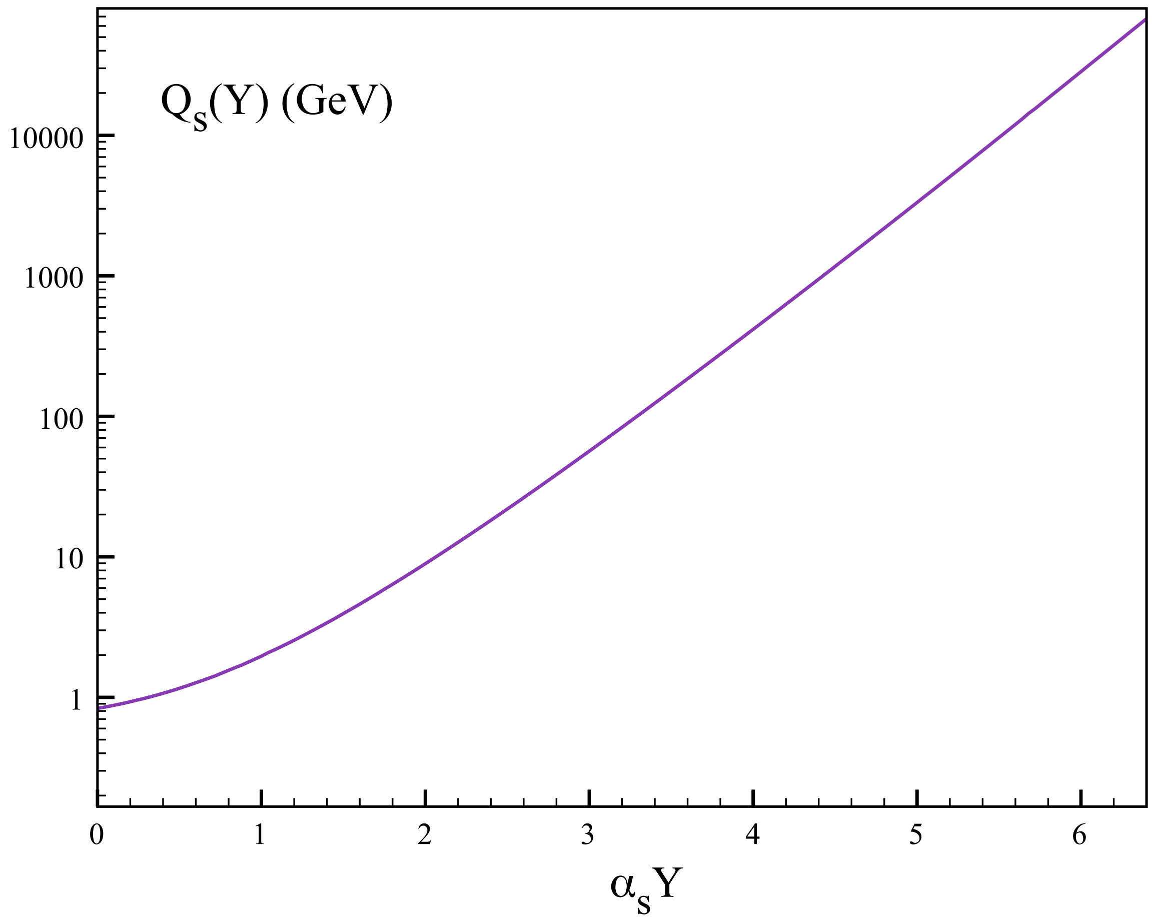


numerical solution
by J. Albacete '03
(earlier solutions were found numerically by Golec-Biernat, Motyka, Stasto, by Braun, and by Lublinsky et al in '01)

BK solution preserves the black disk limit, N<1 always (unlike the linear BFKL equation)

$$\sigma^{q\bar{q}A} = 2 \int d^2b\, N(x_\perp, b_\perp, Y)$$

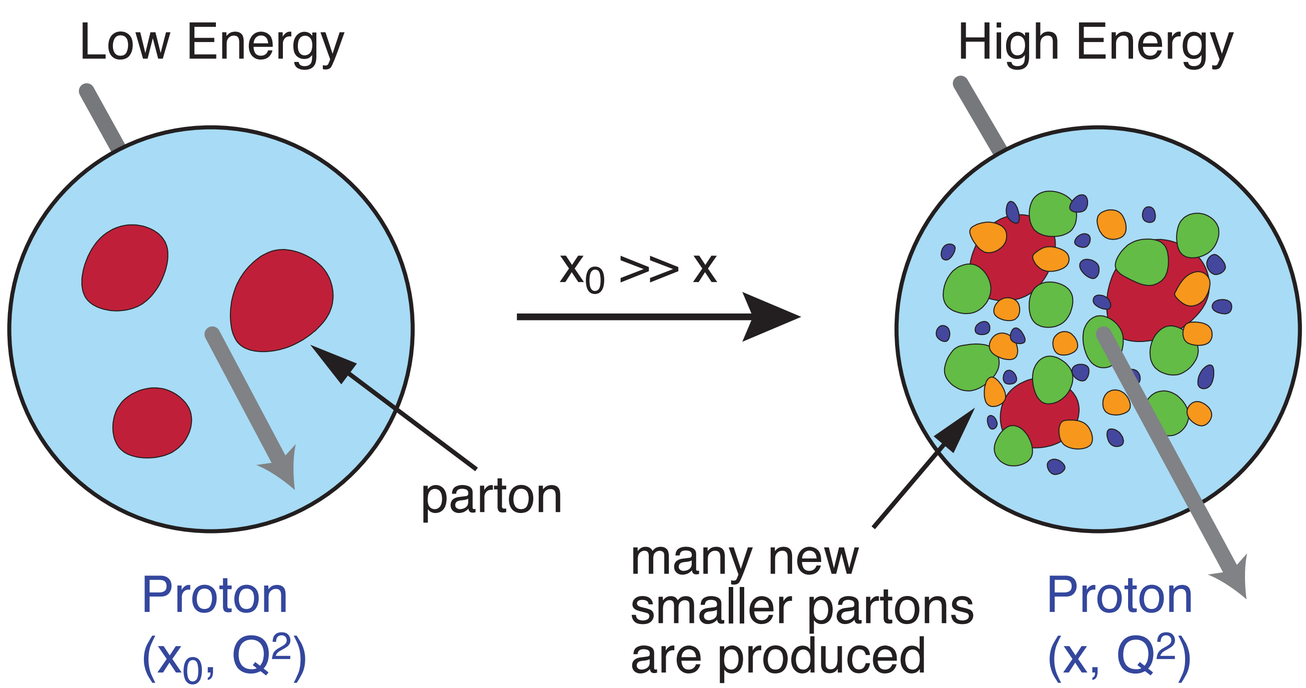
Saturation scale



numerical solution by J. Albacete (ca. 2006)

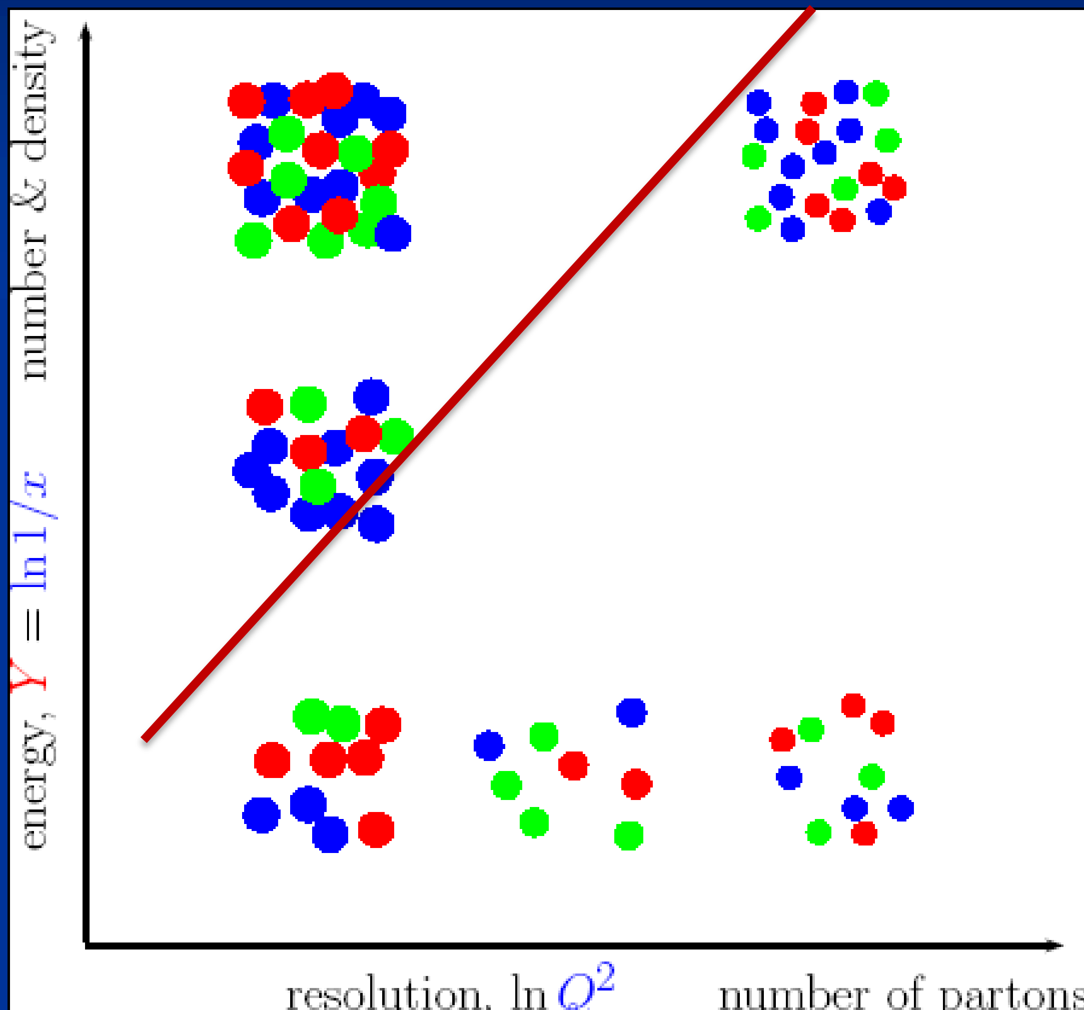
High Density of Gluons

- High number of gluons populates the transverse extent of the proton or nucleus, leading to a very dense saturated wave function known as the Color Glass Condensate (CGC):



"Color Glass Condensate"

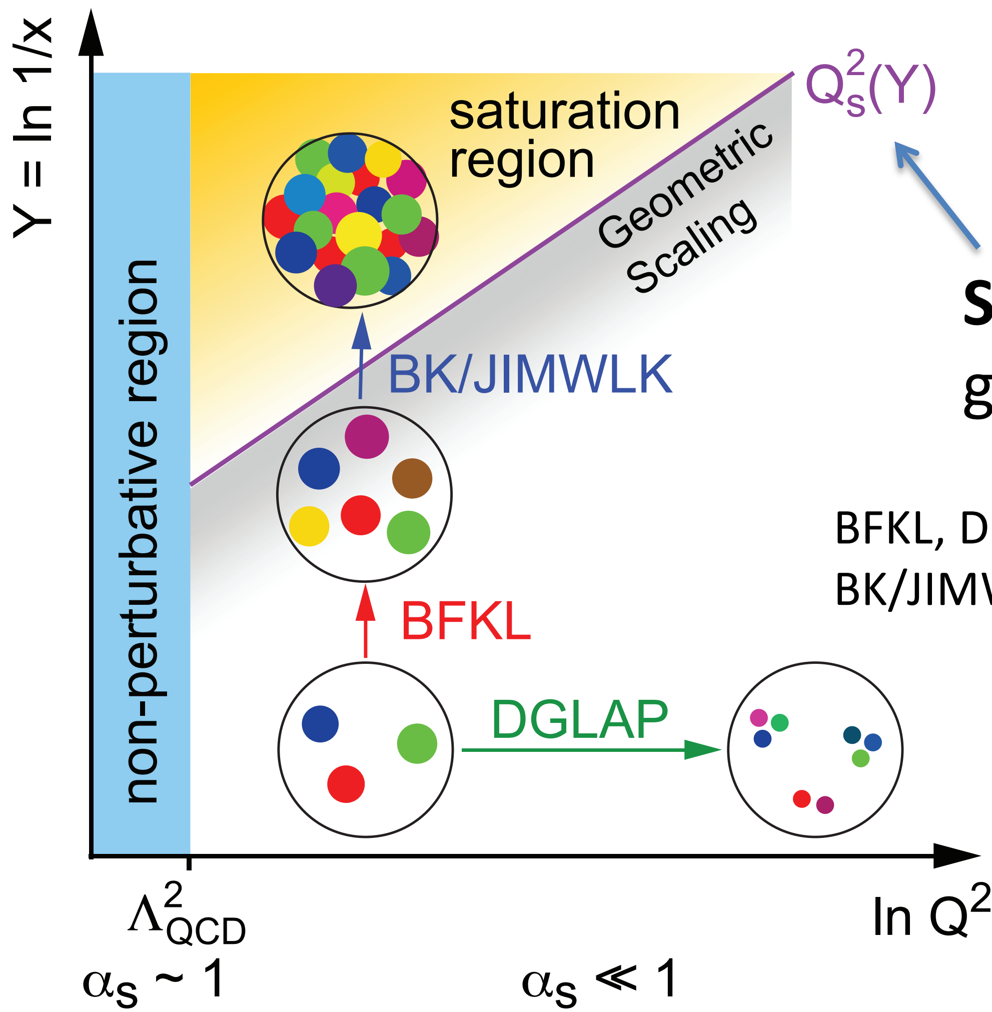
Map of High Energy QCD



energy

size of gluons

Map of High Energy QCD



$$Q_s^2 \sim \left(\frac{1}{x}\right)^{\lambda}$$

Saturation Scale
grows with energy

BFKL, DGLAP – linear equations
BK/JIMWLK – nonlinear

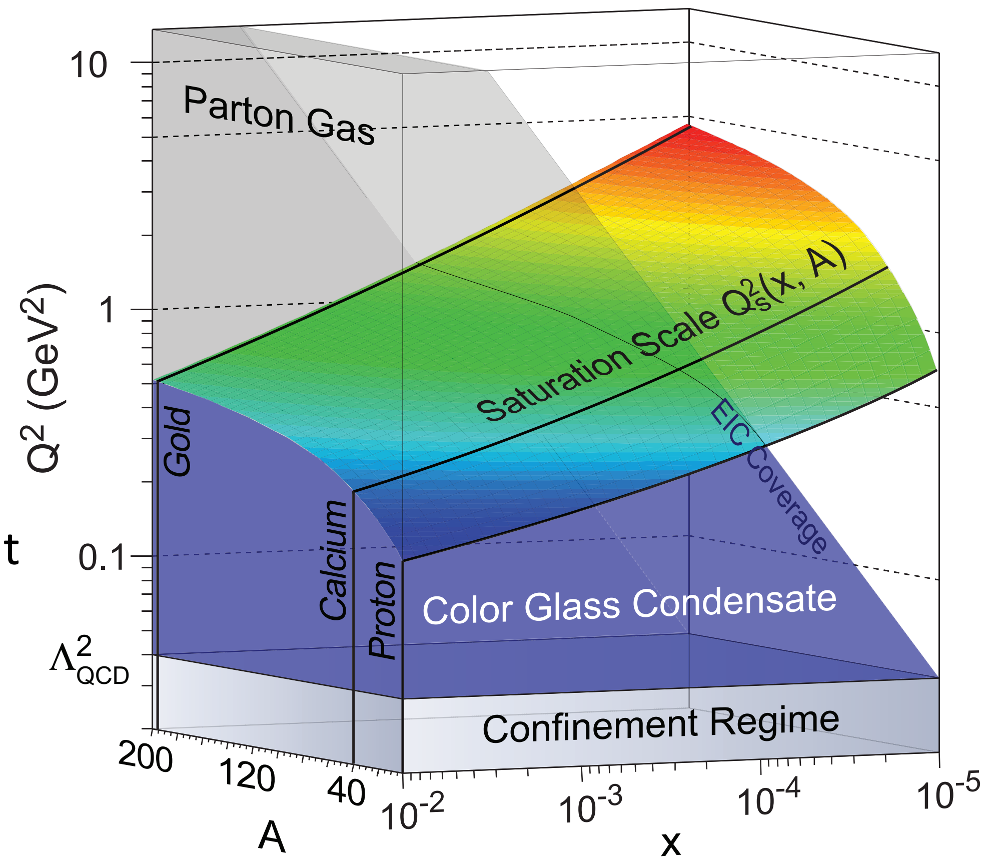
energy

size of gluons

Saturation Scale

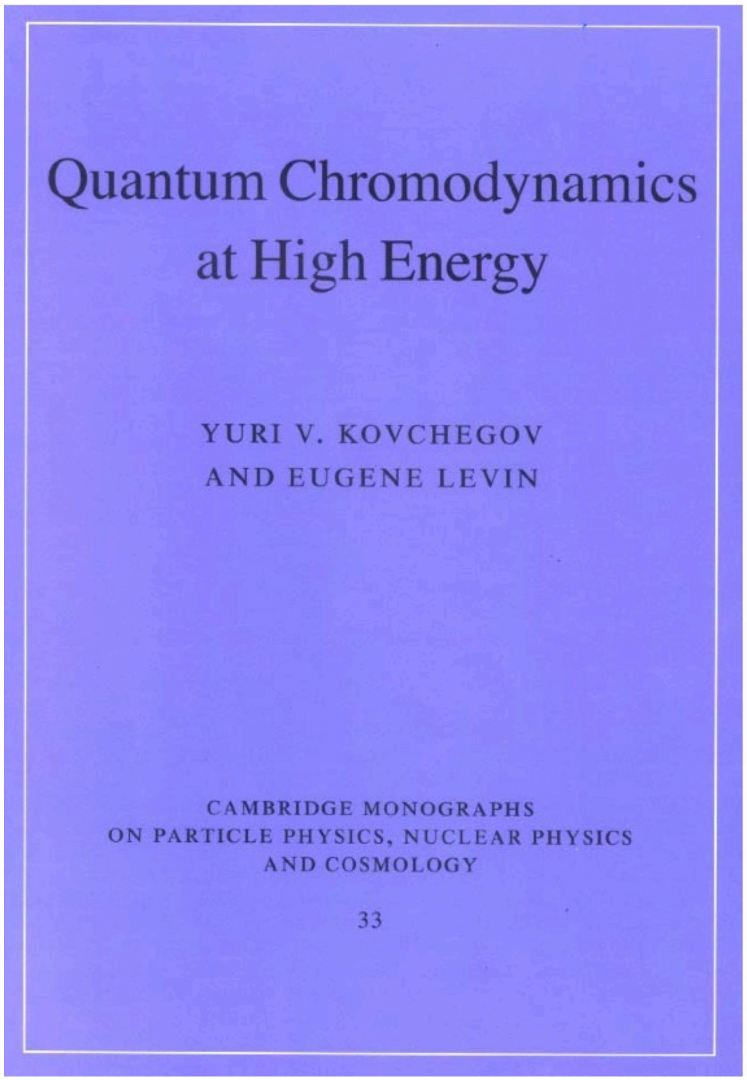
To summarize, saturation scale is an increasing function of both energy (1/x) and A:

$$Q_s^2 \sim \left(\frac{A}{x}\right)^{1/3}$$



Gold nucleus provides an enhancement by $197^{1/3}$, which is equivalent to doing scattering on a proton at 197 times smaller x / higher s!

If you want to know more…



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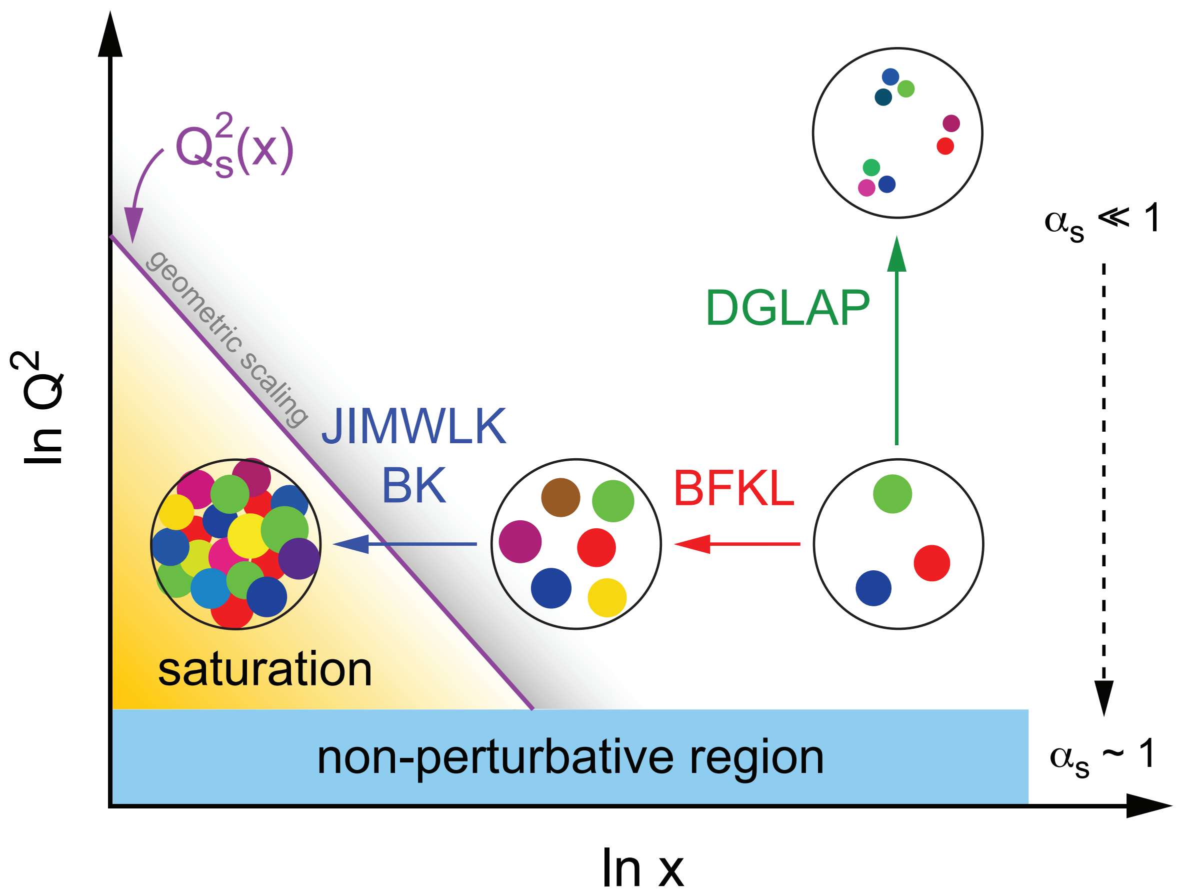
Can Saturation be Discovered at EIC?

EIC has an unprecedented small-x reach for DIS on large nuclear targets, allowing to seal the discovery of saturation physics and study of its properties:



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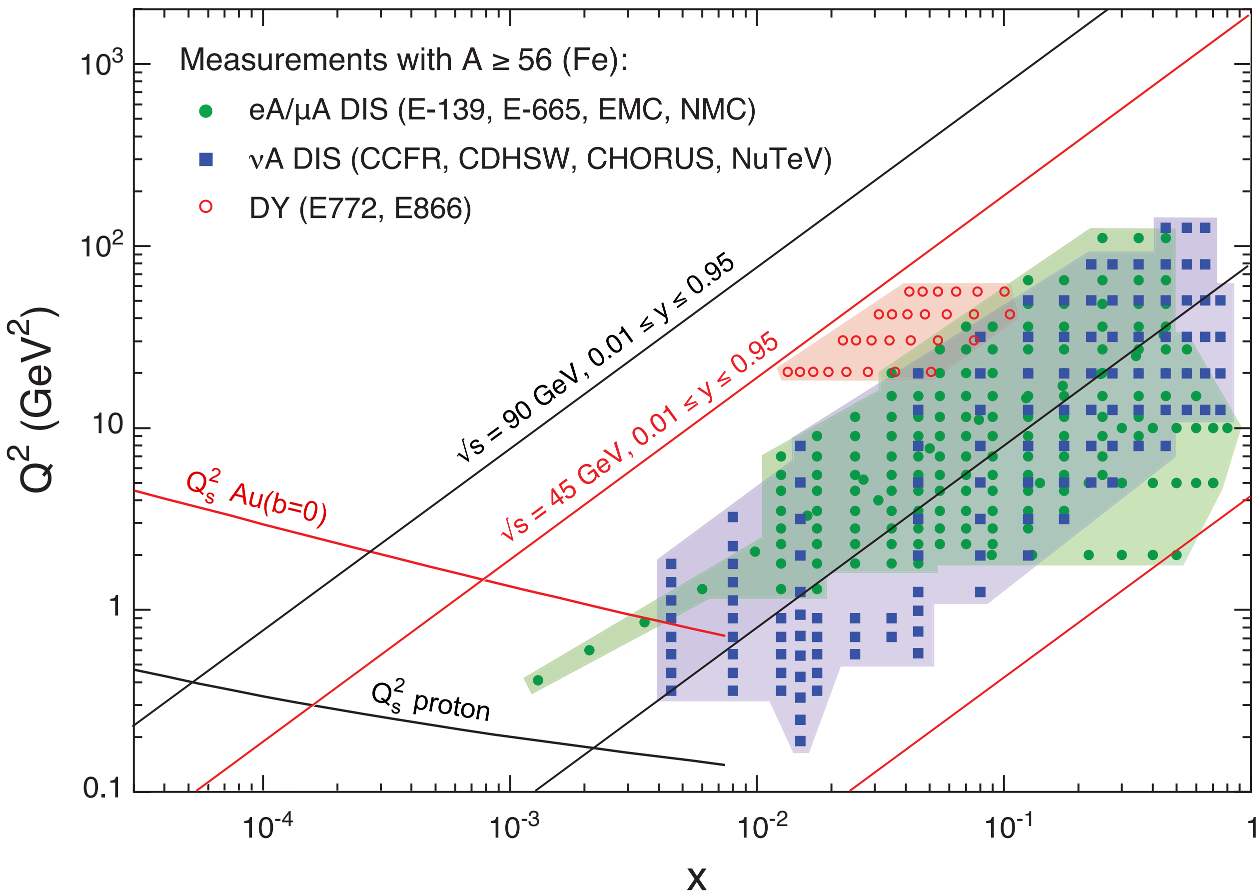
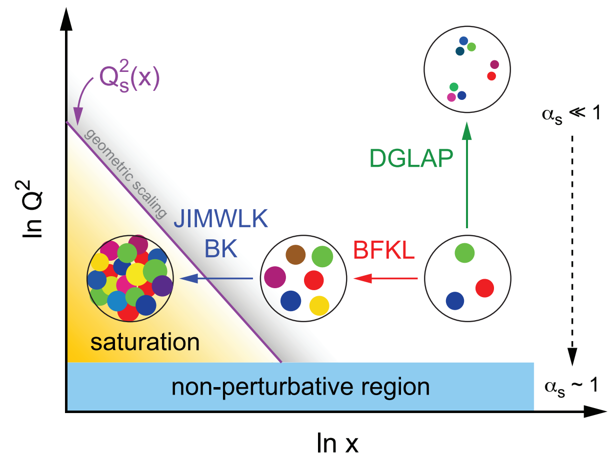




MC Implementation

- Mueller’s dipole model was first studied using MC by Gavin Salam in 1995.

- Mueller’s dipole model has been incorporated in DIPSY (E. Avsar, C. Flensburg, G. Gustafson, L. Lönnblad, ‘05-’11) + more recently in Pythia8 and Herwig7.

- I am not sure whether there are dedicated MC implementations for BK/JIMWLK evolution.

- Other observables to study: inclusive production, 2-particle correlations, diffraction.

Proton Spin at Small x

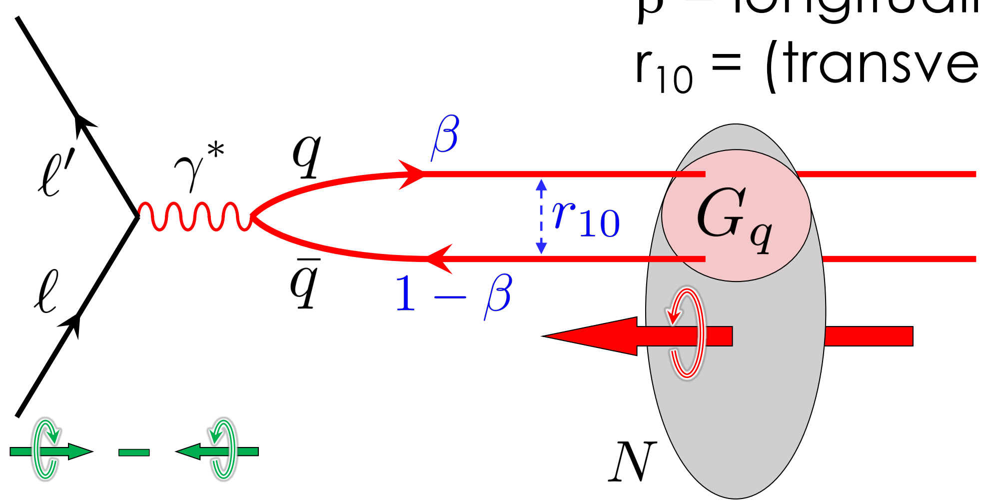
Based on work done with Dan Pitonyak and Matt Sievert (2015-2018, 2021), Florian Cougoulic (2019-2020), Josh Tawabutr (2020), Andrey Tarasov (2021), Daniel Adamiak, Wally Melnitchouk, Nobuo Sato (2021)

Quark Helicity Distribution at Small x

- One can show that the quark helicity PDF ($\Delta\Sigma$) at small-x can be expressed in terms of the polarized dipole amplitude:

$$\Delta\Sigma(x, Q^2) \sim G(r_{10}^2, \beta)$$

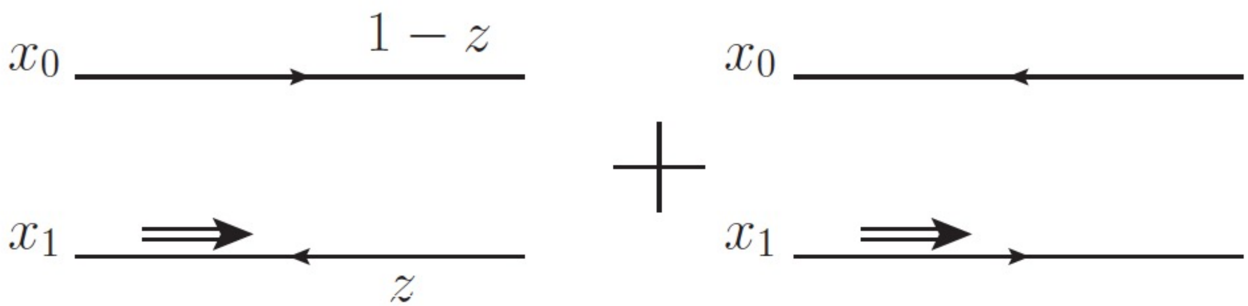
β = longitudinal momentum fraction (aka z);
r_{10} = (transverse) dipole size (aka x_{10})



$$\Delta\Sigma \equiv \Delta u + \Delta\bar{u} + \Delta d + \Delta\bar{d} + \Delta s + \Delta\bar{s}$$

Polarized Dipole

- All flavor-singlet small-x helicity observables depend on one object, "polarized dipole amplitude":



$$G_{10}(z) \equiv \frac{1}{2N_c} \operatorname{Re} \left\langle\!\!\left\langle \mathrm{T}\,\mathrm{tr}\left[V_{\underline{0}} V_{\underline{1}}^{pol\,\dagger}\right] + \mathrm{T}\,\mathrm{tr}\left[V_{\underline{1}}^{pol} V_{\underline{0}}^{\dagger}\right] \right\rangle\!\!\right\rangle(z)$$

unpolarized quark

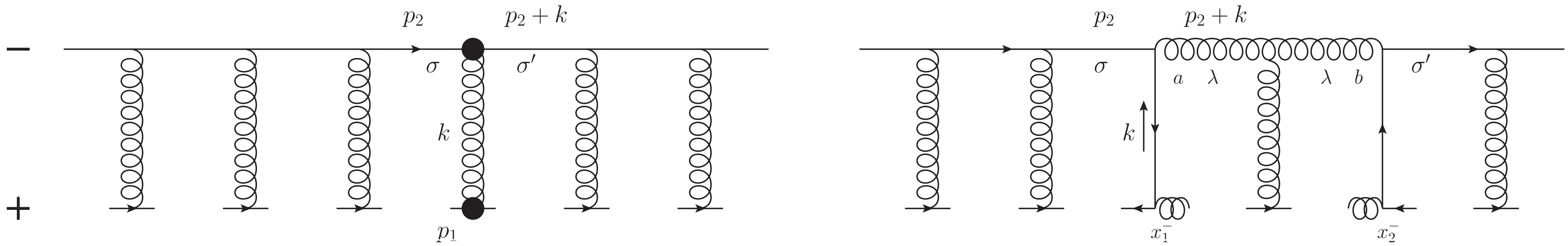
polarized quark: eikonal propagation, non-eikonal spin-dependent interaction

$$V_{\underline{x}} \equiv \mathcal{P} \exp\left[ig \int\limits_{-\infty}^{\infty} dx^+ \, A^-(x^+, 0^-, \underline{x})\right]$$

- Double brackets denote an object with energy suppression scaled out:

$$\left\langle\!\!\left\langle \mathcal{O} \right\rangle\!\!\right\rangle(z) \equiv zs \left\langle \mathcal{O} \right\rangle(z)$$

Polarized fundamental “Wilson line”



- In the end one arrives at (KPS ‘17; YK, Sievert, ‘18; cf. Chirilli ’18; Altinoluk et al, ‘20)

$$V_{\underline{x}}^{pol} = \frac{igp_1^+}{s} \int\limits_{-\infty}^{\infty} dx^- \, V_{\underline{x}}[+\infty, x^-] \, F^{12}(x^-, \underline{x}) \, V_{\underline{x}}[x^-, -\infty]$$

$$- \frac{g^2 \, p_1^+}{s} \int\limits_{-\infty}^{\infty} dx_1^- \int\limits_{x_1^-}^{\infty} dx_2^- \, V_{\underline{x}}[+\infty, x_2^-] \, t^b \, \psi_\beta(x_2^-, \underline{x}) \, U_{\underline{x}}^{ba}[x_2^-, x_1^-] \left[\frac{1}{2} \gamma^+ \gamma^5 \right]_{\alpha\beta} \bar{\psi}_\alpha(x_1^-, \underline{x}) \, t^a \, V_{\underline{x}}[x_1^-, -\infty].$$

- We have employed an adjoint light-cone Wilson line

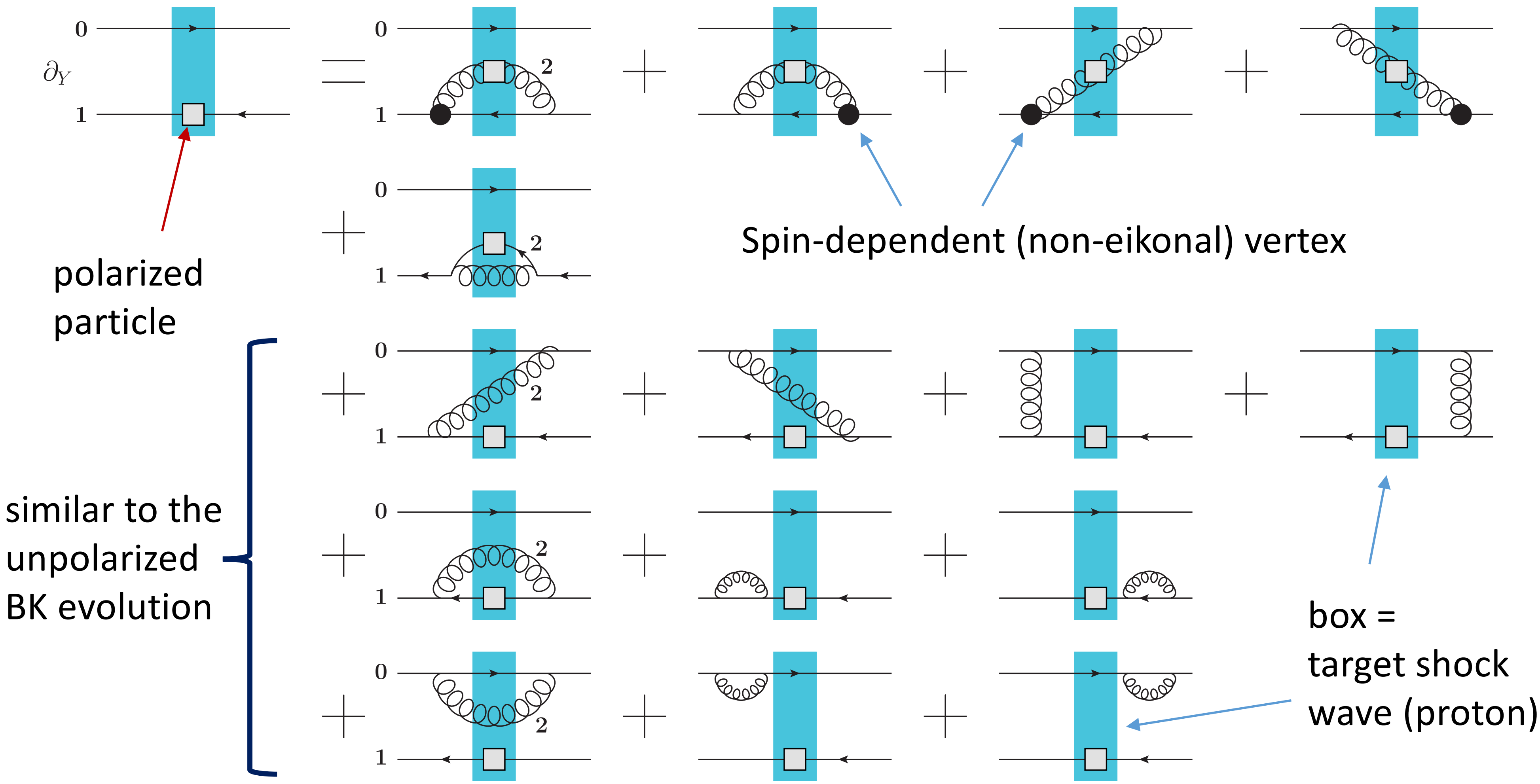
$$U_{\underline{x}}[b^-, a^-] = \mathcal{P} \exp \left[ig \int\limits_{a^-}^{b^-} dx^- \, \mathcal{A}^+(x^+ = 0, x^-, \underline{x}) \right]$$

- Note the simple physical meaning of the first term:

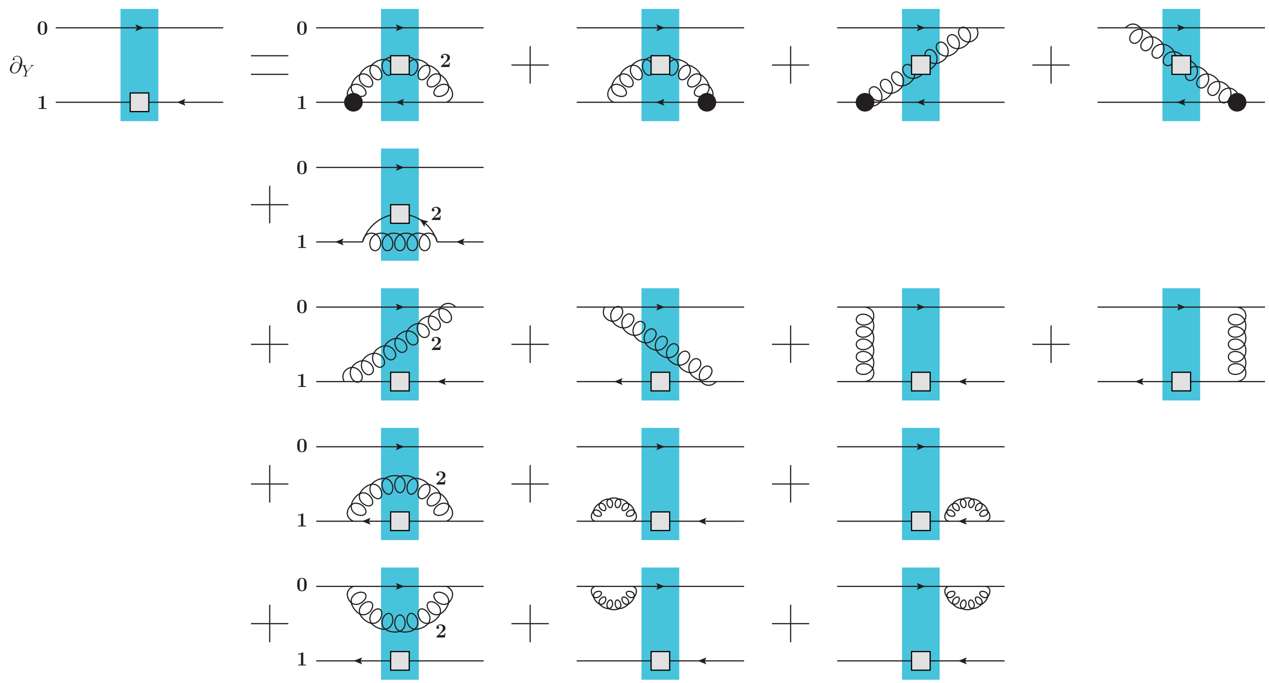
$$-\vec{\mu} \cdot \vec{B} = -\mu_z \, B_z = \mu_z \, F^{12}$$

Evolution for Polarized Quark Dipole

One can construct an evolution equation for the polarized dipole:



Evolution for Polarized Quark Dipole



$$\langle\langle \ldots \rangle\rangle = \frac{1}{z\, s} \langle \ldots \rangle$$

$$\rho'^{\,2} = \frac{1}{z'\, s}$$

$$\frac{1}{N_c} \left\langle\!\!\left\langle \mathrm{tr}\left[V_0^{unp}\, V_1^{pol\,\dagger}\right]\right\rangle\!\!\right\rangle(z) = \frac{1}{N_c} \left\langle\!\!\left\langle \mathrm{tr}\left[V_0^{unp}\, V_1^{pol\,\dagger}\right]\right\rangle\!\!\right\rangle_0(z) + \frac{\alpha_s}{2\pi^2} \int\limits_{z_i}^{z} \frac{dz'}{z'} \int\limits_{\rho'^2} \frac{d^2 x_2}{x_{21}^2}$$

$$\times \left\{ \theta(x_{10} - x_{21})\, \frac{2}{N_c} \left\langle\!\!\left\langle \mathrm{tr}\left[t^b\, V_0^{unp}\, t^a\, V_1^{unp\,\dagger}\right]\, U_2^{pol\, ba} \right\rangle\!\!\right\rangle(z') \right.$$

$$+ \theta(x_{10}^2 z - x_{21}^2 z')\, \frac{1}{N_c} \left\langle\!\!\left\langle \mathrm{tr}\left[t^b\, V_0^{unp}\, t^a\, V_2^{pol\,\dagger}\right]\, U_1^{unp\, ba} \right\rangle\!\!\right\rangle(z')$$

$$\left. + \theta(x_{10} - x_{21})\, \frac{1}{N_c} \left[\left\langle\!\!\left\langle \mathrm{tr}\left[V_0^{unp}\, V_2^{unp\,\dagger}\right]\, \mathrm{tr}\left[V_2^{unp}\, V_1^{pol\,\dagger}\right] \right\rangle\!\!\right\rangle(z') - N_c \left\langle\!\!\left\langle \mathrm{tr}\left[V_0^{unp}\, V_1^{pol\,\dagger}\right] \right\rangle\!\!\right\rangle(z') \right] \right\}$$

Equation does not close!

Large-N_c Evolution

Resummation parameter: $\alpha_s \, \ln^2 \frac{1}{x}$

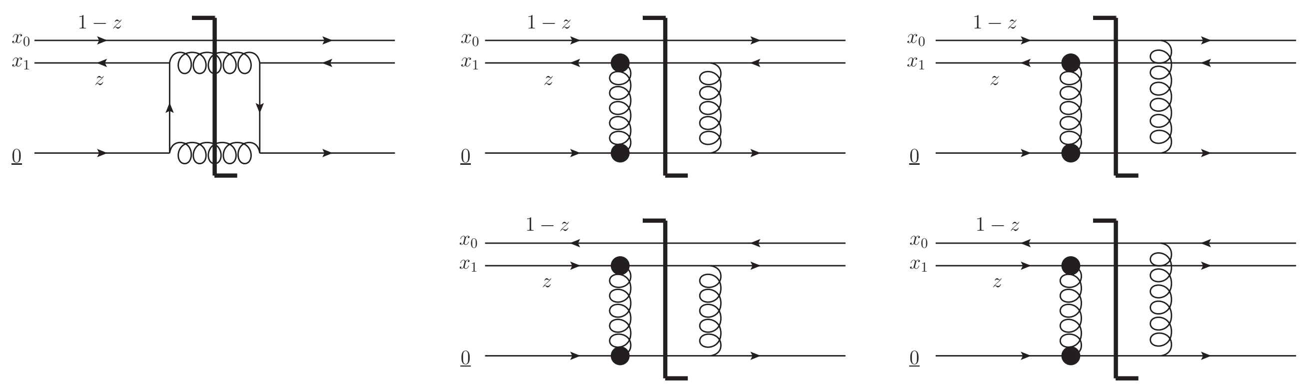
Double-logarithmic approximation (DLA)

- In the strict DLA limit (S=1) and at large N_c we get (here Γ is an auxiliary function we call the 'neighbor dipole amplitude') (KPS '15)

$$G(x_{10}^2, z) = G^{(0)}(x_{10}^2, z) + \frac{\alpha_s \, N_c}{2\pi} \int\limits_{\frac{1}{x_{10}^2 s}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z' s}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[\Gamma(x_{10}^2, x_{21}^2, z') + 3 \, G(x_{21}^2, z') \right]$$

$$\Gamma(x_{10}^2, x_{21}^2, z') = \Gamma^{(0)}(x_{10}^2, x_{21}^2, z') + \frac{\alpha_s \, N_c}{2\pi} \int\limits_{\frac{1}{x_{10}^2 s}}^{z'} \frac{dz''}{z''} \int\limits_{\frac{1}{z'' s}}^{\min\left\{x_{10}^2, x_{21}^2 \frac{z'}{z''}\right\}} \frac{dx_{32}^2}{x_{32}^2} \left[\Gamma(x_{10}^2, x_{32}^2, z'') + 3 \, G(x_{32}^2, z'') \right]$$

- The initial conditions are given by the Born-level graphs

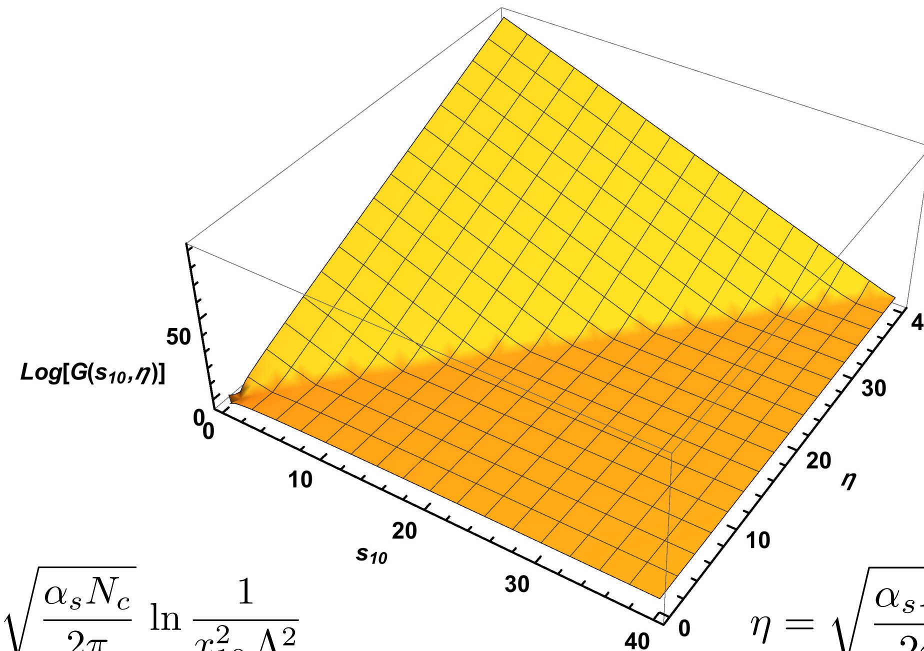


$$\Gamma^{(0)}(x_{10}^2, x_{21}^2, z) = G^{(0)}(x_{10}^2, z)$$

$$G^{(0)}(x_{10}^2, z) = \frac{\alpha_s^2 C_F}{N_c} \pi \left[C_F \ln \frac{zs}{\Lambda^2} - 2 \, \ln(zs \, x_{10}^2) \right]$$

Quark Helicity at Small x

- These equations can be solved both numerically and analytically. (KPS '16-'17)



$$s_{10} = \sqrt{\frac{\alpha_s N_c}{2\pi}} \ln \frac{1}{x_{10}^2 \Lambda^2} \qquad \eta = \sqrt{\frac{\alpha_s N_c}{2\pi}} \ln \frac{zs}{\Lambda^2}$$

- The small-x asymptotics of quark helicity is (at large N_c)

$$\Delta q(x, Q^2) \sim \left(\frac{1}{x} \right)^{\alpha_h^q} \quad \text{with} \quad \alpha_h^q = \frac{4}{\sqrt{3}} \sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 2.31 \sqrt{\frac{\alpha_s N_c}{2\pi}}$$

(Adamiak, Melnitchouk, Pitonyak, Sato, Sievert, YK, 2102.06159 [hep-ph] = JAMsmall*x*)

Small-x Polarized DIS Data

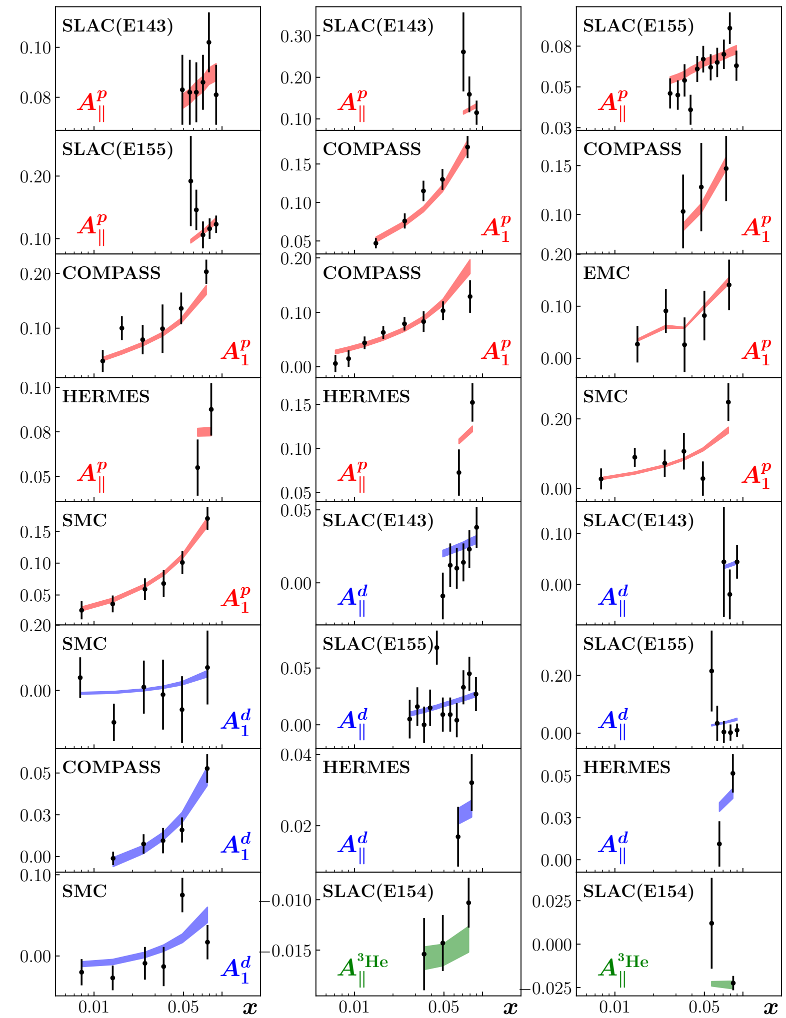
$$A_1 \sim A_{\parallel} = \frac{\sigma_{+-} - \sigma_{++}}{\sigma_{+-} + \sigma_{++}} \sim \frac{g_1}{F_1}$$

- We have analyzed all existing world polarized DIS data with $x<0.1=x_0$, $Q^2 > m_c^2$ (122 data points) using the large-N_C KPS evolution with the Born-inspired initial conditions (8 parameters for 2 flavors, 11 parameters for 3 flavors).

$$G^{(0)}(x_{10}^2, z) \propto a_q \, \ln \frac{zs}{\Lambda^2} + b_q \, \ln \frac{1}{x_{10}^2 \Lambda^2} + c_q$$

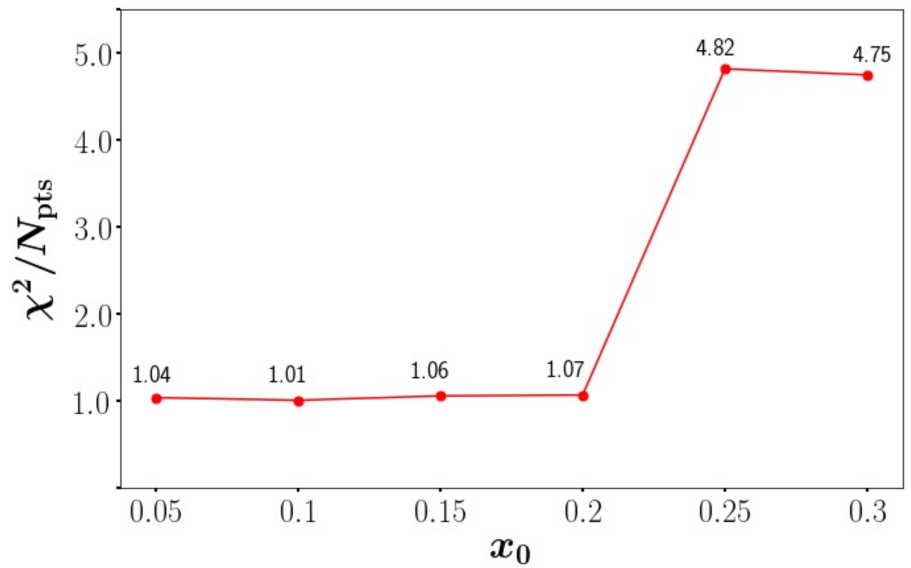
- It worked well, with χ^2/N_{pts} =1.01 (cf. JAM16: χ^2/N_{pts} =1.07)

- Small-x evolution starts at x_0=0.1 ! (cf. x_0=0.01 for unpolarized BK/JIMWLK evolution) Our approach fails at larger *x* as expected (x_0=0.3 gives χ^2/N_{pts} =4.75).

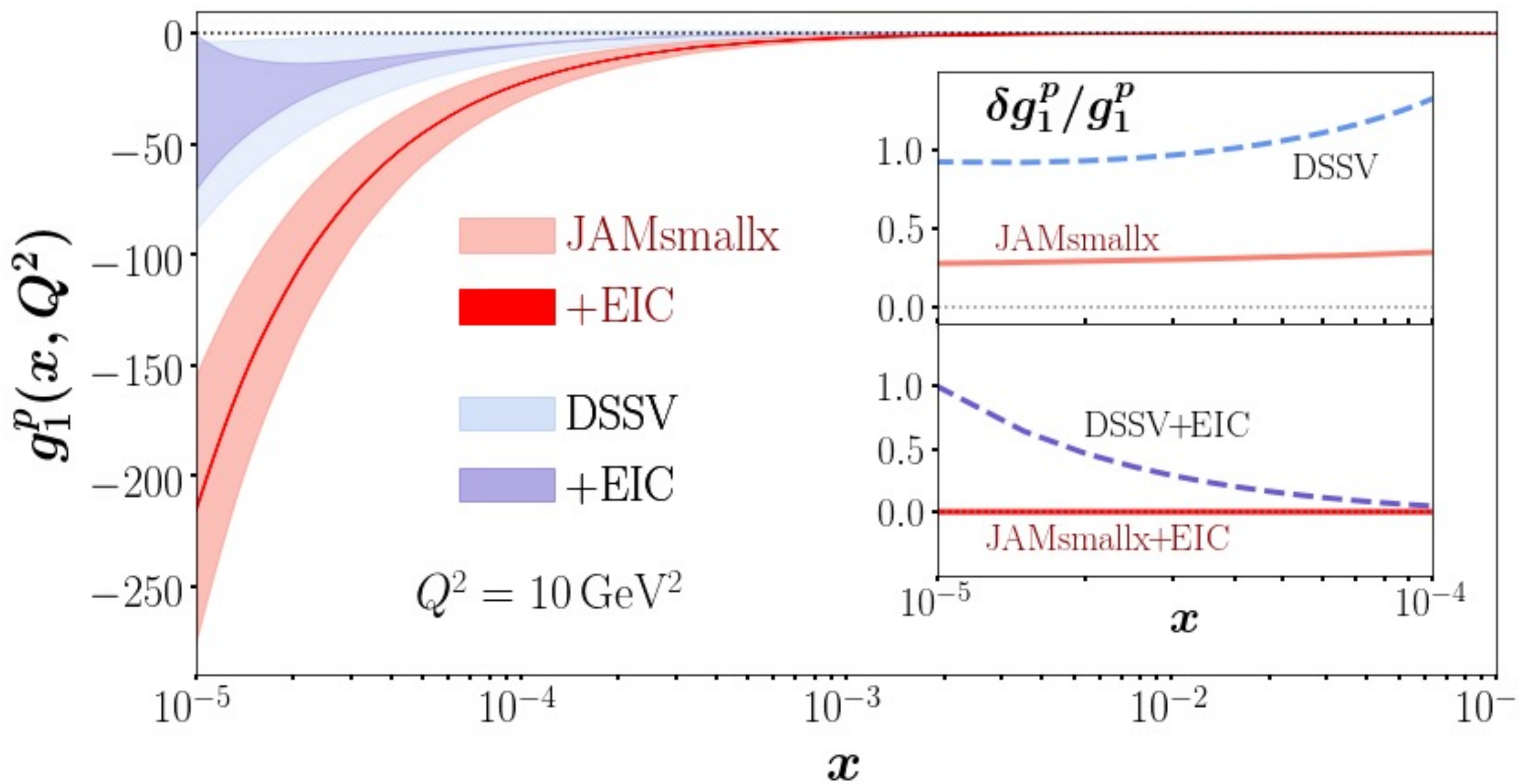


Where to start small-x evolution

- The evolution starts at $x=x_0$, and continues toward smaller x.
- The quality of our fit rapidly deteriorates for $x_0>0.2$, as expected from a small-x approach.
- In unpolarized BK/JIMWLK evolution, typically $x_0=0.01$, so the fact that our fit works up to such a high x_0 is quite remarkable.



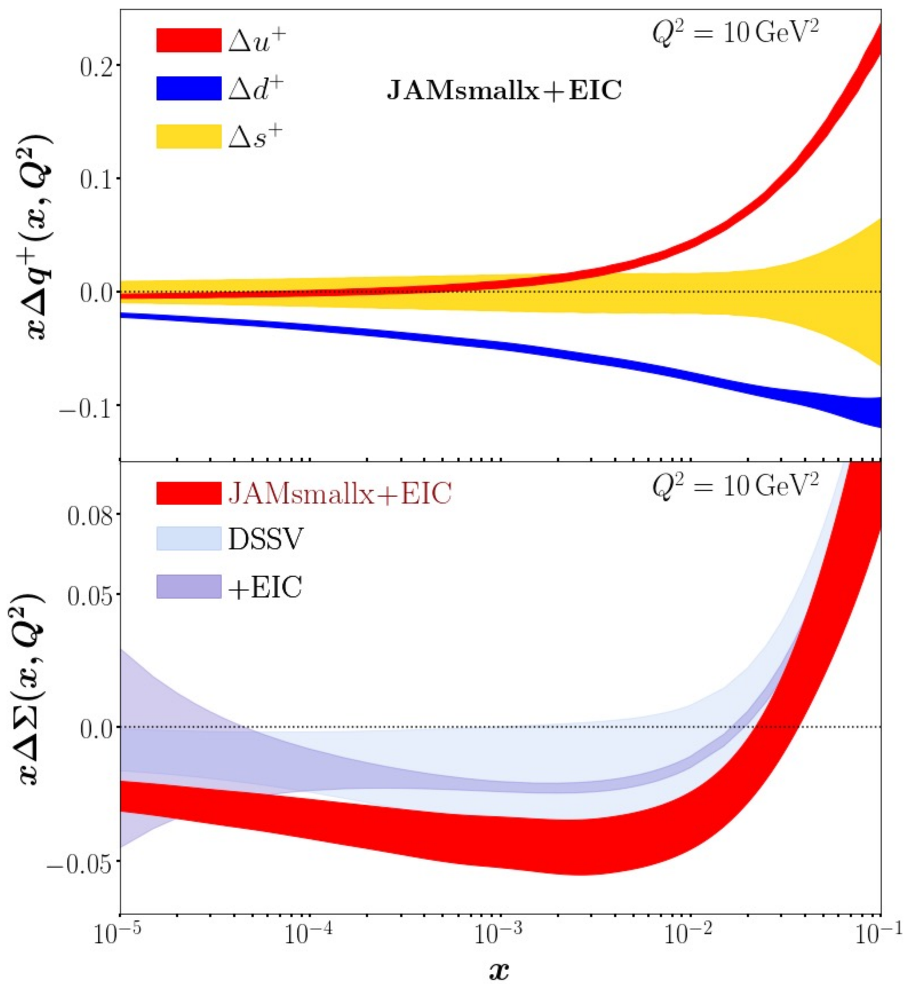
Prediction for g_1 structure function



$$g_1(x, Q^2) = \frac{1}{2} \sum_f e_f^2 \left[\Delta q_f + \Delta q_{\bar{f}}\right]$$

Thick band: 1σ CL; thin band: impact of EIC data. With the EIC pseudo-data we have 1096 data points.

Predictions for helicity PDFs



D. Adamiak, W. Melnitchouk,
D. Pitonyak, N. Sato, M. Sievert & YK,
2102.06159 [hep-ph], in the
JAM Collaboration framework.

$$\Delta q^+ = \Delta q + \Delta \bar{q}$$

$$\Delta\Sigma(x, Q^2) = \sum_f \left[\Delta q_f + \Delta q_{\bar{f}}\right]$$

Our (red) error band does not explode in the unmeasured region.

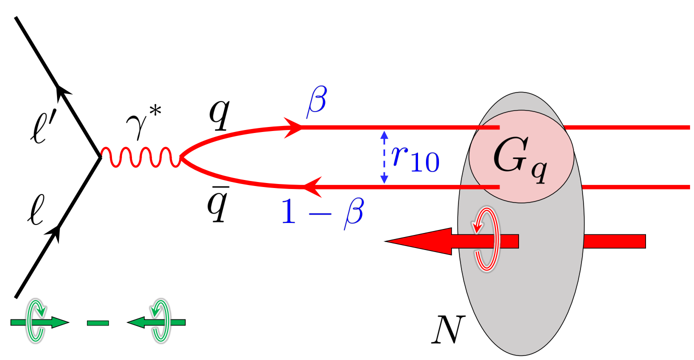
If we plug in $\alpha_s = 0.25$
we get $\alpha_h^q = 0.80$

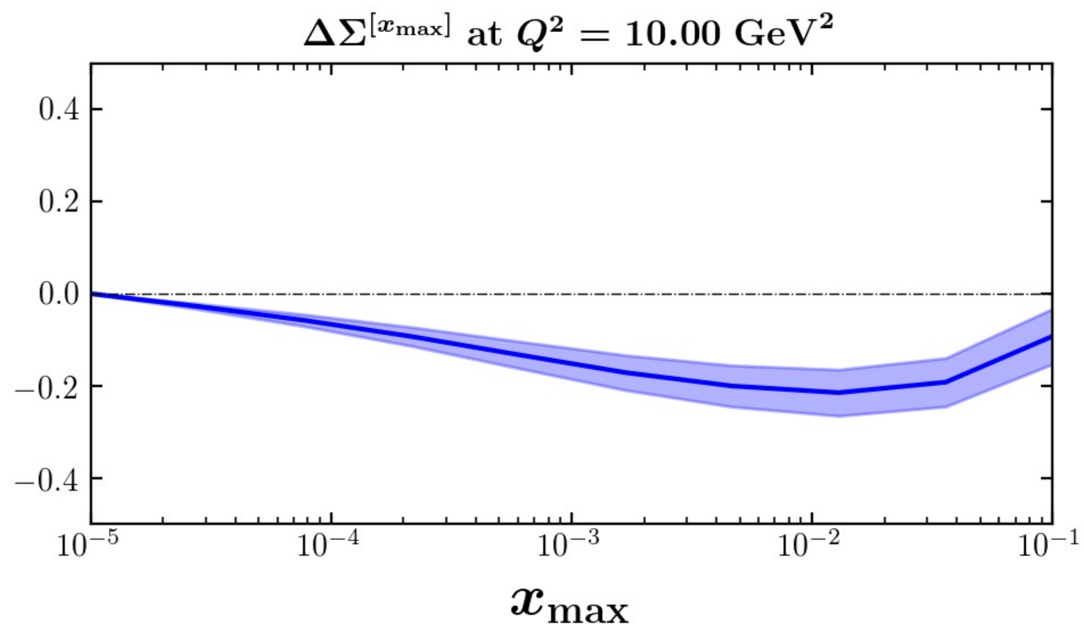
$$\Delta q(x, Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h^q} \quad \text{with} \quad \alpha_h^q = \frac{4}{\sqrt{3}}\sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 2.31\sqrt{\frac{\alpha_s N_c}{2\pi}}$$

Small-*x* quarks impact on the proton spin

- Potentially negative 10-20% of the proton spin may be carried by small-x quarks helicity (JAMsmall*x*, preliminary):

$$\Delta\Sigma^{[x_{max}]}(Q^2) = \int_{10^{-5}}^{x_{max}} dx\, \Delta\Sigma(x, Q^2)$$





Speculation on a path to resolving the spin puzzle

- Above we discussed quark helicity at small *x*. Let's add the orbital angular momentum (OAM) (Hatta & Yang, '18; YK '19):

$$\frac{1}{2}\,\Delta\Sigma(x,Q^2) + L_{q+\bar{q}}(x,Q^2) = -\frac{1}{2}\,\Delta\Sigma(x,Q^2)$$

- So, the net quark (1/2) helicity+OAM = (-1/2) helicity.
- For x<0.001 we thus expect (preliminary!)

$$\left[\frac{1}{2}\Delta\Sigma + L_{q+\bar{q}}\right]_{Q^2=10\,\mathrm{GeV}^2,\ x<0.001} \approx -\frac{1}{2}\,(-0.2) = 0.1$$

JAMsmall*x*, preliminary, Adamiak, Melnitchouk, Pitonyak, Sato, Sievert, YK

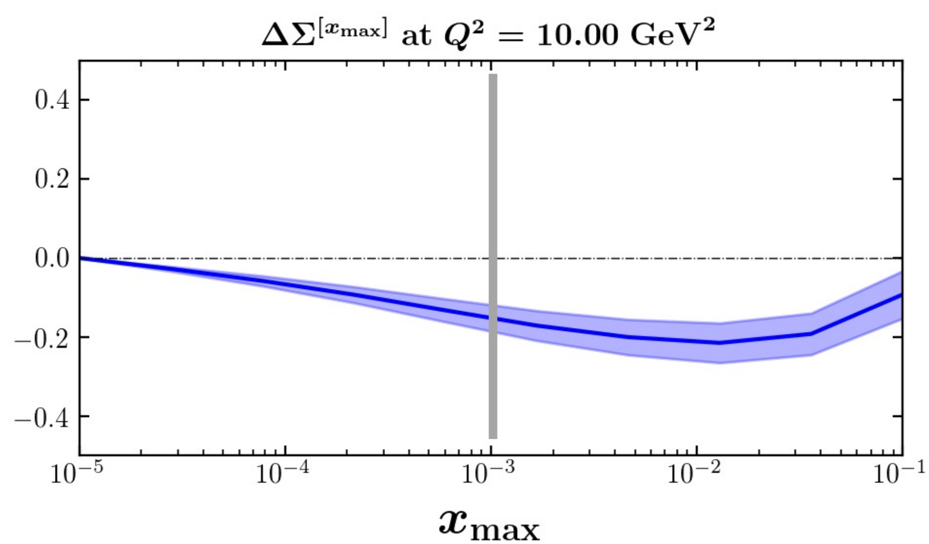
- Add to this the larger-*x* numbers

$$S_q(Q^2=10\,\mathrm{GeV}^2, x>0.001) \approx 0.18$$

$$S_G(Q^2=10\,\mathrm{GeV}^2, x>0.05) \approx 0.2$$

- We get

$$0.18 + 0.2 + 0.1 = \color{red}{0.48}$$



MC Implementation

- Not yet (for our small-x helicity evolution).

Conclusions

- Saturation effects at small *x* come in at two levels:

- Multiple rescattering / quasi—classical fields (GGM/MV). Initial conditions for evolution. Are included into Sartre event generator.

- Non-linear small-*x* evolution equations (BK/JIMWLK) can be thought of as a dipole cascade (at large N_c). Can be incorporated into a MC, probably included in DIPSY (see also Pythia8 and Herwig7).

- Small-*x* evolution for helicity PDFs and TMDs can also be written as a dipole cascade (at large N_c). May also be possible to include in a MC.

Backup Slides

Black Disk Limit

- Start with basic scattering theory: the final and initial states are related by the S-matrix operator,

$$|\psi_f\rangle = \hat{S}\,|\psi_i\rangle$$

- Write it as $$|\psi_f\rangle = |\psi_i\rangle + \left[\hat{S} - 1\right]\,|\psi_i\rangle$$

- The total cross section is

$$\sigma_{tot} \propto \left|\left[\hat{S} - 1\right]\,|\psi_i\rangle\right|^2 = 2 - S - S^*$$

where the forward matrix element of the S-matrix operator is

$$S = \langle\psi_i|\,\hat{S}\,|\psi_i\rangle$$

and we have used unitarity of the S-matrix

$$\hat{S}\,\hat{S}^\dagger = 1$$

Black Disk Limit

- Now, since $$|\psi_f\rangle = |\psi_i\rangle + \left[\hat{S} - 1\right] |\psi_i\rangle$$

  the elastic cross section is

  $$\sigma_{el} \propto \left| \langle\psi_i| \left[\hat{S} - 1\right] |\psi_i\rangle \right|^2 = |1 - S|^2$$

- The inelastic cross section can be found via

  $$\sigma_{tot} = \sigma_{inel} + \sigma_{el}$$

- In the end, for scattering with impact parameter b we write

$$\sigma_{tot} = 2 \int d^2b \, [1 - \mathrm{Re}\, S(b)]$$

$$\sigma_{el} = \int d^2b \, |1 - S(b)|^2$$

$$\sigma_{inel} = \int d^2b \left[1 - |S(b)|^2\right]$$

Unitarity Limit

- Unitarity implies that

$$1 = \langle\psi_i|\,\hat{S}\,\hat{S}^\dagger\,|\psi_i\rangle = \sum_X \langle\psi_i|\,\hat{S}|X\rangle\,\langle X|\hat{S}^\dagger\,|\psi_i\rangle \geq |S|^2$$

- Therefore

$$|S| \leq 1$$

leading to the unitarity bound on the total cross section

$$\sigma_{tot} = 2\int d^2b\;[1 - \mathrm{Re}\,S(b)] \leq 4\int d^2b = 4\pi R^2$$

- Notice that when S=-1 the inelastic cross section is zero and

$$\sigma_{tot} = 2\;\int d^2b\,[1 - \mathrm{Re}\,S(b)]$$

$$\sigma_{tot} = 4\pi R^2 = \sigma_{el}$$

$$\sigma_{el} = \int d^2b\,|1 - S(b)|^2$$

This limit is realized in low-energy scattering!

$$\sigma_{inel} = \int d^2b\left[1 - |S(b)|^2\right]$$

Black Disk Limit

- At high energy inelastic processes dominate over elastic. Imposing

$$\sigma_{inel} \geq \sigma_{el}$$

we get

$$\mathrm{Re}\, S \geq 0$$

- The bound on the total cross section is (aka the **black disk limit**)

$$\sigma_{tot} = 2\int d^2b\,[1 - \mathrm{Re}\, S] \leq 2\int d^2b = 2\pi R^2$$

- The inelastic and elastic cross sections at the black disk limit are

$$\sigma_{inel} = \sigma_{el} = \pi R^2$$

$$\sigma_{tot} = 2\int d^2b\,[1 - \mathrm{Re}\, S(b)]$$

$$\sigma_{el} = \int d^2b\,|1 - S(b)|^2$$

$$\sigma_{inel} = \int d^2b\left[1 - |S(b)|^2\right]$$

Notation

- At high energies $$\mathrm{Im}\, S \approx 0$$

  while the dipole amplitude N is the imaginary part of the T-matrix (S=1+iT), such that
  $$\mathrm{Re}\, S = 1 - N$$

- The cross sections are
  $$\sigma_{tot} = 2 \int d^2b\, N(x_\perp, b_\perp)$$
  $$\sigma_{el} = \int d^2b\, N^2(x_\perp, b_\perp)$$
  $$\sigma_{inel} = \int d^2b \left[2\, N(x_\perp, b_\perp) - N^2(x_\perp, b_\perp)\right]$$

- We see that N=1 is the black disk limit. Hence $N \leq 1$ as we saw above.

Small-x Evolution at large N_c&N_f

- The resulting equations are

$$Q_{10}(zs) = Q_{10}^{(0)}(zs) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int\limits_{1/(z's)}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left\{ \frac{1}{2}\, \Gamma_{02,21}^{adj}(z') + \frac{1}{2}\, G_{21}^{adj}(z') + Q_{12}(z') - \bar{\Gamma}_{01,21}(z') \right\}$$

$$+ \frac{\alpha_s\, N_c}{4\,\pi} \int\limits_{\Lambda^2/s}^{z} \frac{dz'}{z'} \int\limits_{1/(z's)}^{x_{10}^2 z/z'} \frac{dx_{21}^2}{x_{21}^2}\, Q_{21}(z'),$$

$$G_{10}^{adj}(z) = G_{10}^{adj\,(0)}(z) + \frac{\alpha_s\, N_c}{2\pi} \int\limits_{\max\{\Lambda^2, 1/x_{10}^2\}/s}^{z} \frac{dz'}{z'} \int\limits_{1/(z'\,s)}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[\Gamma_{10,21}^{adj}(z') + 3\, G_{21}^{adj}(z') \right]$$

$$- \frac{\alpha_s\, N_f}{2\,\pi} \int\limits_{\Lambda^2/s}^{z} \frac{dz'}{z'} \int\limits_{1/(z'\,s)}^{x_{10}^2 z/z'} \frac{dx_{21}^2}{x_{21}^2}\, \bar{\Gamma}_{02;21}(z'),$$

$$\Gamma_{10,21}^{adj}(z') = \Gamma_{10,21}^{adj\,(0)}(z') + \frac{\alpha_s\, N_c}{2\pi} \int\limits_{\max\{\Lambda^2, 1/x_{10}^2\}/s}^{z'} \frac{dz''}{z''} \int\limits_{1/(z''\,s)}^{\min\{x_{10}^2, x_{21}^2 z'/z''\}} \frac{dx_{32}^2}{x_{32}^2} \left[\Gamma_{10,32}^{adj}(z'') + 3\, G_{32}^{adj}(z'') \right]$$

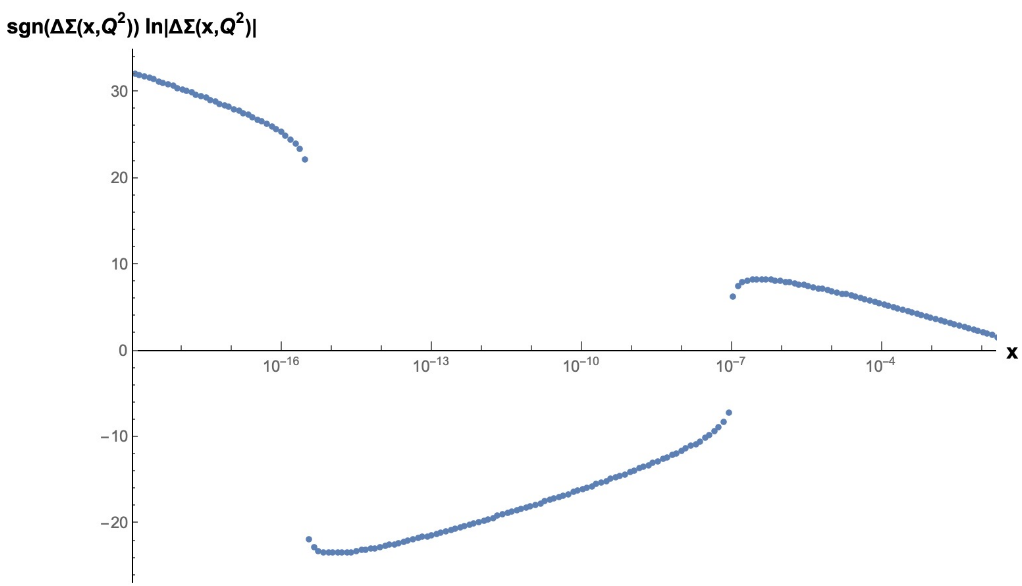
$$- \frac{\alpha_s\, N_f}{2\,\pi} \int\limits_{\Lambda^2/s}^{z'} \frac{dz''}{z''} \int\limits_{1/(z''\,s)}^{x_{21}^2 z'/z''} \frac{dx_{32}^2}{x_{32}^2}\, \bar{\Gamma}_{03;32}(z''),$$

$$\bar{\Gamma}_{10,21}(z') = \bar{\Gamma}_{10,21}^{(0)}(z') + \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z'} \frac{dz''}{z''} \int\limits_{1/(z''s)}^{\min\{x_{10}^2, x_{21}^2 z'/z''\}} \frac{dx_{32}^2}{x_{32}^2} \left\{ \frac{1}{2}\, \Gamma_{03,32}^{adj}(z'') + \frac{1}{2}\, G_{32}^{adj}(z'') + Q_{32}(z'') - \bar{\Gamma}_{01,32}(z \right.$$

$$+ \frac{\alpha_s\, N_c}{4\,\pi} \int\limits_{\Lambda^2/s}^{z'} \frac{dz''}{z''} \int\limits_{1/(z''s)}^{x_{21}^2 z/z'} \frac{dx_{32}^2}{x_{32}^2}\, Q_{32}(z'').$$

Small-x Asymptotics at large N_c&N_f

- Large N_c&N_f can be solved only numerically, due to their complexity.
- This was done by Y. Tawabutr and myself in arXiv:2005.07285 [hep-ph].
- The solution exhibits an interesting qualitative change compared to large-N_c: it oscillates with ln(1/x)!



$$\alpha_h^q \approx 2.3\sqrt{\frac{\alpha_s\, N_c}{2\pi}}$$

$$\omega_q \approx \frac{0.22 N_f}{1+0.1265 N_f}\sqrt{\frac{\alpha_s\, N_c}{2\pi}}$$

$$\Delta\Sigma(x,Q^2)\Big|_{\text{large-}N_c\&N_f} \sim \left(\frac{1}{x}\right)^{\alpha_h^q} \cos\left[\omega_q \ln\frac{1}{x} + \varphi_q\right]$$

Helicity JIMWLK

- To go beyond the large-N_c and large-N_c&N_f limits need to write a helicity analogue of JIMWLK evolution.
- This has been done recently (F. Cougoulic, YK, arXiv:1910.04268 [hep-ph], arXiv:2005.14688 [hep-ph]):

$$W_\tau[\alpha, \beta, \psi, \bar{\psi}] = W_\tau^{(0)}[\alpha, \beta, \psi, \bar{\psi}] + \int d^3\tau' \ \mathcal{K}_h[\tau, \tau'] \cdot W_{\tau'}[\alpha, \beta, \psi, \bar{\psi}]$$

with $\tau \equiv \{z, z\, X_\perp^2, z\, Y_\perp^2\}$ and the kernel

Life-time ordered!

$$\mathcal{K}_h[\tau, \tau'] = \frac{\alpha_s}{\pi^2} \int d^2 w_\perp \frac{\underline{X}' \cdot \underline{Y}'}{X'^2\, Y'^2} \ \theta^{(3)}(\tau - \tau') \theta\left(z' - \frac{\Lambda^2}{s}\right) \theta\left(X'^2 - \frac{1}{z'\, s}\right) \theta\left(Y'^2 - \frac{1}{z'\, s}\right)$$

$$\times \left\{ U_w^{ba} D^+_{x,a,<}\, D^+_{y,b,>} - \frac{1}{2}\left(D^+_{x,a,<} D^+_{y,a,<} + D^+_{x,a,>}\, D^+_{y,a,>}\right) \right.$$ Regular JIMWLK

$$+ \frac{1}{2} U_w^{pol,ba} (D^+_{x,a,<} D^\perp_{y,b,>} + D^\perp_{x,a,<} D^+_{y,b,>})$$ Polarized gluon emissions

$$\left. + \left(\frac{1}{2}\gamma^5\gamma^-\right)_{\beta\alpha} \frac{1}{2}\left((V_w^{pol})_{ij} D^{\bar{\psi}}_{x,j,\alpha,<} D^{\psi}_{y,i,\beta,>} + (V_w^{pol\,\dagger})_{ij} D^{\bar{\psi}}_{x,j,\alpha,>} D^{\psi}_{y,i,\beta,<} \right) \right\}$$ Polarized quark emissions