# Light front methods to compute the initial conditions for small x QCD evolution

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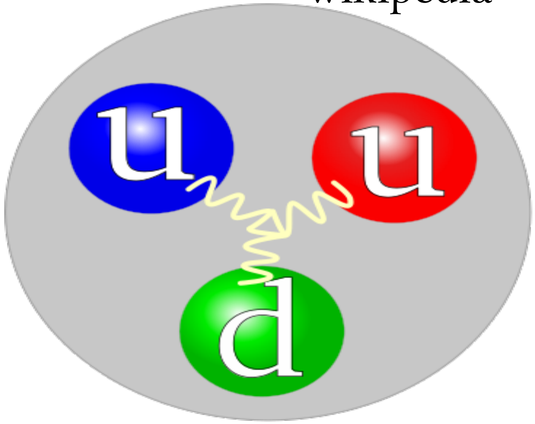
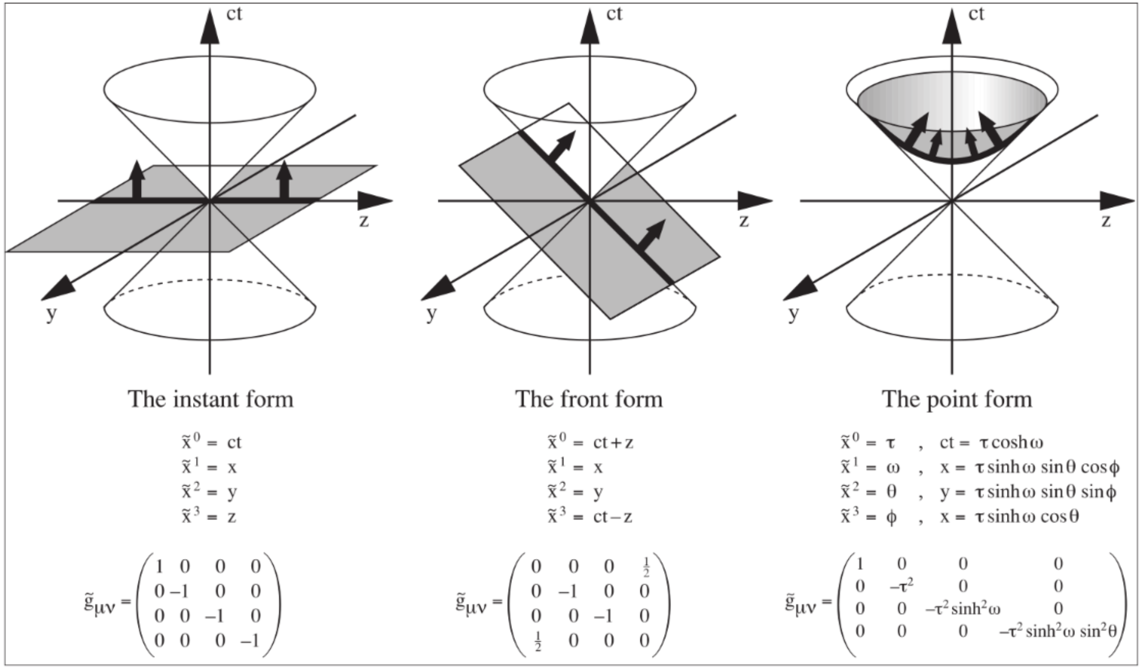
RBRC workshop "Small x physics in the EIC era", December 15-17, 2021

talk based on collaborations with
H. Mäntysaari, R. Paatelainen: 2103.11682
R. Paatelainen: 2010.11245
T. Stebel, V. Skokov: 2001.04516
T. Stebel: 1903.07660
G. Miller, R. Venugopalan: 1808.02501

wikipedia




What's that ?




from Brodsky, Pauli, Pinsky, Phys. Rep. (1998)

# t = 0 quantization: complicated vacuum of interacting theory

$$\langle \text{vac}| = \langle \text{vac}|0\rangle\langle 0| + \sum_n \langle \text{vac}|n\rangle\langle n|$$

eigenstates of free Hamiltonian, arbitrary # of virtual particle anti-particle pairs, momentum =0

$$\langle \text{vac}| \sim \lim_{T\to\infty(1-i\epsilon)} \langle 0|\, e^{iH_0T}\, e^{-iHT}$$

$x^+ = 0$ Light Front: momentum $p^+ > 0$ for all particles
(but: zero modes)

$$P^+ \left|0\right\rangle = 0$$

eigenstates $H_{\mathrm{LF}} \left|M\right\rangle = M^2 \left|M\right\rangle$ , $H_{\mathrm{LF}} = P^- P^+ - P_T^2$

In $A^+ = 0$ gauge

$$\begin{aligned}
P_+ \;=\; & \frac{1}{2}\int dx_+ d^2x_\perp \left(\overline{\widetilde{\Psi}}\gamma^+ \frac{m^2 + (i\nabla_\perp)^2}{i\partial^+}\widetilde{\Psi} + \widetilde{A}^\mu_a (i\nabla_\perp)^2 \widetilde{A}^a_\mu\right) \\
+ \; & g \int dx_+ d^2x_\perp \; \widetilde{J}^\mu_a \widetilde{A}^a_\mu \\
+ \; & \frac{g^2}{4}\int dx_+ d^2x_\perp \; \widetilde{B}^{\mu\nu}_a \widetilde{B}^a_{\mu\nu} \\
+ \; & \frac{g^2}{2}\int dx_+ d^2x_\perp \; \widetilde{J}^+_a \frac{1}{(i\partial^+)^2} \widetilde{J}^+_a \\
+ \; & \frac{g^2}{2}\int dx_+ d^2x_\perp \; \overline{\widetilde{\Psi}}\gamma^\mu T^a \widetilde{A}^a_\mu \, \frac{\gamma^+}{i\partial^+}\left(\gamma^\nu T^b \widetilde{A}^b_\nu \widetilde{\Psi}\right) \; .
\end{aligned}$$

kinetic part

interactions

A. Harindranath, hep-ph/9612244
M. Burkardt, Adv. Nucl. Phys. 23, p. 1-74 (1996)
S.J. Brodsky, H. C. Pauli and S. S. Pinsky, Phys. Rept. 301, p. 299-486 (1998)

# The proton on the light front

The proton on the light front (three quark Fock state; L.C. time $x^+ = 0$)

$$|P\rangle = \frac{1}{\sqrt{6}} \int \frac{dx_1 dx_2 dx_3}{\sqrt{x_1 x_2 x_3}} \delta(1 - x_1 - x_2 - x_3) \int \frac{d^2k_1 d^2k_2 d^2k_3}{(16\pi^3)^3} 16\pi^3 \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3)$$

$$\times\ \psi(k_1, k_2, k_3) \sum_{i_1, i_2, i_3} \epsilon_{i_1 i_2 i_3} |p_1, i_1;\ p_2, i_2;\ p_3, i_3\rangle$$

+ higher Fock states

P. Lepage & Brodsky, 1979 - ....
Brodsky, Pauli, Pinsky, PR (1998)

* Fock space amplitude ψ is gauge invariant, universal, and process independent

* encodes the non-perturbative structure of hadrons (QCD eigenstates)

$\longrightarrow$ Evaluate matrix elements explicitly !

# Quark Wigner distribution over 5D phase space :

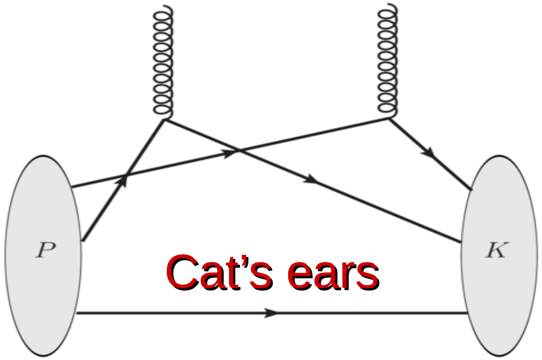
Belitsky, Ji, Yuan, PRD (2004)
Meissner, Metz, Schlegel, JHEP (2009)
Lorce, Pasquini, PRD (2011)
Y. Hatta, PLB (2012), ...

$$W(x,\vec{k}_T,\vec{b}_T) = \int \frac{d^2\Delta_T}{(2\pi)^2}\, e^{i\vec{b}\cdot\vec{\Delta}_T} \int \frac{dz^-\, d^2 z_T}{2(2\pi)^3}\, e^{ixP^+z^- - i\vec{k}_T\cdot\vec{z}_T}$$
$$\left\langle P+\frac{\Delta}{2} \left| \bar{q}(-\frac{z}{2})\,\gamma^+\, q(\frac{z}{2}) \right| P-\frac{\Delta}{2} \right\rangle$$

* Does not factorize into $f(x,\vec{k}_T)\, g(x,\vec{b})$ : correlation of transv. momentum & coordinate $\longrightarrow$ angular momentum

Color charge correlators: "impact parameter dependent" MV (or IPsat model etc):

$$\left\langle \rho^a(\vec{q}_1)\,\rho^b(\vec{q}_2)\right\rangle = g^2\mu^2\,\delta^{ab}\int d^2b\,e^{-i(\vec{q}_1+\vec{q}_2)\cdot\vec{b}}\,T_p(b)$$

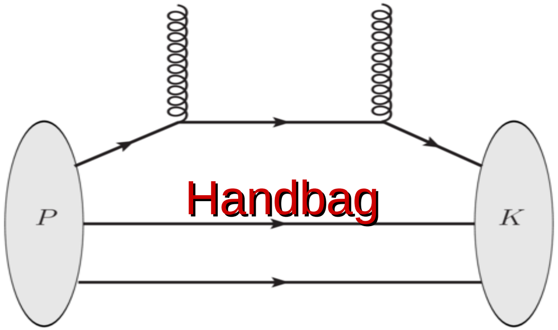



2-body GTMD / Wigner

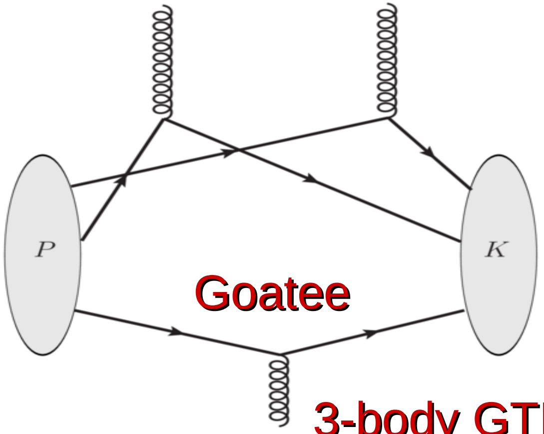
proton :

$$\left\langle \rho^a(\vec{q}_1)\,\rho^b(\vec{q}_2)\right\rangle \sim g^2\,\delta^{ab}\,G_2(\vec{q}_1,\vec{q}_2)$$

LO diagrams:




1-body GTMD / Wigner




3-body GTMD / Wigner

# Motivation :

1) Two- (and three-) body correlations in the proton LCwf at x ~ 0.1 – 0.01;
b-dependence, $q_T$ dependence, angular dependence (!)

2) Small-x evolution fits to HERA:
- ad hoc initial conditions at $x_0$ = 0.01, parameters adjusted so that BK fit is optimal
- how do they depend on $x_0$ ? (important for NLO BK; Ducloue et al, 1902.06637)
- no b, r*b dependence (sometimes modelled via MV / IPsat etc)

Our goal is to relate (*x-dependent*) initial condition to light-front w.f. of proton, take advantage of "proton imaging" at EIC $\longrightarrow N_0(\vec{r}, \vec{b}; x)$

# $<\rho^a\rho^b>$ correlator

with G. Miller, R. Venugopalan: 1808.02501

$$\rho^a(\vec{x}) = g(t^a)_{ij}\int \mathrm{d}x^-\,\overline{\psi}_i(x)\gamma^+\psi_j(x)$$

$$\begin{aligned}
\langle\,\rho^a(\vec{q})\,\rho^b(\vec{k})\,\rangle_{K_\perp} \quad &= \quad g^2\,\mathrm{tr}\;t^a t^b \int [dx_i][d^2p_i] \\
&\quad \Big\{\psi^*\left(\vec{p}_1 + (1-x_1)\vec{K}_T, \vec{p}_2 - x_2\vec{K}_T, \vec{p}_3 - x_3\vec{K}_T\right) \\
&\quad -\psi^*\left(\vec{p}_1 - \vec{q} - x_1\vec{K}_T, \vec{p}_2 - \vec{k} - x_2\vec{K}_T, \vec{p}_3 - x_3\vec{K}_T\right)\Big\} \\
&\quad \psi\left(\vec{p}_1, \vec{p}_2, \vec{p}_3\right) \\
&\sim \quad g^2\,\delta^{ab}\,G_2(\vec{q},\vec{k})
\end{aligned}$$

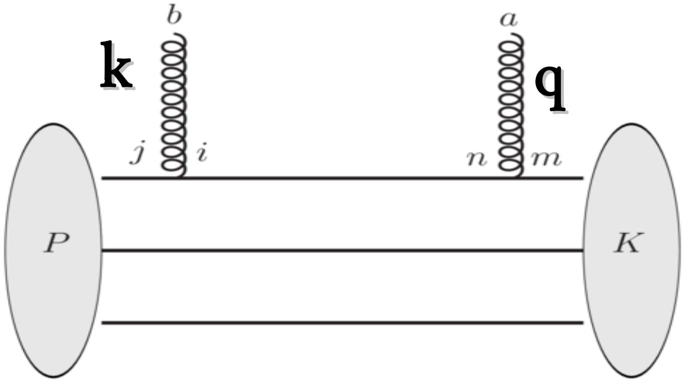
$$(\vec{q} + \vec{k} + \vec{K}_T = 0)$$

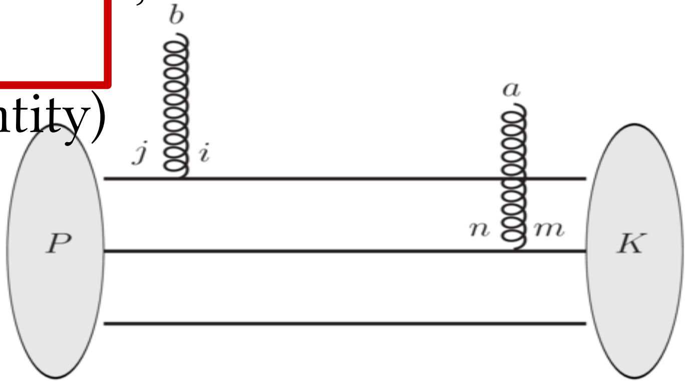
involves 1- and 2-particle GTMDs,
sum vanishes in IR

(color neutrality / Ward identity)

$\leftarrow$ dominates when $q^2,\, k^2 \gg K_T{}^2$

$\rightarrow$ dominates when $\vec{q} \sim \vec{k} \sim -\vec{K}_T/2$



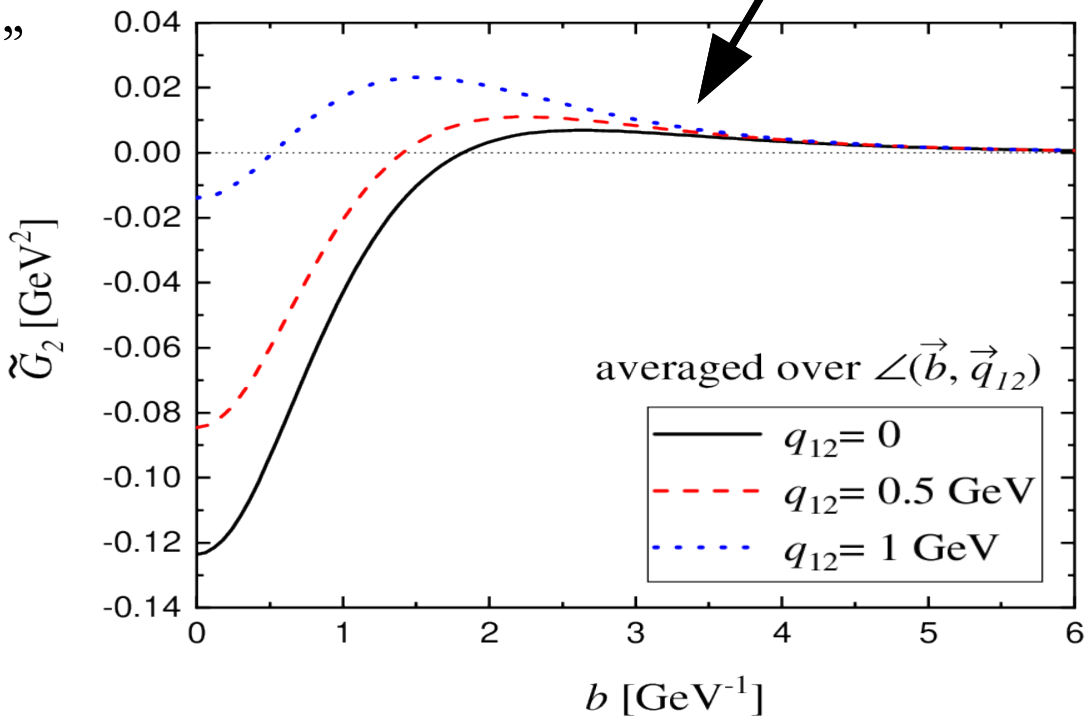
# Numerical results for $G_2$ correlator at LO

(using Brodsky & Schlumpf LFwf)

w/ T. Stebel, V. Skokov: 2001.04516

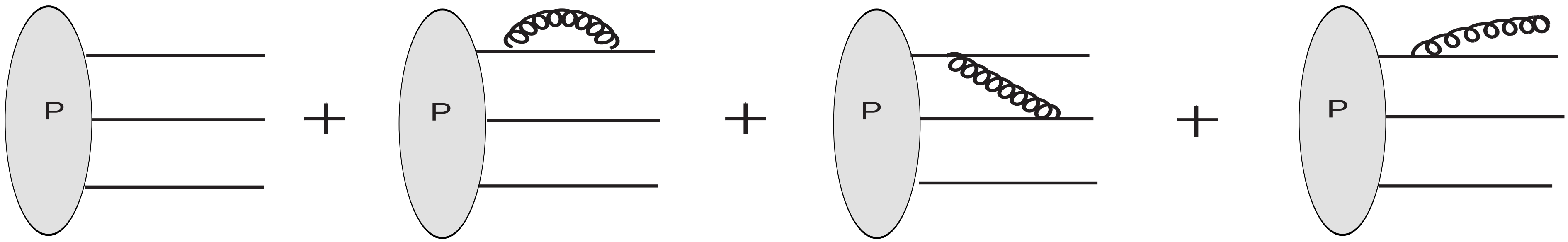
- depends on b as well as on $q_{12}$ = q1-q2 (resp. r)
- and on their relative angle
- note: small b is large $K_T$, "cat's ears" dominates !
- **not a Gaussian** in b

here it resembles a "shape function"



# Now to $|P\rangle \sim \psi_{qqq}|qqq\rangle + \psi_{qqqg}|qqqg\rangle$

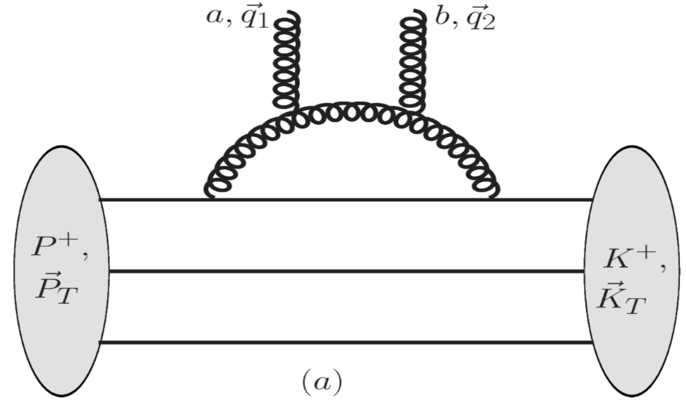
computed in perturbation theory, 1-gluon emission / exchange,
*w/o employing $x_g \to 0$ approximation*



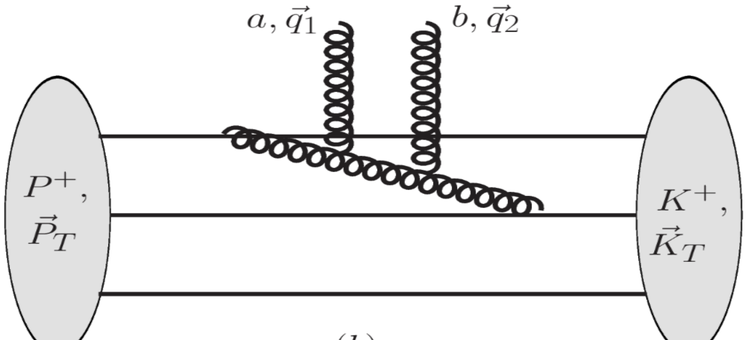
# NLO color charge correlators

with Risto Paatelainen, 2010.11245

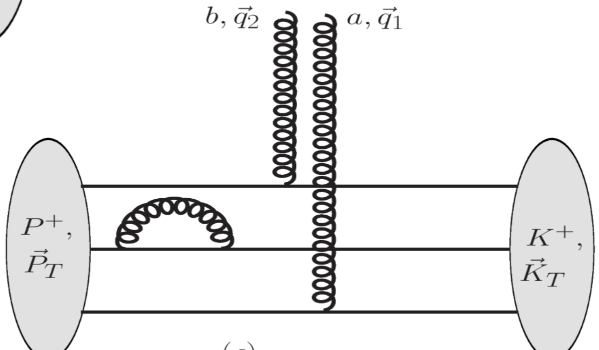
notes:
* soft & collinear div.
* UV divergences cancel
* Ward satisfied



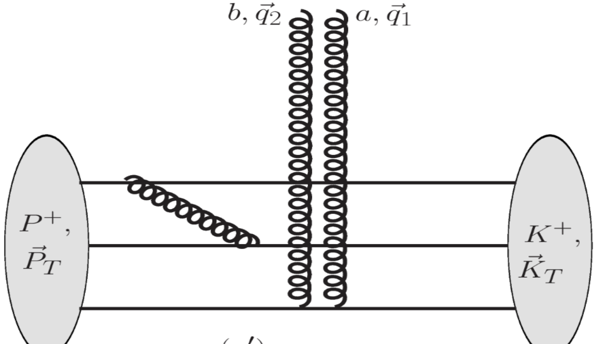

$(a)$



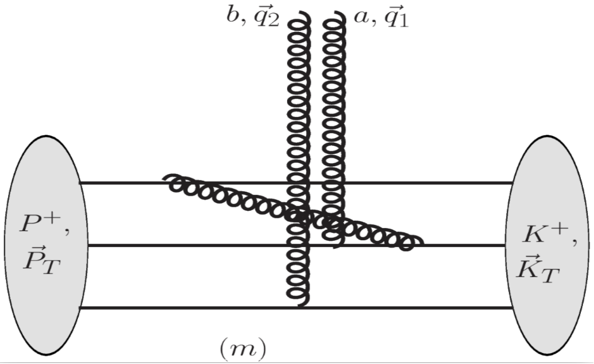

$(b)$




$(c)$




$(g')$

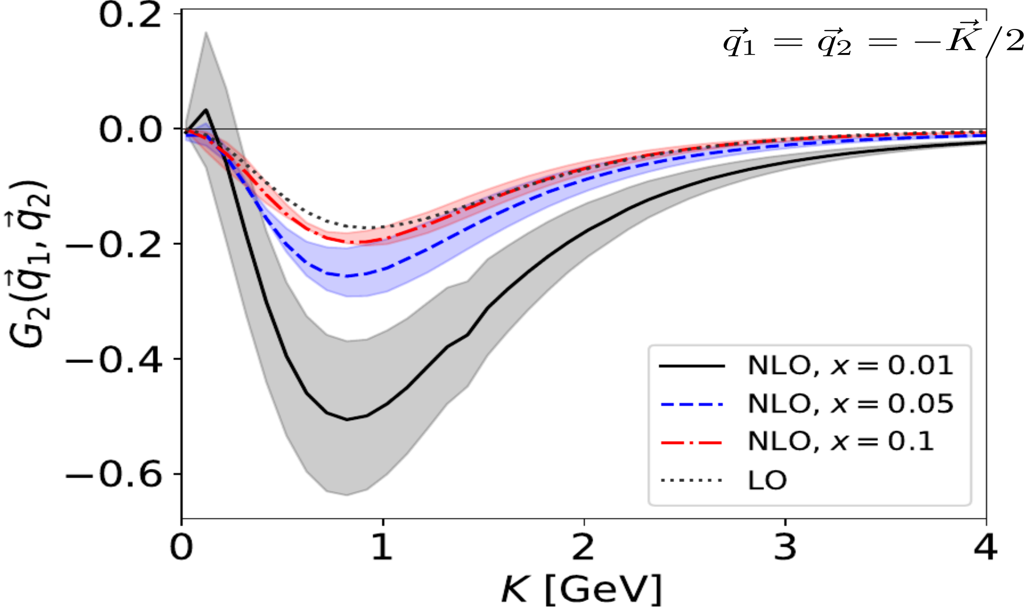



$(m)$

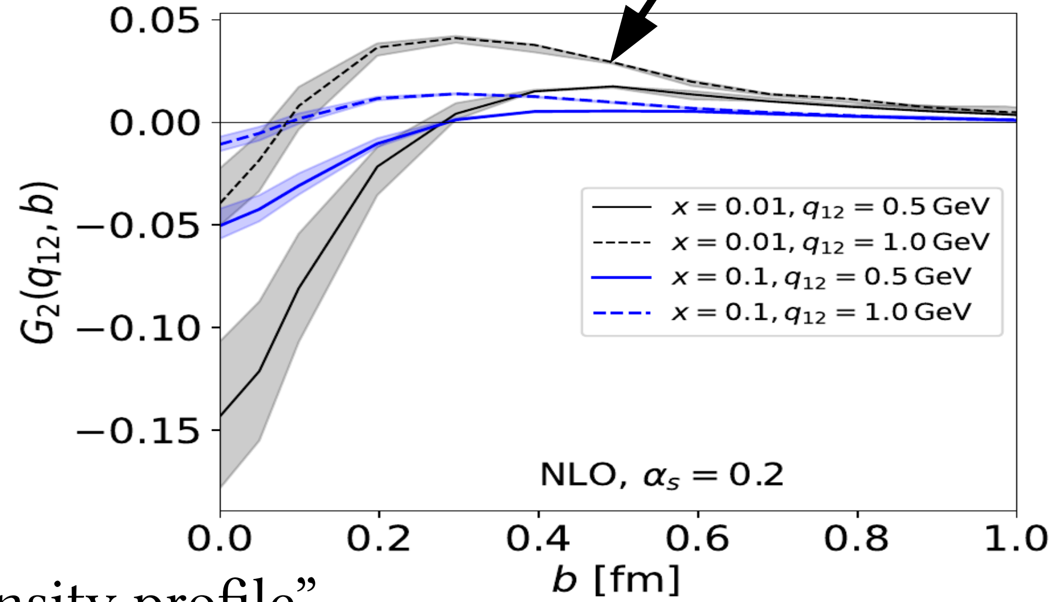
+ a bunch more

# The effect of adding the gluon:

with Heikki Mäntysaari, 2103.11682

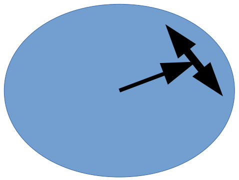



~ $T_p(b)$ profile fct.




* modest effect at x=0.1
* big effect at x = 0.01 !
* $G_2$ at b $\longrightarrow$ 0 even more negative than LO cat’s ears diagram, not a “density profile”

# Azimuthal anisotropy of dipole scatt. amplitude

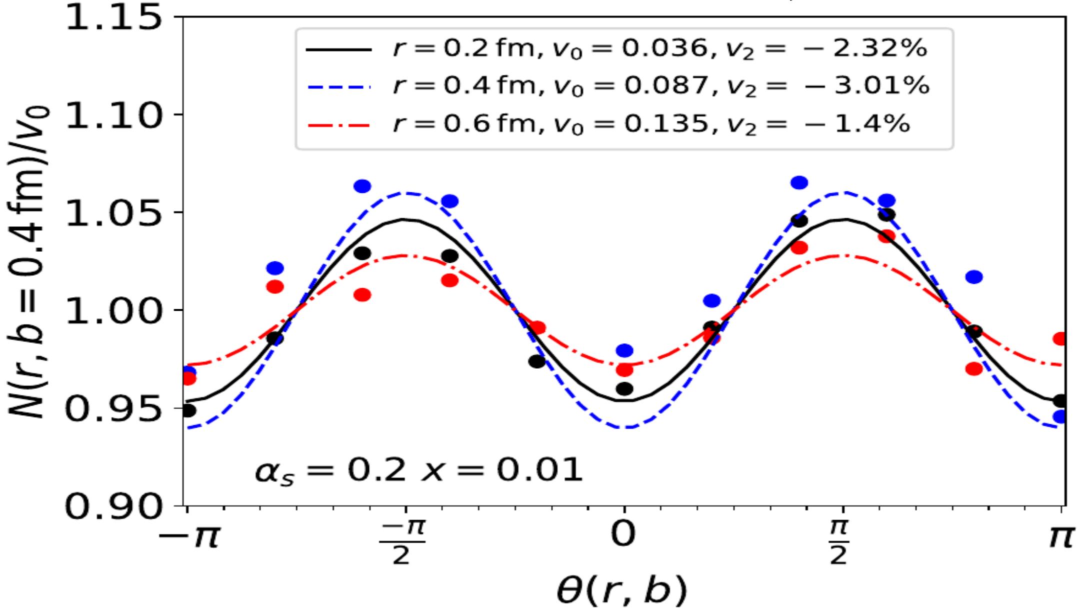


If the color charge correlator is simply proportional to the "proton shape function" then

$$N(\vec{r},\vec{b}) = f(r,b)\ \left(1 + c\, r^2\, b^2\ \cos 2\theta\right)$$

at small r, b; with $c > 0$ !

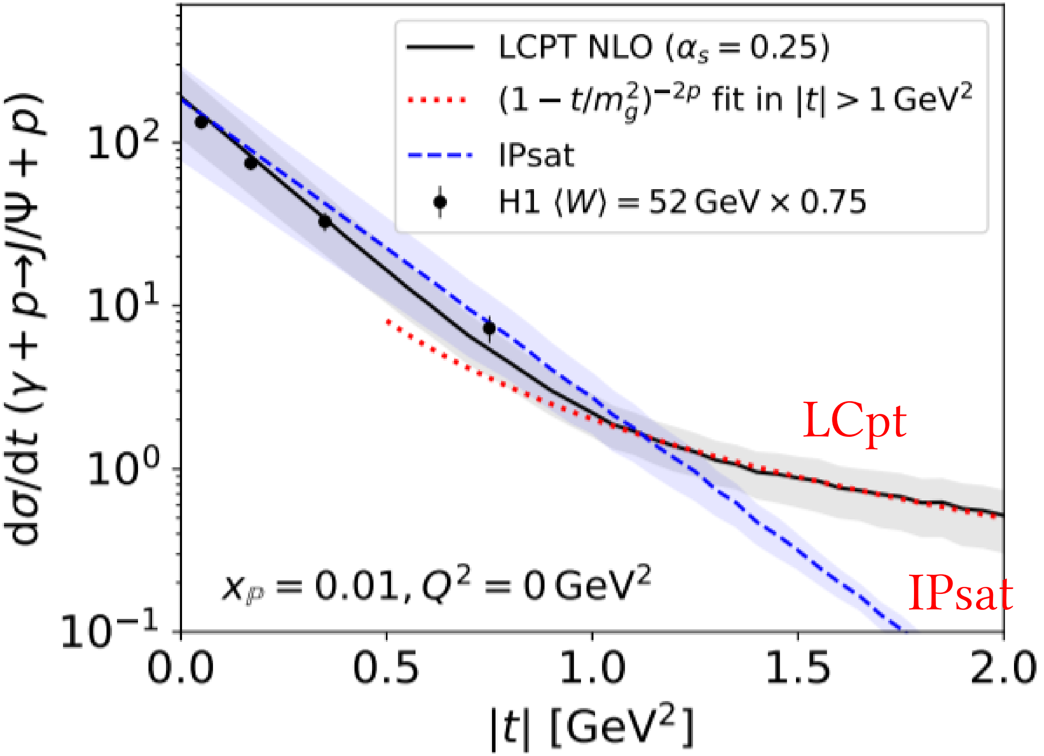
e.g. A. Rezaeian & E. Iancu, 1702.03943
Kovner & Lublinsky, 1211.1928
E. Levin & A. Rezaeian, 1105.3275



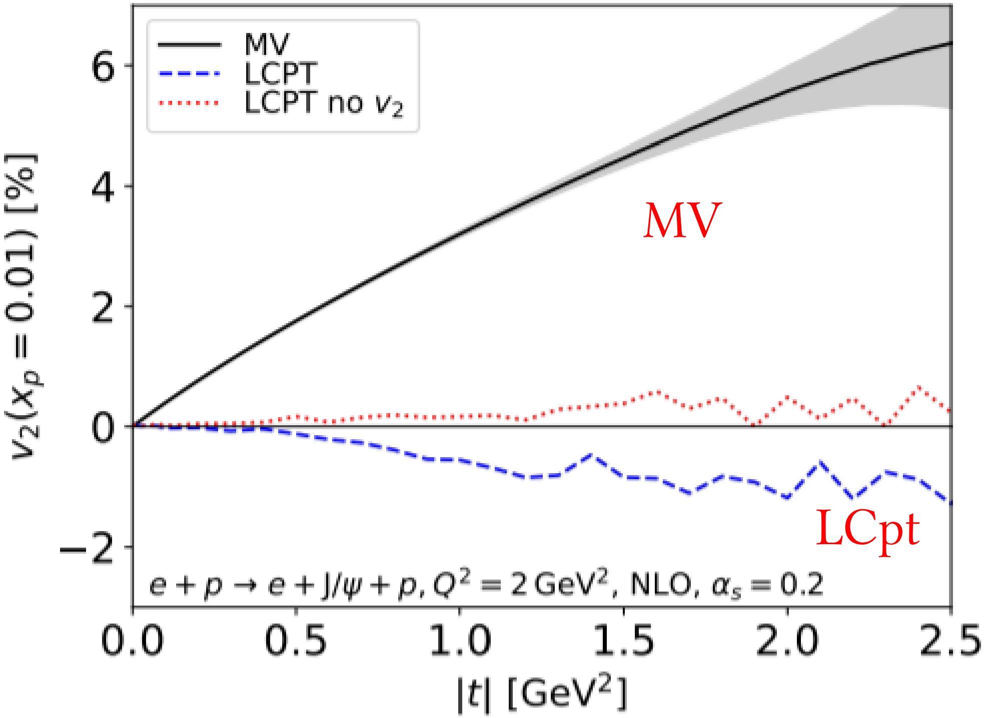

- Recall that "cat's ears" diagram involves 2-body correlations and breaks <ρρ> ~ $T_p(b)$
- v2 from our $N(\vec{r},\vec{b})$ is *negative*

# Potential observables:

J/ψ photoproduction in γp



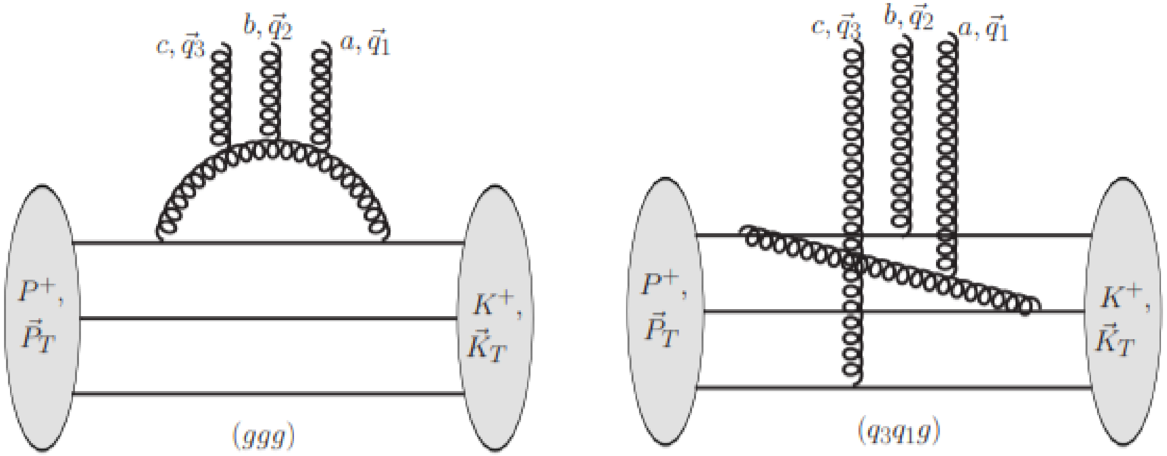

J/ψ − e angular correl. in DIS




H. Mäntysaari et al: 2105.10144, 2105.08503

# $\langle \rho^a \rho^b \rho^c \rangle$ correlator

with Risto Paatelainen, 2106.12623



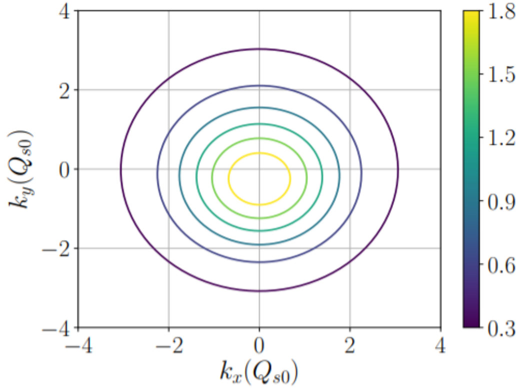

* first correction to Gaussian color charge fluctuations

* C-odd contribution to dipole scattering amplitude

(small-x evol: Kovchegov, Szymanowski, Wallon, PLB 2004; Hatta, Iancu, Itakura, L. McLerran, NPA2005; Lappi, Ramnath, Rummukainen, Weigert, PRD 2016)

* Gluon Sivers function for transv. pol. proton $f_{1T}^{\perp g}(x, k_T^2)$

(Yao, Hagiwara, Hatta: 1812.03959)



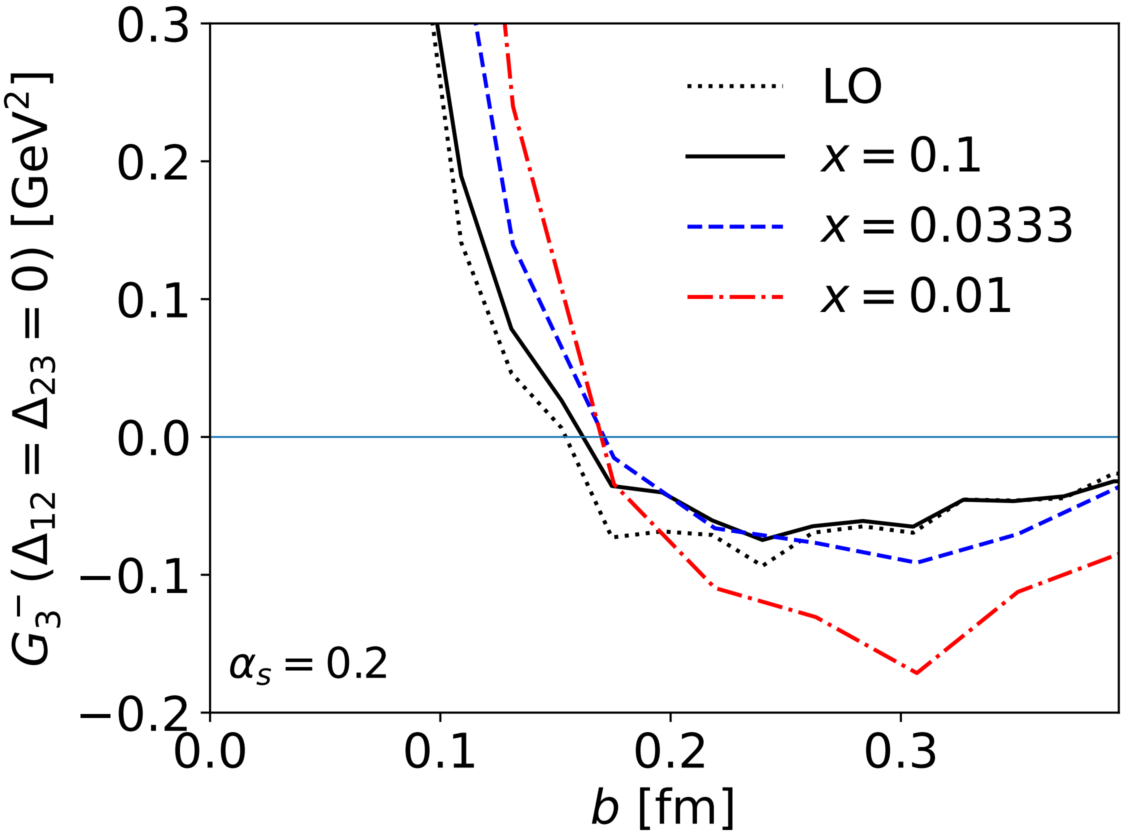
# $G_3^-$ at NLO (|qqq> + |qqqg>)

H. Mäntysaari

- again NLO correction small at
  x ~ 0.1

- not ~ 1-body "parton density" / thickness function
  resulting O(r,b) should differ from quasi-cl. result
  $O(\vec{r},\vec{b}) \sim \alpha_s^3\, r^2\, \vec{r}\cdot\vec{\nabla}T(b)$
  Kovchegov & Sievert, 1201.5890

- n-body quantum correlations at small b !



# Quark entanglement:

small x glue: Kharzeev & Levin, Kovner & Lublinsky, Kovner & Skokov, Armesto et al, Dvali & Venugopalan, Hentschinski & Kutak, ...

color max entangled: proton ~ $\epsilon_{i_1 \cdots i_{N_c}} \left| i_1, \cdots, i_{N_c} \right\rangle \to \rho_{ij} = N_c^{-1}\, \delta_{ij}$

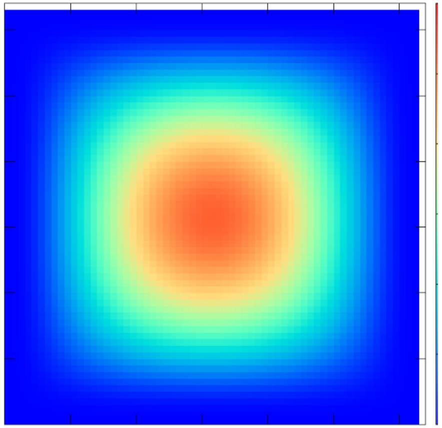
$$\mathrm{tr}\, \rho^2 = N_c^{-1} \,, \quad S_{\mathrm{vN}} = \log N_c$$

spatial dof: density matrix $\rho_{\alpha\alpha'} = \Psi^*_{q_1 \cdots q_{N_c}}(\alpha')\, \Psi_{q_1 \cdots q_{N_c}}(\alpha)$

* at Nc $\longrightarrow \infty$, $\Psi$ factorizes (Witten 1979), so no entanglement, $S_{vN}$=0

* model calculation at Nc=3 (with Eric Kolbusz):
percent level entanglement

calculation of NLO correction is under way



| d.o.f. | w.f. | purity $\mathrm{tr}\, \rho^2$ | $S_{\mathrm{vN}}$ |
|---|---|---|---|
| $\xi$ | HO | 0.983 | 0.052 |
| $\xi$ | PWR | 0.992 | 0.029 |
| $\eta$ | HO | 0.946 | 0.14 |
| $\eta$ | PWR | 0.962 | 0.10 |
| $Q_x$ | HO | 0.985 | 0.046 |
| $Q_x$ | PWR | 0.980 | 0.058 |
| $q_x$ | HO | 0.985 | 0.046 |
| $q_x$ | PWR | 0.980 | 0.058 |

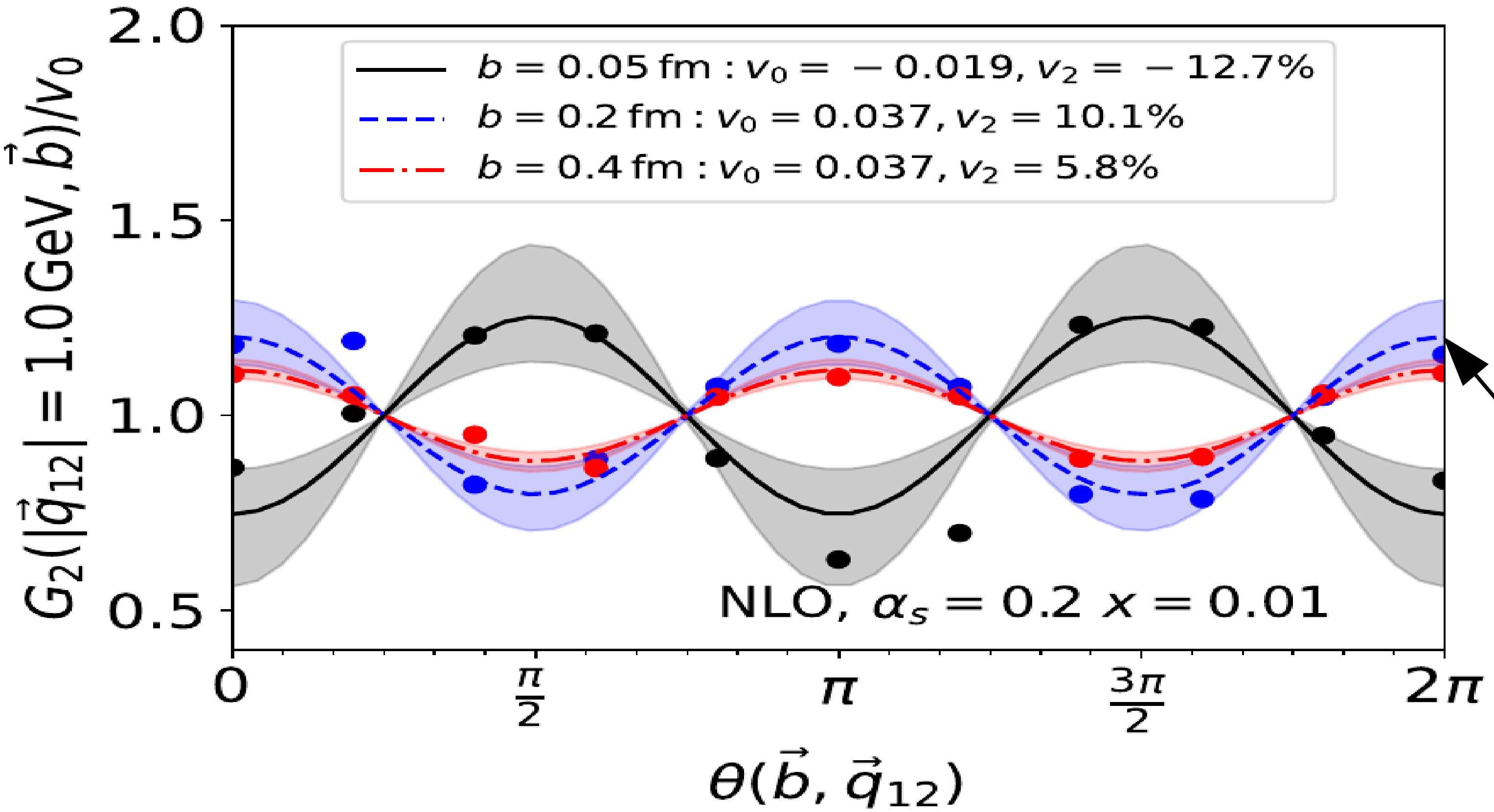
# Future directions, outstanding theory challenges

- …
- …   (improved accuracy: better initial conditions, NLO evol., NLO X-sections, sub-eikonal corrections, …)
- **n-body quantum correlations** in the LCwf of the proton, (Bose-Einstein, entanglement, emerging non-Abelian Glauber coherence?)

  The usual GPD / Wigner function limit (high $Q^2$, low $|t|$) may be insufficient

# Backup Slides

# Color charge correlator exhibits angular dependence:




fits:
$v_0\,(1+2\,v_2 \cos 2\theta)$

* band = variation of coll. regulator 0.1 – 0.4 GeV

* sign and magnitude of <cos 2θ> = $v_2$ changes drastically with b, $q_{12}$

# NLO BK: evolution in terms of target rapidity (i.e. in x) :

B. Ducloué et al: 1902.06637

$$
\begin{aligned}
\frac{\partial \bar{S}_{\boldsymbol{xy}}(\eta)}{\partial \eta} = & \frac{\bar{\alpha}_s}{2\pi} \int \frac{\mathrm{d}^2\boldsymbol{z}\,(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2}\, \Theta(\eta-\delta_{\boldsymbol{xyz}})\left[\bar{S}_{\boldsymbol{xz}}(\eta-\delta_{\boldsymbol{xz};r})\bar{S}_{\boldsymbol{zy}}(\eta-\delta_{\boldsymbol{zy};r})-\bar{S}_{\boldsymbol{xy}}(\eta)\right] \\
& -\frac{\bar{\alpha}_s^2}{4\pi} \int \frac{\mathrm{d}^2\boldsymbol{z}\,(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2} \ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2} \ln\frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2} \left[\bar{S}_{\boldsymbol{xz}}(\eta)\bar{S}_{\boldsymbol{zy}}(\eta) - \bar{S}_{\boldsymbol{xy}}(\eta)\right] \\
& +\frac{\bar{\alpha}_s^2}{2\pi^2} \int \frac{\mathrm{d}^2\boldsymbol{z}\,\mathrm{d}^2\boldsymbol{u}\,(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{u}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2} \left[\ln\frac{(\boldsymbol{u}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{y})^2} + \delta_{\boldsymbol{uy};r}\right] \bar{S}_{\boldsymbol{xu}}(\eta)\left[\bar{S}_{\boldsymbol{uz}}(\eta)\bar{S}_{\boldsymbol{zy}}(\eta) - \bar{S}_{\boldsymbol{uy}}(\eta)\right] \\
& + \bar{\alpha}_s^2 \times \text{“regular”}, \qquad (6.4)
\end{aligned}
$$

non-local in rapidity, involves S at rapidities $\eta < \eta_0 = \log 1/x_0$ !

# Aside: $\langle\rho^a\rho^b\rho^c\rangle$ correlator (C odd part, LO)

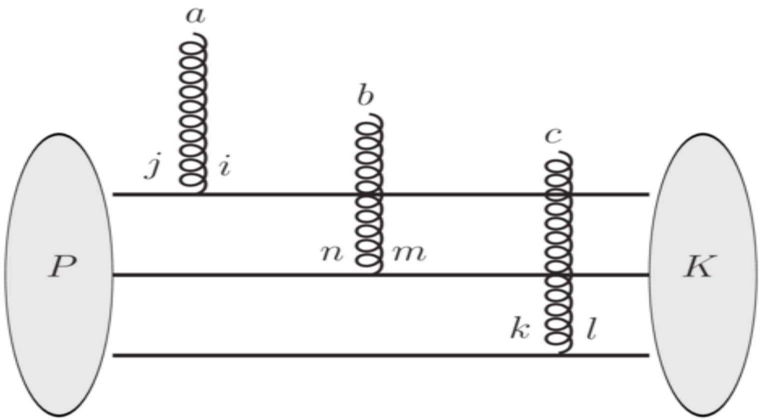
does not vanish (color charge fluct. not Gaussian) :

$$\left\langle \rho^a(\vec{q}_1)\,\rho^b(\vec{q}_2)\,\rho^c(\vec{q}_3)\,\right\rangle_{K_\perp}\Big|_{\mathcal{C}=-} \equiv \frac{g^3}{4}\,d^{abc}\,G_3^-(\vec{q}_1,\vec{q}_2,\vec{q}_3)$$

$$\begin{aligned} G_3^-(\vec{q}_1,\vec{q}_2,\vec{q}_3) = & \int [dx_i][dp_i] \\ & \Big[\psi^*(\vec{p}_1+(1-x_1)\vec{K}_\perp,\vec{p}_2-x_2\vec{K}_\perp,\vec{p}_3-x_3\vec{K}_\perp) \\ & -\psi^*(\vec{p}_1-\vec{q}_1-x_1\vec{K}_\perp,\vec{p}_2+\vec{q}_1+(1-x_2)\vec{K}_\perp,\vec{p}_3-x_3\vec{K}_\perp) \\ & -\psi^*(\vec{p}_1+\vec{q}_2+(1-x_1)\vec{K}_\perp,\vec{p}_2-\vec{q}_2-x_2\vec{K}_\perp,\vec{p}_3-x_3\vec{K}_\perp) \\ & -\psi^*(\vec{p}_1-\vec{q}_1-\vec{q}_2-x_1\vec{K}_\perp,\vec{p}_2+\vec{q}_1+\vec{q}_2+(1-x_2)\vec{K}_\perp,\vec{p}_3-x_3\vec{K}_\perp) \\ & +2\,\psi^*(\vec{p}_1-\vec{q}_1-x_1\vec{K}_\perp,\vec{p}_2+\vec{q}_1+\vec{q}_2+(1-x_2)\vec{K}_\perp,\vec{p}_3-\vec{q}_2-x_3\vec{K}_\perp) \\ & \Big]\,\psi(\vec{p}_1,\vec{p}_2,\vec{p}_3) \end{aligned}$$

* 1-, 2- and 3-particle operators, sum vanishes when either $q_i \longrightarrow 0$ (Ward identities)

* "3-body" diagrams are not (power-) suppressed when $\vec{q}_1 \sim \vec{q}_2 \sim \vec{q}_3 \sim -\vec{K}_T/3 \gg \Lambda_{\mathrm{QCD}}$ but actually dominant !



# Model LFwf for the proton (Brodsky & Schlumpf, PLB 329, 1994)

$$\psi_{\mathrm{H.O.}}(x_1,\vec{k}_1;x_2,\vec{k}_2;x_3,\vec{k}_3) = N_{\mathrm{H.O.}}\exp(-\mathcal{M}^2/2\beta^2)\ ,$$

$$\psi_{\mathrm{Power}}(x_1,\vec{k}_1;x_2,\vec{k}_2;x_3,\vec{k}_3) = N_{\mathrm{Power}}(1+\mathcal{M}^2/\beta^2)^{-p}\ .$$

$$\mathcal{M}^2 = \sum_{i=1}^{3}\frac{\vec{k}^2_{\perp i}+m^2}{x_i}$$

m = 0.26 GeV, β = 0.55 for H.O. wf

m = 0.263, β = 0.607, p = 3.5 for PWR wf

With these parameters they fit:

- proton radius $R^2 = -6\,\dfrac{dF_1(Q^2)}{dQ^2}\Big|_{Q^2=0} = (0.76\ \mathrm{fm})^2$

- proton / neutron magnetic moments $1+F_2(Q^2\to 0) = 2.81\ /\ -1.66$

- axial vector coupling $g_A$ = 1.25