x - dependent unintegrated gluon distribution from Regge to Bjorken kinematics

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(In collaboration with Renaud Boussarie)

2006.14569 [hep-ph]   2112.01412 [hep=ph]

@ RBRC Workshop: Small-x Physics in the EIC Era (BNL)
December 15 - 17, 2021

BROOKHAVEN NATIONAL LABORATORY

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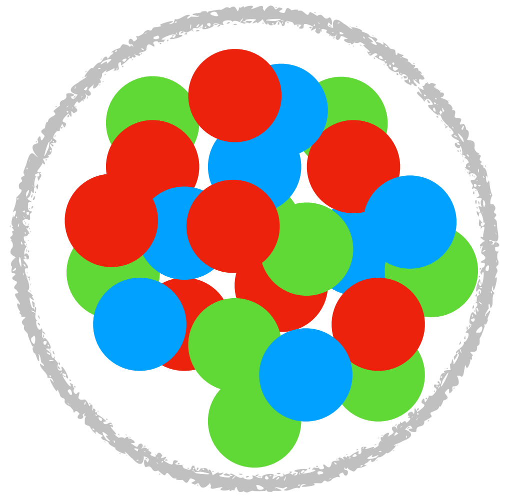
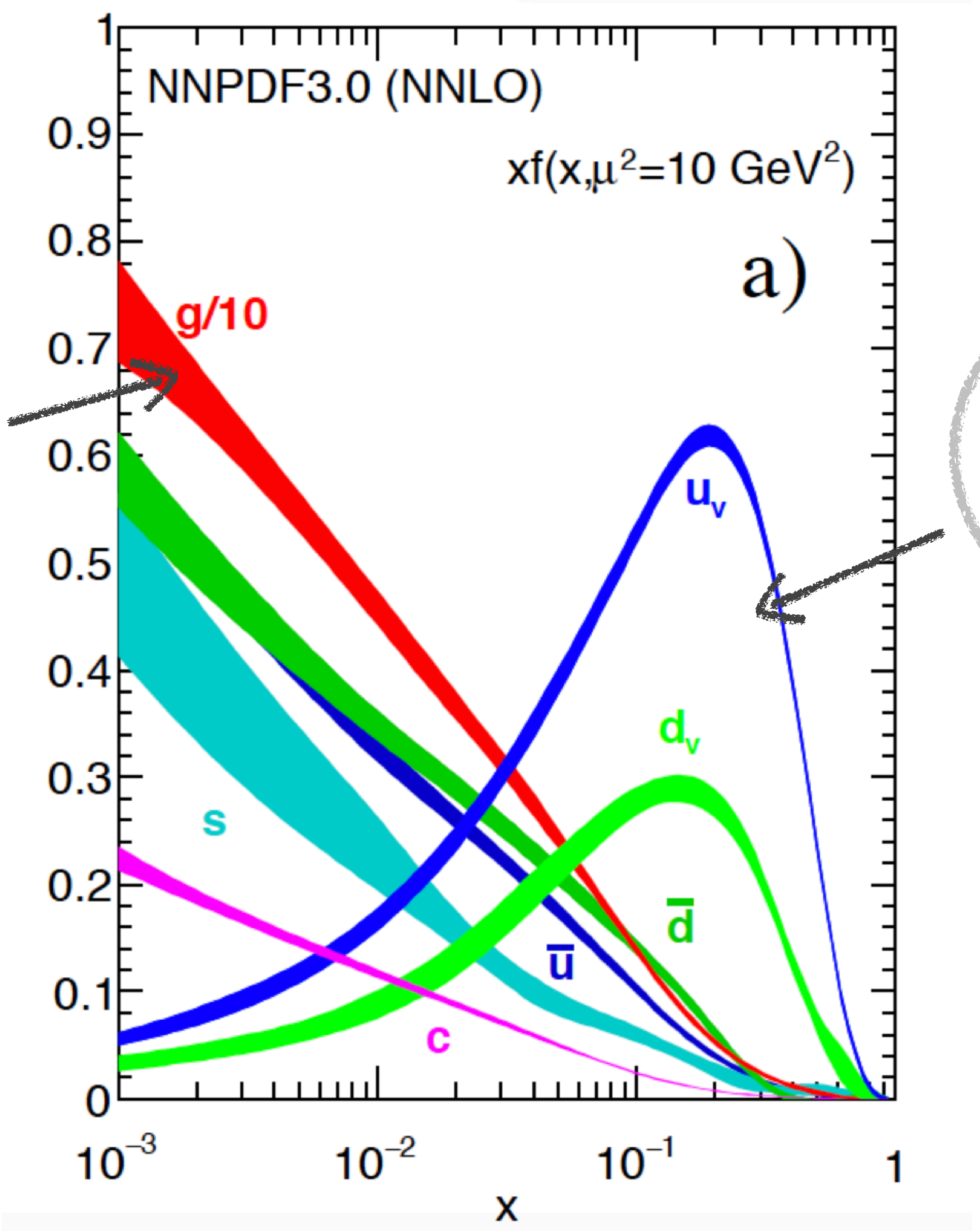
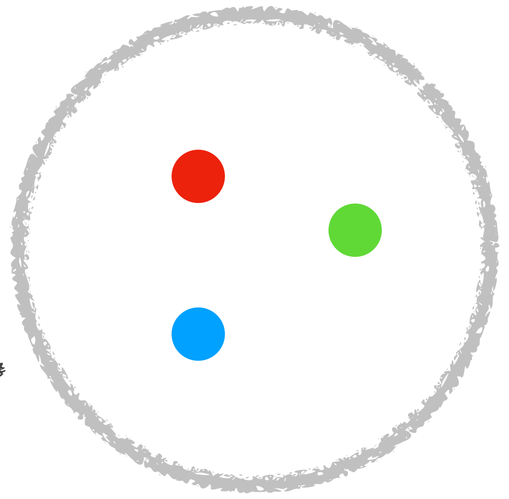
Reconciling two pictures of the proton

CGC: BK - JIMWLK (BFKL)

DGLAP







Strong fields

Partons

Gluon density rises rapidly at small x: large occupation numbers →

Regime of strong classical fields: breakdown of the parton picture

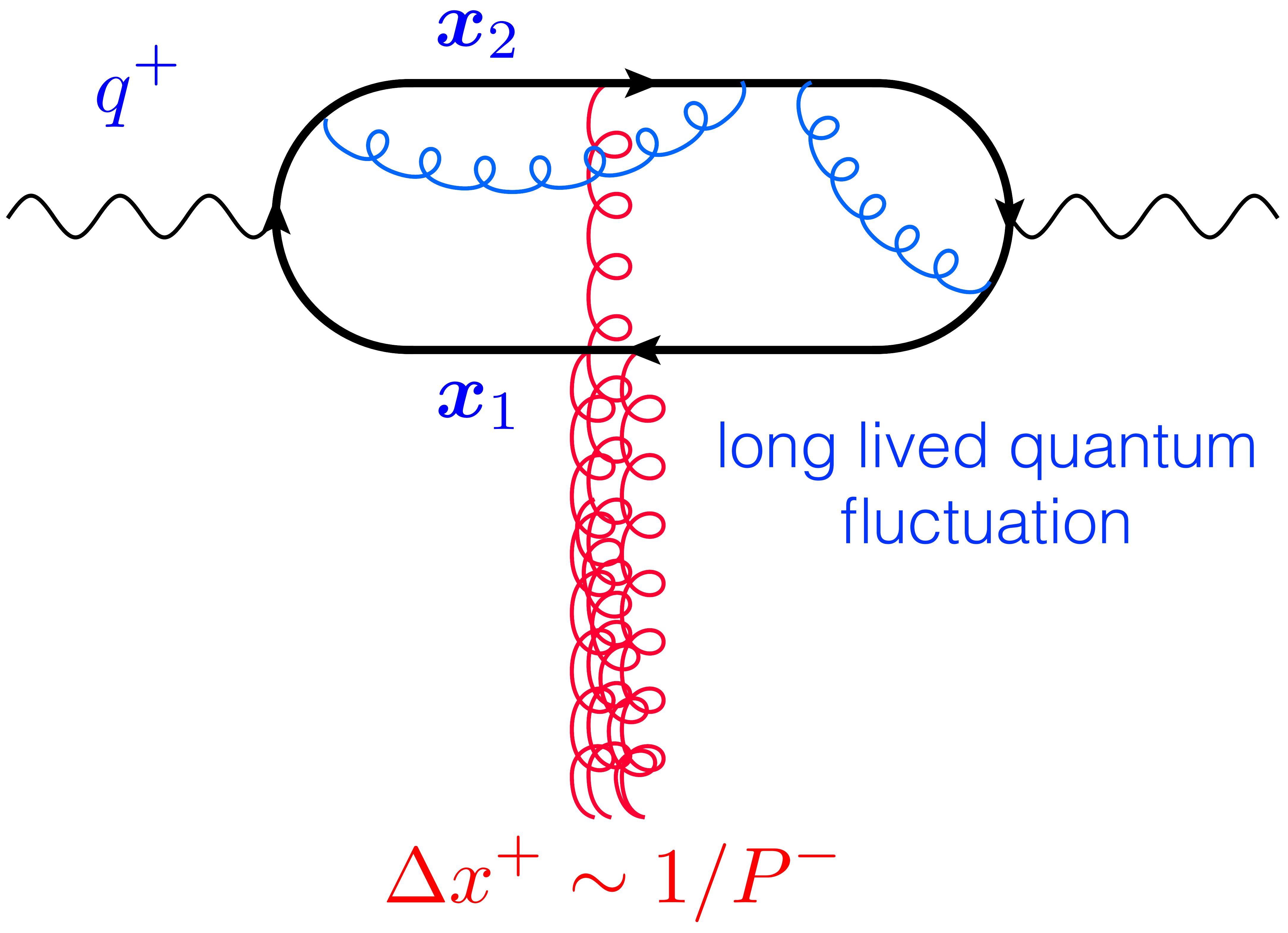
Rapidity factorization at small x - coherent scattering

Step 1: Split the gluon fields into fast and slow gluons

$$A^\mu(k) \equiv A^\mu(k^+ < \Lambda^+) + a^\mu(k^+ > \Lambda^+)$$

Step 2: shock wave approximation at small x the - component of the momenta must also be strongly separated

$$q^- \equiv x_{\rm Bj} P^- \ll P^-$$



$$x_{\rm Bj} = Q^2/s \to 0$$

[Gribov, Levin, Ryskin, 1983- Mueller, Qiu, 1986, Venugopalan, McLerran (MV), Balitsky, Kovchegov (BK) Jallilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner (JIMWLK) (1993-2001)]

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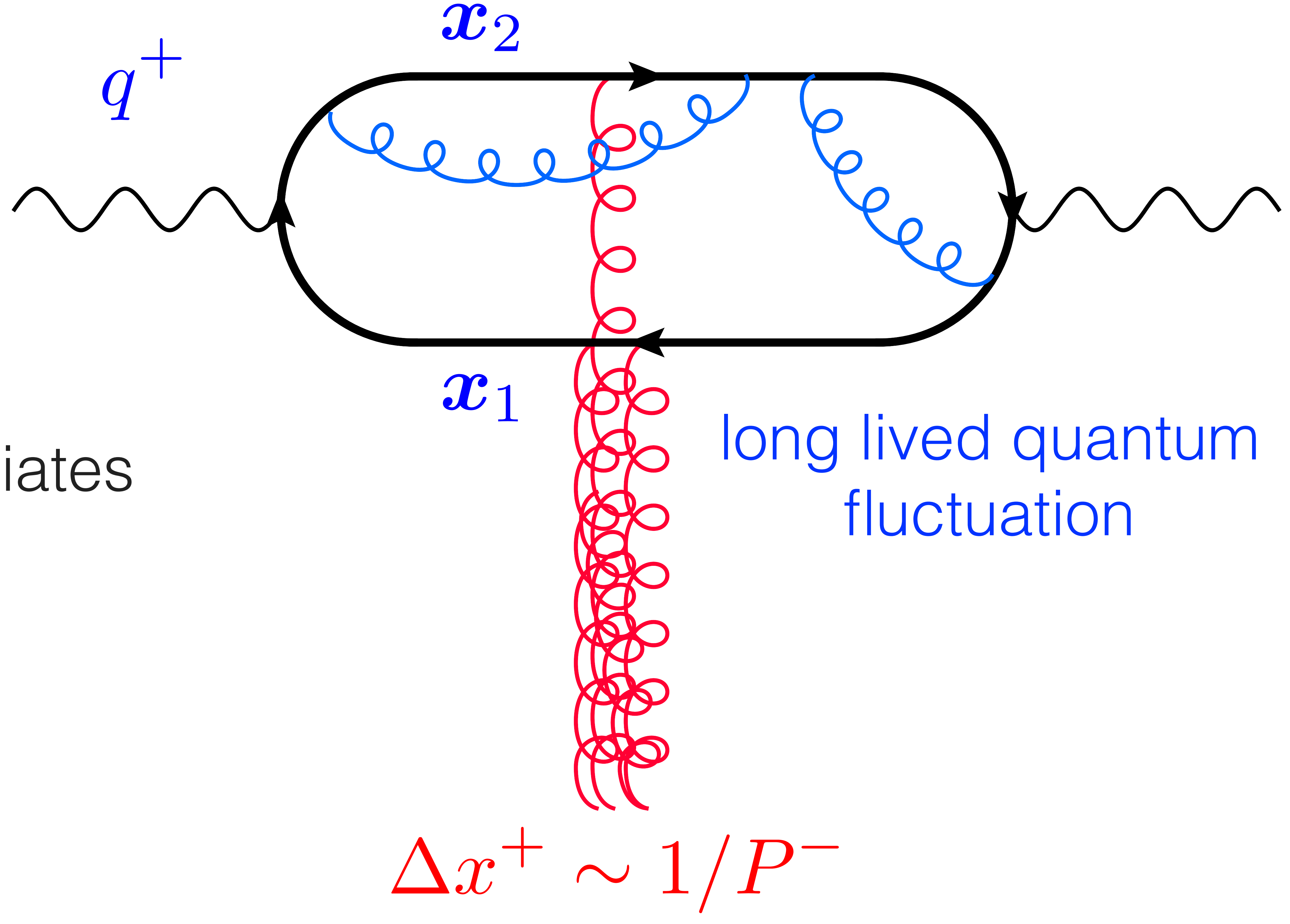
The interaction with the background field exponentiates and light cone time integrations decouple

$$U_{\boldsymbol{x}} \equiv \mathcal{P}_+ \exp\left[ig \int_{-\infty}^{+\infty} {\rm d}z^+ A^-(z^+, \boldsymbol{x}_\perp)\right]$$



Factorization at small x:

$$\sigma \sim \int_{\boldsymbol{x}_1,\boldsymbol{x}_2} \mathcal{H}(\boldsymbol{x}_2 - \boldsymbol{x}_1) \otimes \langle P| \,{\rm Tr}\, U_{\boldsymbol{x}_1} U^\dagger_{\boldsymbol{x}_2} |P\rangle$$

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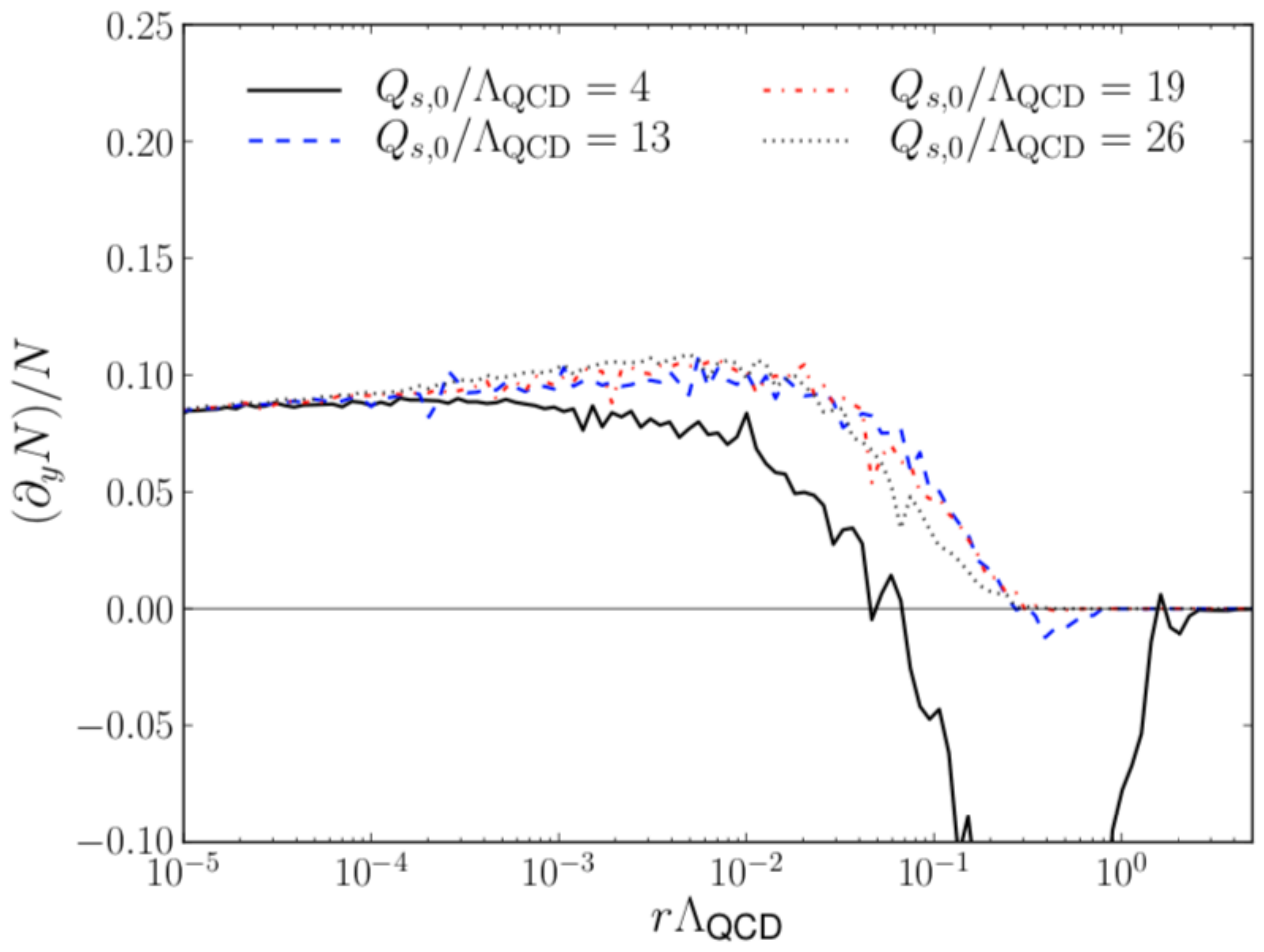
BK and the NLO crisis

- Balitsky-Kovchegov (1996-1999) equation describes the non-linear evolution of the dipole scattering amplitude as function of the rapidity $Y \equiv \log q^+/\Lambda^+$

$$\frac{\partial}{\partial Y} S_Y(\boldsymbol{x}-\boldsymbol{y}) = \bar{\alpha}\mathcal{K}_{NLO} \otimes [S_Y(\boldsymbol{x}-\boldsymbol{z})S_Y(\boldsymbol{z}-\boldsymbol{y}) - S_Y(\boldsymbol{x}-\boldsymbol{y})] + \text{non}-\text{dipole}$$

- At NLO BK equation [Balitsky and Chirilli (2008)] was found to be numerically unstable



[Lappi and Mäntysaari (2015)]

- This is due to the fact that the rapidity variable $Y \equiv \log \frac{q^+}{\Lambda^+}$ evolves independently from $\boldsymbol{x}_\perp$ violating $k^- = xP^-$ ordering $\rightsquigarrow$ produces large collinear logs

[Beuf (2014) Ducloué, Iancu, Mueller, Soyez, Triantafyllopoulos (2015-2019)]

BK and the NLO crisis

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    - they alter the renormalization picture established at LO
    - exhibit scheme dependence
    - no operator definition for systematic order by order calculations
    - not discussed at the level of observables: factorization scheme.

x dependence at small x

- In the Regge limit distributions (operators) evaluated strictly at x=0

$$f(k_\perp, x = 0)$$

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The origin of the issue at NLO in the collinear corner of phase space can be traced back to neglecting x in the distribution

Going beyond shock wave approximation

- Sub-eikonal expansion around the shock wave $\delta(x^+)$ [Agostini, Altinoluk, Armesto, Beuf, Martinez, Moscoso, Salgado]
- Expansion in the boost parameter [Chirilli] ; [Altinoluk, Beuf, Czajka, Tymowska]
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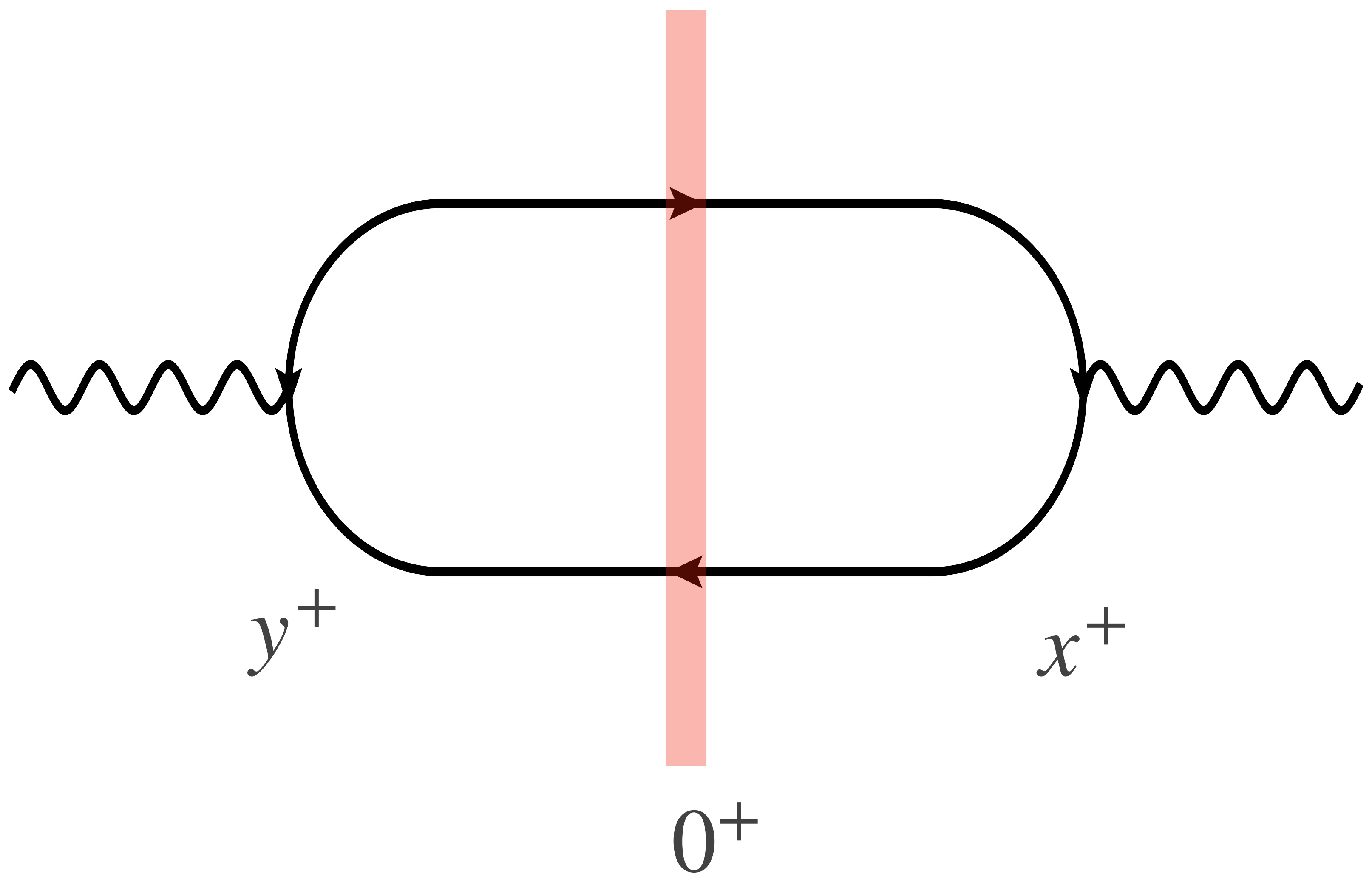
Our approach:

- revisit the shock wave factorization scheme to restore the x dependence of the gluon distribution
- perform a partial twist expansion to connect Regge and Bjorken limits

$$\rightsquigarrow \quad f(k_\perp, x) + O\left(\frac{x_{\mathrm{Bj}}}{Q^2}\right)$$

Inclusive DIS beyond shock wave

- **Decoupling of time integrals:** In the shock wave approximation the times of the photon splitting into quark antiquark pair are integrated form $0 < x^+ < +\infty$ and $-\infty < y^+ < 0$



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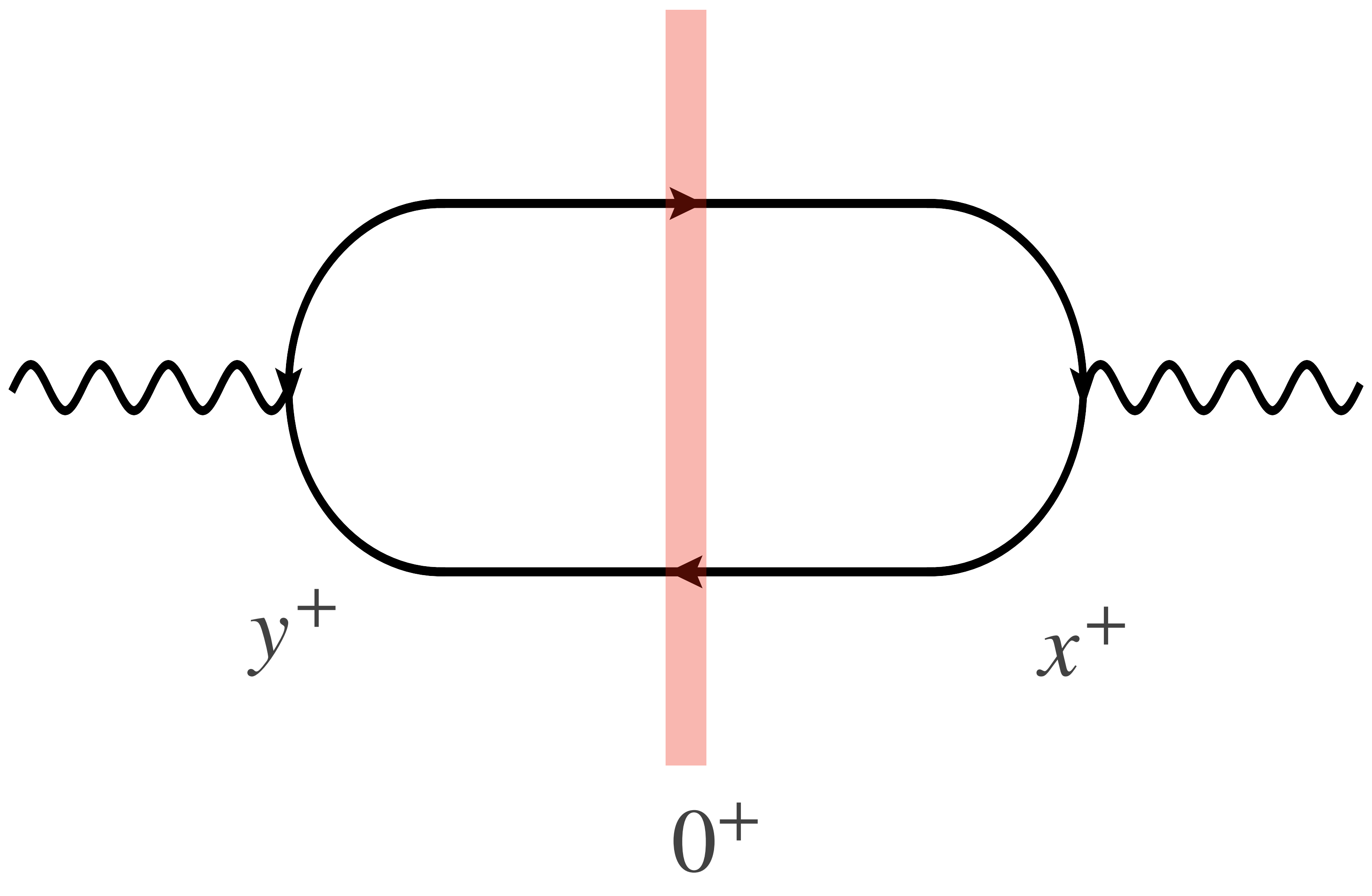
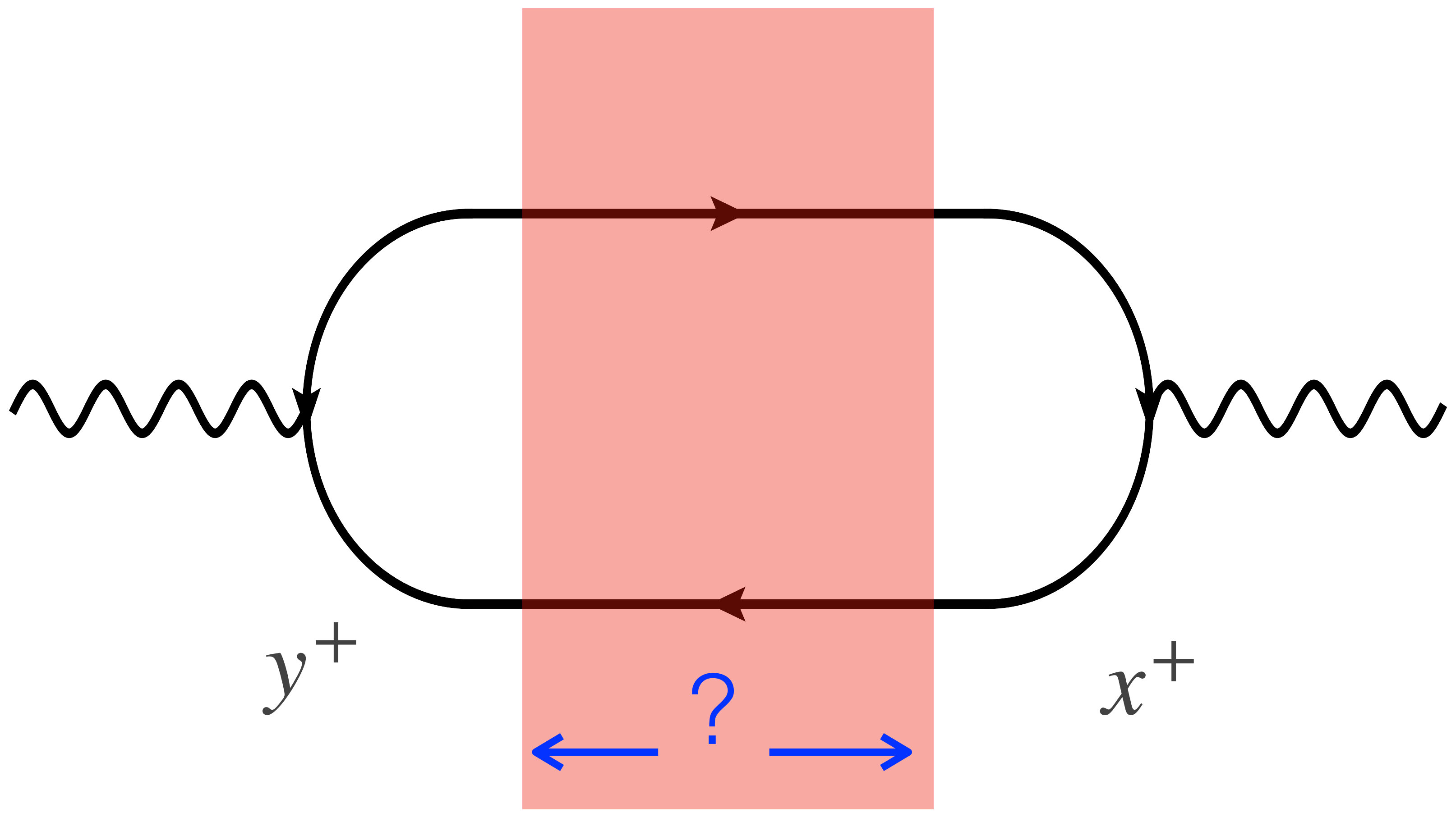


- What are the integration limits of the vertices if one relaxes the shock wave approximation?
- What is the longitudinal extent of the shock wave ?

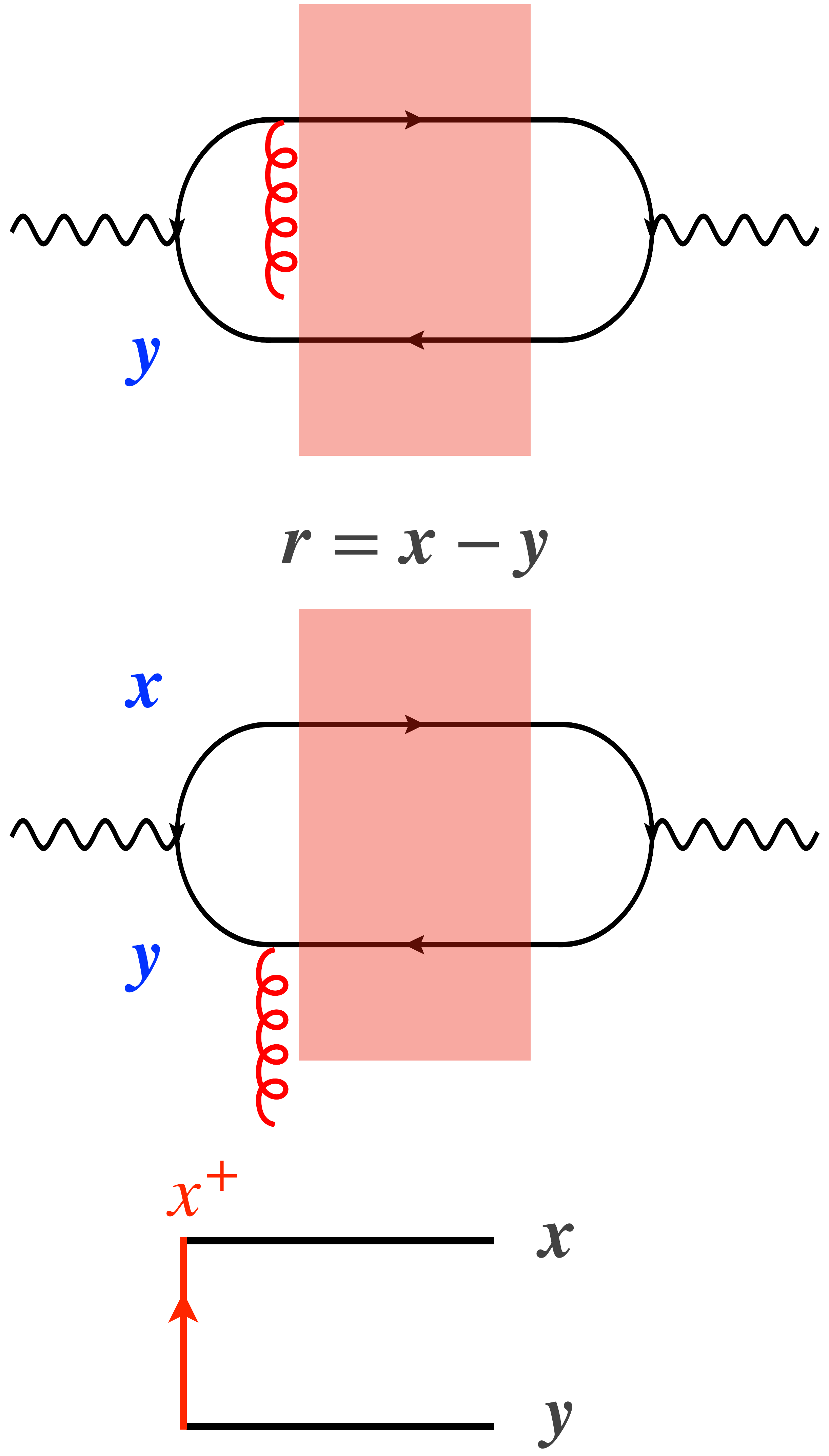


Inclusive DIS beyond shock wave

- **Solution**: extracting the first and last interactions provides a physical boundary to the shock wave
- 4 contributions that combines into to one
- Consider the left side of the diagram first
- The two gluon fields combine to generate a field strength tensor

$$A^-(\boldsymbol{x}) - A^-(\boldsymbol{y}) = \int_0^1 \mathrm{d}s\, r^i \partial^i A^-(\boldsymbol{y} + s\boldsymbol{r}) = \int_0^1 \mathrm{d}z^i\, F^{i-}(\boldsymbol{z})$$

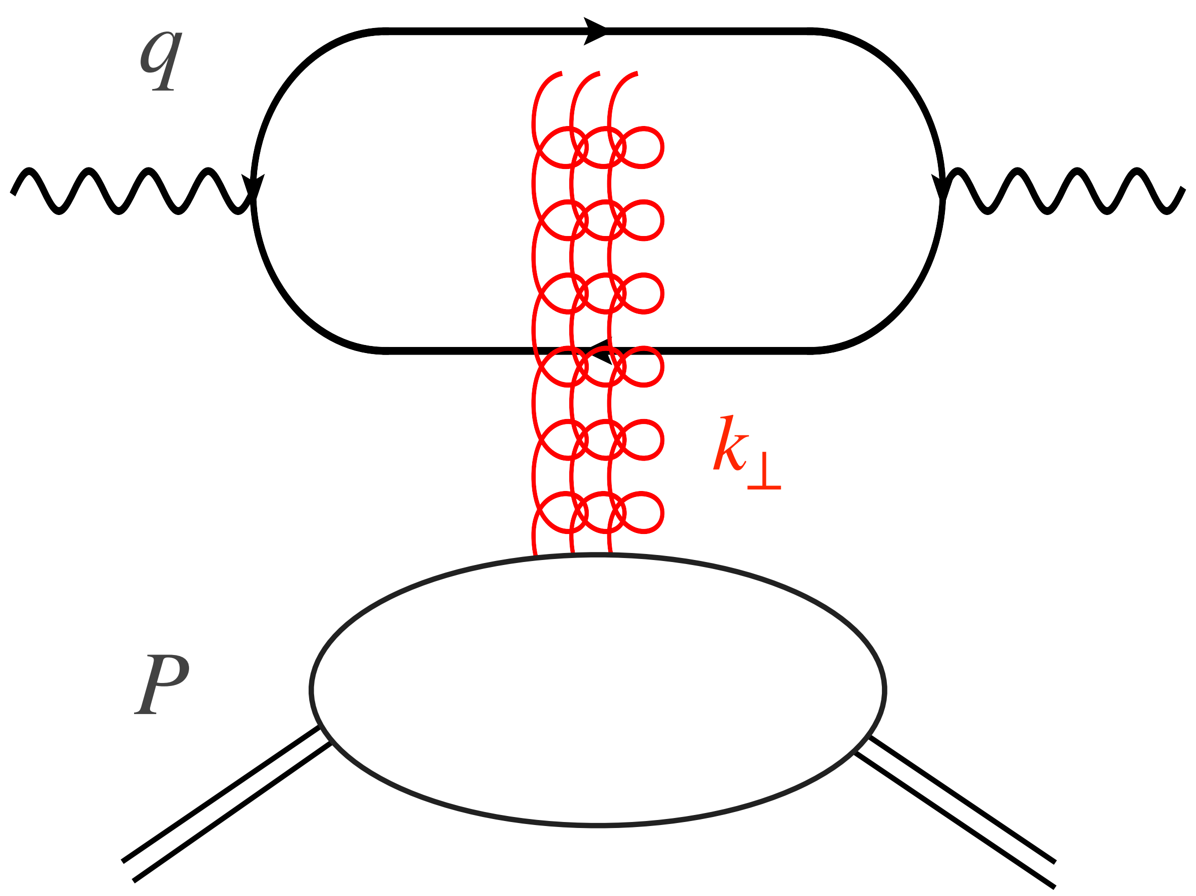
- More generally: $\frac{\partial^+}{\partial x^+}[y^+, x^+]_{\boldsymbol{x}}[\boldsymbol{x}, \boldsymbol{y}]_{x^+}[x^+, y^+]_{\boldsymbol{y}}$



$\rightsquigarrow$ Preserves LC time x^+ ordering and hence k^- ordering is built in.

Inclusive DIS beyond shock wave

- DIS cross-section takes a similar form to that of the shock wave: identical wave functions



Shock wave factorization:

$$\sigma(x_{Bj}, Q^2) \sim e^2 \int_0^1 \mathrm{d}\, z\, P(z) \int_{\boldsymbol{r},\boldsymbol{b}} \mathrm{d}\, \boldsymbol{r}\ |\varphi(z(1-z)|\boldsymbol{r}|^2 Q^2)|^2\ \langle \mathrm{Tr} U(\boldsymbol{r}) U^\dagger(0)\rangle_Y$$

Beyond shock wave:

$$\sigma(x_{Bj}, Q^2) \sim e^2 \int_0^1 \mathrm{d}\, z \int_{\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{b}_1,\boldsymbol{b}_2} \partial^i \varphi(z(1-z)|\boldsymbol{r}_1|^2 Q^2)\ \partial^j \varphi^*(z(1-z)|\boldsymbol{r}_2|^2 Q^2)$$
$$\times\, \mathrm{e}^{i\, x_{Bj} P^-(x_2^+ - x_2^+)}\ \langle P\,|\, U^{ij}(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{b}_1, \boldsymbol{b}_2, x_1^+, x_2^+\,|\,z)\,|\,P\rangle$$

x-dependent Fourier phase

gauge invariant dipole operator

- NB: factorization in k^+ [Balitsky-Tarasov]

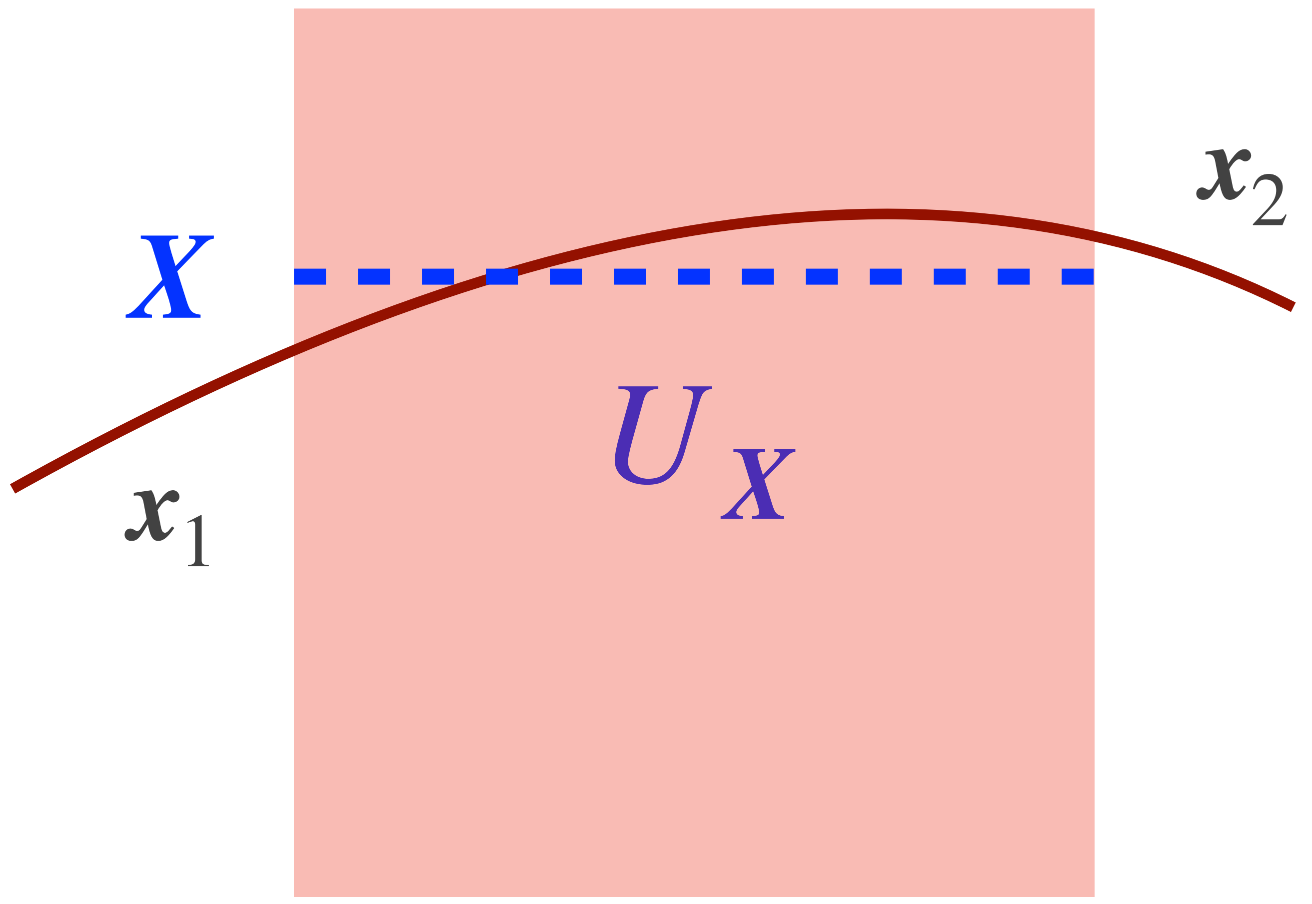
Partial twist expansion

- Leading power in s and Q^2 can be obtained by neglecting transverse recoil of fast partons

$O(x_{\mathrm{Bj}})$ - suppressed in the Regge limit

$O(Q^2)$ - suppressed in the Bjorken limit

$\Delta \boldsymbol{x}^2 \sim x_{\mathrm{Bj}}/Q^2$ ↕



$$\mathcal{G}_{p^+}(x^+, \boldsymbol{x}_2;\, y^+, \boldsymbol{x}_1) = \mathcal{G}_0(\boldsymbol{x}_2 - \boldsymbol{x}_1, x_2^+ - y_1^+)\, U_{\boldsymbol{X}}(x_2^+, x_1^+) + \ldots$$

Altinoluk, Armesto, Beuf, Martinez, Salgado (2015)

- x encoded in quantum diffusion: FT w.r.t. $\boldsymbol{r} = \boldsymbol{x}_2 - \boldsymbol{x}_1$

$$\mathcal{G}_{p^+}(x_2, x_1^+, \boldsymbol{X}; \boldsymbol{\ell}) = \mathrm{e}^{i\frac{\ell^2}{2zq^+}\Delta x^+}\, U_{\boldsymbol{X}}(x_2^+, x_1^+) + \ldots$$

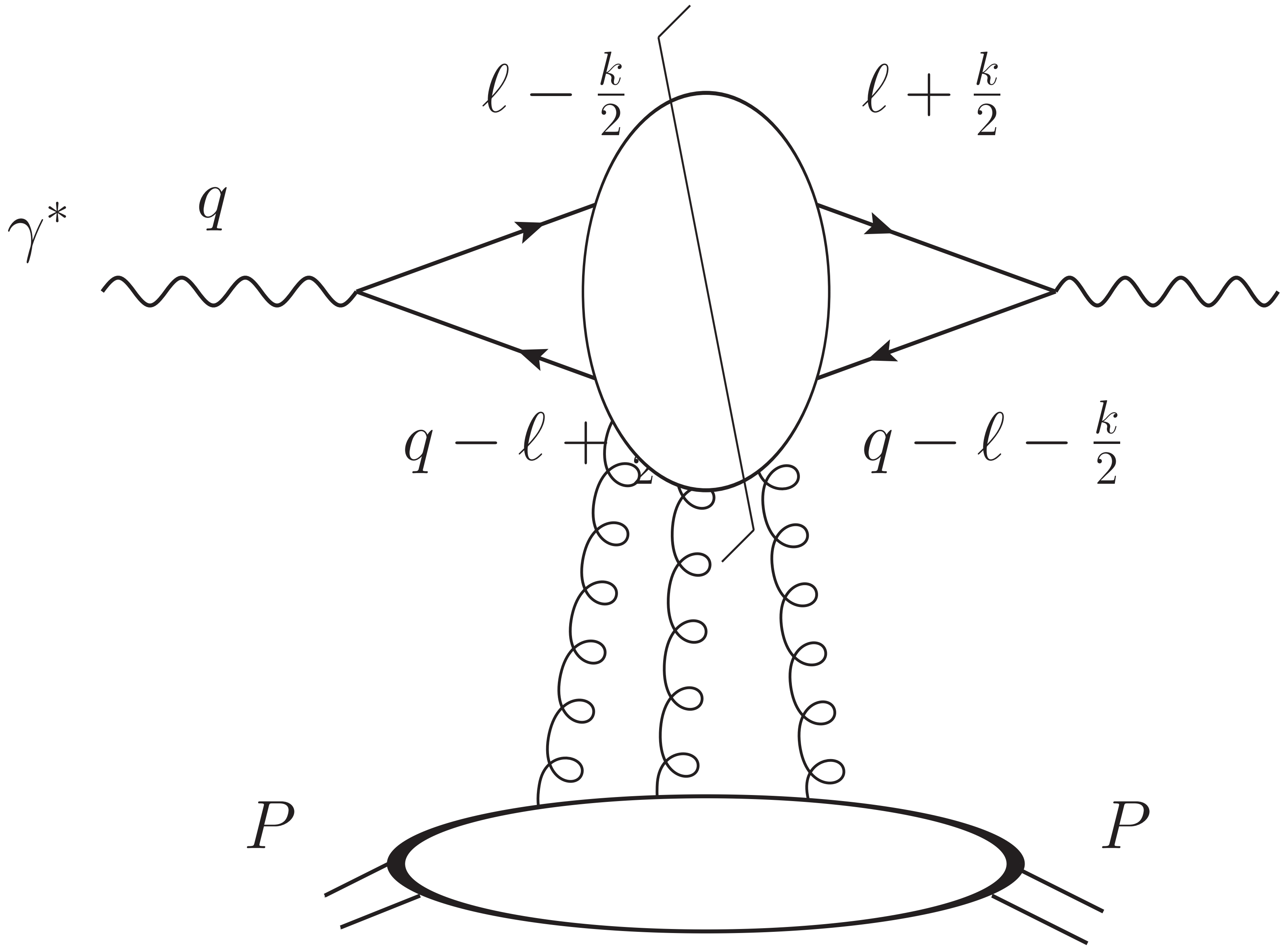
shock wave (eikonal limit) $\quad \lim_{p^+\to+\infty} \mathcal{G}_{p^+}(x^+, \boldsymbol{x};\, y^+, \boldsymbol{y}) = \delta(\boldsymbol{x} - \boldsymbol{y})\, U_{\boldsymbol{x}}(x^+, y^+)$

Factorization formula for DIS at arbitrary x

- After applying partial PTE to leading power we obtain the factorization formula (for the transverse photon cross-section), in momentum space,

$$\sigma(x_{Bj}, Q^2) \sim e^2 \int_0^1 \mathrm{d}\, z \int_0^1 \mathrm{d}x \int_{\boldsymbol{\ell},\boldsymbol{k}} \partial^i \varphi\left(\boldsymbol{\ell} - \frac{\boldsymbol{k}}{2}\right) \partial^j \varphi^* \left(\boldsymbol{\ell} + \frac{\boldsymbol{k}}{2}\right) \delta\left(x - x_{Bj} - \frac{\boldsymbol{\ell}^2}{2z\bar{z}q^+}\right)$$

$$\times\, xG^{ij}(x, \boldsymbol{k}) \; + O\left(k_\perp^2/s\right)$$



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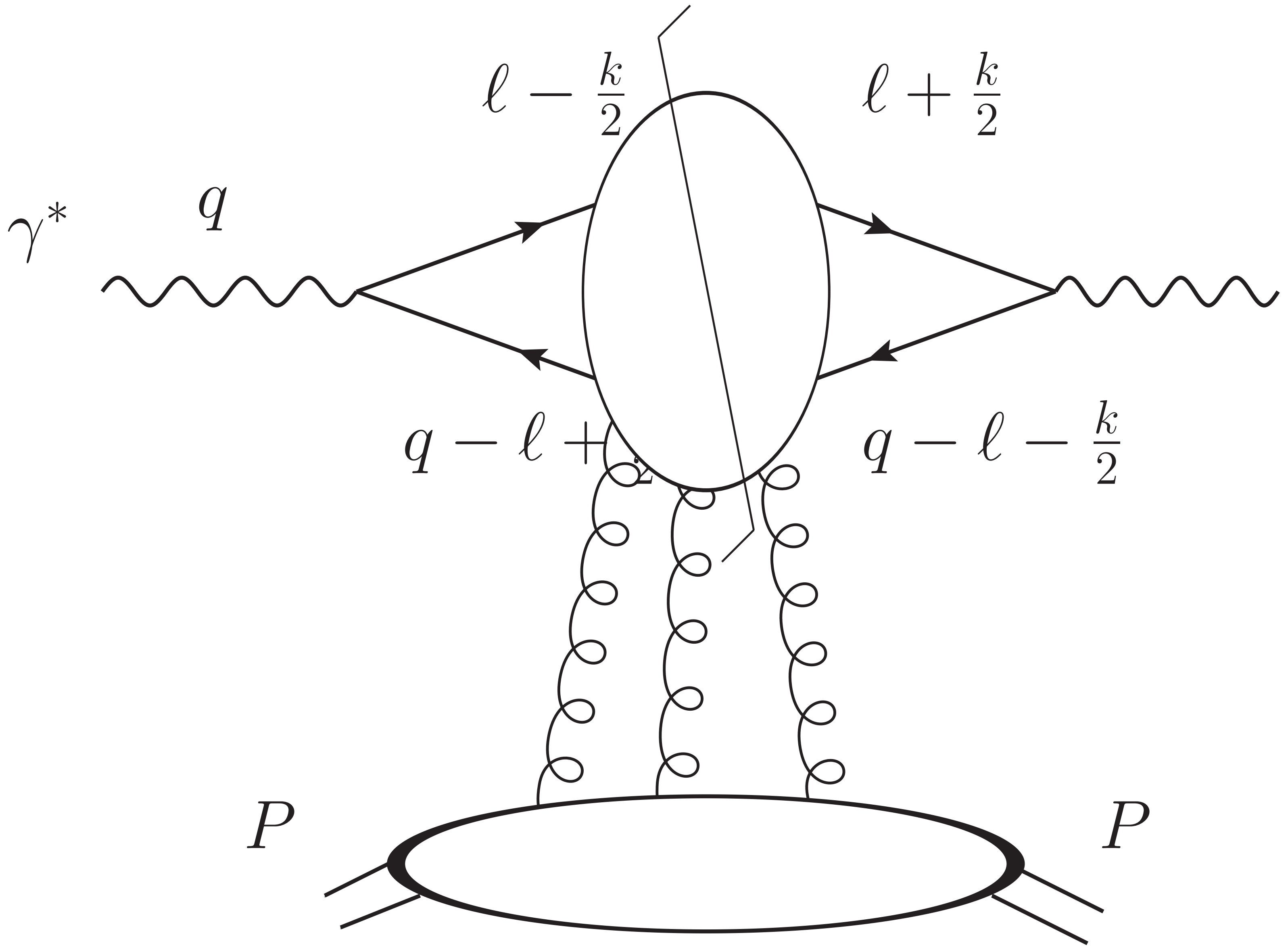

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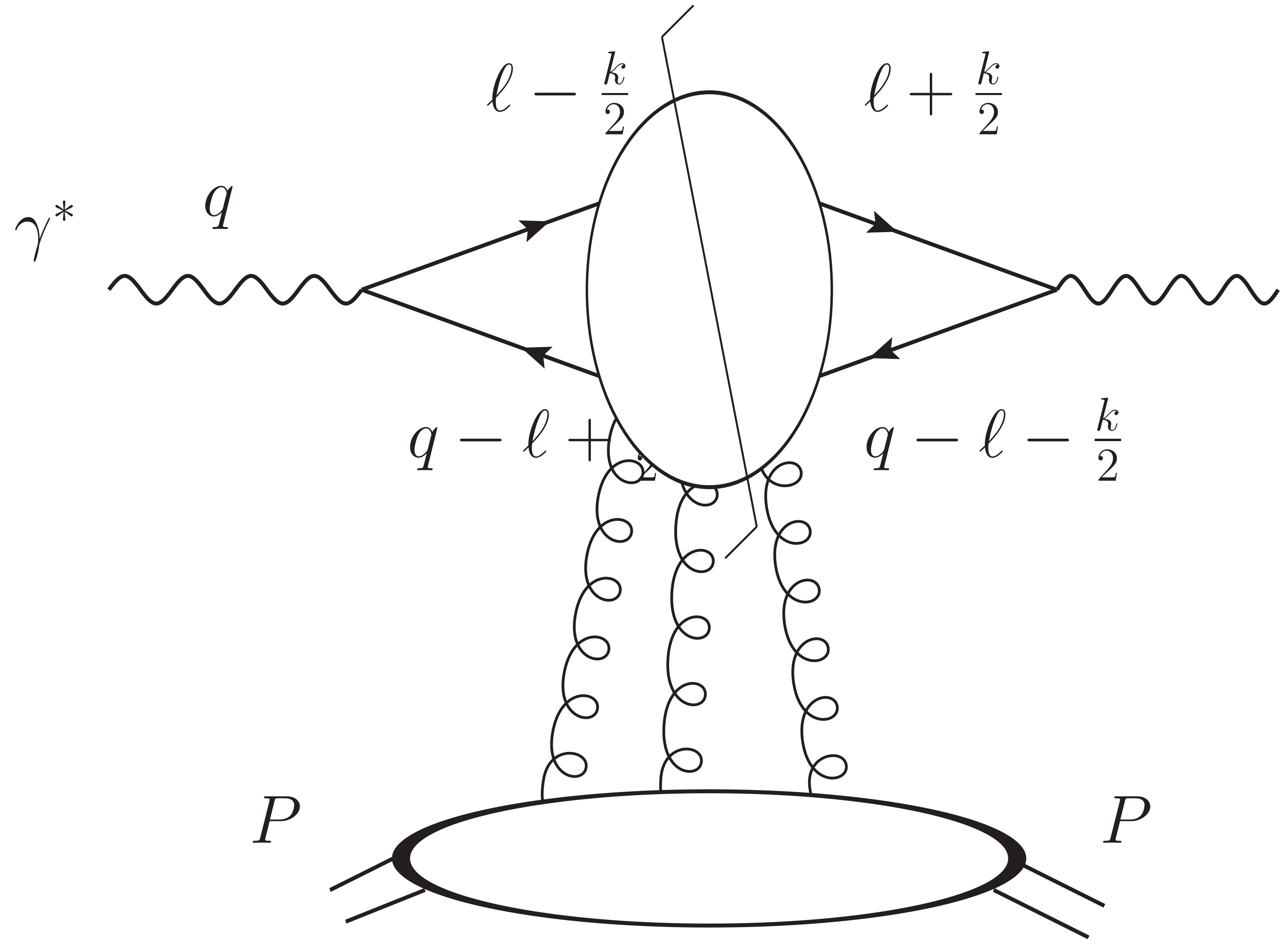

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- The delta function relates x in the gluon distribution to x_{Bj} (kinematic constraint in momentum space)

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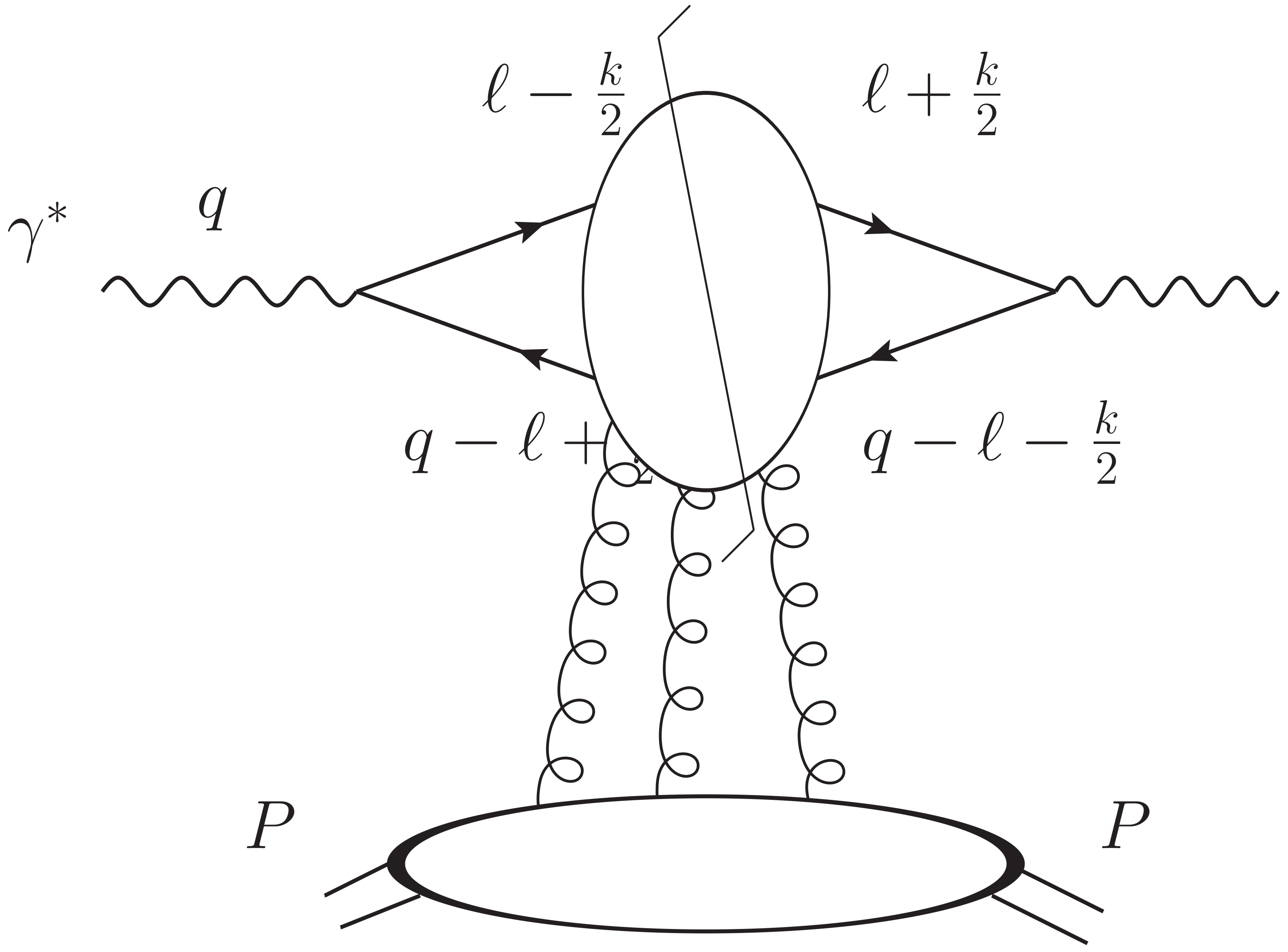
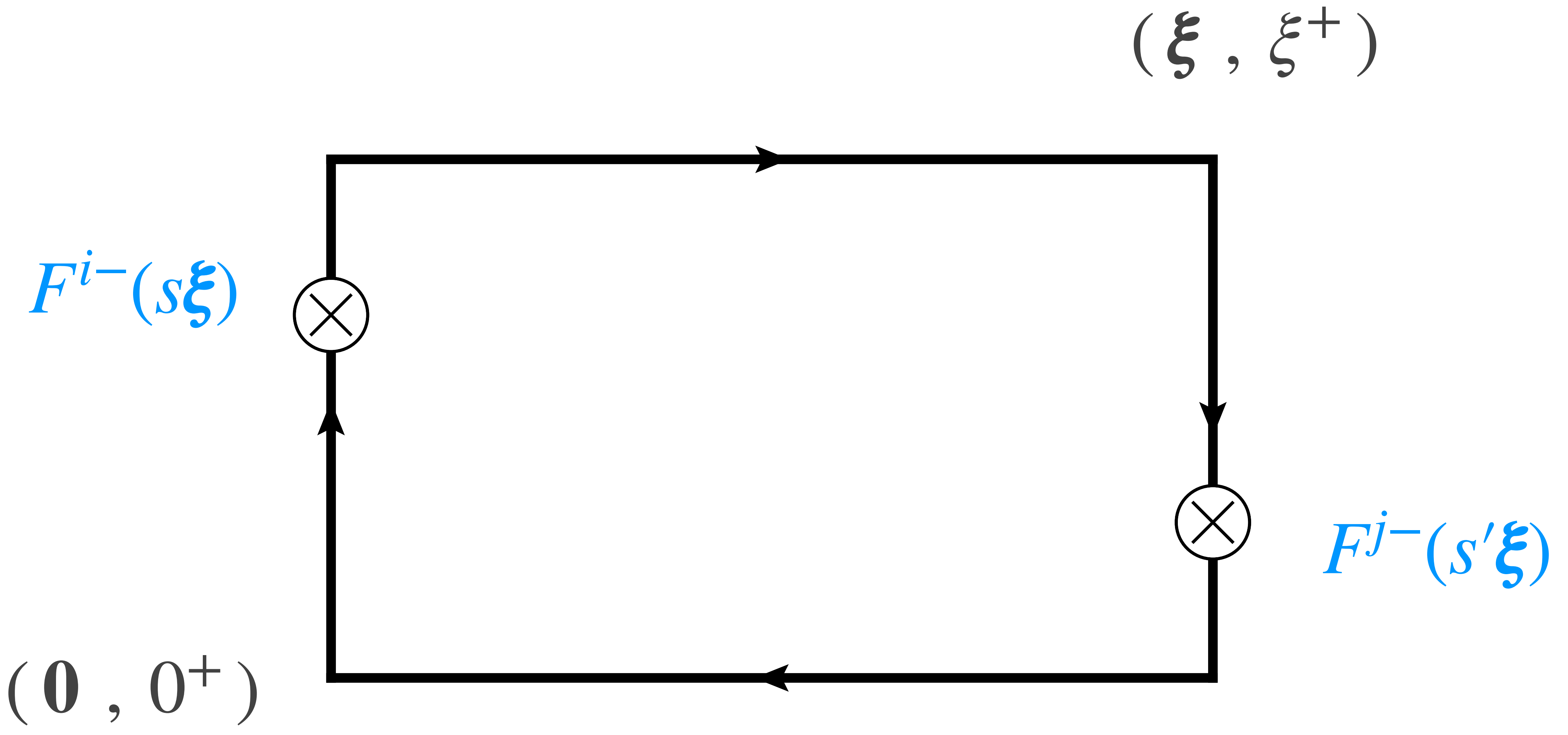


- Same wave functions as in small x
- The delta function relates x in the gluon distribution to x_{Bj} (kinematic constraint in momentum space)
- Gluon distribution different than small x

x-dependent and gauge invariant unintegrated gluon distribution

$$xG^{ij}(x, k_\perp) \equiv 2 \int_{s,s'} \int \frac{\mathrm{d}\xi^+ \mathrm{d}\boldsymbol{\xi}}{(2\pi)^3 P^-} \mathrm{e}^{\,i\,x\,P^- \xi^+ - i\boldsymbol{k}\cdot\boldsymbol{\xi}} \left\langle P \,|\, \mathrm{Tr}\,[\,0\,,\xi^+]_{\boldsymbol{\xi}}\, F^{j-}(\xi^+, s'\boldsymbol{\xi})\,[\,\xi^+\,,0]_{\boldsymbol{0}}\, F^{i-}(\,0\,,s\boldsymbol{\xi})\,|\, P \right\rangle$$

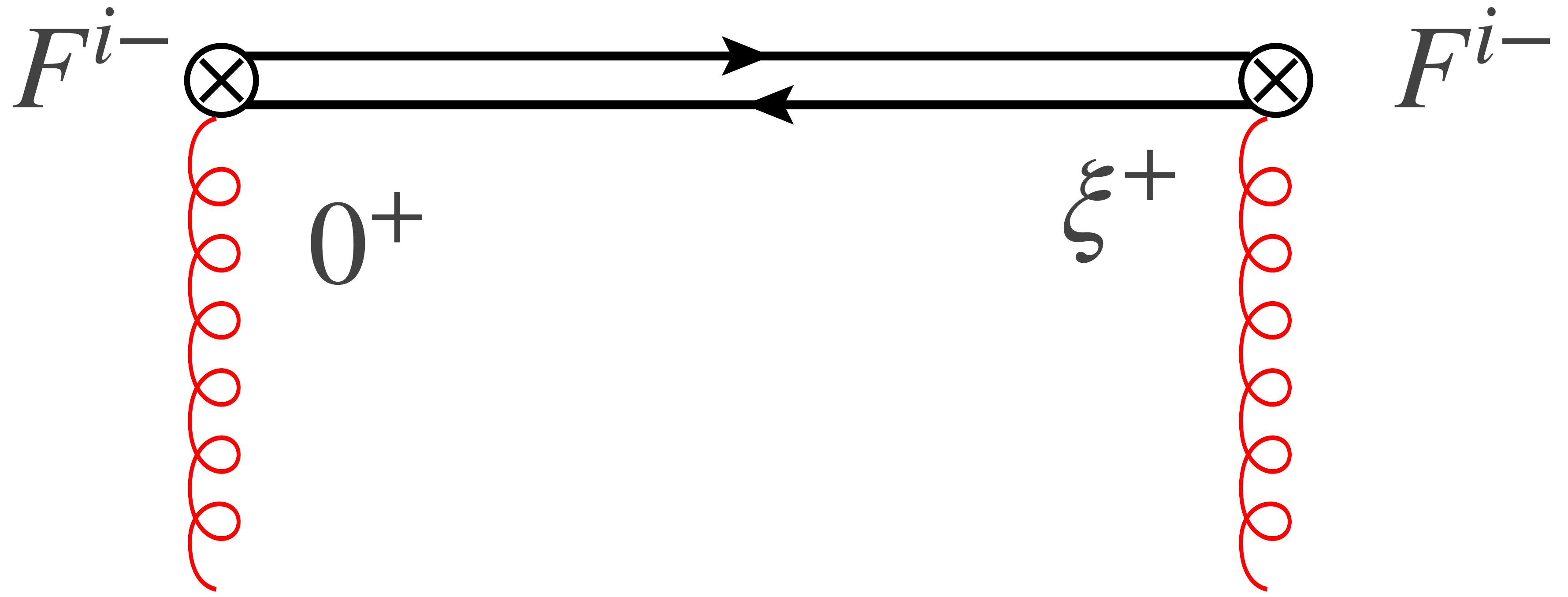


- Note that this uPDF involves finite Wilson lines in contrast to gluon TMDs such as Weizsacker-Williams

Bjorken and Regge limits of the uPDF

- **Collinear limit:** Integrating over $k_\perp$ yields $\xi_\perp = 0$ and we recover the gluon PDF

$$xg(x, \mu^2) = 2\int \frac{\mathrm{d}\xi^+}{(2\pi)P^-} \mathrm{e}^{ixP^-\xi^+} \langle P | \mathrm{Tr}[0, \xi^+] F^{i-}(\xi^+)[\xi^+, 0] F^{i-}(0) | P \rangle$$



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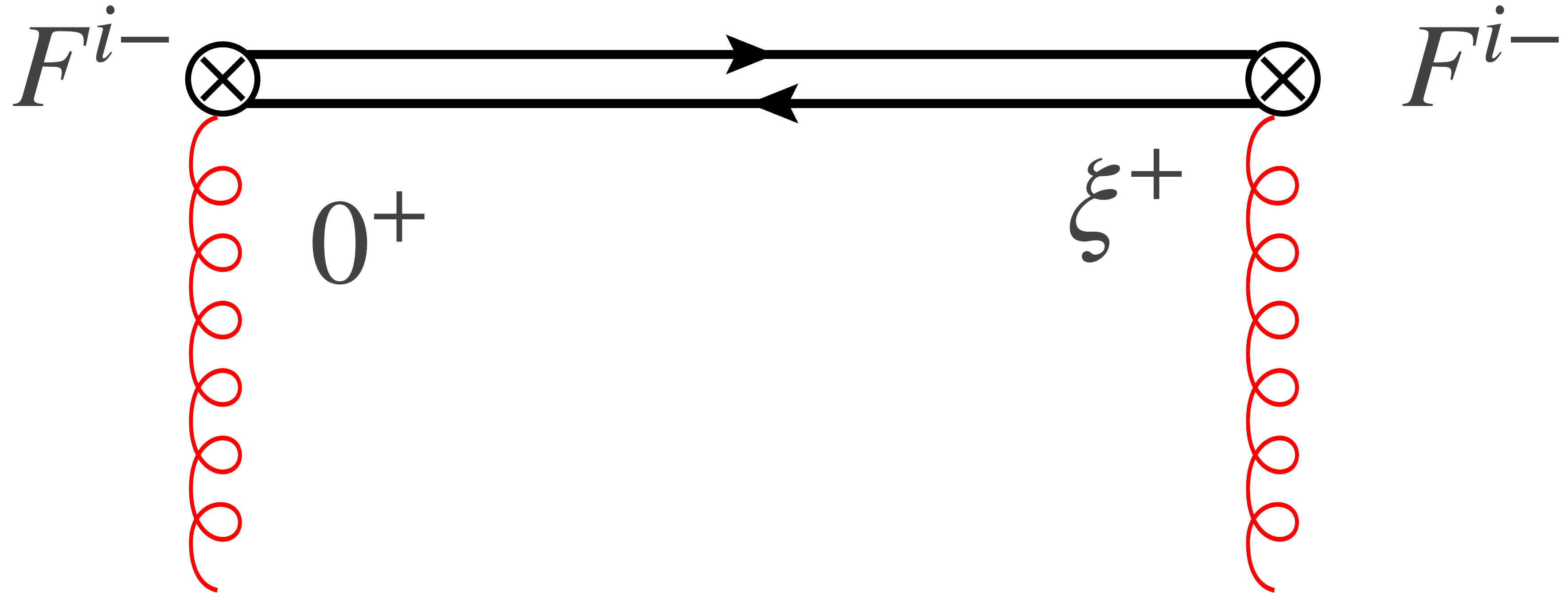

- **Small-x limit:** At $x = 0$ we recover the small x dipole operator

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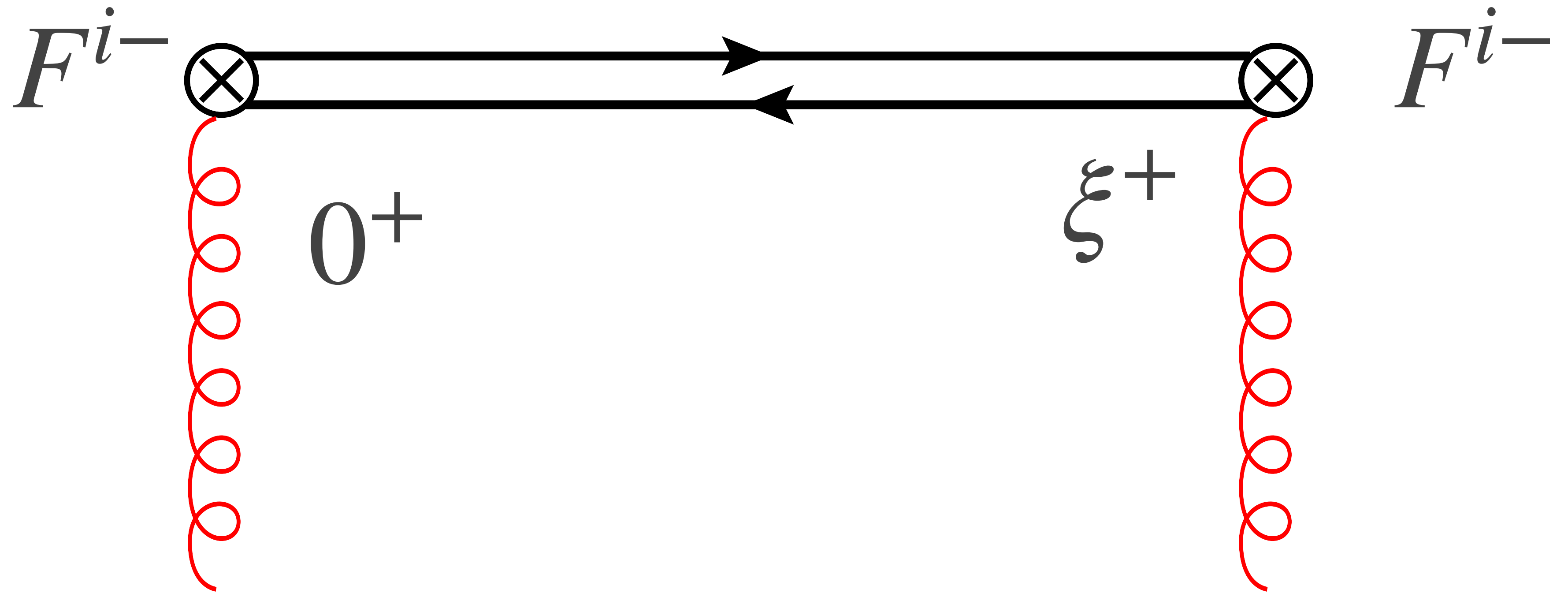

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Provides the interpolation between the leading twist term in the Bjorken limit and the eikonal term in the Regge limit

Summary and outlook

- **Minimal correction** of the semi-classical approach to small x to restore x dependence using a partial twist expansion instead of the shock wave approximation.
- In the case of inclusive DIS: while the hard part is unchanged a new (gauge invariant) x-dependent unintegrated gluon distribution that interpolates between the dipole operator at small x and the gluon PDF at leading twist
- Outlook: quantum evolution, other processes (eg. DVCS (in preparation)). Explore on the Lattice the saturation scale?

Backup

Complete DIS at one-loop order

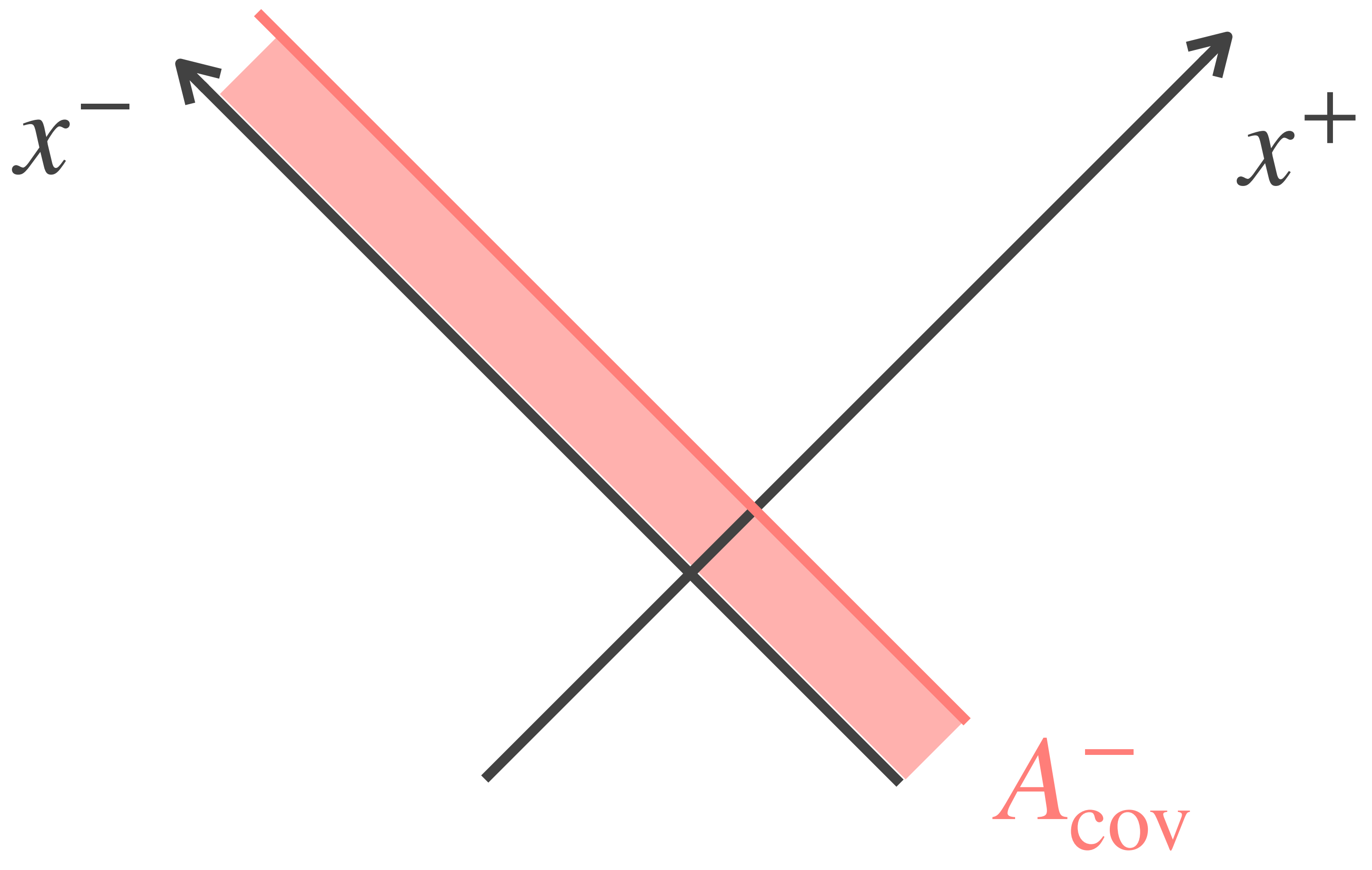
- In the Bjorken limit $Q^2 \to \infty$, we reproduce the 1-loop contribution to the DIS structure function

$$F_T(x_{Bj}, Q^2) = \frac{\alpha_s}{\pi} \sum_f q_f^2 \int_{x_{Bj}}^{1} dy \; xg(x_{Bj}/y, \mu^2)$$

$$\times \left[\frac{1}{\epsilon} \left(\frac{\mathrm{e}^{\gamma_E}}{4\pi} \right)^{\epsilon} P_{qg}(y) + \left[(1-y)^2 + y^2 \right] \log \left[\frac{Q^2(1-y)}{\mu^2 y} \right] - 1 + 4y(1-y) \right]$$

J. Collins, Foundations of pQCD 2011

Background field and transverse gauge links



target sources

$$J^\nu(x) \rightarrow J^-(x^+, x_\perp)$$

and $\quad J^+ = J_\perp = 0$

- consider a target boosted along the $-z$ direction close to the light cone. Due to time dilation the target color sources are "frozen" in the - direction

- Yang-Mills equations $[D_\mu, F^{\mu\nu}] = J^\nu$ can be solved exactly (together with the continuity equation $[D_\mu, J^\mu] = 0$) in covariant gauge $\partial \cdot A = 0$ (or light-cone gauge $A^+ = 0$)

$$A^-_{\rm cov} = -\frac{1}{\partial_\perp^2} J^- \quad \text{and} \quad A^+ = A_\perp = 0$$

Background field and transverse gauge links

- under an arbitrary gauge rotation $\Omega(x^+, x_\perp)$ the target field transforms as

$$A^- \;\rightarrow\; \Omega_{\boldsymbol{x}}(x^+)\; A^-_{\rm cov}\,(x^+,\boldsymbol{x})\; \Omega^{-1}_{\boldsymbol{x}}(x^+) \;-\; \frac{1}{ig}\Omega_{\boldsymbol{x}}(x^+)\;\partial^-\;\Omega^{-1}_{\boldsymbol{x}}(x^+)$$

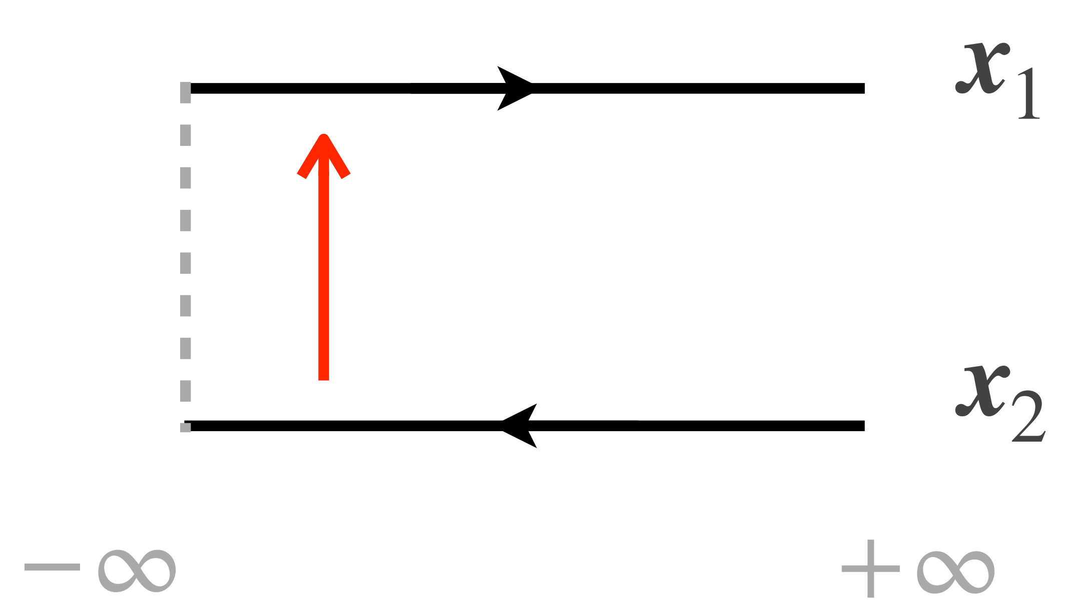
$$\boldsymbol{A}^i \;\rightarrow -\frac{1}{ig}\;\Omega_{\boldsymbol{x}}(x^+)\;\partial^i\;\Omega^{-1}_{\boldsymbol{x}}(x^+)$$

- exploiting the residual gauge freedom we can generate a transverse pure gauge
- N.B.: the partonic picture is manifest in the LC-gauge $A^- = 0$ (with $A_\perp \neq 0$)
- small x observables are (in the dilute/dense limit) more naturally expressed in the wrong LC-gauge $A^- \neq 0$ (with $A_\perp = 0$).
- in order to connect to the partonic interpretation one needs to deal with transverse fields

Background field and transverse gauge links

- geometric interpretation of the all twist resummation

$$U_{x_1} = U_{x_2} - r^i \int_0^1 \mathrm{d}s \, (\partial^i U_{x_2+sr})$$



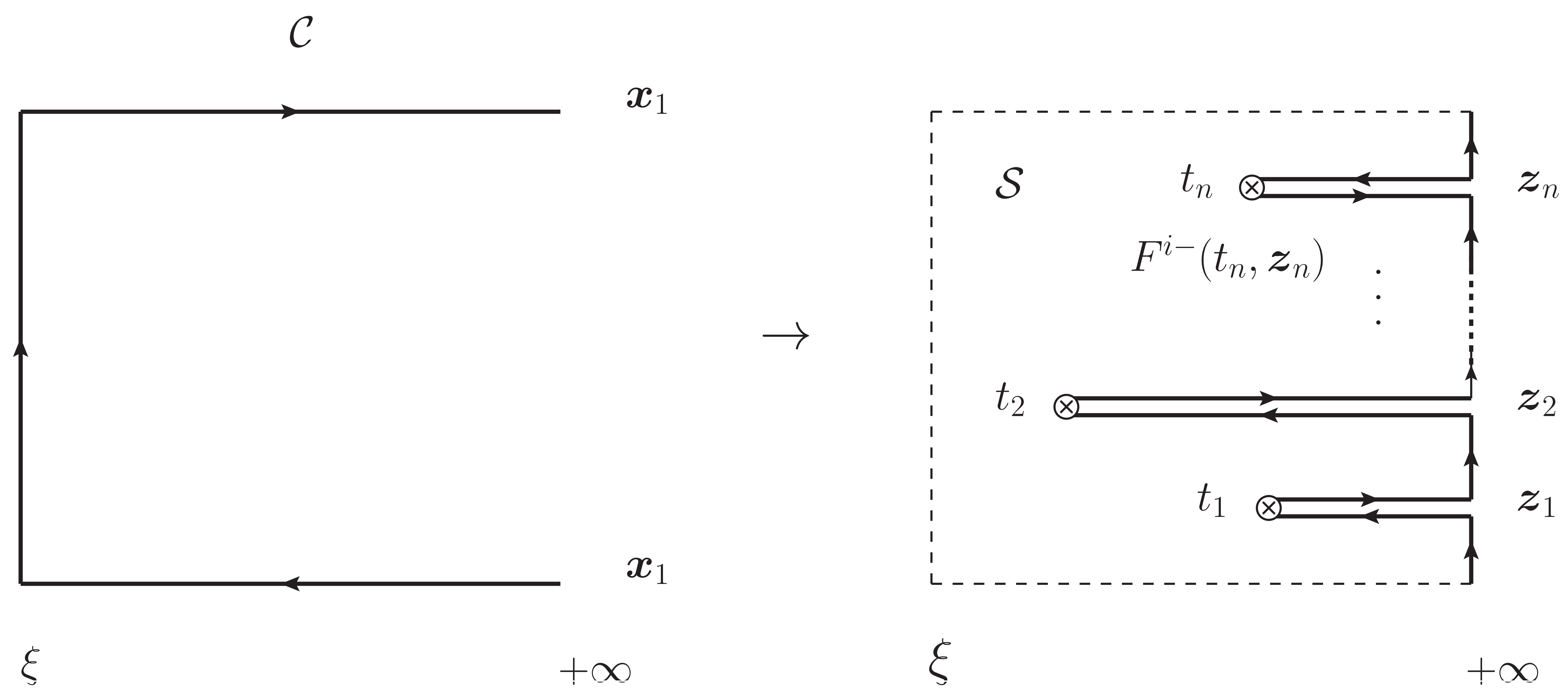
- and noticing that $\frac{1}{ig}(\partial^i U_x) U_x \equiv A^i(x)$, one can express the dipole operator (in the background field A^-) as a transverse gauge link:

$$U_{x_1} U^\dagger_{x_2} = [x_1, x_2] = 1 - ig \int_{x_2}^{x_1} \mathrm{d}z \, A^i(z) \, [z, x_2]$$

- dipole operator can be expessed in terms of tranverse link operators

Background field and transverse gauge links

- non-Abelian Stokes' theorem: more generally, the dipole operator can be written as a path ordered tower of "twisted" field strength tensor (i.e. dressed with future pointing Wilson lines)

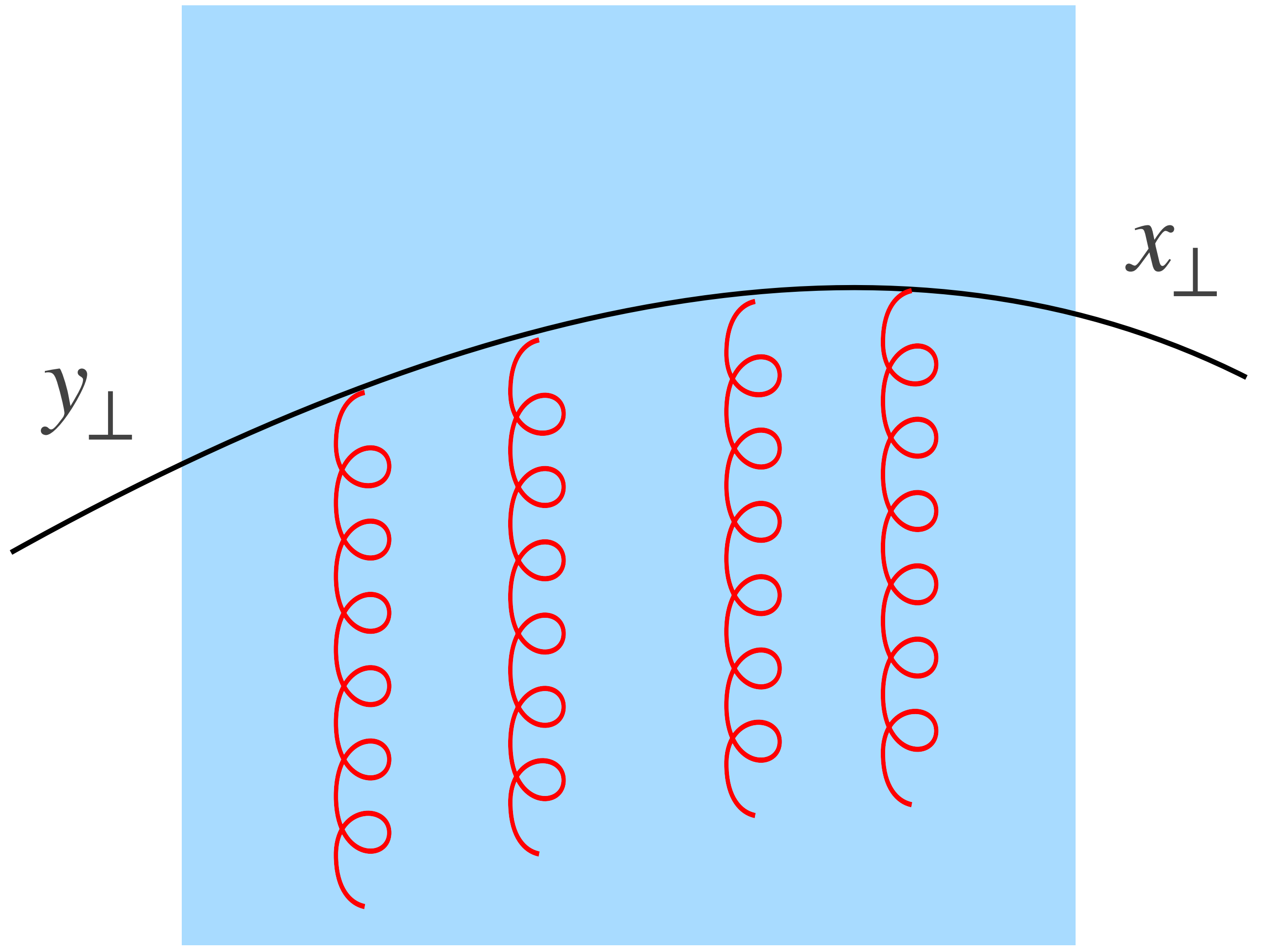


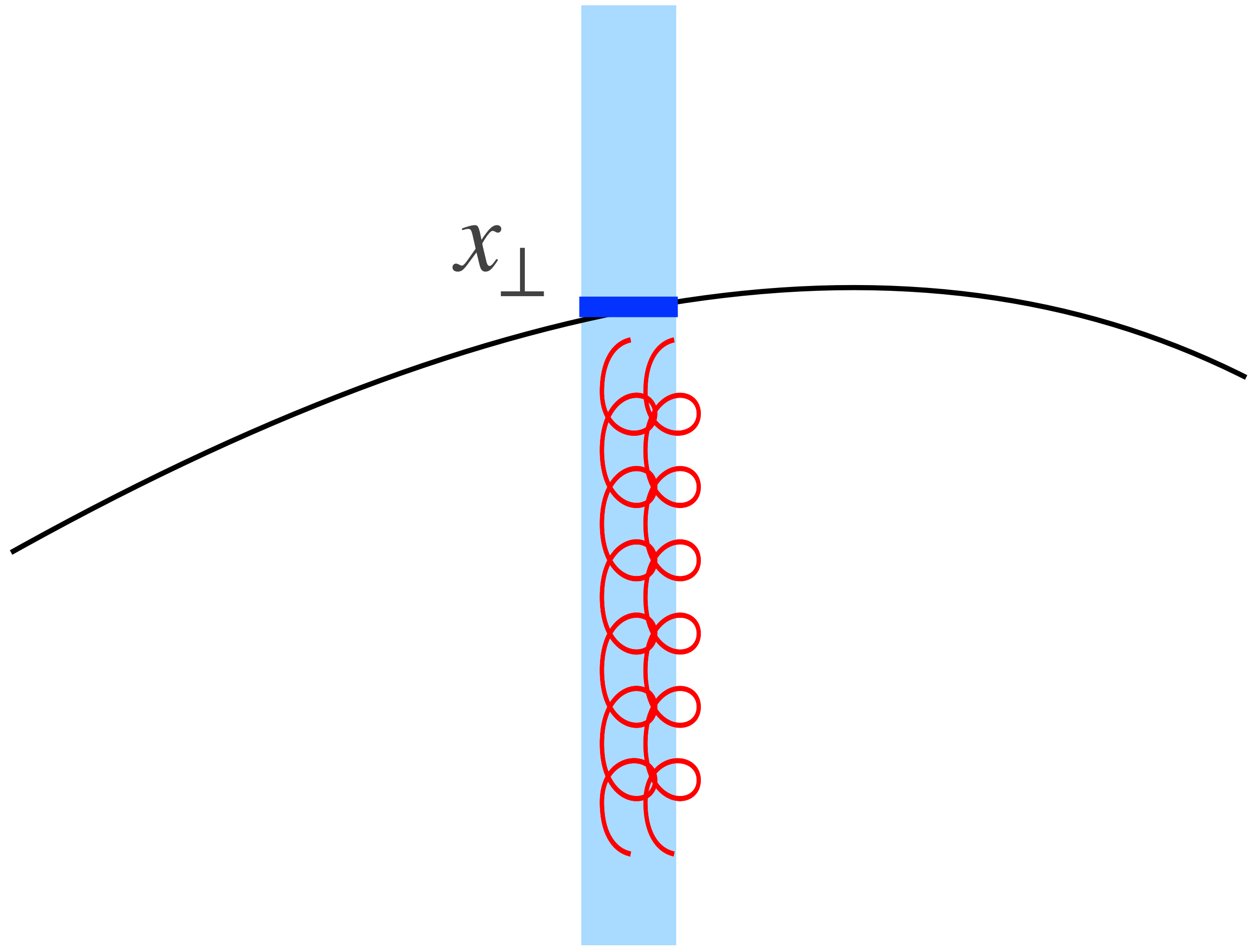
[Fishbane, Gasiorowicz, Kaus (1981) Wiedemann (2000)
YMT, Boussarie (2020)]

$$U_{\boldsymbol{x}_2} U^{\dagger}_{\boldsymbol{x}_1} \equiv P \exp\left[-ig \int_S \mathrm{d}t \mathrm{d}\boldsymbol{z} \, [+\infty, x^+]_{\boldsymbol{x}} \, F^{i-}(x^+, \boldsymbol{x}) \, [x^+, +\infty]_{\boldsymbol{x}}\right]$$

Revisiting the Shock Wave Approximation

- Under the assumption that all transverse momenta are of same order along the ladder, the leading power $1/s$ is obtained letting $p^+ \to \infty$ for any particle propagating inside the shock wave





Non-eikonal propagator

$$\mathscr{G}_{p^+}(x^+, y^+) = \left[i\frac{\partial}{\partial x^+} - \frac{\hat{p}_\perp^2}{2p^+} - gA^- \right]^{-1}$$

shock wave (eikonal limit)

$$\lim_{p^+ \to +\infty} \mathscr{G}_{p^+}(x^+, \boldsymbol{x};\, y^+, \boldsymbol{y}) \;=\; \delta(\boldsymbol{x} - \boldsymbol{y})\, U_{\boldsymbol{x}}(x^+, y^+)$$

Revisiting the Shock Wave Approximation

- This limit neglects quantum diffusion

$$\mathcal{G}^{0}_{p^+}(x^+, \boldsymbol{x}; y^+, \boldsymbol{y}) = \frac{p^+}{2i\pi\,\Delta x^+}\, \mathrm{e}^{\,i\frac{(\boldsymbol{x}-\boldsymbol{y})^2\, p^+}{\Delta x^+}}$$

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- In effect, the phase relates the transverse dynamics to longitudinal dynamics, this is the phase that appears in the definition of PDF's
- It encodes the information about k^-'s in the target. It is expected to be non-negligible away from the strongly ordered region in k^-.

Factorization formula for DIS at arbitrary x

- Combining all phases we obtain

$$ik^-\Delta x^+ \equiv i\,\frac{\ell^2 + z\bar{z}Q^2}{2z\bar{z}\,q^+}\Delta x^+$$

- This is nothing but Feynman x that we encounter when deriving the DGLAP limit

$$x_F \equiv \frac{\ell^2 + z\bar{z}Q^2}{2z\bar{z}\,s}$$

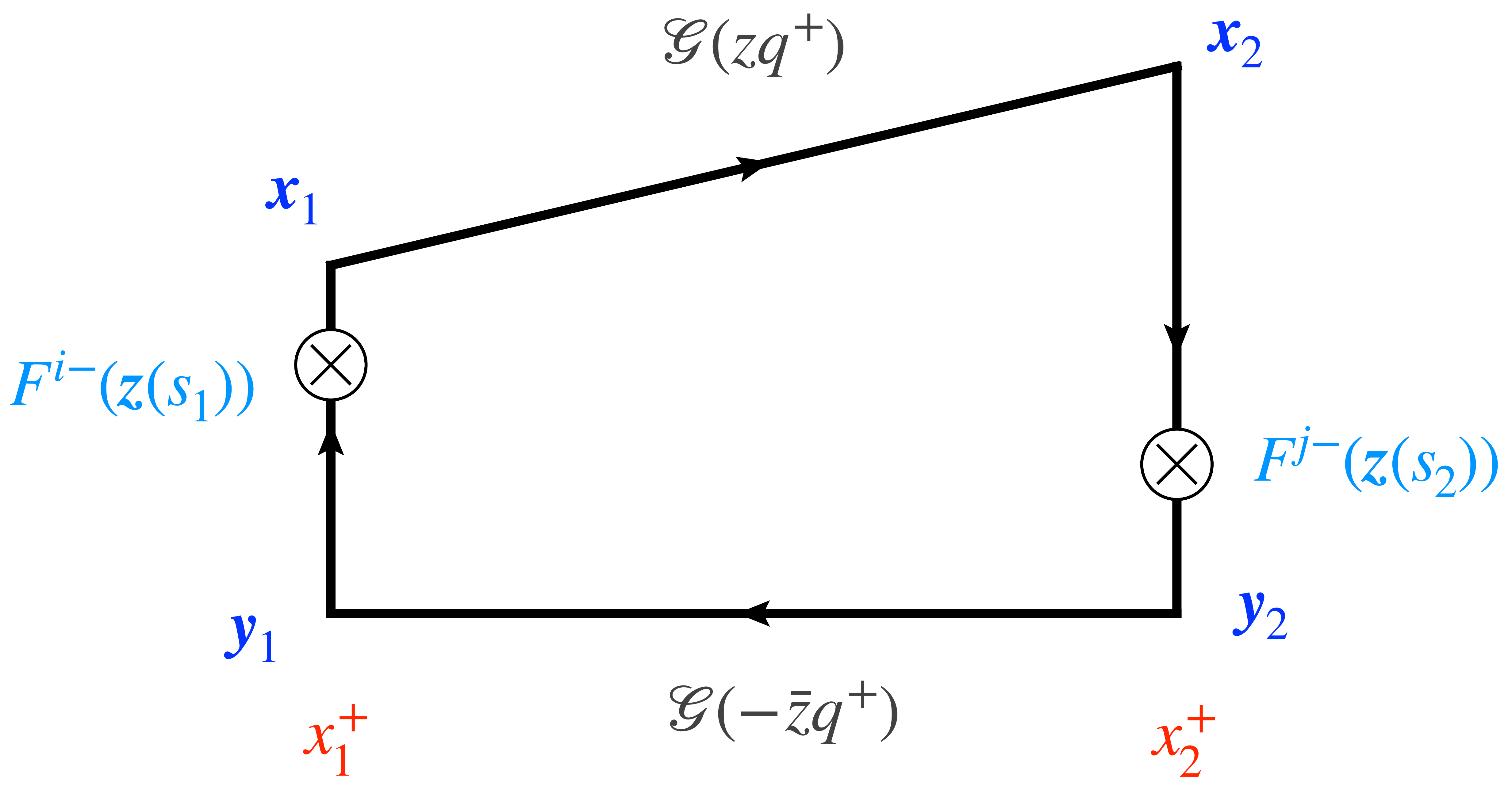
Inclusive DIS beyond shock wave

- applying the same trick to the r.h.s. we obtain the following hadronic operator

$$O_{\text{dipole}}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2, x_1^+, x_2^+) \equiv \int_{\boldsymbol{y}_1}^{\boldsymbol{x}_1} \mathrm{d}z_1^i \int_{\boldsymbol{y}_2}^{\boldsymbol{x}_2} \mathrm{d}z_2^j$$

$$\left\langle P \right| \mathrm{Tr}\, \mathcal{G}(\boldsymbol{y}_2, x_2^+ ; \boldsymbol{y}_1\, x_1^+ \,|\, (1-z)q^+)\, F^{j-}(x_1^+, \boldsymbol{z}_1)\, \mathcal{G}(\boldsymbol{x}_1, x_1^+ ; \boldsymbol{x}_2, x_2^+ \,|\, zq^+)\, F^{i-}(x_2^+, \boldsymbol{z}_2) \left| P \right\rangle$$

- performing a gauge rotation leads to transverse gauge links: explicit gauge invariance
- dependence on + momenta of the dipole zq^+ and $(1-z)q^+$ can be factorized with further approximations



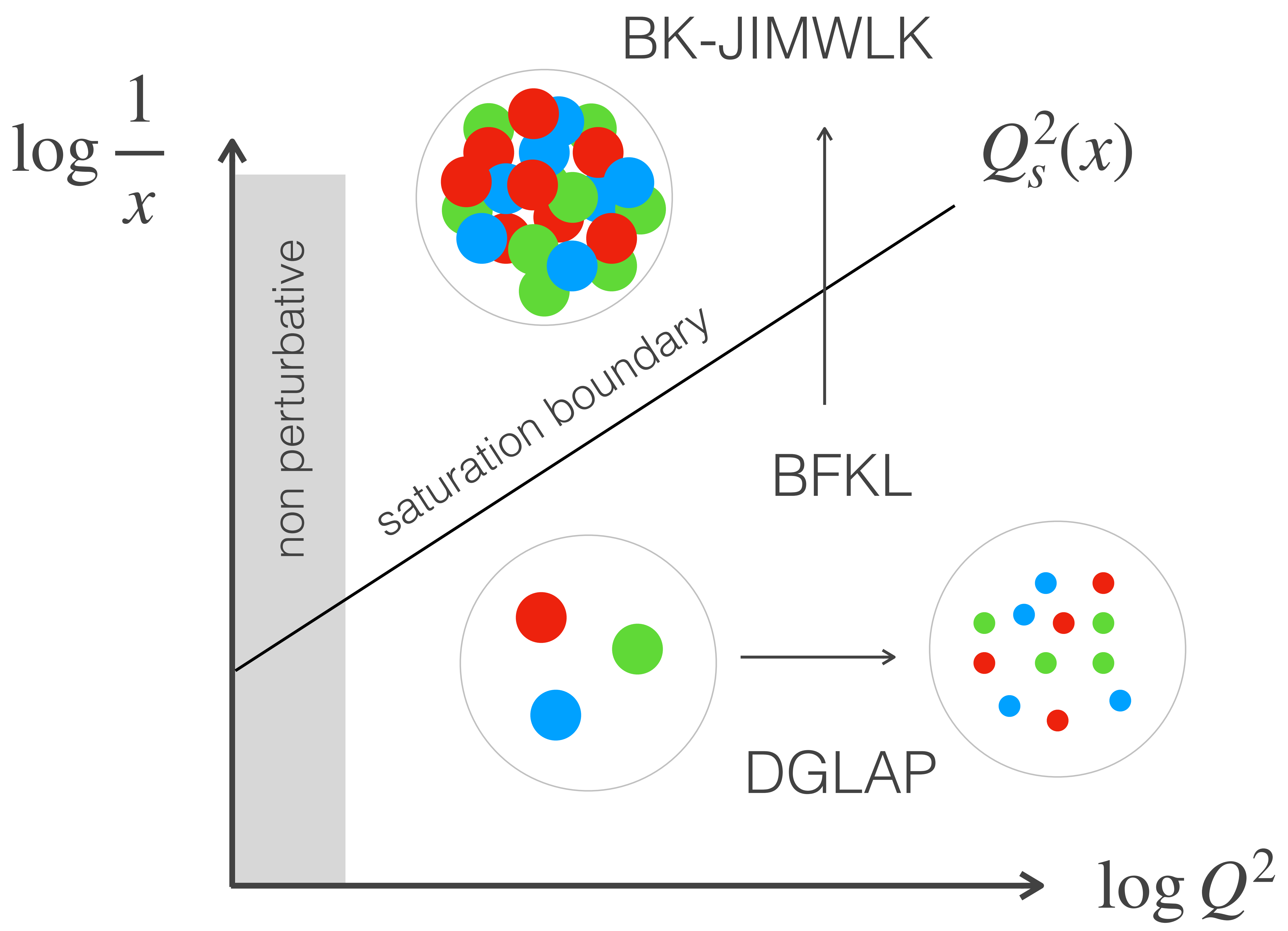
Gluon saturation at small x

- Gluon saturation criterion: large number of gluons populate the transverse extend of the proton leading to saturation when

$$S_\perp \sim \frac{\alpha_s}{Q^2} \times xg(x, Q^2)$$

- Defining the saturation scale

$$Q^2 \sim Q_s^2(x) \sim x^{-\lambda}$$

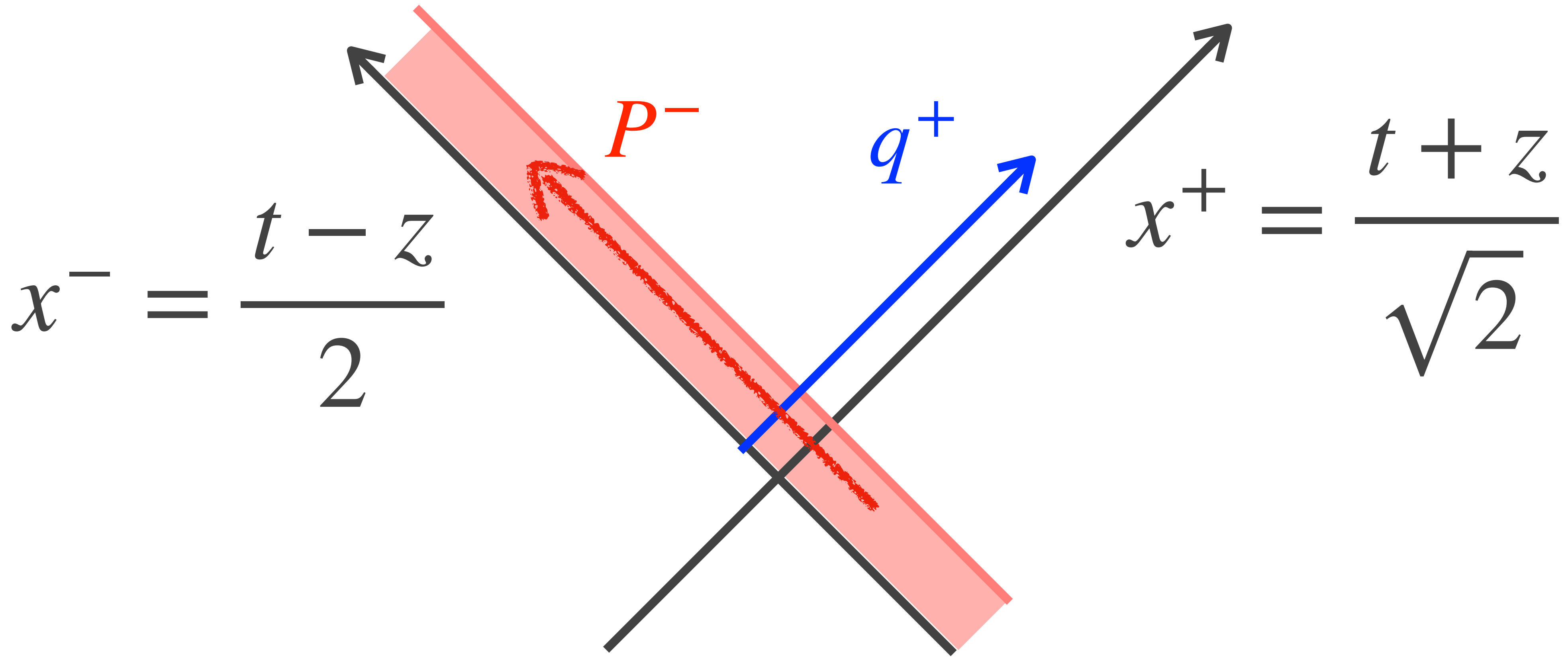


[Gribov, Levin, Ryskin, 1983- Mueller, Qiu, 1986, Venugopalan, McLerran (MV), Balitsky, Kovchegov (BK) Jallilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner (JIMWLK) (1993-2001)]

Rapidity factorization at small x - coherent scattering

- **Regge limit:** $s \to \infty$ with Q^2 fixed

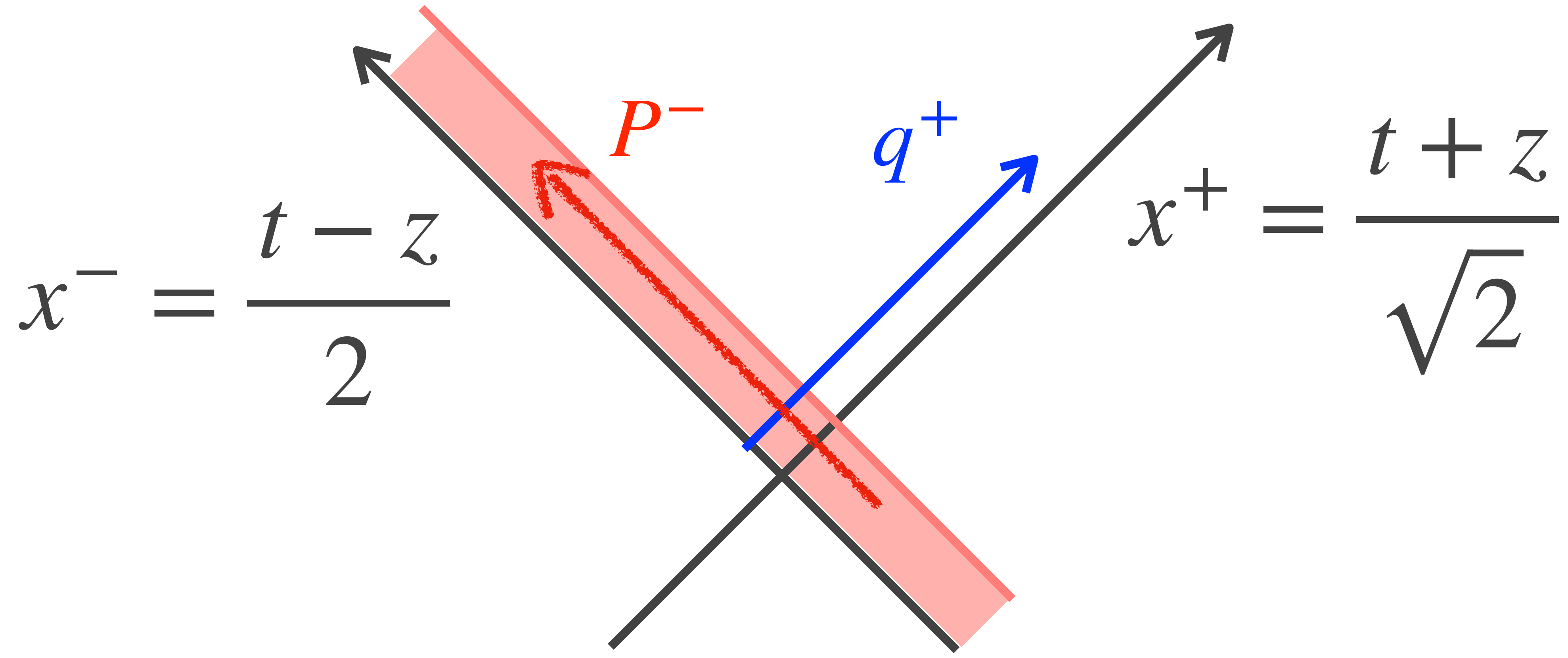
$$s = 2q \cdot P = 2q^+ P^-$$

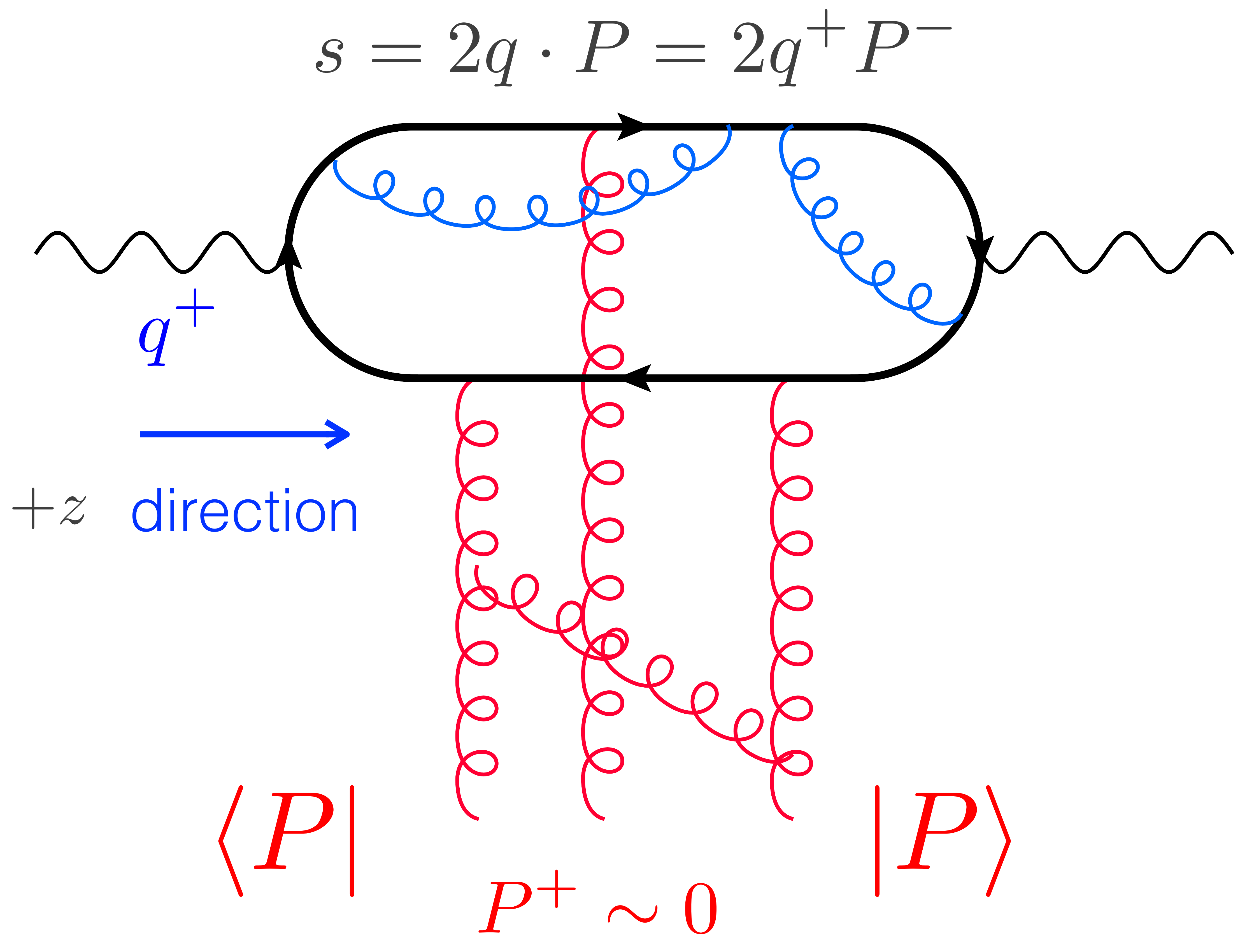


$$x_{\mathrm{Bj}} = Q^2/s \to 0$$

Rapidity factorization at small x - coherent scattering

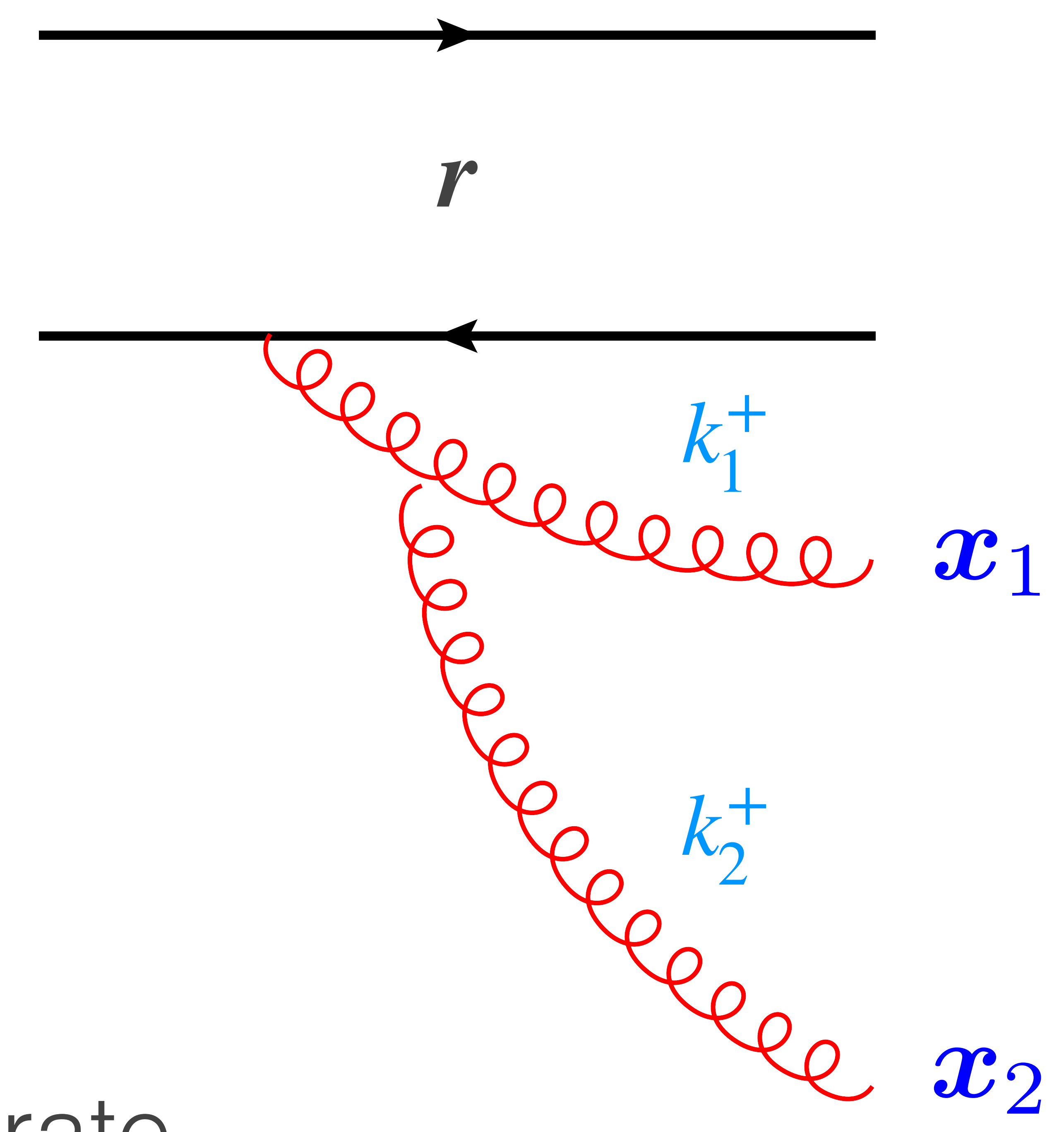
- **Regge limit:** $s \to \infty$ with Q^2 fixed
- Although gluons contribute only at NLO they dominate the cross section at small x. Dominant diagram in DIS: scattering of quark dipole moving in the +z direction off longitudinally polarized gluons in the target





$$x_{\rm Bj} = Q^2/s \to 0$$

BK and the NLO crisis



- This is due to the fact that the rapidity variable $Y \equiv \log \frac{q^+}{\Lambda^+}$ evolves independently from $x_\perp$ violating $k^- = xP^-$ ordering

  formation time ordering implies

  $$k_1^- = x_1^2 k_1^+ < k_2^- = x_2^2 k_2^+ \qquad \Rightarrow \qquad \frac{x_1^2}{x_2^2} < \frac{k_2^+}{k_1^+} \ll 1$$

  $$k_1^+ \gg k_2^+$$

- Small dipoles that radiate larger dipoles $x_1 \ll x_2$ generate large collinear logarithms when $k_1^- \sim k_2^-$

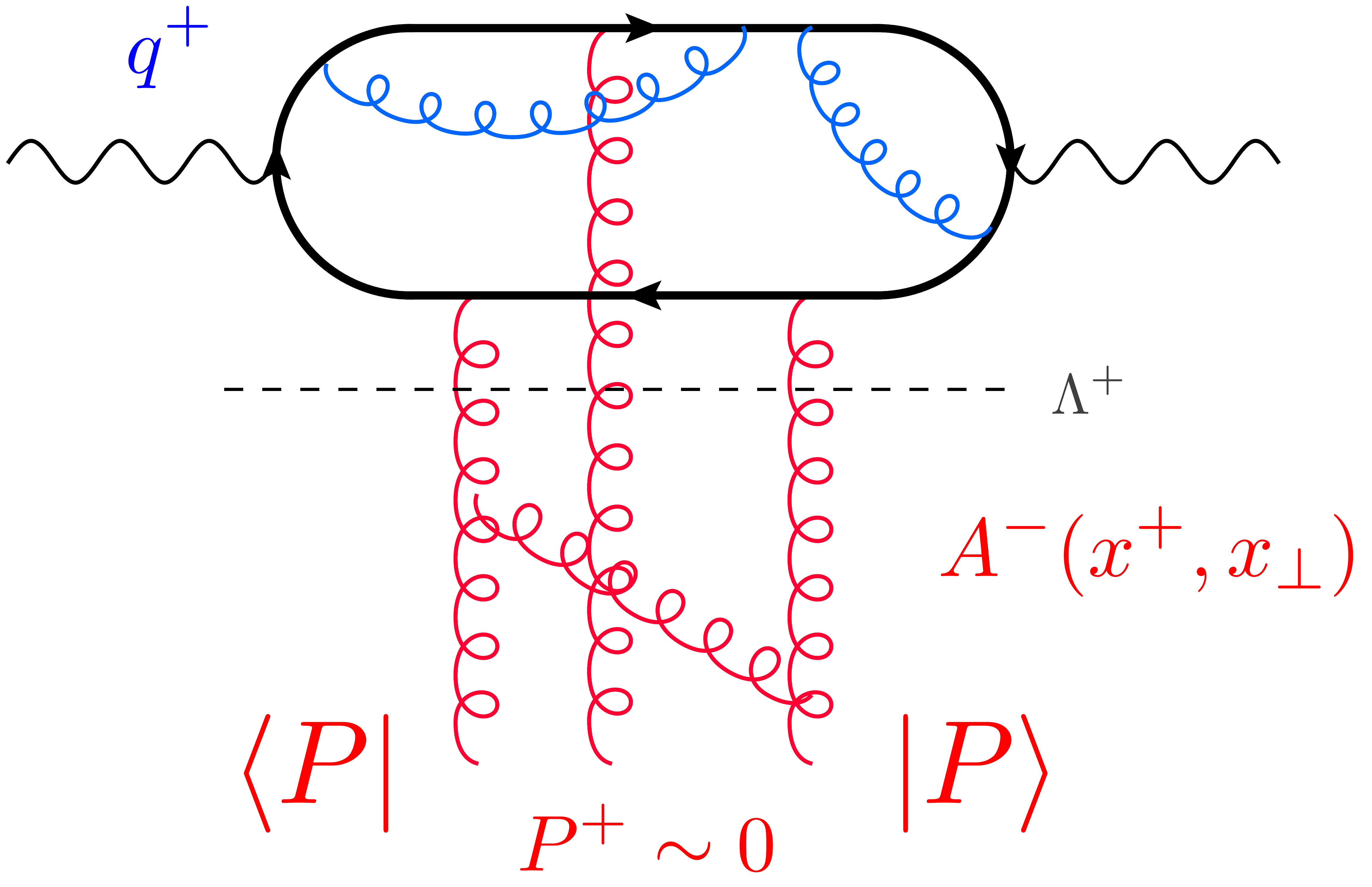
$\Rightarrow$ potentially large double logs : NLO/ LO ~ $\alpha_s \log^2 \frac{1}{r_\perp^2} > 1$

[Beuf (2014) Ducloué, Iancu, Mueller, Soyez, Triantafyllopoulos (2015-2019)]

Rapidity factorization at small x - coherent scattering

Step 1: Split the gluon fields into
fast and slow gluons

$$A^\mu(k) \equiv A^\mu(k^+ < \Lambda^+) + a^\mu(k^+ > \Lambda^+)$$



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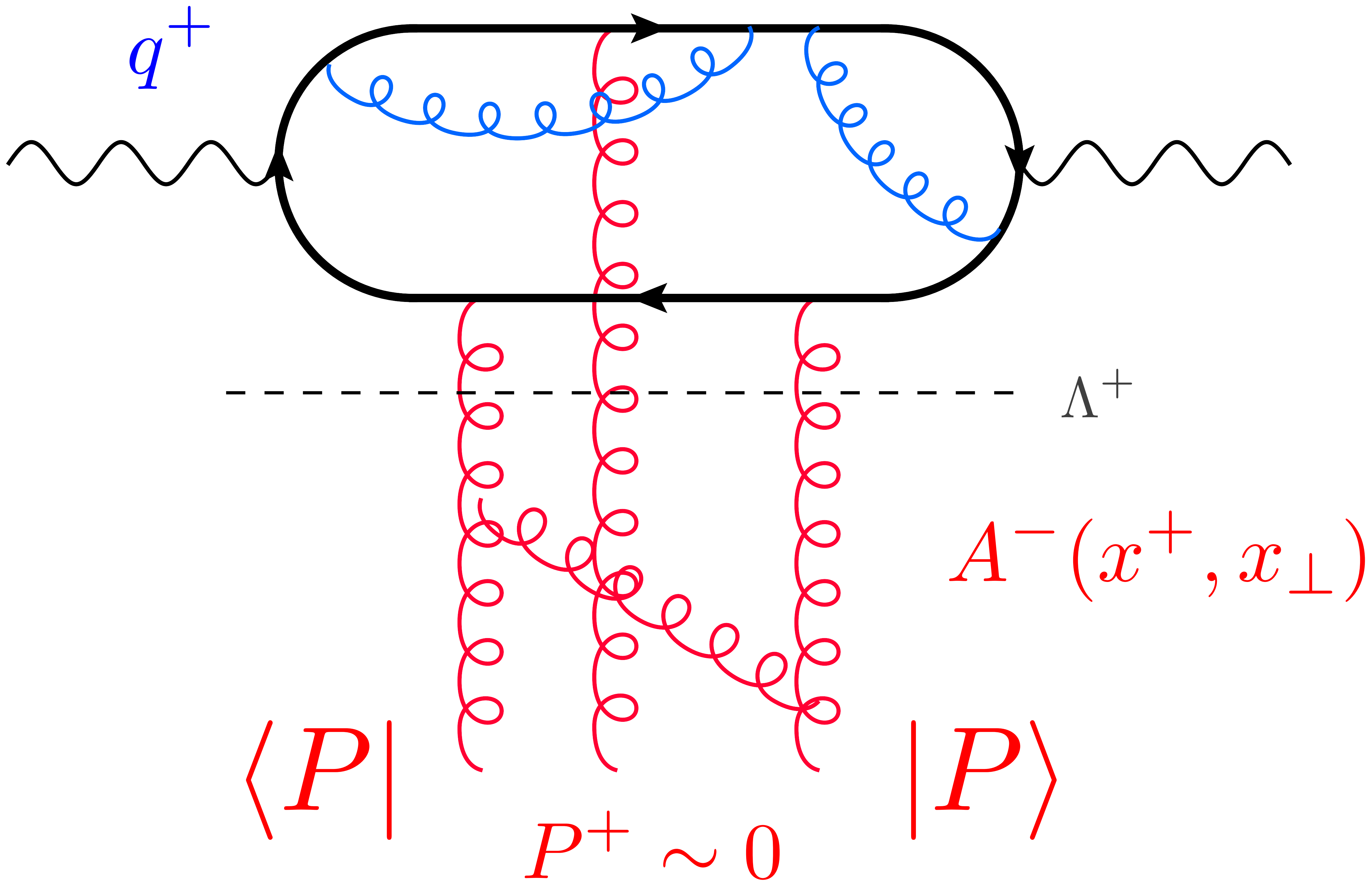

- The relevant d.o.f. in the saturation regime are strong classical fields $g\,A^- \sim 1$: boosted target field dominated by its - component

$$A^\mu(x) \quad \to \quad \gamma\, A^-(\gamma x^+, \frac{x^-}{\gamma}, \boldsymbol{x}) \quad A^+ \sim O(1/\gamma) \quad A_\perp \sim O(1)$$