Resummation at small x

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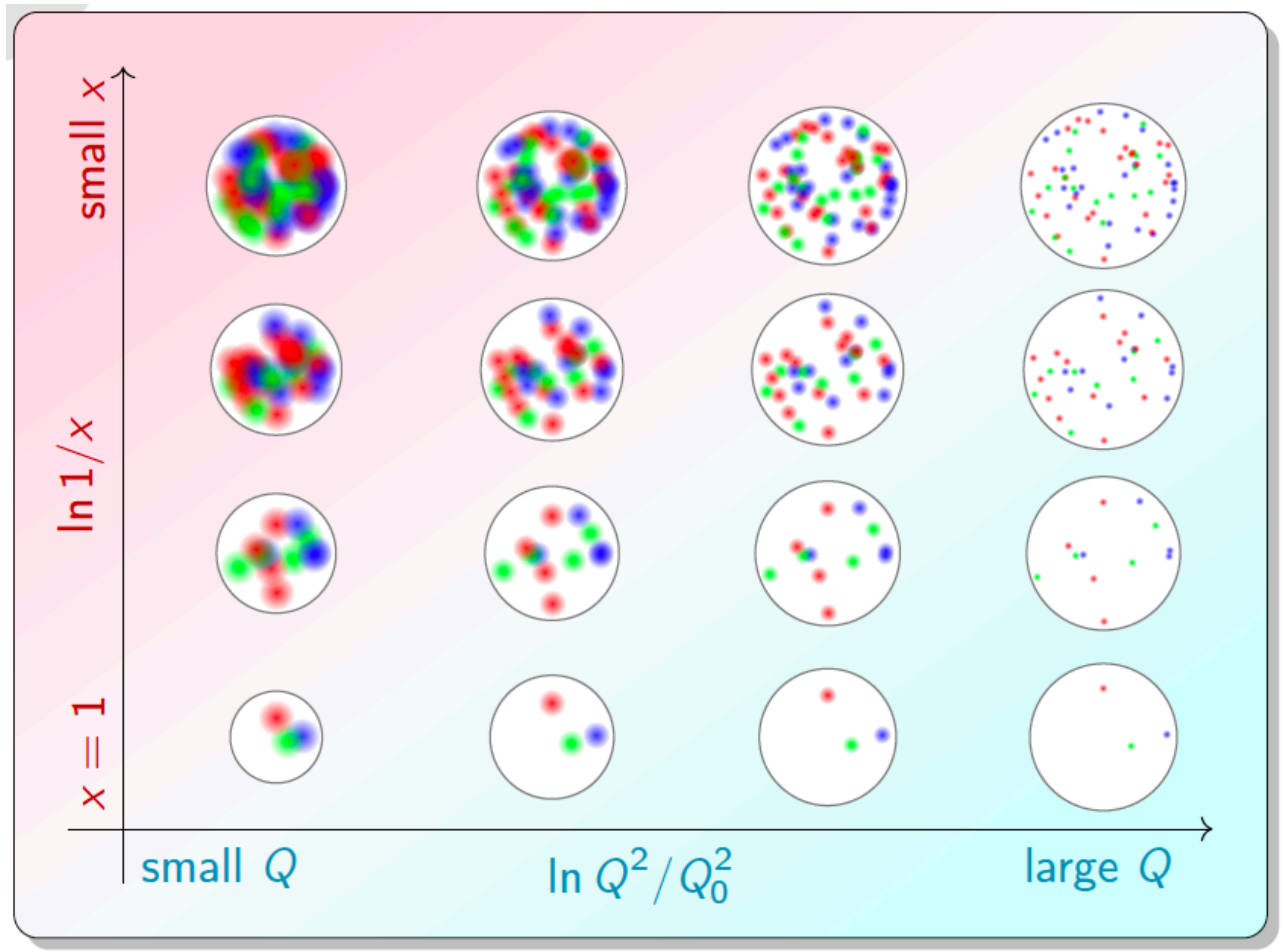
RBRC Workshop, Small x Physics in the EIC era, December 15, 2021

Obligatory small-x physicist plot: (x,Q) plane



Question: *what is the range of applicability of standard collinear formalism with DGLAP evolution and the calculations with low x effects (including saturation)?*

One possible answer: *it depends on the process*

Another answer: *it depends on the accuracy of calculation in both cases. Is it possible to extend the region of validity of any of these approaches through the resummation?*

High energy limit

$$\boxed{\sqrt{s} \to \infty, x \to 0}$$

Energy much larger than any other scale in the process

At small x there are large logs. Splitting function :

$$xP_{gg} \sim \alpha_s^n \ln^{n-1}(1/x)$$

At high energy, or small x we can have:

$$\boxed{\alpha_S \ln 1/x \sim 1}$$

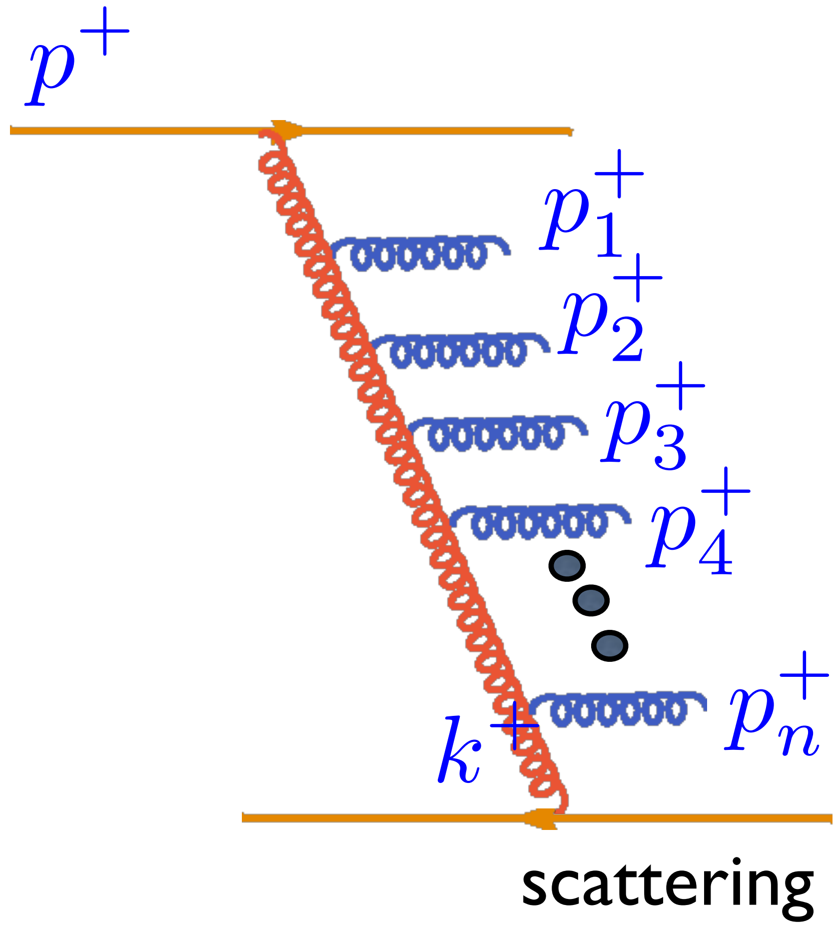
Need to resum them as well to all orders:

$$\boxed{(\alpha_S \ln 1/x)^n}$$

Any fixed order here would not be sufficient, potentially very large corrections.

Many soft gluon emissions in small x limit

Cascade of the n soft gluons



Strong ordering (in longitudinal momenta)

$$p^+ \gg p_1^+ \gg p_2^+ \gg \cdots \gg p_n^+ \gg k^+$$

Note: transverse momenta are not ordered

$$k^+ = xp^+$$

$$\frac{\alpha_s N_c}{\pi} \int_{k^+}^{p^+} \frac{dp_1^+}{p_1^+} = \frac{\alpha_s N_c}{\pi} \ln \frac{1}{x}$$

Large logarithm

Nested logarithmic integrals

$$\left(\frac{\alpha_s N_c}{\pi} \ln \frac{1}{x} \right)^n$$

Resummation of the gluon emissions performed by the equation

$$\frac{df_g(x, k_T^2)}{d \ln 1/x} = \frac{\alpha_s N_c}{\pi} \int d^2 k'_T \mathcal{K}(k_T, k'_T) \, f_g(x, k'_T)$$

integral over transverse momenta

kernel describing branching of gluons

unintegrated gluon density

I.Balitsky, V.Fadin, E.Kuraev, L.Lipatov

Evolution equation in longitudinal momenta

$$\frac{df_g(x,k_T^2)}{d\ln 1/x} = \frac{\alpha_s N_c}{\pi}\int d^2k'_T \mathcal{K}(k_T,k'_T)\, f_g(x,k'_T)$$

Solution:

$$f_g(x,k_T) \sim x^{-\omega_P}$$

$$\omega_P = j-1 = \frac{\alpha_s N_c}{\pi} 4\ln 2$$

Leading exponent(spin)

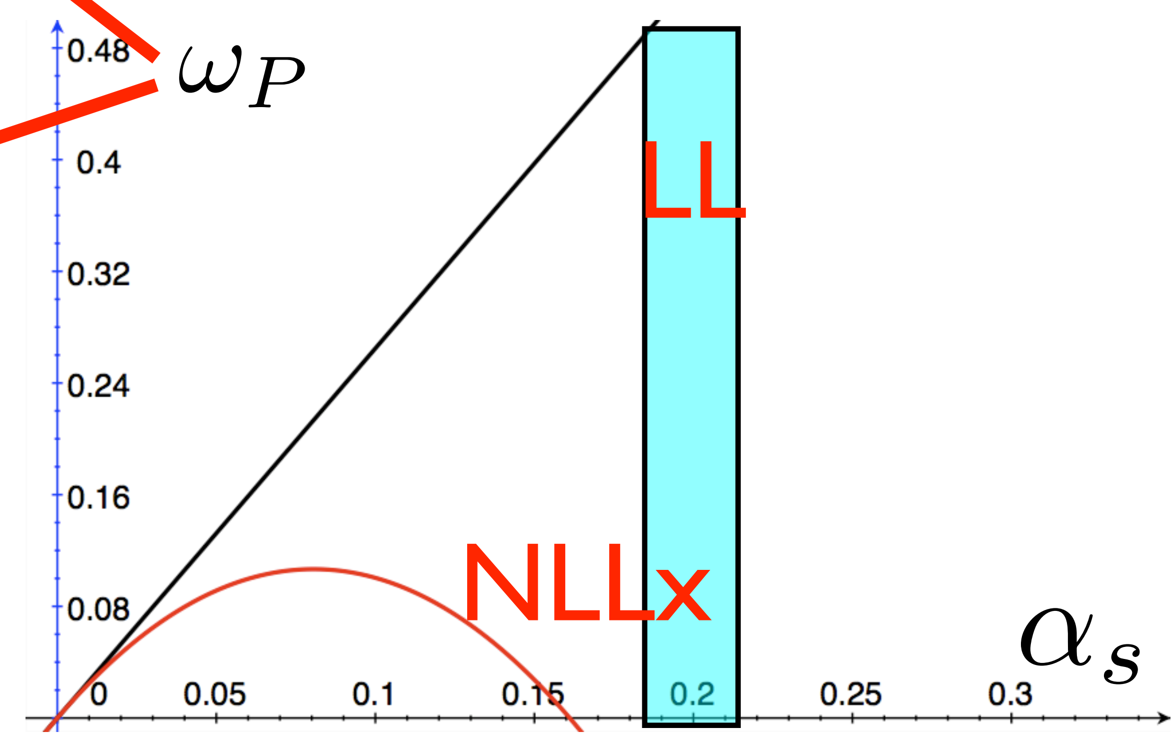
$$\sigma^{DIS}_{\gamma^* p} \sim s^{\omega_P}$$

Rise too strong for the data!

Take higher order corrections.

$$\alpha_s \mathcal{K}_0 + \alpha_s^2 \mathcal{K}_1 + \dots$$

V.Fadin,L.Lipatov, G.Camici,M.Ciafaloni



$$\omega_P \simeq \bar{\alpha}_s 4\ln 2(1-6.5\bar{\alpha}_s)$$

relevant values of α_s

Very large next-to-leading correction! Problems with convergence.

Why NLLx is so large in BFKL?

- Strong coupling constant is not a naturally small parameter in the Regge limit: $s \gg |t|, \Lambda^2_{QCD}$ but $\alpha_s(\mu^2),\ \mu^2 \neq s$

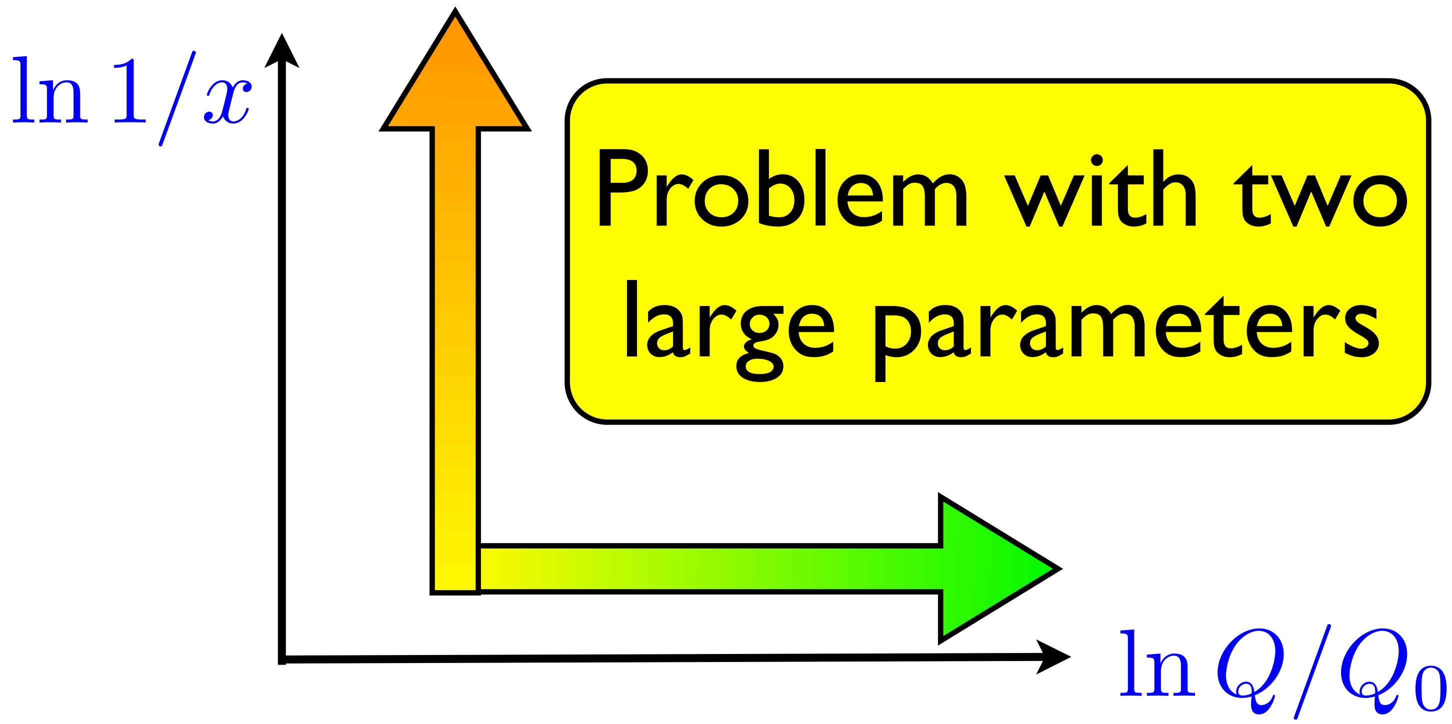
- Regge limit is inherently nonperturbative.

- Compare DGLAP (collinear approach): $Q^2 \gg \Lambda^2$ and $\alpha_s(Q^2) \ll 1$

- No momentum sum rule, since the evolution is local in x. In DGLAP: momentum sum rule satisfied at each order due to the initial assumption of the collinearity of the partons and the non-locality of the evolution in x.

- Approximations in the phase space (multi-Regge kinematics, quasi multi-Regge kinematics, etc..) cannot be recovered by the (fixed number of) the higher orders of expansion in the coupling constant.

Resummation



$$\left(\frac{\alpha_s N_c}{\pi}\ln\frac{1}{x}\right)^n$$ energy

$$\left(\frac{\alpha_s N_c}{\pi}\ln\frac{Q}{Q_0}\right)^n$$ scale (related to transverse momentum)

Mellin variables: $\gamma \leftrightarrow \ln k_T^2$ $\qquad \omega \leftrightarrow \ln 1/x$

Kernel in Mellin space $$\chi(\gamma) = \int \frac{dk'^2}{k^2} K(k^2, k'^2) \left(\frac{k'^2}{k^2}\right)^\gamma$$

Anomalous dimension $$\gamma(\omega) = \int dz P(z) z^{-\omega}$$

Resummation

linear case

Anderson,Gustafson,Kharraziha,Samuelson Z.Phys. C71(1996) 613

Kwiecinski, Martin, Sutton Z.Phys. C71(1996) 585; Kwiecinski,Martin,AS Phys.Rev. D56 (1997) 3991

Salam JHEP 9807 (1998) 19; Ciafaloni,Colferai,Salam,AS Phys.Rev. D68(2003) 114003

Altarelli,Ball,Forte Nucl.Phys. B575(2000) 313; Bonvini,Marzani,Perano Eur. Phys. J C76(2016) 597.

Thorne Phys. Rev. D64 (2001) 074005

Sabio-Vera Nucl. Phys. B722 (2005) 65.

Brodsky,Fadin,Kim,Lipatov,Pivovarov JETP Lett. 70 (1999) 155.

nonlinear case

selected literature…

Motyka,AS Phys. Rev.D79(2009) 085016;

Beuf Phys.Rev.D89(2014) 074039

Iancu,Madrigal,Mueller,Soyez; Phys.Lett. B744 (2015) 293;

Ducloue, Iancu,Mueller,Soyez; JHEP 04 (2019) 081;

Lappi,Mantysaari Phys.Rev.D93(2016) 094004

General setup

Ciafaloni, Colferai.
Salam, AS

- Kinematical constraint.
- DGLAP splitting function at LO and NLO.
- NLLx BFKL with suitable subtraction of terms included above.
- Momentum sum rule.
- Running coupling.
- Calculations done in momentum space, even though Mellin space used as a guidance.

LLx + NLLx

Representation of the kernel $\quad \mathcal{K} = \sum_{n=0}^{\infty} \bar{\alpha}_s^{n+1} \mathcal{K}_n \qquad \bar{\alpha}_s \equiv \frac{N_c \alpha_s}{\pi}$

Mellin variables: $\quad \gamma \leftrightarrow \ln k_T^2 \qquad \omega \leftrightarrow \ln 1/x$

LLx kernel in Mellin space

$$\chi_0(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma)$$

running coupling

triple poles

double poles

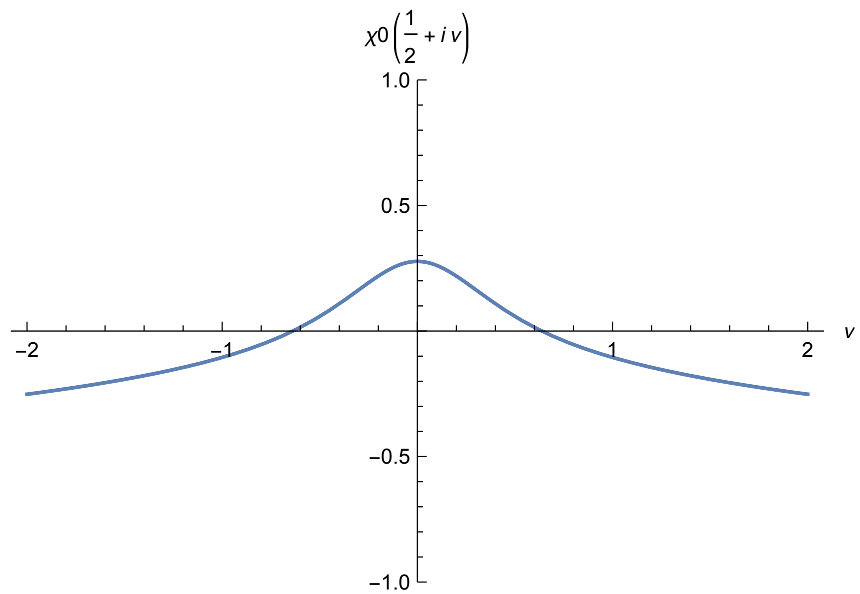
NLLx kernel in Mellin space

$$\chi_1(\gamma) = \boxed{-\frac{b}{2}[\chi_0^2(\gamma) + \chi_0'(\gamma)]} - \boxed{\frac{1}{4}\chi_0''(\gamma)} - \boxed{\frac{1}{4}\left(\frac{\pi}{\sin\pi\gamma}\right)^2} \frac{\cos\pi\gamma}{3(1-2\gamma)}\left(11 + \frac{\gamma(1-\gamma)}{(1+2\gamma)(3-2\gamma)}\right)$$
$$+ \left(\frac{67}{36} - \frac{\pi^2}{12}\right)\chi_0(\gamma) + \frac{3}{2}\zeta(3) + \frac{\pi^3}{4\sin\pi\gamma}$$
$$- \sum_{n=0}^{\infty}(-1)^n \left[\frac{\psi(n+1+\gamma) - \psi(1)}{(n+\gamma)^2} + \frac{\psi(n+2-\gamma) - \psi(1)}{(n+1-\gamma)^2}\right]$$

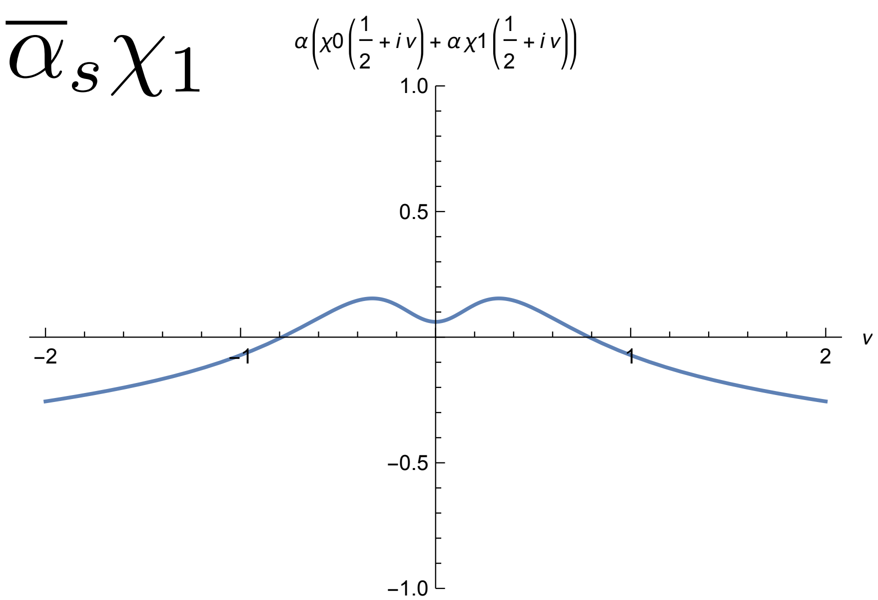
LLx + NLLx kernel

Kernel on imaginary axis

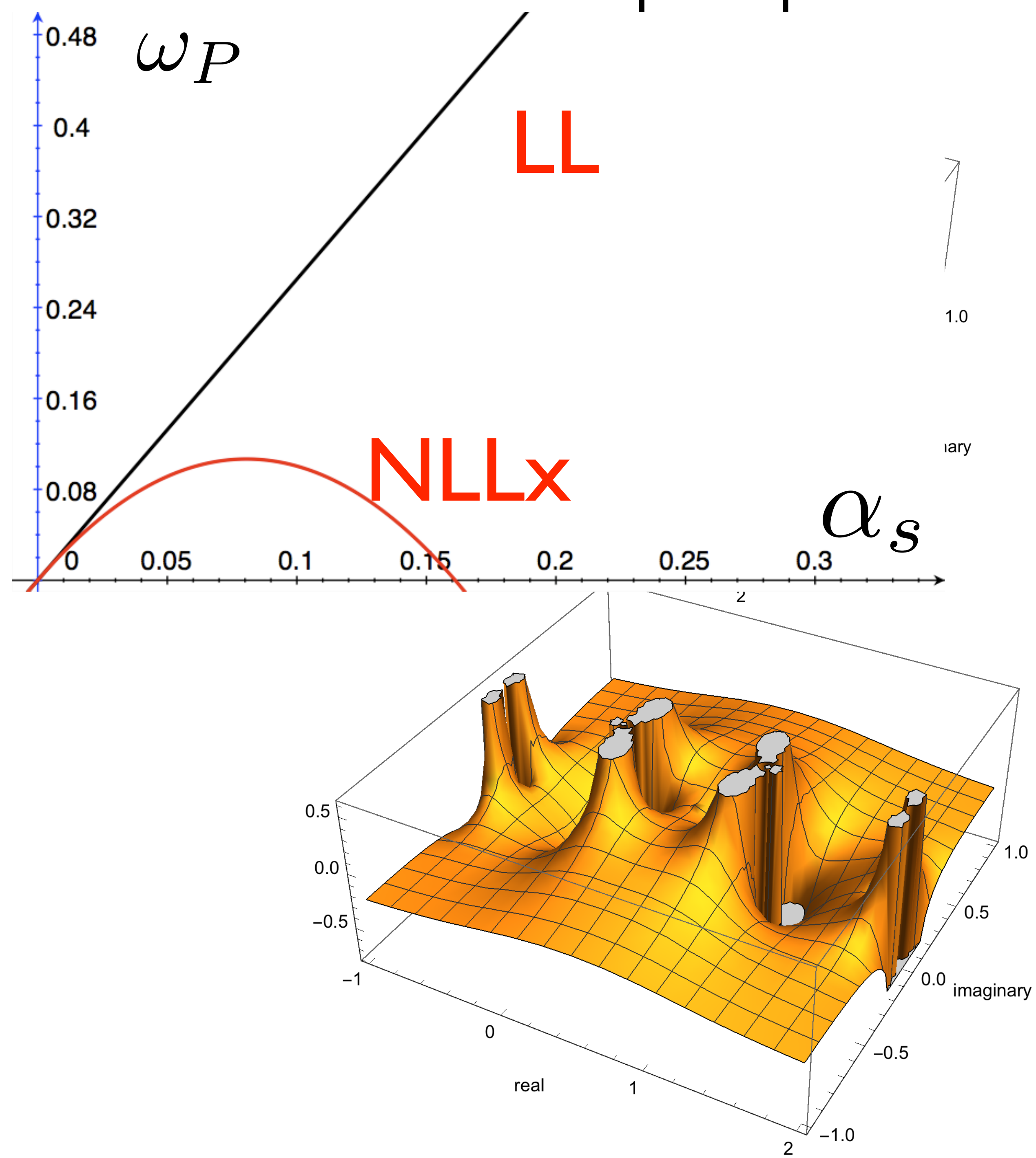
χ_0



$\chi_0 + \overline{\alpha}_s \chi_1$



Kernel in complex space



Two saddle points on complex plane at higher order: oscillating cross section

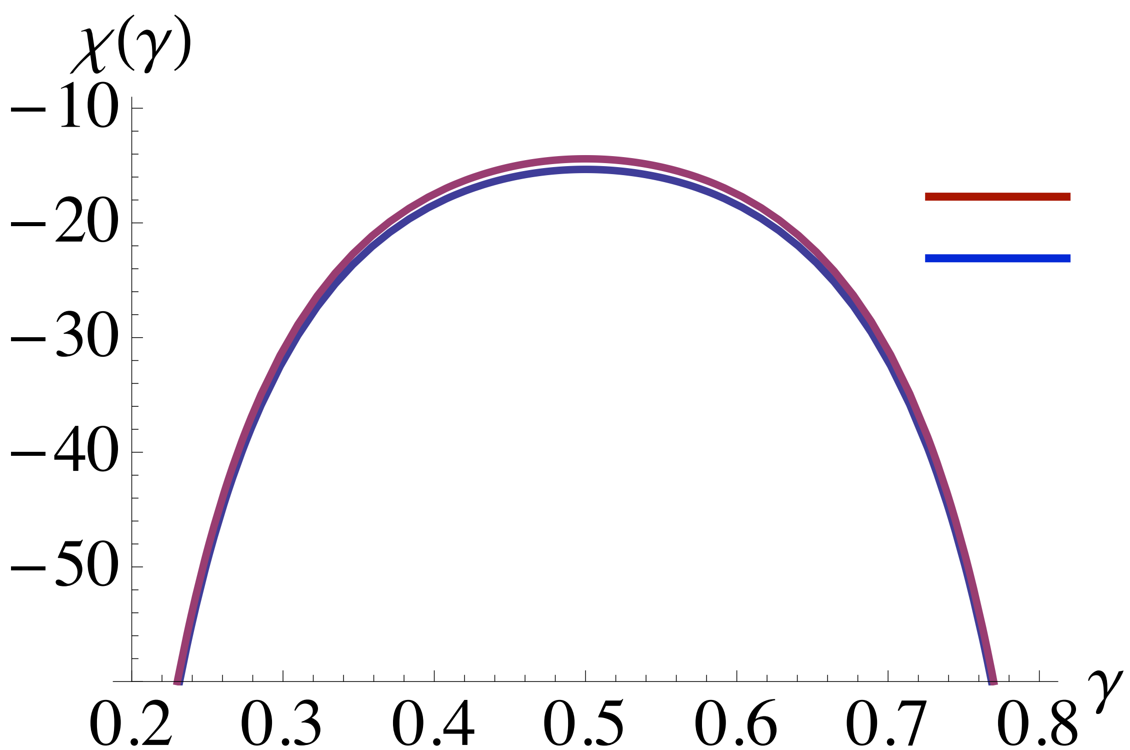
Collinear poles

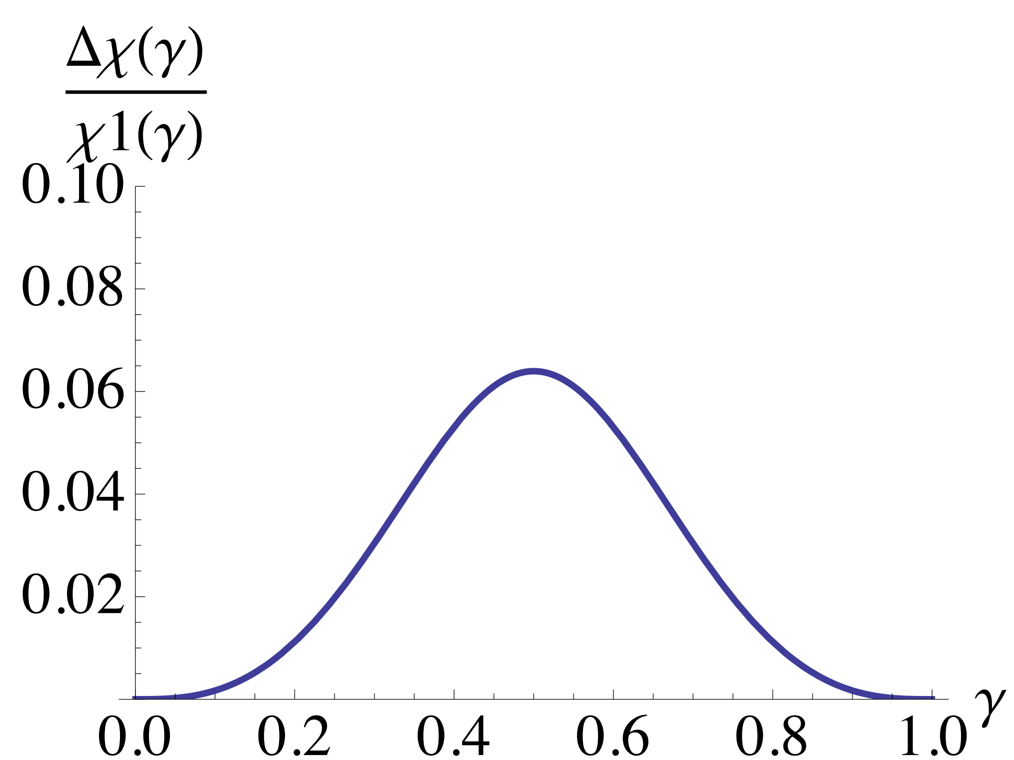
$$\chi_1^{\mathrm{coll}}(\gamma) = \boxed{-\frac{1}{2\gamma^3} - \frac{1}{2(1-\gamma)^3}} + \boxed{\frac{A_1(0)}{\gamma^2} + \frac{A_1(0)-b}{(1-\gamma)^2}}$$

double and triple poles of the NLL part

LO DGLAP anomalous dimension $\gamma_{gg}^{(0)}(\omega) = \frac{\bar{\alpha}_s}{\omega} + \bar{\alpha}_s A_1(\omega)$ $A_1(\omega) = -\frac{11}{12} + \mathcal{O}(\omega)$

Difference of about 7% at most





Scale choices

HE factorization for the cross section

$$\sigma_{AB}(s;Q,Q_0) = \int \frac{d\omega}{2\pi i}\,\frac{d^2\boldsymbol{k}}{\boldsymbol{k}^2}\,\frac{d^2\boldsymbol{k}_0}{\boldsymbol{k}_0^2}\left(\frac{s}{QQ_0}\right)^{\omega} h_\omega^A(Q,\boldsymbol{k})\;\mathcal{G}_\omega(\boldsymbol{k},\boldsymbol{k}_0)\;h_\omega^B(Q_0,\boldsymbol{k}_0)$$

BFKL equation for the gluon Green's function

$$\omega\mathcal{G}_\omega(\boldsymbol{k},\boldsymbol{k}_0) = \delta^2(\boldsymbol{k}-\boldsymbol{k}_0) + \int \frac{d^2\boldsymbol{k}'}{\pi}\,\mathcal{K}_\omega(\boldsymbol{k},\boldsymbol{k}')\;\mathcal{G}_\omega(\boldsymbol{k}',\boldsymbol{k}_0)$$

<u>Different possible scale choices:</u>

symmetric (ex. two jets) $\qquad \nu_0 = kk_0 \qquad k \sim k_0$

DIS type configuration

$\nu_0 = k^2 \qquad k \gg k_0$

$\nu_0 = k_0^2 \qquad k \ll k_0$

Similarity transformation

$$\mathcal{G}_\omega \to \left(\frac{k_>}{k_<}\right)^{\omega}\mathcal{G}_\omega$$

$$\mathcal{K}_\omega(k,k') \;\to\; \mathcal{K}_\omega^u(k,k') \;=\; \mathcal{K}_\omega(k,k')\left(\frac{k}{k'}\right)^{\omega}, \qquad \nu_0 = k^2\,,$$

$$\mathcal{K}_\omega(k,k') \;\to\; \mathcal{K}_\omega^l(k,k') \;=\; \mathcal{K}_\omega(k,k')\left(\frac{k'}{k}\right)^{\omega}, \qquad \nu_0 = k'^2\,,$$

Shift of poles

Shift of poles (symmetric case)

$$\chi_n^{\omega}(\gamma) = \chi_{nL}^{\omega}(\gamma + \tfrac{\omega}{2}) + \chi_{nR}^{\omega}(1 - \gamma + \tfrac{\omega}{2})$$

LL case with shifts

$$\chi_0^{\omega} = 2\psi(1) - \psi(\gamma + \tfrac{\omega}{2}) - \psi(1 - \gamma + \tfrac{\omega}{2})$$

Shift of poles (symmetric case) reproduces highest poles up to NNLO in sYM
(highest poles the same in QCD)

Exact result

$$\chi_1^{sYM} = -\frac{1}{2\gamma^3} - 1.79 + \mathcal{O}(\gamma) \, ,$$

$$\chi_2^{sYM} = \frac{1}{2\gamma^5} - \frac{\zeta(2)}{\gamma^3} - \frac{9\zeta(3)}{4\gamma^2} - \frac{29\zeta(4)}{8\gamma} + \mathcal{O}(1) \, .$$

Gromov, Levkovich-Maslyuk, Sizov;Velizhanin;
Caron-Huot, Herranen

From shifts

$$\chi_1(\gamma) = -\frac{1}{2\gamma^3} - \frac{0}{\gamma^2} + \dots$$

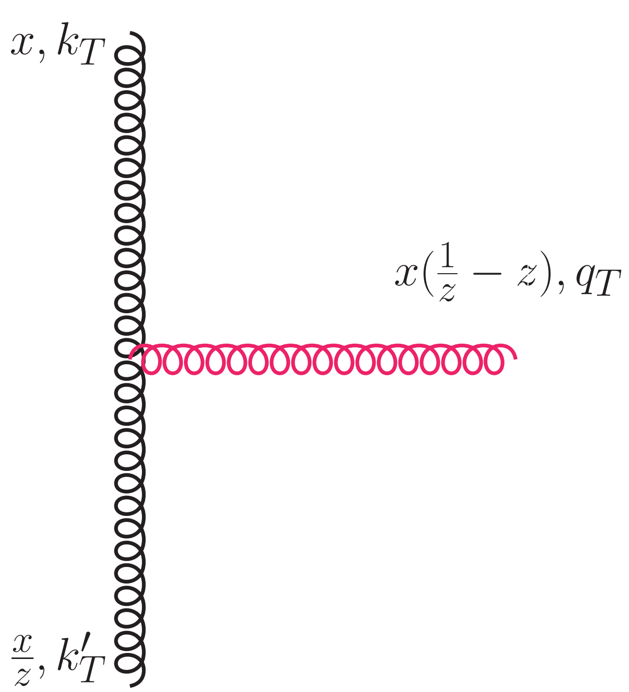
$$\chi_2(\gamma) = \frac{1}{2\gamma^5} - \frac{0}{\gamma^4} + \dots$$

Highest poles reproduced, lack of next-to-highest poles.

Deak, Kutak, Li, AS

Kinematical constraint

Shifts are equivalent to the kinematical constraints imposed on the transverse momenta in the ladder



Ciafaloni
Kwiecinski, Martin, Sutton;
Anderson, Gustafson, Kharazziha, Samuelson

$$k = (k^+, k^-, \mathbf{k}_T)$$

$$k^2 = k^+ k^- - k_T^2$$

Virtualities dominated by transverse components

$$|k^2| \simeq k_T^2$$

Kinematical constraint

$$k_T'^2 < \frac{k_T^2}{z}$$

Leads to the shift of the poles in the kernel

Resumed kernel

$$\tilde{\mathcal{K}}_\omega = \bar{\alpha}_s(\boldsymbol{q}^2) K_0^\omega + \omega \bar{\alpha}_s(k_>^2) K_c^\omega + \bar{\alpha}_s^2(k_>^2)\tilde{K}_1^\omega$$

LL with shifts

non-singular DGLAP

NLL with subtractions

$$\chi_0^\omega = 2\psi(1) - \psi(\gamma + \tfrac{\omega}{2}) - \psi(1 - \gamma + \tfrac{\omega}{2})$$

$$\chi_c^\omega(\gamma) = \frac{A_1(\omega)}{\gamma + \frac{\omega}{2}} + \frac{A_1(\omega)}{1 - \gamma + \frac{\omega}{2}} \ ,$$

$$\begin{aligned}\tilde{\chi}_1(\gamma) &= \chi_1(\gamma) - \chi_0^0(\gamma)[\chi_0^1(\gamma) + \chi_c^0(\gamma)] - \chi_0^{\mathrm{run}}(\gamma) \\ &= \chi_1(\gamma) + \frac{1}{2}\chi_0(\gamma)\frac{\pi^2}{\sin^2(\pi\gamma)} - \chi_0(\gamma)\frac{A_1(0)}{\gamma(1-\gamma)} + \frac{b}{2}(\chi_0' + \chi_0^2)\end{aligned}$$

Additional subtraction needed to satisfy the momentum sum rule.

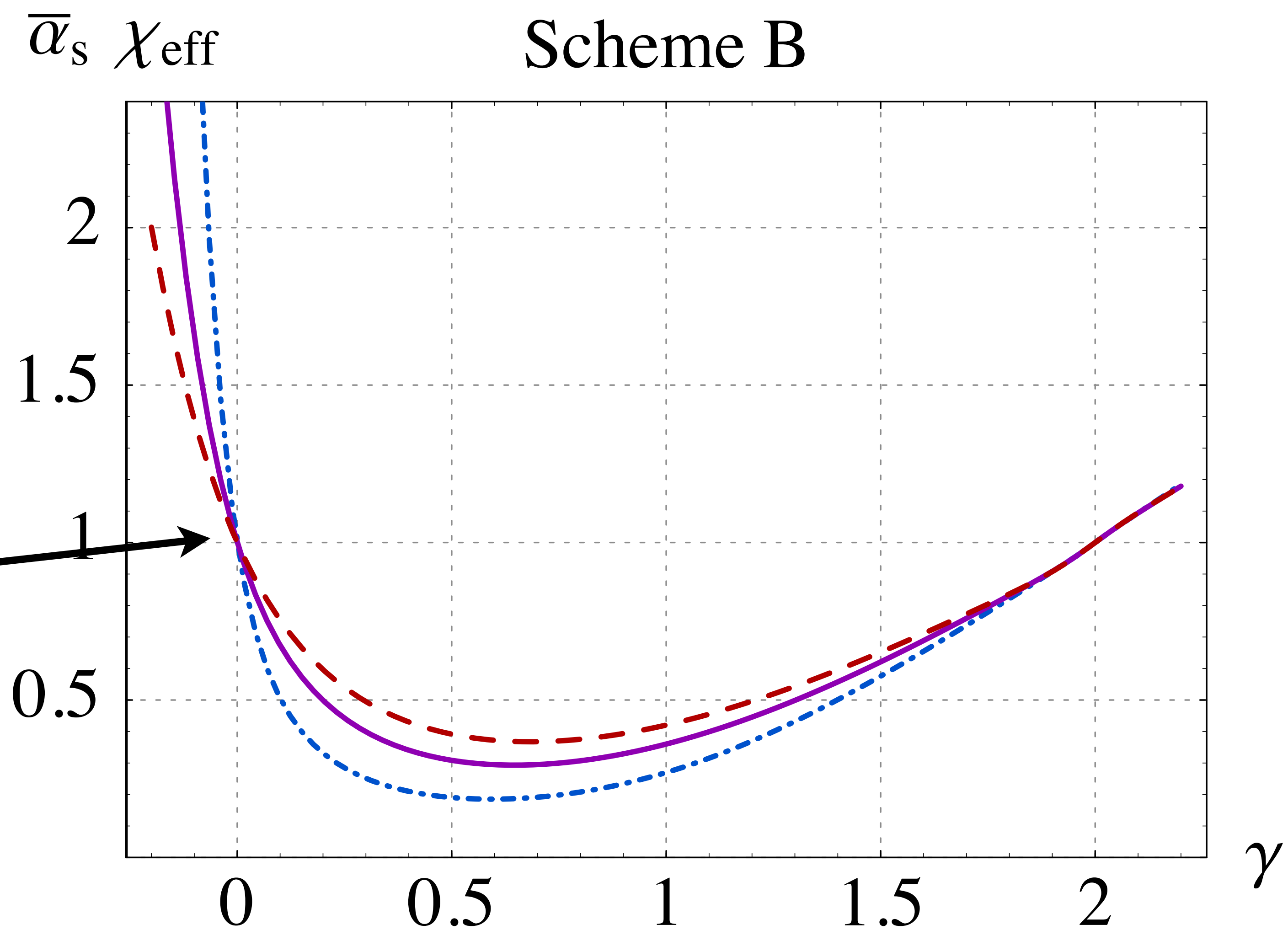
Most of the calculations are actually done in momentum space

Frozen coupling features

$$\bar{\alpha}_s \chi_\omega(\gamma, \bar{\alpha}_s) = \bar{\alpha}_s(\chi_0^\omega + \omega \chi_c^\omega) + \bar{\alpha}_s^2 \tilde{\chi}_1^\omega$$

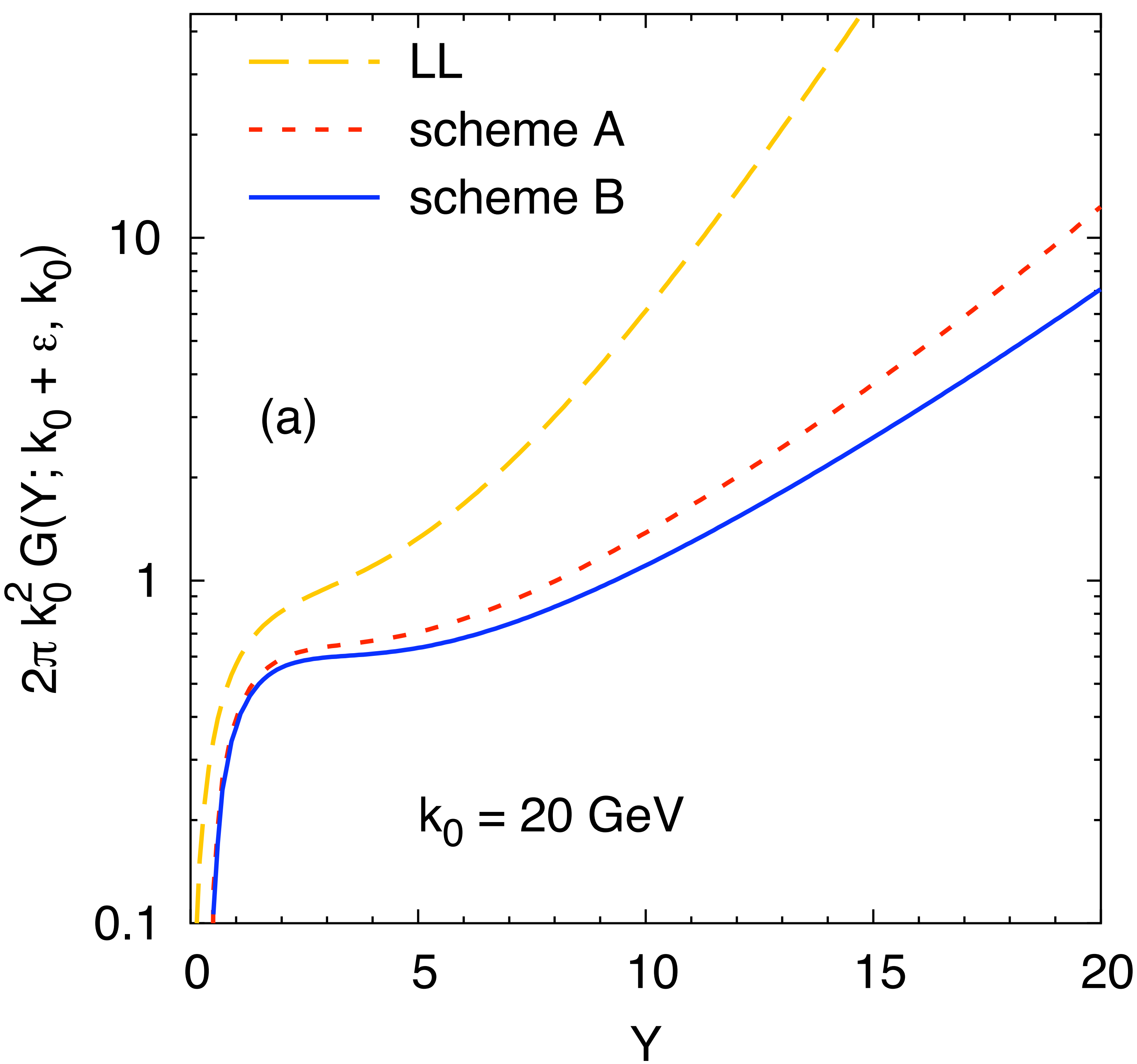
Effective characteristic function: $\omega = \bar{\alpha}_s \chi_{\text{eff}}^{(0)}(\gamma, \bar{\alpha}_s)$

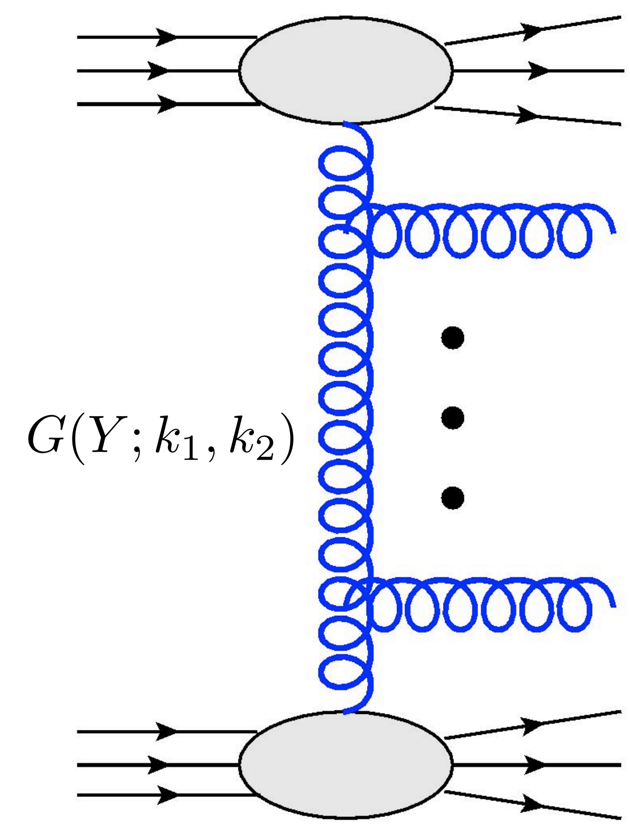
Fixed point (independent of the coupling). Energy momentum conservation



Gluon Green's function

Solution to the BFKL equation (gluon Green's function)
Single channel: gluons only.

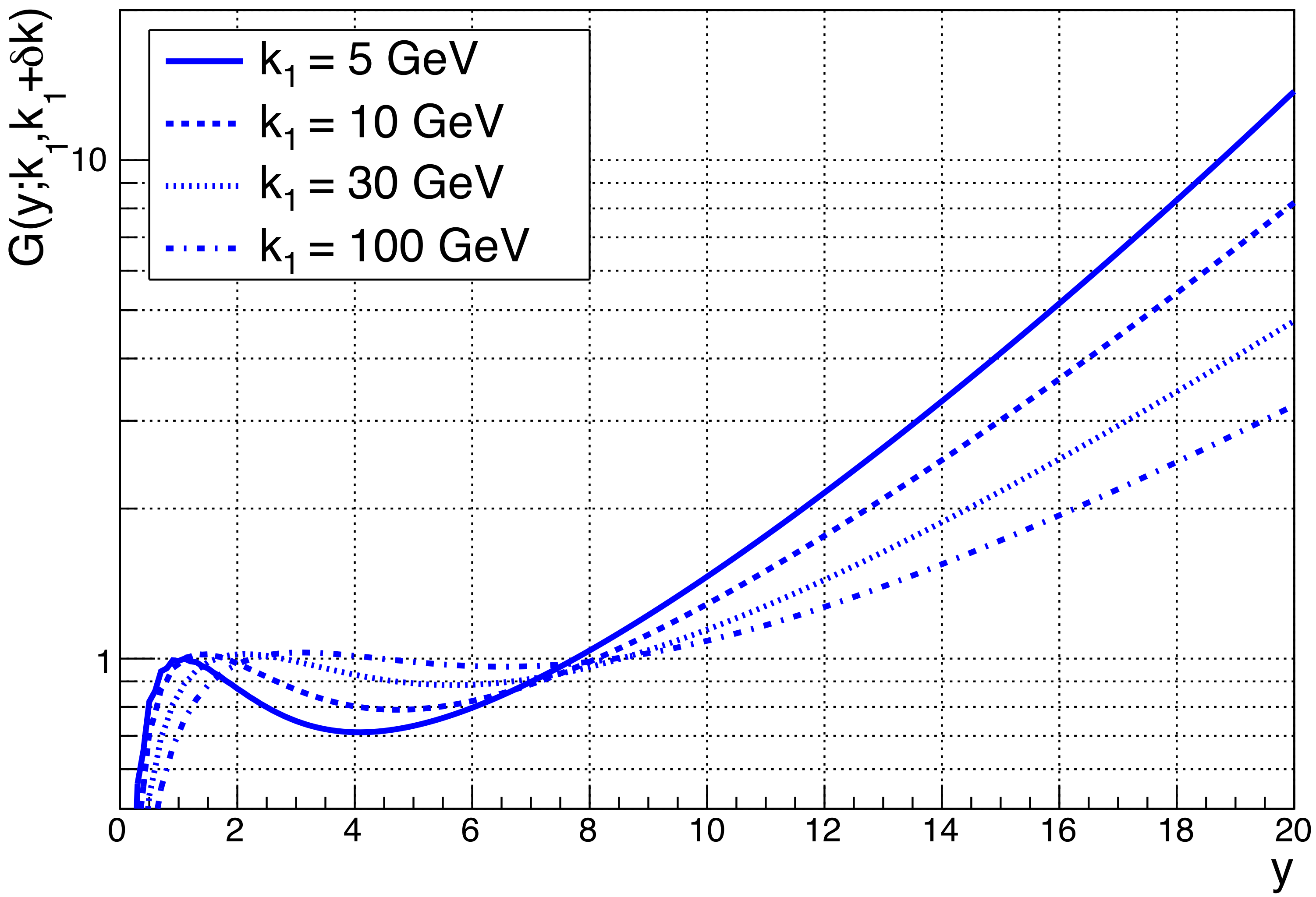




Large suppression as compared to LLx.

Two schemes, small differences.

Gluon Green's function



$$G(y; k, k + \delta k)$$

(Almost) equal scales
Small shift introduced to mitigate the effects of the numerical implementation of the 'delta' fuction

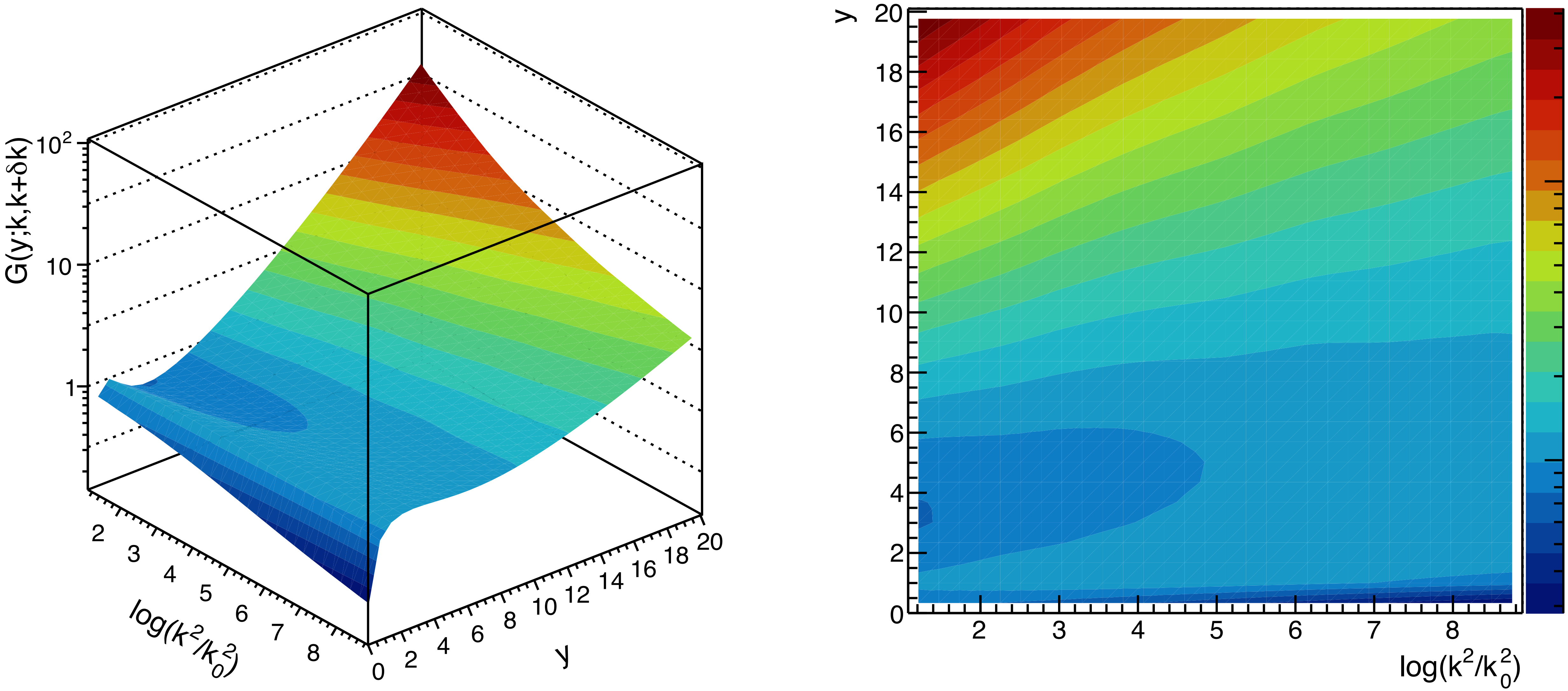
Effects of resummation:

Lowering effective power

Onset of small x rise delayed

Dip or plateau

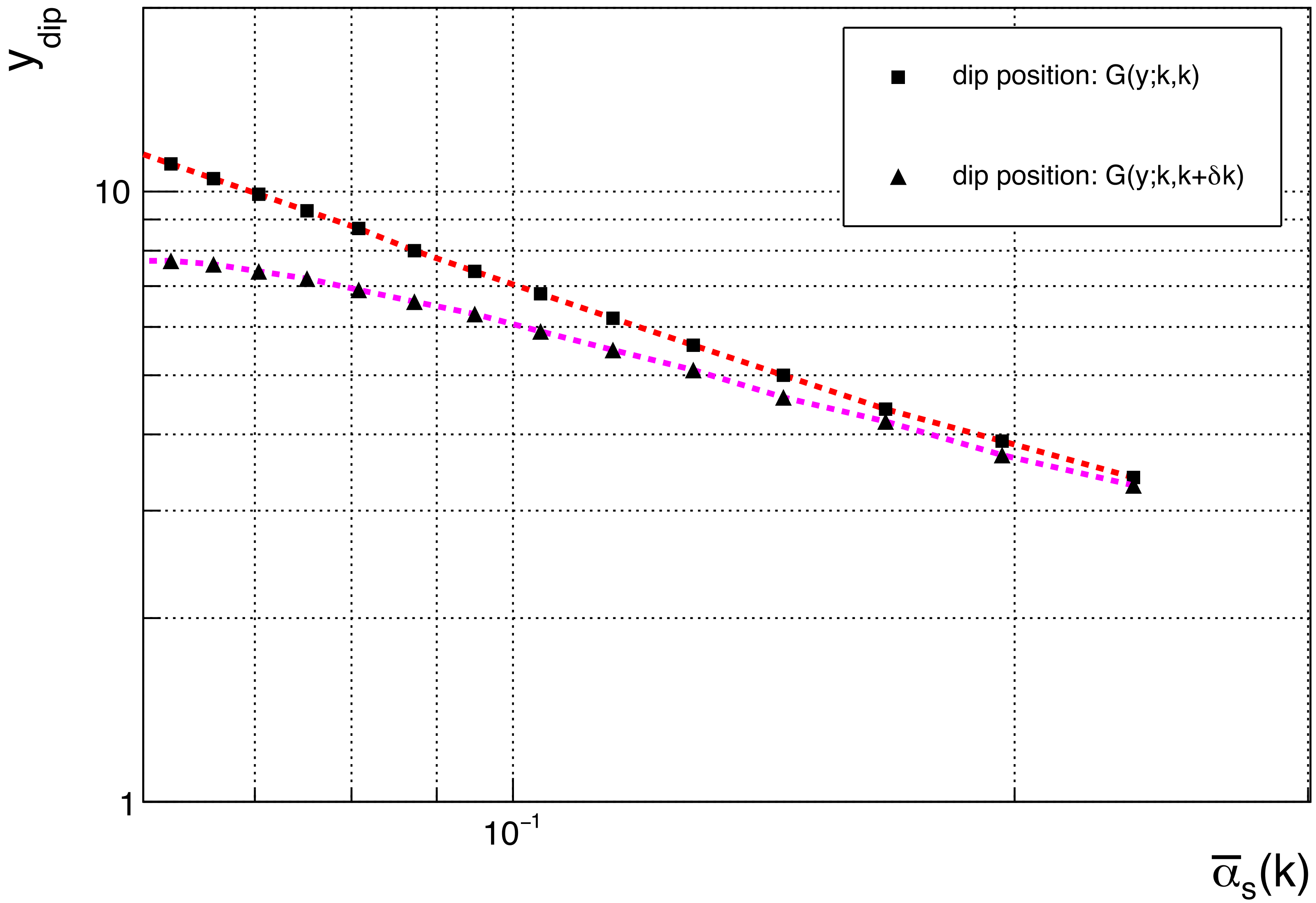
Gluon Green's function



Strong preasymptotic effects, which delay the onset of growth towards small x / large y

Dip or a plateau in y

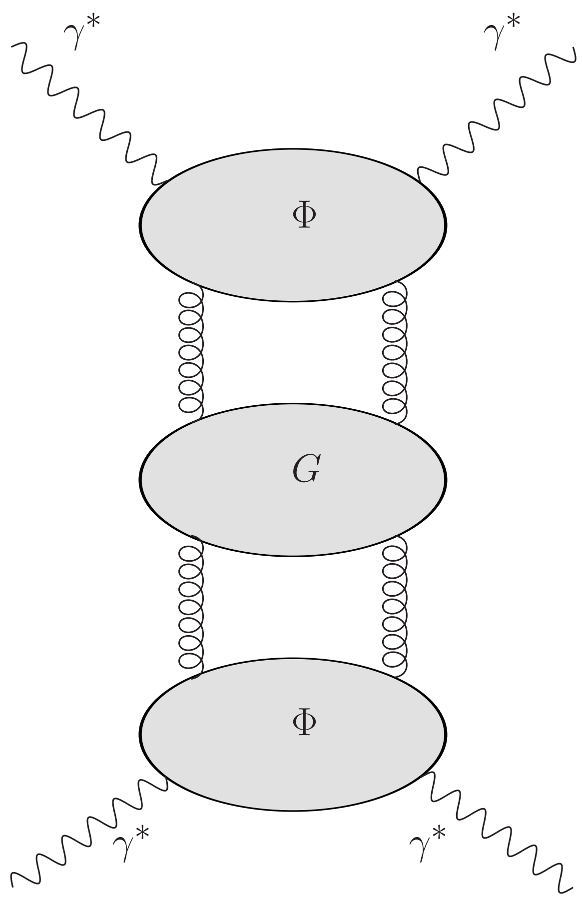
Dip in gluon Green's function



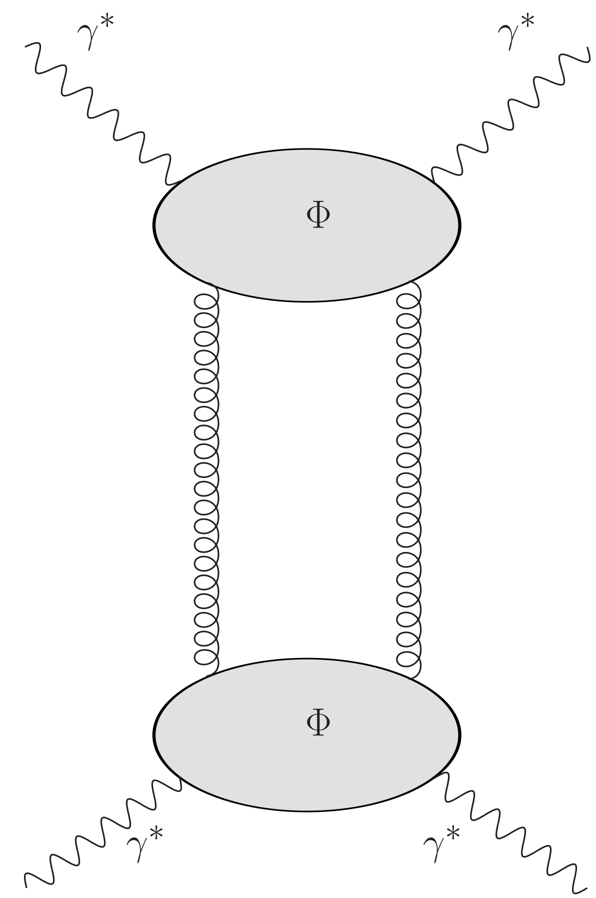
Position of 'dip' in the gluon Green's function
Inverse relation with strong coupling

$$y_{\mathrm{dip}}\overline{\alpha}_s(k) \simeq 0.7 - 0.8$$

Impact on two-scale processes



BFKL

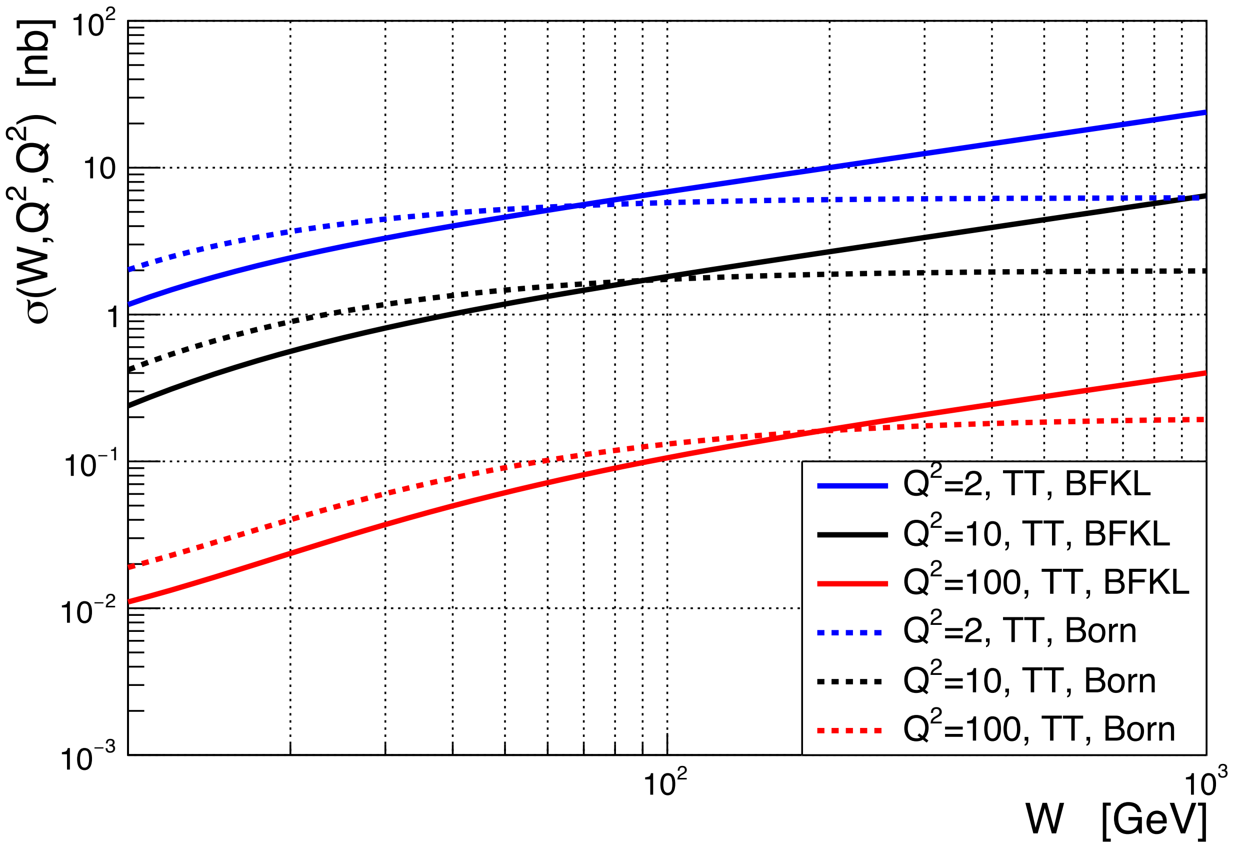


Born

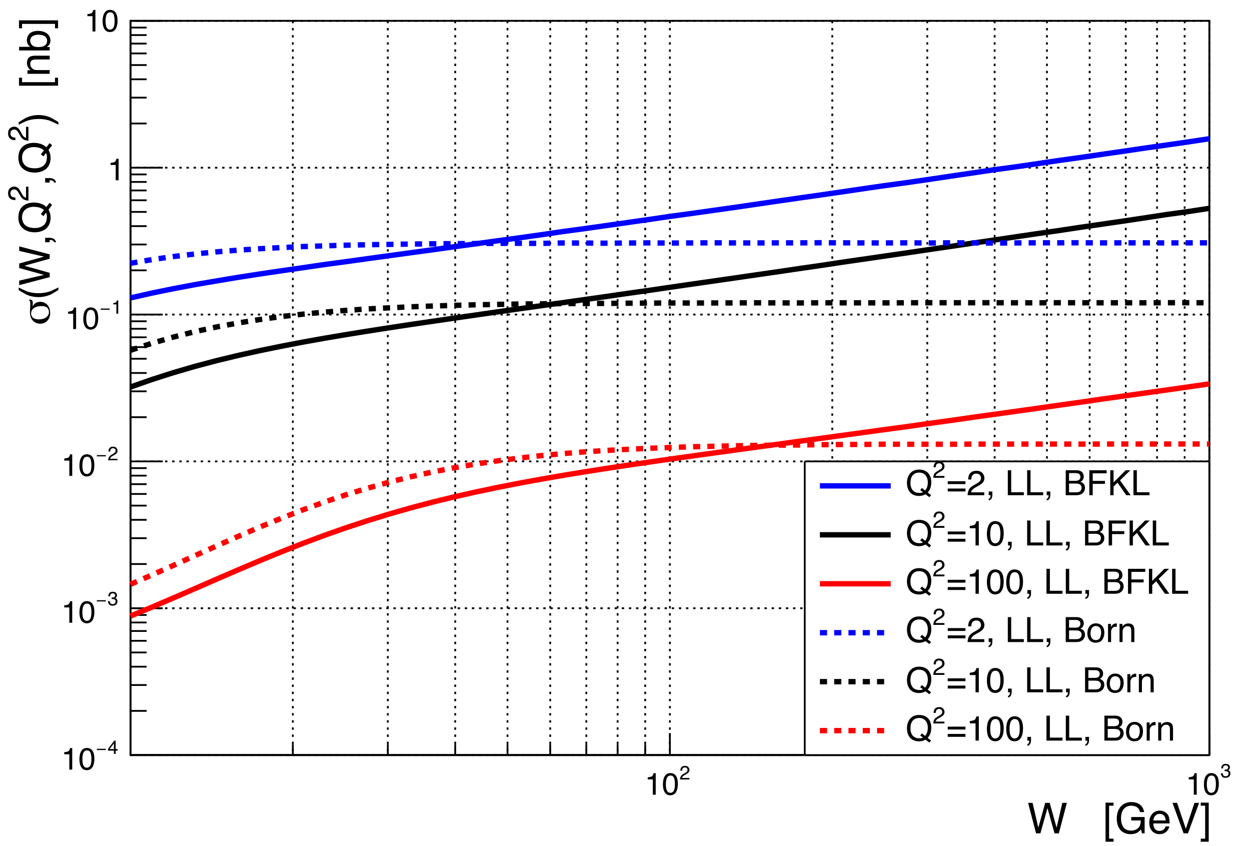
Virtual photon scattering with equal scales

Impact on two-scale processes

Transverse-transverse



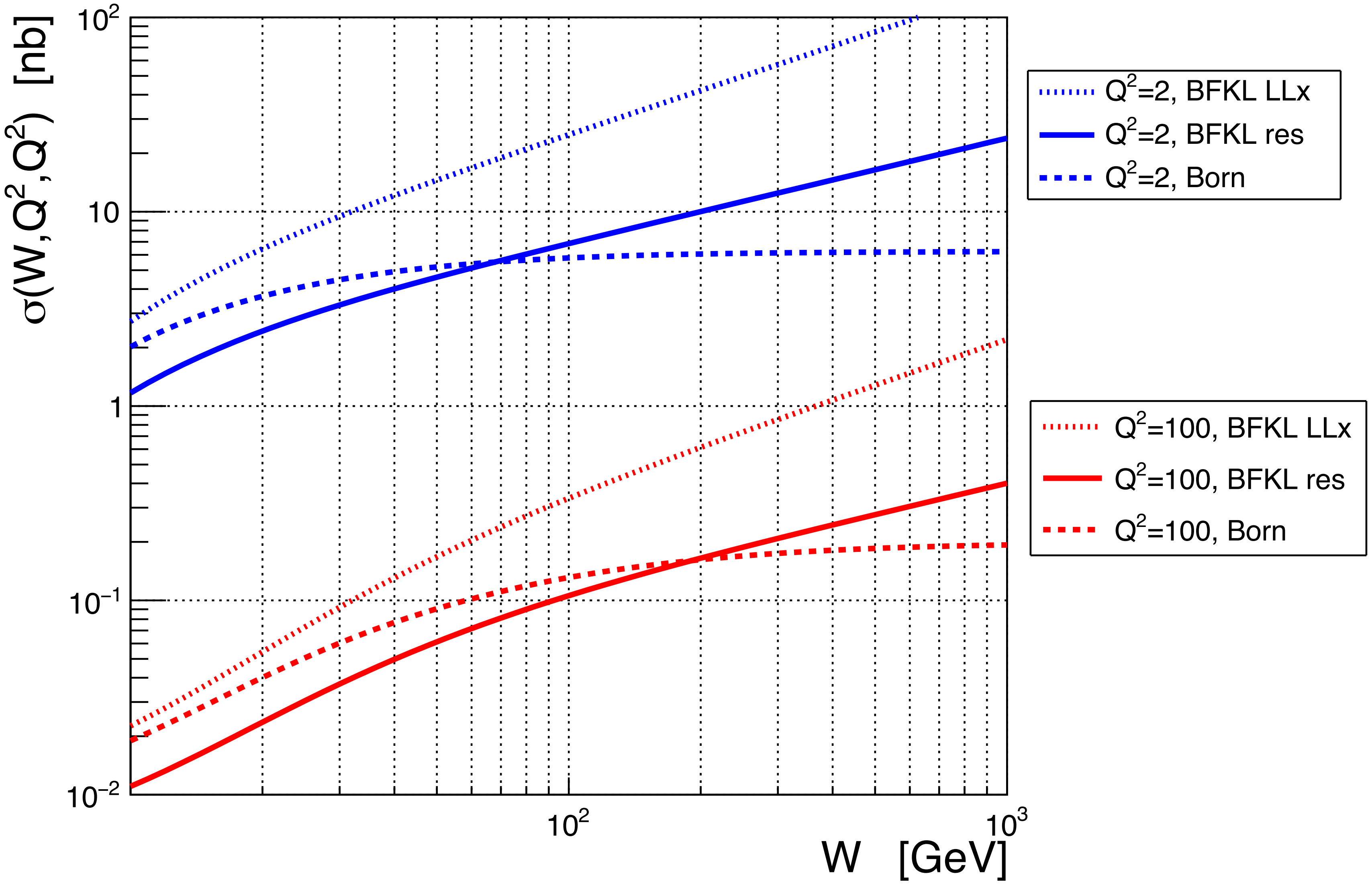
Longitudinal-longitudinal



Energy dependence of BFKL vs flat behavior at large W

Preasymptotic effects : resummed BFKL lower than Born calculation at low W. Observed previously.

Impact on two-scale processes



LLx calculation (with running coupling) always larger than Born
Preasymptotic effects of resummation having large impact onto W behavior

Splitting function

Gluon-gluon splitting function has logarithmic enhancements at small x

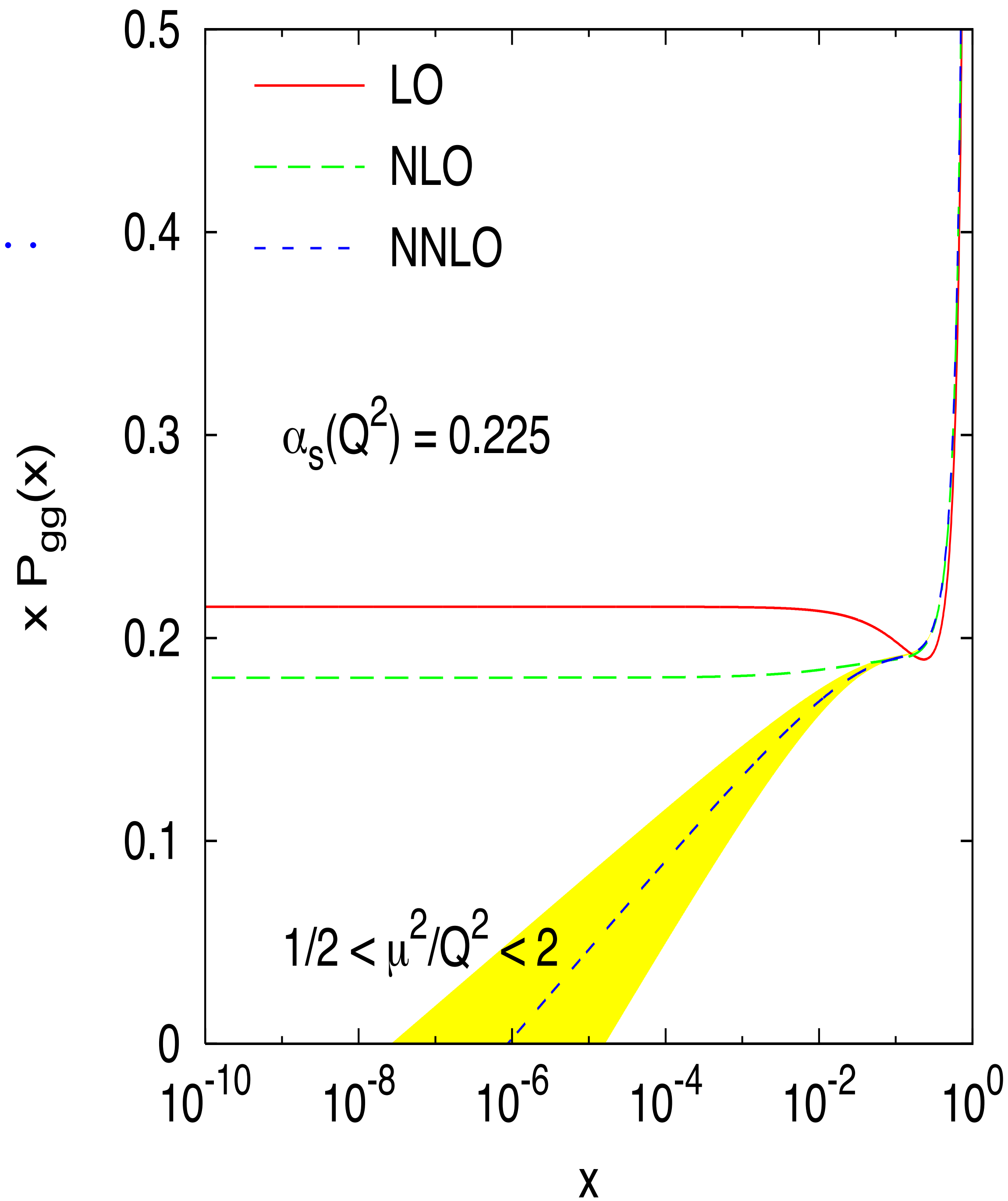
$$xP_{gg}(x) = \sum_{n=1} a_n \alpha_s^n \ln^{n-1}\frac{1}{x} + \sum_{n=2} b_n \alpha_s^n \ln^{n-2}\frac{1}{x} + \dots$$

LLx NLLx

First small x logarithmic term which belongs to NLLx hierarchy recovered at NNLO

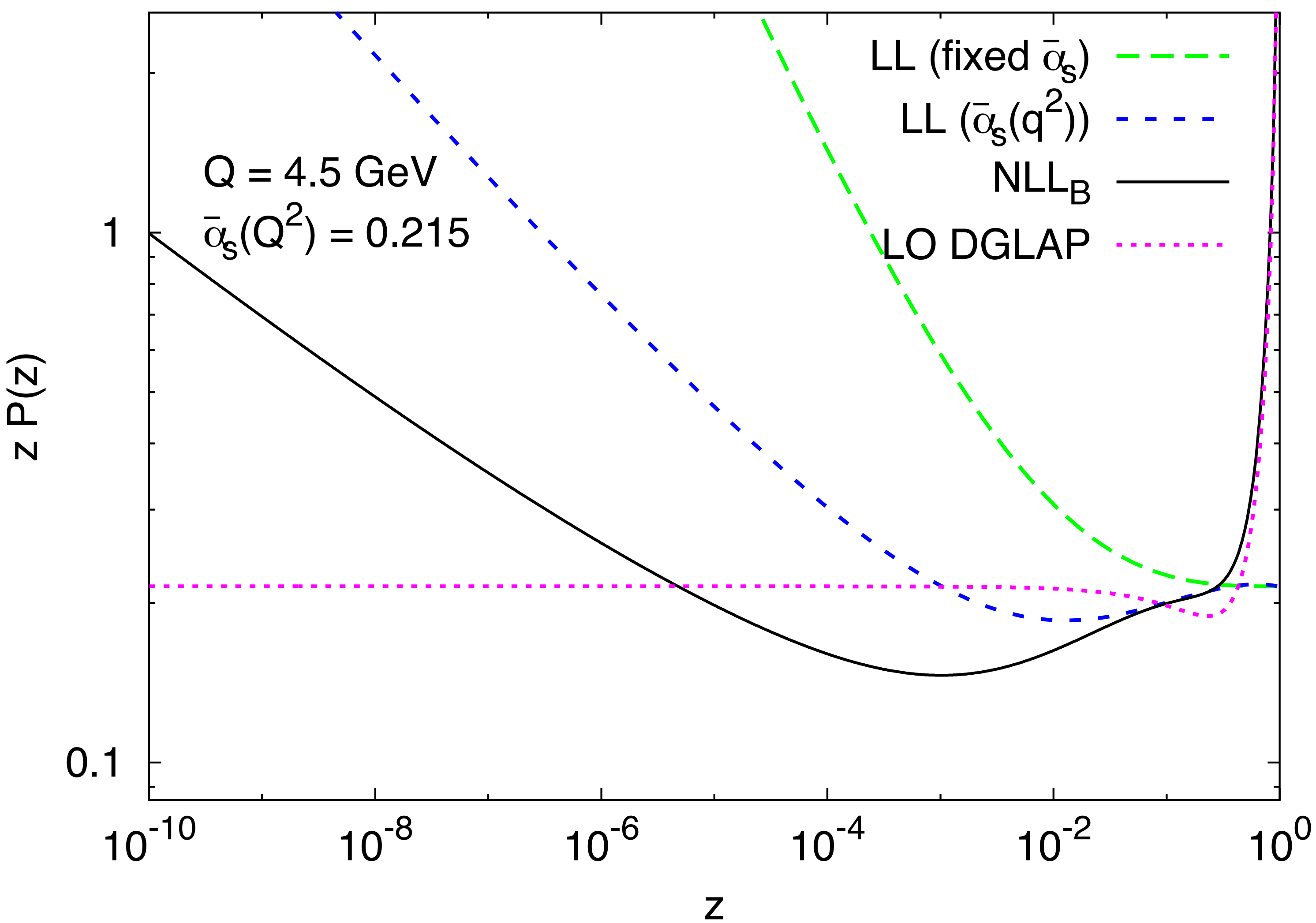
$$-1.54\bar{\alpha}_s^3 \ln 1/x$$

Resummation at small x is inevitable.



Moch, Vermaseren, Vogt

Resummed splitting function



- Small x growth delayed to much smaller values of x (beyond HERA)
- Interesting feature: a dip seen at around $x \simeq 10^{-3}$
- Is this universal feature ?
- Need to understand the origin of the dip in splitting function.

Understanding the structure of the splitting function

Perturbative terms in the splitting function

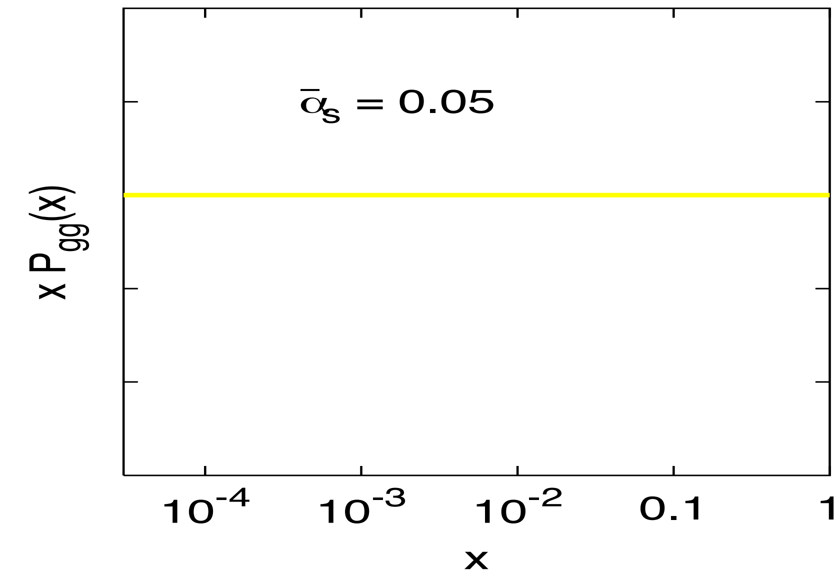
	LLx	NLLx	NNLLx	...
α_s	x	–	–	
α_s^2	0	n_f	–	
α_s^3	0	x	x	
α_s^4	x	x	x	
α_s^5	0	x	x	
$\vdots$				

const.

ln 1/x

$\ln^2 1/x$

$\ln^3 1/x$



$$-1.54\bar{\alpha}_s^3 \ln 1/x \;+\; 0.401\bar{\alpha}_s^4 \ln^3 1/x$$

There is a minimum when

$$\alpha_s \ln^2 \frac{1}{x} \sim 1 \longrightarrow \ln \frac{1}{x} \sim \frac{1}{\sqrt{\alpha_s}}$$

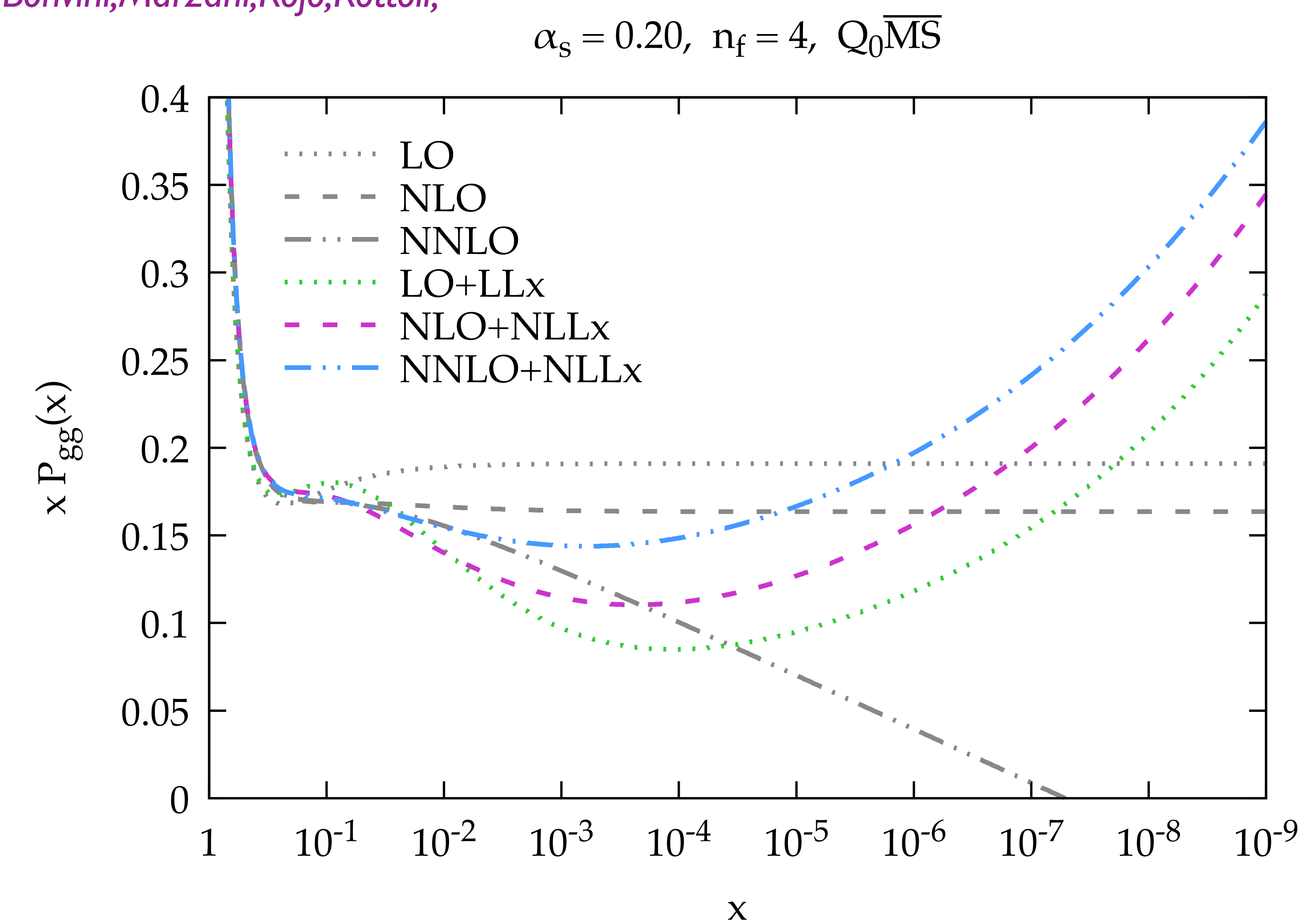
This is valid at small coupling. For larger values another regime

$$\ln \frac{1}{x_{\min}} \simeq \frac{3}{2\omega(\alpha_s)}$$

In general: dip comes from the interplay between NNLO and the resummation.

Resummed splitting function

Ball,Bertoni,Bonvini,Marzani,Rojo,Rottoli;



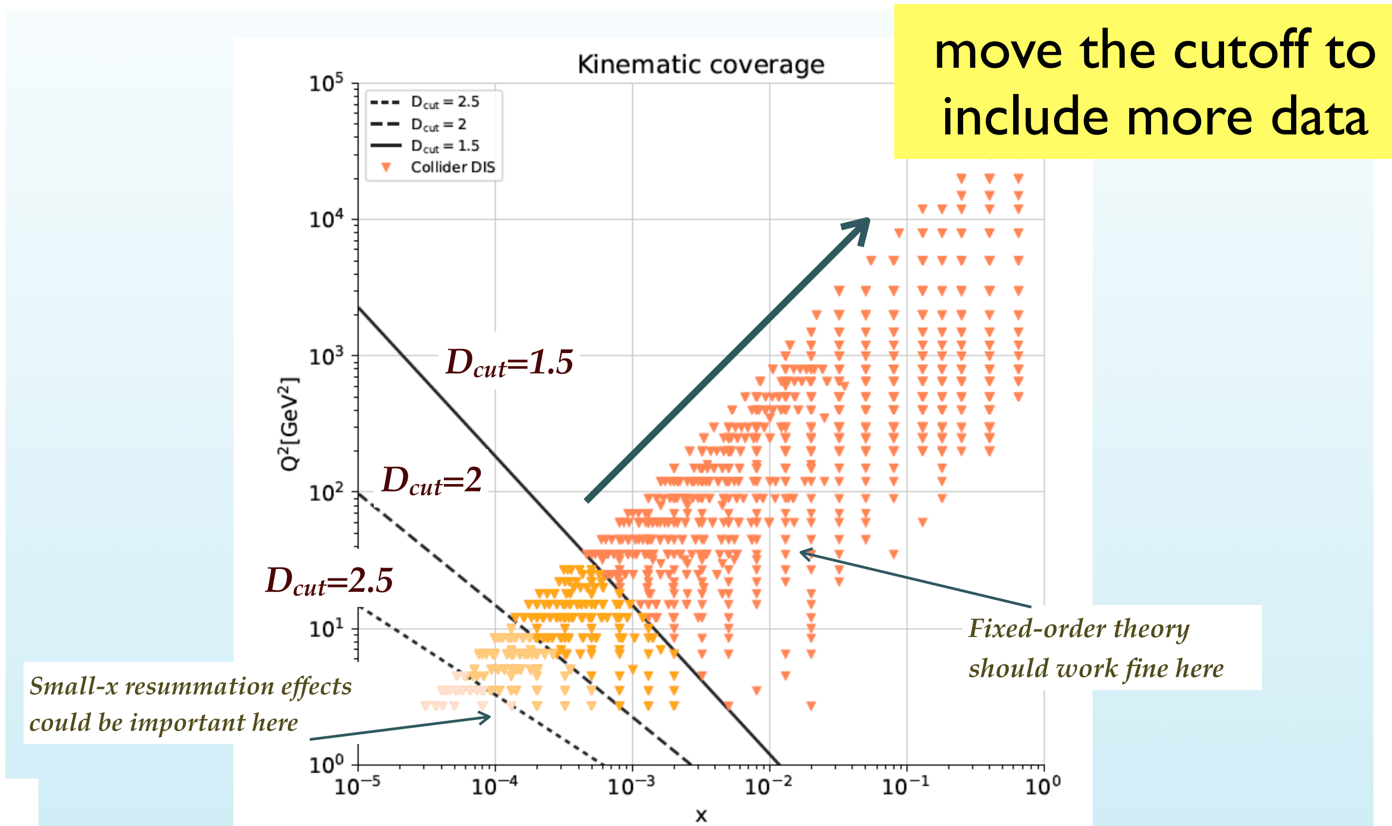
Dip in the splitting function visible in other resummation approaches

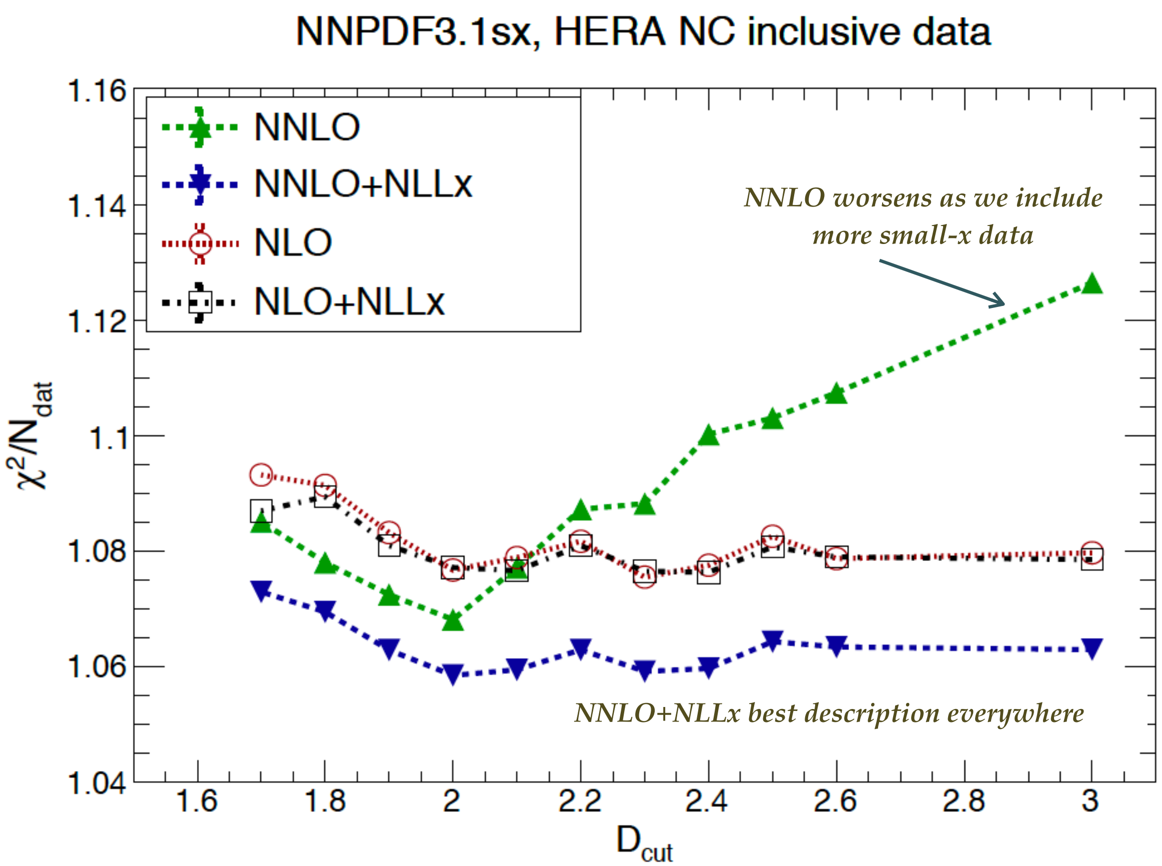
Small x resummation and HERA data

Ball,Bertoni,Bonvini,Marzani,Rojo,Rottoli

- Perform fits to data with the cut on small x/small Q^2 region

- Observe the variation or lack of variation in χ^2





- χ2 changes for DGLAP at NNLO when more small x data are included

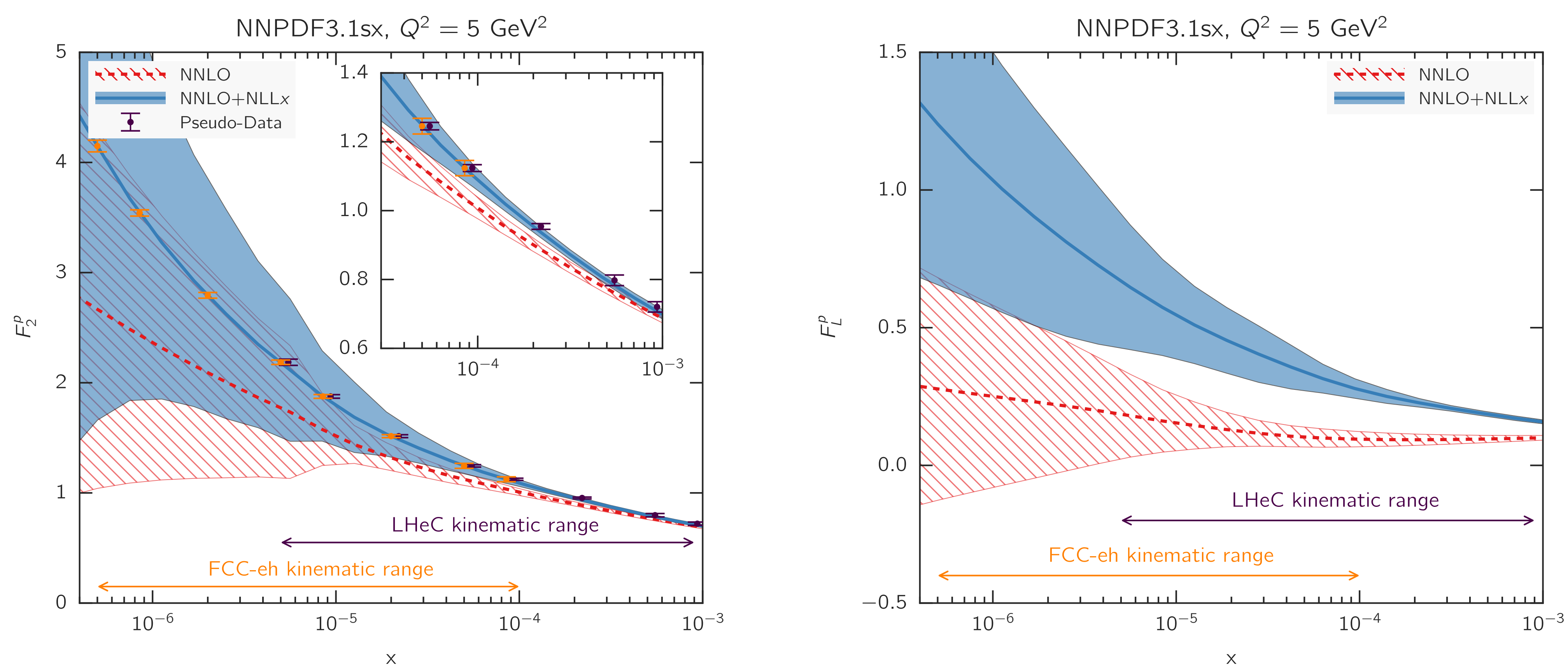
- NNLO+NNLLx gives best description

- Interestingly NLO and NLO+NLLx do not differ by a lot (flat splitting function at NLO?)

Small x resummation: future colliders

*Ball,Bertone,Bonvini,
Marzani,Rojo,Rottoli*

- Perform extrapolation of the calculations to the higher energy range (smaller x).
- Simulations with and without the resummation
- Compared with the pseudodata

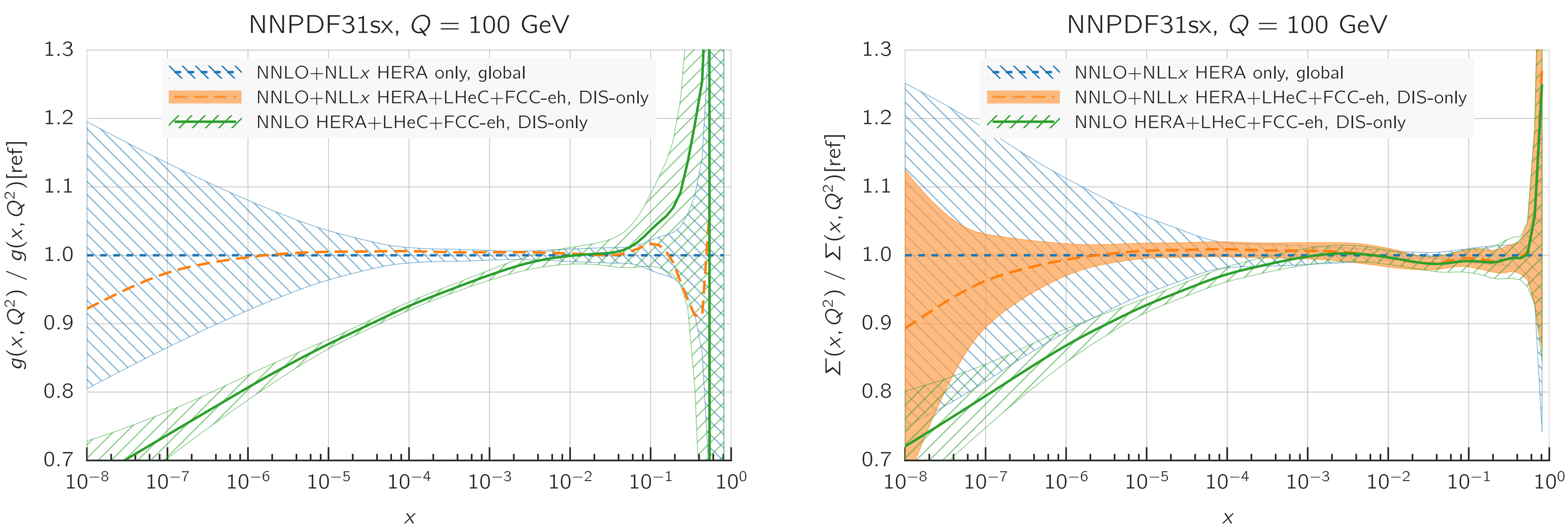


- Structure function in the LHeC/FCC-eh range can discriminate between different scenarios
- Longitudinal structure function particularly sensitive to the resummation vs fixed order
- EIC: lower energy, so likely in preasymptotic regime, but can measure longitudinal structure function **with precision**

Impact of resummation on future machines

Ball,Bertone,Bonvini, Marzani,Rojo,Rottoli

- Perform fits and extraction of PDFs using HERA data supplemented by pseudodata from LHeC+FCC-eh colliders
- Pseudodata restrict the uncertainties in PDFs
- Large differences in the extrapolation of the PDFs towards small x with and without the resummation



Important consequences for the LHeC and FCC-eh: large differences!

Summary and outlook

- Resummation schemes at low x based on collinear improvements: kinematical effects, matching to DGLAP
- Stability of the results demonstrated for scale changes and model changes.
- Characteristic features: reduced Pomeron intercept and small x growth delayed by several units of rapidity.
- Preasymptotic effects: dip of the splitting function and dip/plateau in the Green's function.
- Impact on saturation: lowering the saturation scale.
- EIC : kinematic range where strong preasymptotic effects present. Still, increased luminosity and possibility of precision F_L measurement can help. Other colliders (like LHeC/FCC-eh): very important
- Needed: resummation of impact factors, off shell matrix elements for other processes

Backup

Resummed kernel in x,k_T

$$\int_x^1 \frac{dz}{z} \int dk'^2\, \tilde{\mathcal{K}}(z;k,k') f(\frac{x}{z},k')$$
$$= \int_x^1 \frac{dz}{z} \int dk'^2 \left[\bar{\alpha}_s(\boldsymbol{q}^2) K_0^{\rm kc}(z;\boldsymbol{k},\boldsymbol{k}') + \bar{\alpha}_s(k_>^2) K_{\rm c}^{\rm kc}(z;k,k') + \bar{\alpha}_s^2(k_>^2)\tilde{K}_1(k,k')\right] f(\frac{x}{z},k')$$

LL BFKL with consistency constraint

$$\int_x^1 \frac{dz}{z} \int dk'^2 \ \left[\bar{\alpha}_s(\boldsymbol{q}^2) K_0^{\rm kc}(z;\boldsymbol{k},\boldsymbol{k}')\right] f(\frac{x}{z},k')$$
$$= \int_x^1 \frac{dz}{z} \int \frac{d^2\boldsymbol{q}}{\pi \boldsymbol{q}^2}\, \bar{\alpha}_s(\boldsymbol{q}^2) \left[f(\frac{x}{z},|\boldsymbol{k}+\boldsymbol{q}|)\Theta(\frac{k}{z}-k')\Theta(k'-kz) - \Theta(k-q) f(\frac{x}{z},k)\right]$$

non-singular DGLAP with consistency constraint

$$\int_x^1 \frac{dz}{z} \int dk'^2\, \bar{\alpha}_s(k_>^2) K_{\rm c}^{\rm kc}(z;k,k') f(\frac{x}{z},k')$$
$$= \int_x^1 \frac{dz}{z} \int_{(kz)^2}^{k^2} \frac{dk'^2}{k^2}\, \bar{\alpha}_s(k^2) z \frac{k}{k'} \tilde{P}_{gg}(z\frac{k}{k'}) f(\frac{x}{z},k')$$
$$+ \int_x^1 \frac{dz}{z} \int_{k^2}^{(k/z)^2} \frac{dk'^2}{k'^2}\, \bar{\alpha}_s(k'^2) z \frac{k'}{k} \tilde{P}_{gg}(z\frac{k'}{k}) f(\frac{x}{z},k')\ ,$$

Resummed kernel in x,k_T

NLL BFKL with subtractions

$$
\begin{aligned}
&\int_x^1 \frac{dz}{z} \int dk'^2\, \bar{\alpha}_s^2(k_>^2)\tilde{K}_1(k,k')f(\frac{x}{z},k') \\
&\qquad = \frac{1}{4}\int_x^1 \frac{dz}{z}\int dk'^2\, \bar{\alpha}_s^2(k_>^2)\Bigg\{ \\
&\left(\frac{67}{9}-\frac{\pi^2}{3}\right)\frac{1}{|k'^2-k^2|}\left[f(\frac{x}{z},k'^2)-\frac{2k_<^2}{(k'^2+k^2)}f(\frac{x}{z},k^2)\right]+ \\
&\left[-\frac{1}{32}\left(\frac{2}{k'^2}+\frac{2}{k^2}+\left(\frac{1}{k'^2}-\frac{1}{k^2}\right)\log\left(\frac{k^2}{k'^2}\right)\right)+\boxed{\frac{4\mathrm{Li}_2(1-k_<^2/k_>^2)}{|k'^2-k^2|}}\right. \\
&\boxed{-4A_1(0)\mathrm{sgn}(k^2-k'^2)\left(\frac{1}{k^2}\log\frac{|k'^2-k^2|}{k'^2}-\frac{1}{k'^2}\log\frac{|k'^2-k^2|}{k^2}\right)} \\
&-\left(3+\left(\frac{3}{4}-\frac{(k'^2+k^2)^2}{32k'^2k^2}\right)\right)\int_0^\infty\frac{dy}{k^2+y^2k'^2}\log|\frac{1+y}{1-y}| \\
&\left.+\frac{1}{k'^2+k^2}\left(\frac{\pi^2}{3}+4\mathrm{Li}_2(\frac{k_<^2}{k_>^2})\right)\right]f(\frac{x}{z},k')\Bigg\} \\
&+\frac{1}{4}6\zeta(3)\int_x^1\frac{dz}{z}\,\bar{\alpha}_s^2(k^2)f(\frac{x}{z},k)\ .
\end{aligned}
$$