Domain-wall fermion and Atiyah-Patodi-Singer index

M. F. Atiyah
(1929-2019)

* photo from Wikipedia

V. K. Patodi
(1945-1976)

* photo from mathshistory.st-andrews.ac.uk

I. M. Singer
(1924-2021)

* photo from Wikipedia

Hidenori Fukaya (Osaka U.)

OSAKA UNIVERSITY
Live Locally, Grow Globally

F, Onogi, Yamaguchi, PRD96(2017) no.12, 125004 [arXiv: 1710.03379],
F, Furuta, Matsuo, Onogi, Yamaguchi, Yamashita, *Commun.Math.Phys.* 380 (2020) 3, 1295-1311 [arXiv:1910.01987],
F, Kawai, Matsuki, Mori, Nakayama, Onogi, Yamaguchi, PTEP 2020 (2020) 4, 043B04 [arXiv:1910.09675].
F, Furuta, Matsuki, Matsuo, Onogi, Yamaguchi, Yamashita, arXiv:2012.03543,
F, IJMPA 36 (2021) 26, 2130015 [arXiv: 2109.11147]

Supported by Kobo-Kenkyu budget 18H04484 in JSPS Grant-in-Aid for Scientific Research on Innovative Areas “Discrete Geometric Analysis for Materials Design” Grant Number 17H06460

Domain-wall fermion and my works

My first paper in 2003 is

PHYSICAL REVIEW D **68**, 074503 (2003)

Lattice study of the massive Schwinger model with a θ term under Lüscher's "admissibility" condition

Hidenori Fukaya and Tetsuya Onogi
Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan
(Received 31 May 2003; published 23 October 2003)

Lüscher's "admissibility" condition on the gauge field space plays an essential role in constructing lattice gauge theories which have exact chiral symmetries. We apply the gauge action proposed by Lüscher with the domain-wall fermion action to the numerical simulation of the massive Schwinger model. We find this action can generate configurations in each topological sector separately without any topology changes. By developing a new method to sum over different topological sectors, we calculate the meson masses in the nonzero θ vacuum.

DOI: 10.1103/PhysRevD.68.074503 PACS number(s): 12.38.Gc

Since then, 70% of my works are with domain-wall fermion.

Most of them are on numerical simulations.

But today I discuss its mathematics (index theorem).

Which index theorem?

I am NOT talking about

the Atiyah-Singer index $= \mathrm{Ind} D^{4\mathrm{D}}_{\mathrm{overlap}} = \mathrm{Ind} D^{5\mathrm{D}}_{\mathrm{domain-wall}}$

I will show that

the Atiyah-Patodi-Singer index on manifold with boundary

$$= -\frac{1}{2}\eta(D^{4\mathrm{D}}_{\mathrm{domain-wall}})$$

on a closed manifold.

(one of our referees misunderstood this.)

Atiyah-Singer index theorem [1968] on a manifold without boundary

$$D\!\!\!/\,\psi = 0 \qquad D\!\!\!/ := \gamma^\mu(\partial_\mu + iA_\mu)$$

Index theorem

$$\underbrace{n_+ - n_-}_{\mathrm{Ind}(D\!\!\!/)} = \frac{1}{32\pi^2}\int d^4x \epsilon^{\mu\nu\rho\sigma}\mathrm{tr}(F_{\mu\nu}F_{\rho\sigma})$$

n_+: #sol with + chirality

n_-: #sol with - chirality

This text-book level theorem is physicist-friendly.

Atiyah-Patodi-Singer (APS) index theorem [1975]

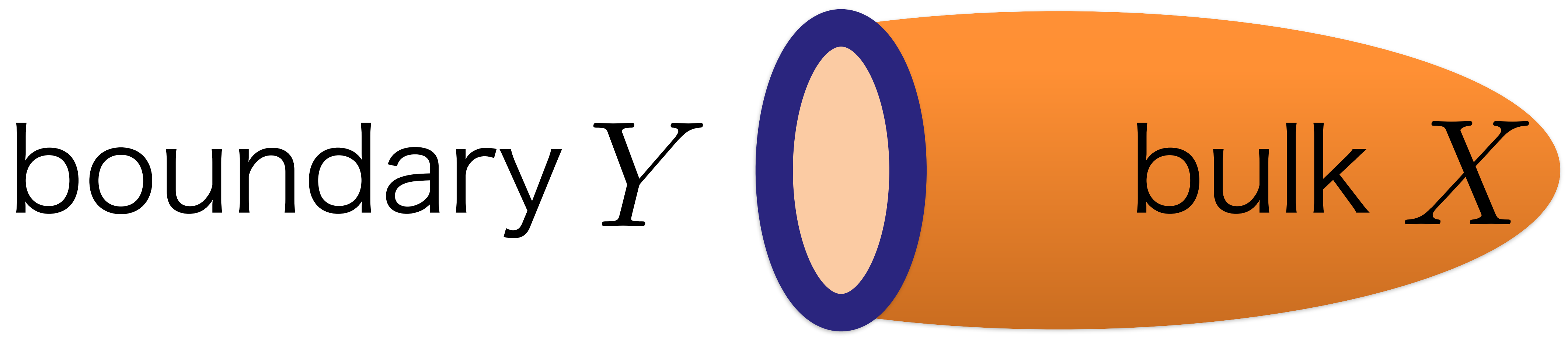
is less known

$$Ind(D_{\rm APS}) = \frac{1}{32\pi^2}\int_X d^4x\,\epsilon_{\mu\nu\rho\sigma}{\rm tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD_Y)}{2}$$

$$\eta(H) = \sum_{\lambda\geq 0}^{reg} - \sum_{\lambda<0}^{reg}$$

curvature

λ : eigenvalues of H



(because we were not very interested in manifolds with boundary until very recently).

APS index in topological insulator

Witten 2015 : APS index is a key to understand bulk-edge correspondence in symmetry protected topological insulator:

gapped material in the bulk but conductor on boundary (edge).

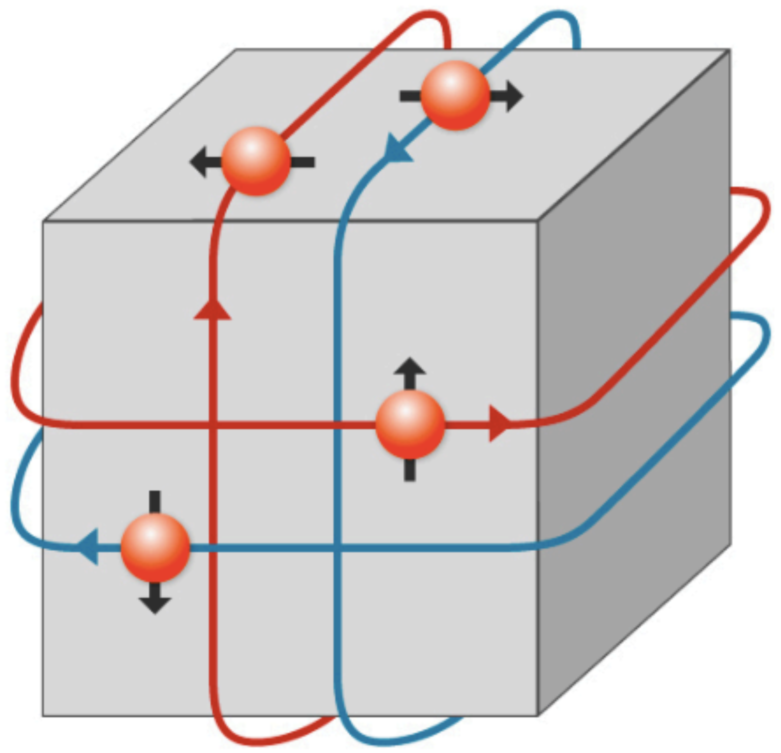
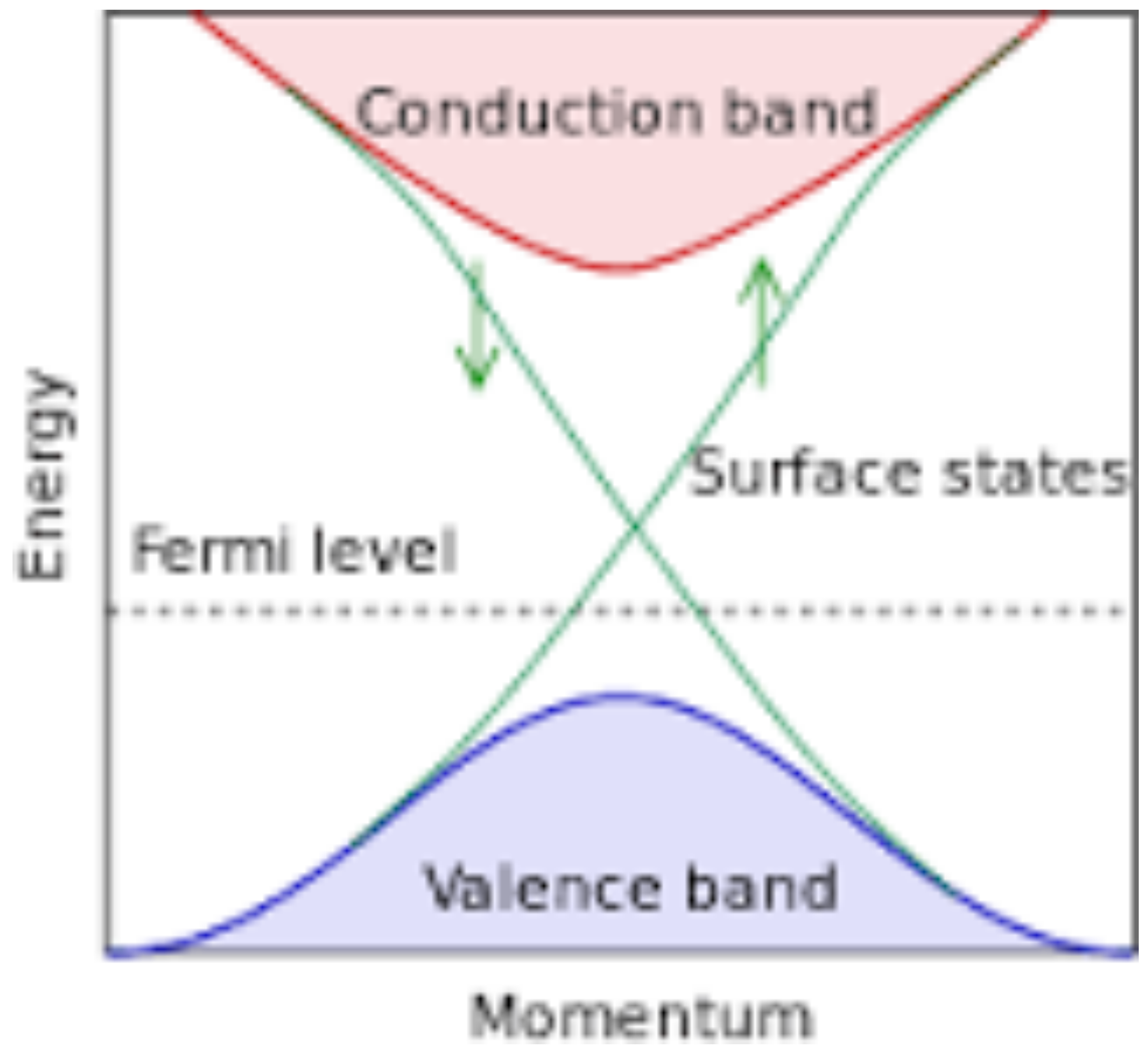


Figure from Wikipedia



2005 predicted by Kane et al.

2007 discovered [Koenig et al.].

T anomaly cancellation

[Related works: Metlitski 15, Seiberg-Witten 16, Tachikawa-Yonekura 16&18, Freed-Hopkins 16, Witten 16, Yonekura 16&19, Witten-Yonekura 19…]

The APS index protects the Time reversal (T) symmetry.

fermion path integrals

$$Z_{\rm edge} \propto \exp(-i\pi\eta(iD^{\rm 3D})/2)$$ T-anomalous

$$Z_{\rm bulk} \propto \exp\left(i\pi\frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}{\rm tr}[F^{\mu\nu}F^{\rho\sigma}]\right)$$ T-anomalous

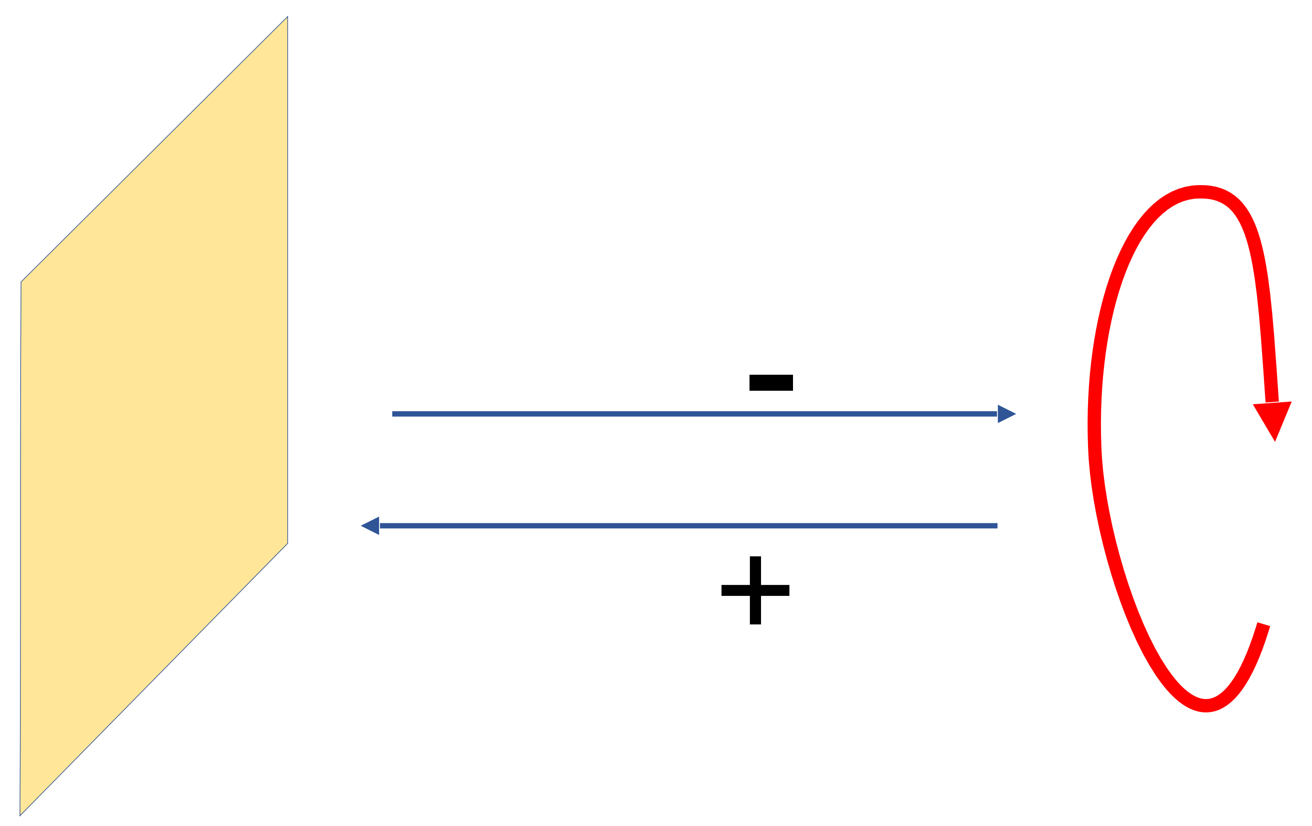
$$Z_{\rm edge}Z_{\rm bulk} \propto (-1)^{\mathfrak{I}} = (-1)^{-\mathfrak{I}}$$ ➡ T is protected !

$$\mathfrak{I} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}{\rm tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{\rm 3D})}{2}$$

But the LHS $\mathfrak{I} = {\rm Ind}D_{\rm APS}$ of massless Dirac with non-local boundary condition is physicist-unfriendly.

Difficulty with boundary

If we impose **local** and **Lorentz (rotation)** invariant boundary condition, + and – chirality sectors do not decouple any more.



For reflecting particle, momentum flips, but angular momentum does not → chirality flips.

n_+, n_- and the index do not make sense.

Atiyah-Patodi-Singer boundary condition

[Atiyah, Patodi, Singer 75]

gives up the locality and rotational symmetry to keep the chirality.

Eg. 4 dim $x^4 \geq 0$ $A_4 = 0$ gauge

$$D = \gamma^4 \partial_4 + \gamma^i D_i = \gamma^4 (\partial_4 + \underbrace{\gamma^4 \gamma^i D_i}_{A})$$

They impose a non-local b.c.

$$(A + |A|)\psi|_{x^4=0} = 0$$

$$\Rightarrow \quad \text{index} = n_+ - n_-$$

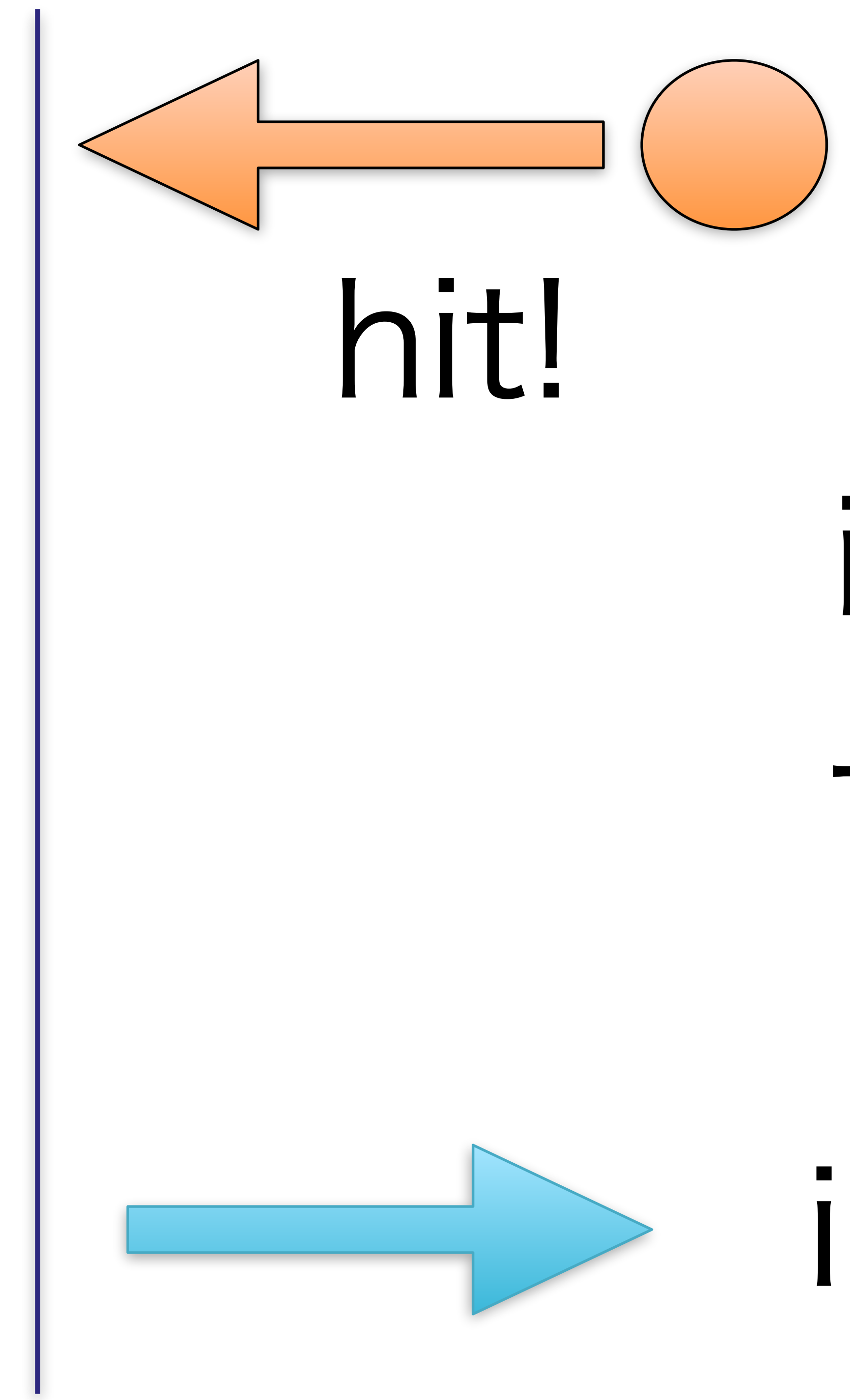
Mathematically beautiful!

But physicist-unfriendly.

Locality >> chirality for physicists

Locality (=causality) is essential.

We cannot accept APS condition even if it is beautiful.



information propagates
faster than speed of light.

Locality >> chirality for physicists

Locality (=causality) is essential.

We cannot accept APS condition even if it is beautiful.

→ need to give up chirality and consider L/R mixing

(massive case)

$$\cancel{n_+ - n_-} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{\mathrm{3D}})}{2}$$

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Can we still make a fermionic integer (even if it is ugly)?

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Can we still make a fermionic integer (even if it is ugly)?

Our answer is “Yes, we can”.

References

In 2017, we proposed “A physicist-friendly reformulation of the Atiyah-Patodi-Singer index”.

[F, Onogi, Yamaguchi, arXiv:1710.03379]

In 2018, 3 mathematicians joined and we succeeded in a mathematical proof.

[F, Furuta, Matuso, Onogi, Yamaguchi, Yamashita, arXiv:1910.01987]

Lattice version:

[F, Kawai, Matsuki, Mori, Nakayama, Onogi, Yamaguchi, arXiv:1910.09675]

Curved lattice version: Shoto Aoki and F, arXiv:2111.11649

Application to odd dimensions (mod-two index):

[F, Furuta, Matsuki, Matuso, Onogi, Yamaguchi, Yamashita, arXiv:2012.03543]

A physicist-friendly review paper on the whole project

[F, IJMPA 36 (2021) 26, 2130015 arXiv: 2109.11147].

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2. Massive Dirac operator index without boundary

3. New index with boundary [F, Onogi, Yamaguchi 2017]

4. Mathematical proof [F, Furuta, Matsuo, Onogi, Yamaguchi, Yamashita, 2019]

5. APS index on a lattice [F, Kawai, Matsuki, Mori, Nakayama, Onogi, Yamaguchi 2019]

6. Summary

Massive Dirac fermion

Let us consider a Dirac fermion in continuum theory with negative mass (compared to regulator),

$$\frac{\det(D+m)}{\det(D-M)} \longrightarrow \text{Pauli-Villars}$$

with SU(N) gauge fields on an even-dimensional closed flat Euclidean manifold.

Axial U(1) rotation

In the large mass limit, $m \to M \gg 0$, let us perform an axial U(1) rotation with angle π,

$$M\bar{\psi}\psi \to M\bar{\psi}e^{\frac{i\pi}{2}\gamma_5}e^{\frac{i\pi}{2}\gamma_5}\psi = -M\bar{\psi}\psi$$

to flip the sign of mass.

$$\frac{\det(D+M)}{\det(D-M)} = \frac{\det(D-M)}{\det(D-M)} = 1?$$

Atiyah-Singer index appears

Taking the axial U(1) anomaly into account,

$$\frac{\det(D+M)}{\det(D-M)} = \frac{\det(D-M)}{\det(D-M)} \times \exp\left(i\pi \underbrace{\frac{1}{32\pi^2}\int d^4x F\tilde{F}}_{=I} \right) = (-1)^I.$$

I =Atiyah-Singer index

Our proposal: Why don't we use massive Dirac operator to "define" the index?

(We use anomaly rather than symmetry.)

“New” Atiyah-Singer index

$$\frac{\det(D+M)}{\det(D-M)} = \frac{\det i\gamma_5(D+M)}{\det i\gamma_5(D-M)} = \frac{\prod_{\lambda_{+M}} i\lambda_{+M}}{\prod_{\lambda_{-M}} i\lambda_{-M}} = \exp\left[\frac{i\pi}{2}\left(\sum_{\lambda_{+M}} \mathrm{sgn}\lambda_{+M} - \sum_{\lambda_{-M}} \mathrm{sgn}\lambda_{-M}\right)\right]$$

$\lambda_{\pm M}$: eigenvalues of $\gamma_5(D \pm M)$.

$$I = \frac{1}{2}\left[\eta(\gamma_5(D+M)) - \eta(\gamma_5(D-M))\right].$$

$$\equiv \frac{1}{2}\eta(\gamma_5(D+M))^{reg.}$$

$$\eta(H) = \sum_{\lambda\geq 0}^{reg} - \sum_{\lambda<0}^{reg}$$

Chirality is not important.

No more written by zero modes only.

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6. Summary

More physical set-up?

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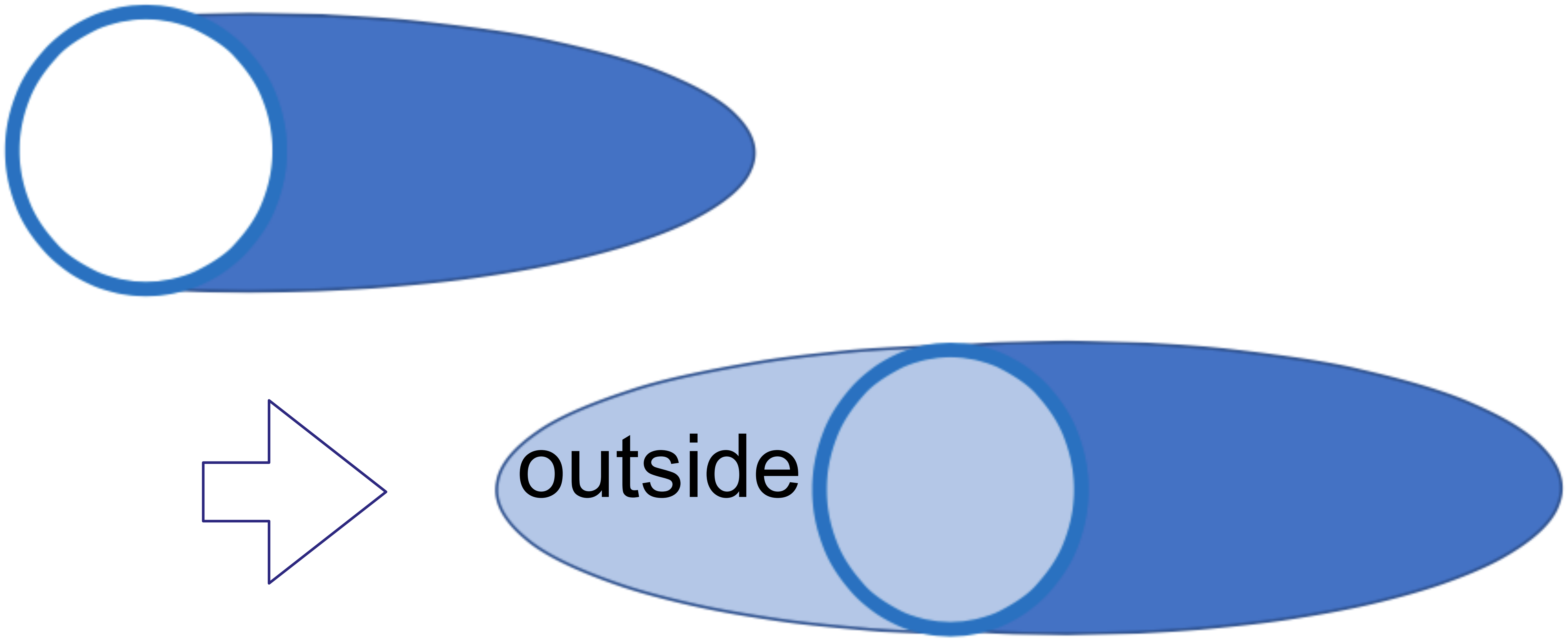
In physics,

1. Any boundary has "outside":

~~manifold + boundary~~ → domain-wall.



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In physics,

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   massless → massive (in bulk)

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   → but automatically chosen.

4. Edge-localized modes play the key role.

Domain-wall fermion Dirac operator

[Jackiw-Rebbi 1976, Callan-Harvery 1985, Kaplan 1992 …]

Let us consider

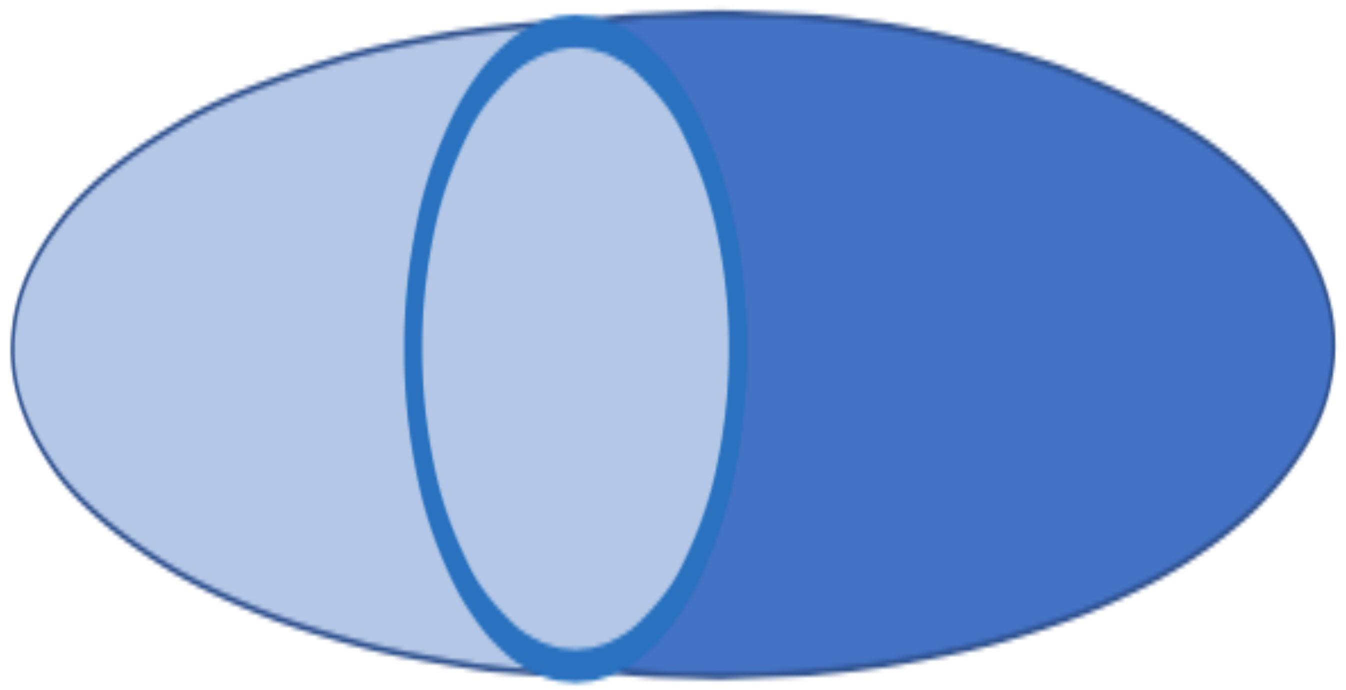
$$D_{\rm 4D} + M\epsilon(x_4), \quad \epsilon(x_4) = {\rm sgn} x_4$$

on a closed manifold

with sign flipping mass,

without assuming any boundary condition

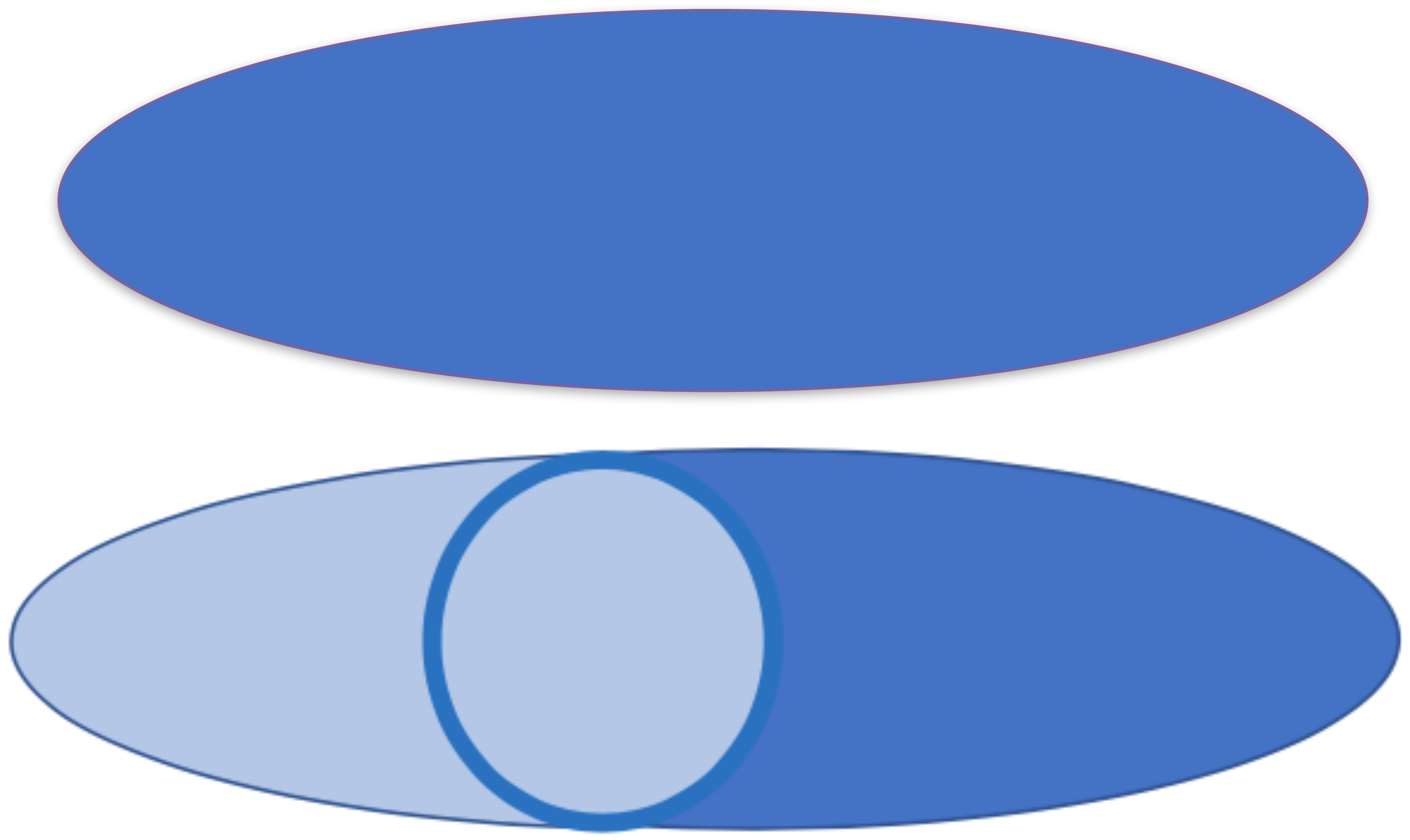
(we expect it dynamically given.).



Here our “domain-wall fermion” is in 4D continuum. (not 5D lattice)

“new” APS index [F-Onogi-Yamaguchi 2017]

$\frac{1}{2}\eta(\gamma_5(D+M))^{reg}$ = AS index



$$\frac{1}{2}\eta(\gamma_5(D+M\epsilon(x_4)))^{reg}$$

$$=\frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}]-\frac{\eta(iD^{\mathrm{3D}})}{2}$$

which can be shown by Fujikawa-method.

Fujikawa method:

$$\frac{1}{2}\eta(H_{DW}) = \frac{1}{2}\mathrm{Tr}\frac{\gamma_5(D+M\varepsilon(x_4))}{\sqrt{\{\gamma_5(D+M\varepsilon(x_4))\}^2}}$$

1. choose regularization
2. choose complete set to evaluate trace
3. perturbation

Fujikawa method:

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1. choose regularization

   Pauli-Villars: $-\frac{1}{2}\mathrm{Tr}\frac{\gamma_5(D-M_2)}{\sqrt{\{\gamma_5(D-M_2)\}^2}}$ $\quad M_2 \gg M$

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Fujikawa method:

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1. choose regularization

   Pauli-Villars: $-\frac{1}{2}\mathrm{Tr}\frac{\gamma_5(D-M_2)}{\sqrt{\{\gamma_5(D-M_2)\}^2}}$ $M_2 \gg M$

2. choose complete set to evaluate trace

   eigen set of $\{\gamma_5(D^{\mathrm{free}}+M\varepsilon(x_4))\}^2$

3. perturbation

Complete set in the free case

Solutions to

$$\{\gamma_5(D^{\rm free}+M\varepsilon(x_4))\}^2\phi=\left[-\partial_\mu^2+M^2-2M\gamma_4\delta(x_4)\right]\phi=\lambda^2\phi$$

are $\varphi(x_4)\otimes e^{i\boldsymbol{p}\cdot\boldsymbol{x}}$ where

$$\varphi^{\omega}_{\pm,o}(x_4)=\frac{1}{\sqrt{4\pi}}\left(e^{i\omega x_4}-e^{-i\omega x_4}\right),$$

$$\varphi^{\omega}_{\pm,e}(x_4)=\frac{1}{\sqrt{4\pi(\omega^2+M^2)}}\left((i\omega\mp M)e^{i\omega|x_4|}+(i\omega\pm M)e^{-i\omega|x_4|}\right),$$

$$\varphi^{\rm edge}_{+,e}(x_4)=\sqrt{M}e^{-M|x_4|},$$

→ Edge mode appears !

Here, $\omega=\sqrt{\boldsymbol{p}^2+M^2-\lambda_{4D}^2}$ and $\gamma_4\varphi^{\omega,{\rm edge}}_{\pm,e/o}=\pm\varphi^{\omega,{\rm edge}}_{\pm,e/o}$

3D direction = conventional plane waves.

“Automatic” boundary condition

We didn’t put any boundary condition by hand. But

$$\left[\frac{\partial}{\partial x_4} \pm M\epsilon(x_4)\right] \varphi^{\omega,\mathrm{edge}}_{\pm,e}(x_4)\Big|_{x_4=0} = 0, \quad \varphi^{\omega}_{\pm,o}(x_4=0) = 0.$$

is automatically satisfied due to the domain-wall.

This condition is LOCAL and PRESERVES angular-momentum in x_4 direction but DOES NOT keep chirality.

Bulk & edge contributions

By a simple perturbation, we obtain

$$\frac{1}{2}\eta(H_{DW})^{bulk} = \frac{1}{2}\sum_{bulkmodes}(\phi^{bulk})^{\dagger}\mathrm{sgn}(H_{DW})\phi^{bulk} = \frac{1}{64\pi^2}\int d^4x\epsilon(x_4)\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}_c F^{\mu\nu}F^{\rho\sigma}(x) + O(1/M).$$

$$\frac{1}{2}\eta(H_{DW})^{edge} = \frac{1}{2}\sum_{edgemodes}\phi^{edge}(x)^{\dagger}\mathrm{sgn}(H_{DW})\phi^{edge}(x) = -\frac{1}{2}\eta(iD^{\mathrm{3D}})|_{x_4=0}$$

$$-\frac{1}{2}\eta(H_{PV}) = \frac{1}{64\pi^2}\int d^4x\ \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}_c F^{\mu\nu}F^{\rho\sigma}(x) + O(1/M).$$

⇨

$$\frac{1}{2}\eta(\gamma_5(D + M\epsilon(x_4)))^{reg} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{\mathrm{3D}})}{2}$$

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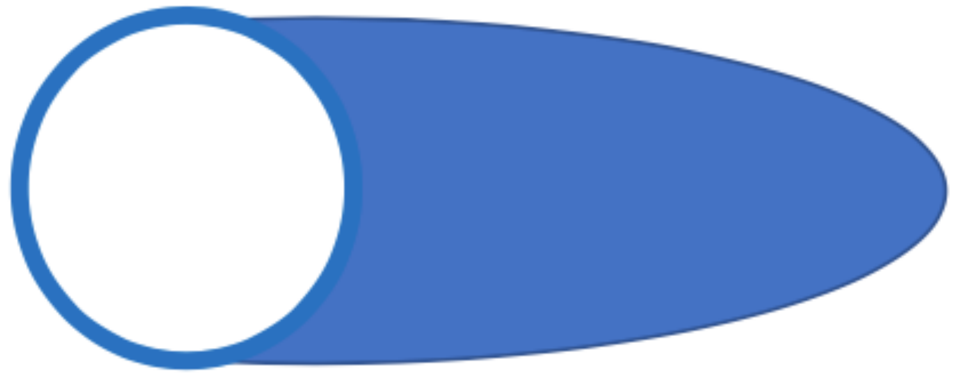
Just a coincidence?

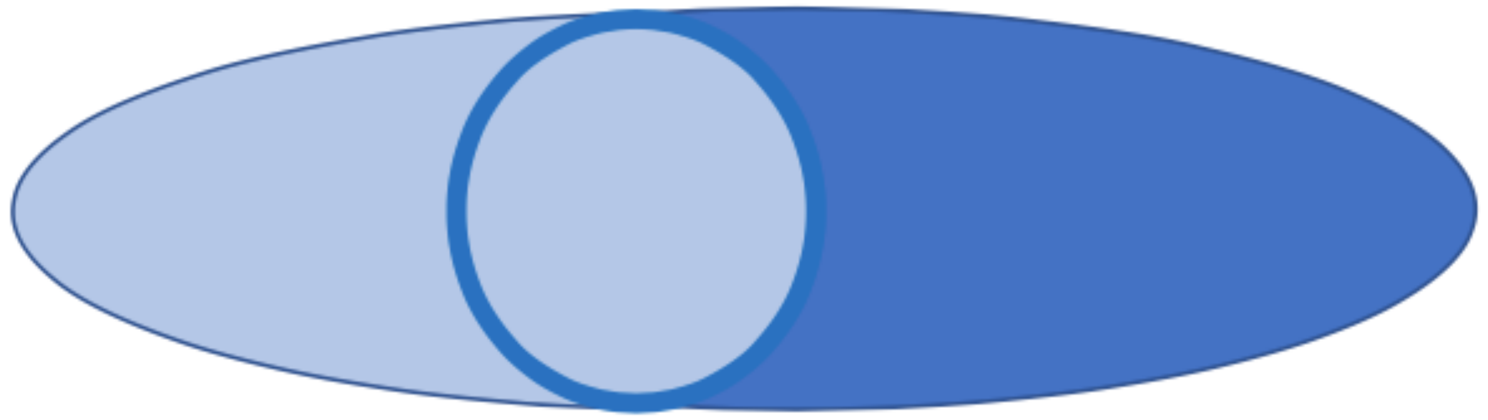
$$Ind(D_{\mathrm{APS}}) = \frac{1}{2}\eta(H_{DW}^{reg})$$

on general even-dimensional manifolds ?



APS

1. massless Dirac (even in bulk)
2. non-local boundary cond.
   (depending on gauge fields)
3. SO(3) rotational sym. on boundary is lost.
4. no edge mode appears.
5. manifold + boundary



Domain-wall fermion

1. massive Dirac in bulk (massless mode at edge)
2. local boundary cond.
3. SO(3) rotational sym. on boundary is kept.
4. Edge mode describes eta-invariant.
5. closed manifold + domain-wall

Mathematician's response

In 2018, I gave a talk in a workshop

organized by Mikio Furuta (U. Tokyo).

He said "This must be a challenge for us."

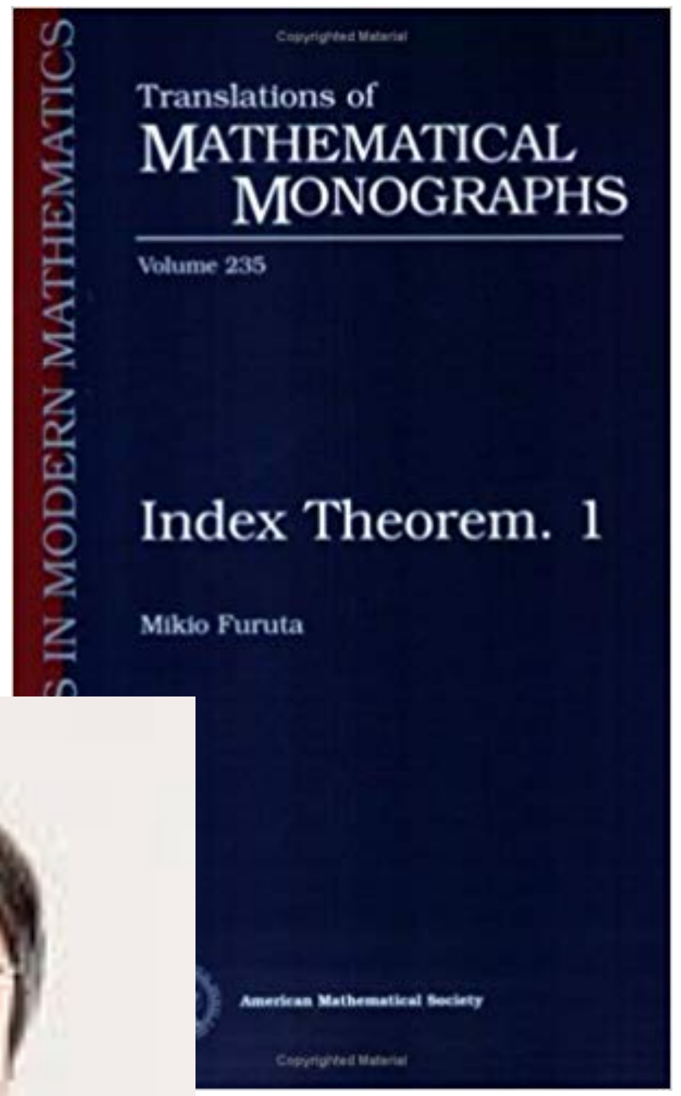
Moreover, only 1 week later,

he proposed a sketch of proof for

$$\frac{1}{2}\eta(H_{DW}^{reg}) = Ind(D_{\rm APS})$$

[F, Furuta, Matsuo, Onogi,

Yamaguchi, and Yamashita, arXiv:1910.01987]



Theorem

(F-Furuta-Matsuo-Onogi-Yamaguchi-Yamashita 2019)

For any APS index of a massless Dirac operator on a even-dim. Riemannian manifold X with boundary, there exists a massive (domain-wall) Dirac operator on a closed manifold, sharing its half with X, and its eta invariant is equal to the original index.

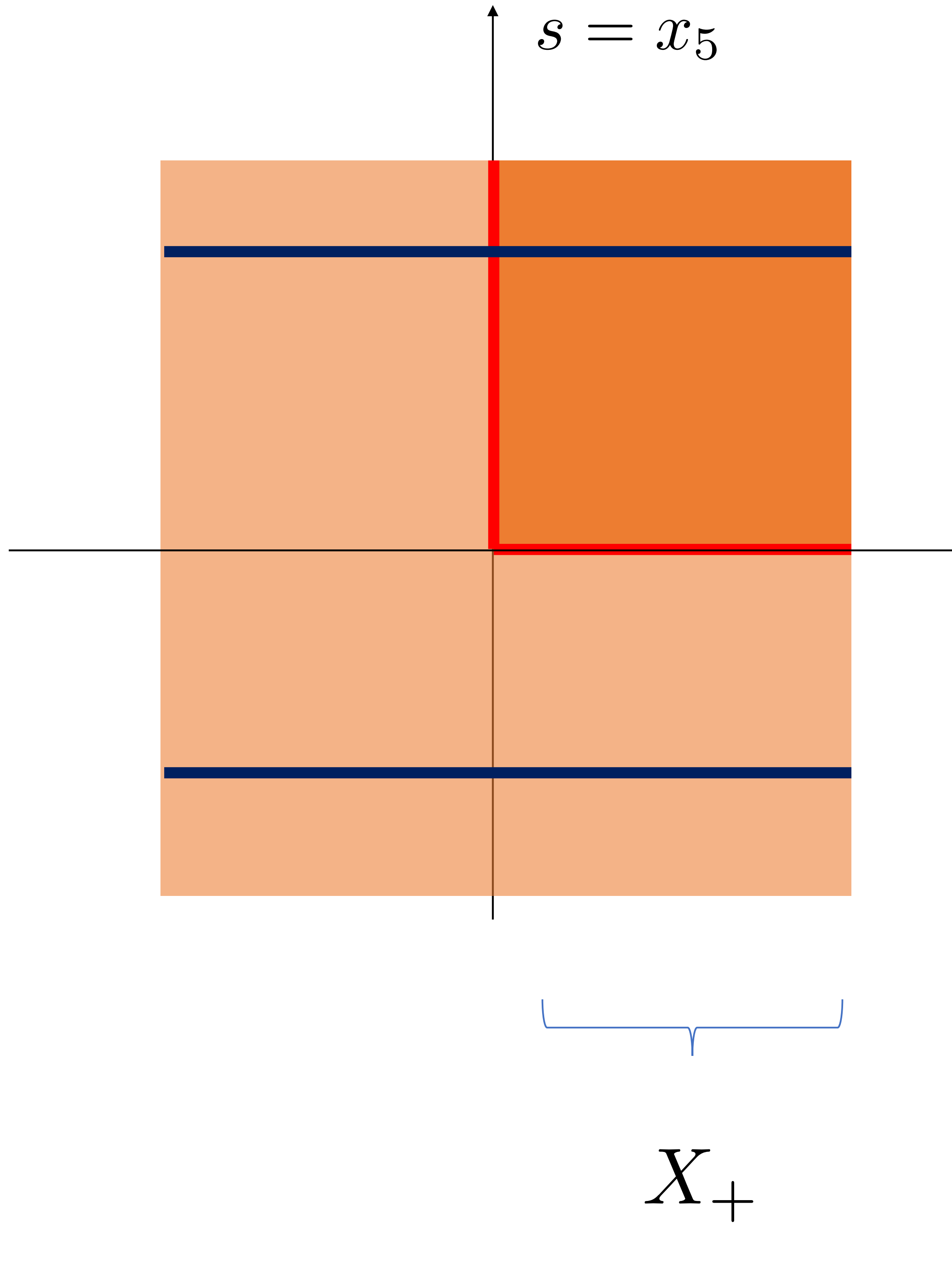
Sketch of the proof

(F-Furuta-Matsuo-Onogi-Yamaguchi-Yamashita 2019)

We introduce an extra dimension and consider a Dirac operator on the higher dim. manifold.

$$D^{5\mathrm{D}} = \begin{pmatrix} 0 & \partial_5 + \gamma_5(D^{4\mathrm{D}} + m(x_4, x_5)) \\ -\partial_5 + \gamma_5(D^{4\mathrm{D}} + m(x_4, x_5)) & 0 \end{pmatrix}$$

$$m(x_4, x_5) = \begin{cases} M & \text{for } x_4 > 0 \;\&\; x_5 > 0 \\ 0 & \text{for } x_4 = 0 \;\&\; x_5 = 0 \\ -M_2 & \text{otherwise} \end{cases}$$



With 2 different evaluations, we can show

$$Ind(D^{5D}) = Ind(D_{\mathrm{APS}}) = \frac{1}{2}\eta(H_{DW})$$

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$\mathfrak{I} = \eta(\gamma_5(D+M))^{reg}/2$ coincides with the AS index.

✔ 3. New index with ~~boundary~~ domainwall [FOY 2017]

$\mathfrak{I} = \eta(\gamma_5(D+M\epsilon(x_4)))^{reg}/2$ coincides with the APS index.

✔ 4. Mathematical proof [F, Furuta, Matsuo, Onogi, Yamaguchi, Yamashita, 2019]

$Ind(D_{\rm APS})$ and $\eta(\gamma_5(D+M\epsilon(x_4)))^{reg}/2$ are different expressions of the same 5D Dirac index.

5. APS index on a lattice [F, Kawai, Matsuki, Mori, Nakayama, Onogi, Yamaguchi 2019]

6. Summary

Atiyah-Singer index on a lattice

Overlap fermion action $S = \sum_x \bar{q}(x) D_{ov} q(x)$

[Neuberger 1998]

is invariant under

$$q \to e^{i\alpha\gamma_5(1-aD_{ov})} q, \quad \bar{q} \to \bar{q} e^{i\alpha\gamma_5}.$$

[Luescher 1998]

but fermion measure transforms

as $D q\bar{q} \to \exp\left[2i\alpha \mathrm{Tr}(\gamma_5 + \gamma_5(1 - aD_{ov}))/2\right] D q\bar{q}$

which reproduces U(1)A anomaly.

Moreover, $\mathrm{Tr}\gamma_5 \left(1 - \frac{aD_{ov}}{2}\right)$ is AS index !

[Hasenfratz et al. 1998]

On the lattice, AS is O.K. but APS is not.

Atiyah-Singer index can be formulated by overlap Dirac operator, $D_{ov} = \frac{1}{a}\left(1 + \gamma_5 \frac{H_W}{\sqrt{H_W^2}}\right)$

but APS was not known.

1. Lattice version of APS condition impossible, as D_{ov} does not have a form $D_{\mathrm{normal}} + D_{\mathrm{horizontal}}$.

2. Any boundary condition breaks Ginsparg-Wilson relation [Luescher 2006].

Cf. Kikukawa, “Suri-kagaku” 2020 Jan.

But the lattice AS index theorem "knew" the eta invariant!

$$Ind(D_{ov}) = \frac{1}{2}\mathrm{Tr}\gamma_5\left(1 - \frac{aD_{ov}}{2}\right) \qquad D_{ov} = \frac{1}{a}\left(1 + \gamma_5 \frac{H_W}{\sqrt{H_W^2}}\right)$$

$$H_W = \gamma_5(D_W - M)$$
$$M = 1/a$$

$$= -\frac{1}{2}\mathrm{Tr}\frac{H_W}{\sqrt{H_W^2}} = -\frac{1}{2}\eta(\gamma_5(D_W - M))!$$

Cf. Itoh-Iwasaki-Yoshie 1982, Adams 2001

The lattice index theorem "knew"

1. index can be given with massive Dirac.

2. chiral symmetry is not important.

Wilson Dirac operator is enough.

Unification of index theorems

The standard formulation of index with massless Dirac

	continuum	lattice
AS	$\mathrm{Tr}\gamma^5 e^{-D^2/M^2}$	$\mathrm{Tr}\gamma^5(1 - aD_{ov}/2)$
APS	$\mathrm{Tr}\gamma^5 e^{-D^2/M^2}$ w/ APS b.c.	not known.

New formulation of index with massive Dirac

	continuum	lattice
AS	$-\frac{1}{2}\eta(\gamma_5(D-M))$	$-\frac{1}{2}\eta(\gamma_5(D_W-M))$
APS	$-\frac{1}{2}\eta(\gamma_5(D-M\epsilon(x)))$	$-\frac{1}{2}\eta(\gamma_5(D_W-M\epsilon(x)))$?

APS index on a lattice

F, Kawai, Matsuki, Mori, Nakayama, Onogi, Yamaguchi, arXiv:1910.09675

On 4-dimensional Euclidean lattice with periodic boundaries (T4), we have perturbatively shown

$$-\frac{1}{2}\eta(\gamma_5(D_W - M\varepsilon(x_4))) = \frac{1}{32\pi^2}\int_{0<x_4<L_4} d^4x\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}F_{\mu\nu}F_{\rho\sigma} - \frac{1}{2}\eta(iD^{\mathrm{3D}})|_{x_4=0} + \frac{1}{2}\eta(iD^{\mathrm{3D}})|_{x_4=L_4},$$

$$+O(a)$$

$$\varepsilon(x_4) = \mathrm{sgn}(x_4 - a/2)\mathrm{sgn}(T - x_4 - a/2)$$

* Bulk part is similar to that of AS index [H.Suzuki 1998].

Note that LHS is always an integer.

Contents

✔ 1. Introduction

Original definition of APS index is physicist-unfriendly.

✔ 2. Massive Dirac operator index without boundary

$\mathfrak{I} = \eta(\gamma_5(D+M))^{reg}/2$ coincides with the AS index.

✔ 3. New index with ~~boundary~~ domainwall [FOY 2017]

$\mathfrak{I} = \eta(\gamma_5(D+M\epsilon(x_4)))^{reg}/2$ coincides with the APS index.

✔ 4. Mathematical proof [F, Furuta, Matsuo, Onogi, Yamaguchi, Yamashita, 2019]

$Ind(D_{\rm APS})$ and $\eta(\gamma_5(D+M\epsilon(x_4)))^{reg}/2$ are different expressions of the same 5D Dirac index.

✔ 5. APS index on a lattice [F, Kawai, Matsuki, Mori, Nakayama, Onogi, Yamaguchi 2019]

can be defined by $-\eta(\gamma_5(D_W - M\epsilon(x_4+a/2)))/2$

6. Summary

Summary

Massive (domain-wall) fermion is physicist-friendly:

APS index can be formulated (even on a lattice).

Moreover, it is mathematically rich:

The eta inv. of massive Dirac on a closed manifold unifies the index theorems.

	continuum	lattice
AS	$-\frac{1}{2}\eta(\gamma_5(D-M))$	$-\frac{1}{2}\eta(\gamma_5(D_W-M))$
APS	$-\frac{1}{2}\eta(\gamma_5(D-M\epsilon(x)))$	$-\frac{1}{2}\eta(\gamma_5(D_W-M\epsilon(x)))$

Backup slides

Spectral flow gives a bigger unification.

F, Furuta, Matsuki, Matsuo, Onogi, Yamaguchi and Yamashita arXiv:2012.03543.

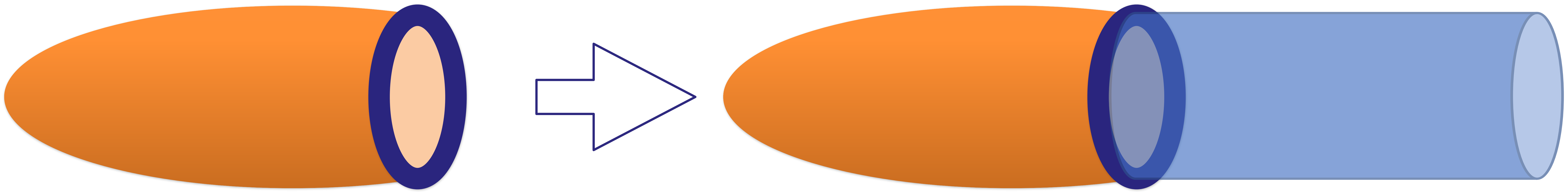
$$-\frac{1}{2}\eta(H_1) + \frac{1}{2}\eta(H_1) = \text{Spectral flow of } H_t, \quad t \in [0,1]$$

	continuum	lattice
AS	$\mathrm{Sf}(\gamma_5(D-M))$	$\mathrm{Sf}(\gamma_5(D_W-M))$
APS	$\mathrm{Sf}(\gamma_5(D-\varepsilon M))$	$\mathrm{Sf}(\gamma_5(D_W-\varepsilon M))$
mod-two AS	$\mathrm{Sf}'\begin{pmatrix} & D-M \\ -(D-M)^\dagger & \end{pmatrix}$	$\mathrm{Sf}'\begin{pmatrix} & D_W-M \\ -(D_W-M)^\dagger & \end{pmatrix}$
mod-two APS	$\mathrm{Sf}'\begin{pmatrix} & D-\varepsilon M \\ -(D-\varepsilon M)^\dagger & \end{pmatrix}$	$\mathrm{Sf}'\begin{pmatrix} & D_W-\varepsilon M \\ -(D_W-\varepsilon M)^\dagger & \end{pmatrix}$

$\mathrm{Sf}' = $ mod-two spectral flow : counting zero-crossing pairs from PV op.

Theorem 1:
APS index = index with infinite cylinder

In original APS paper, they showed



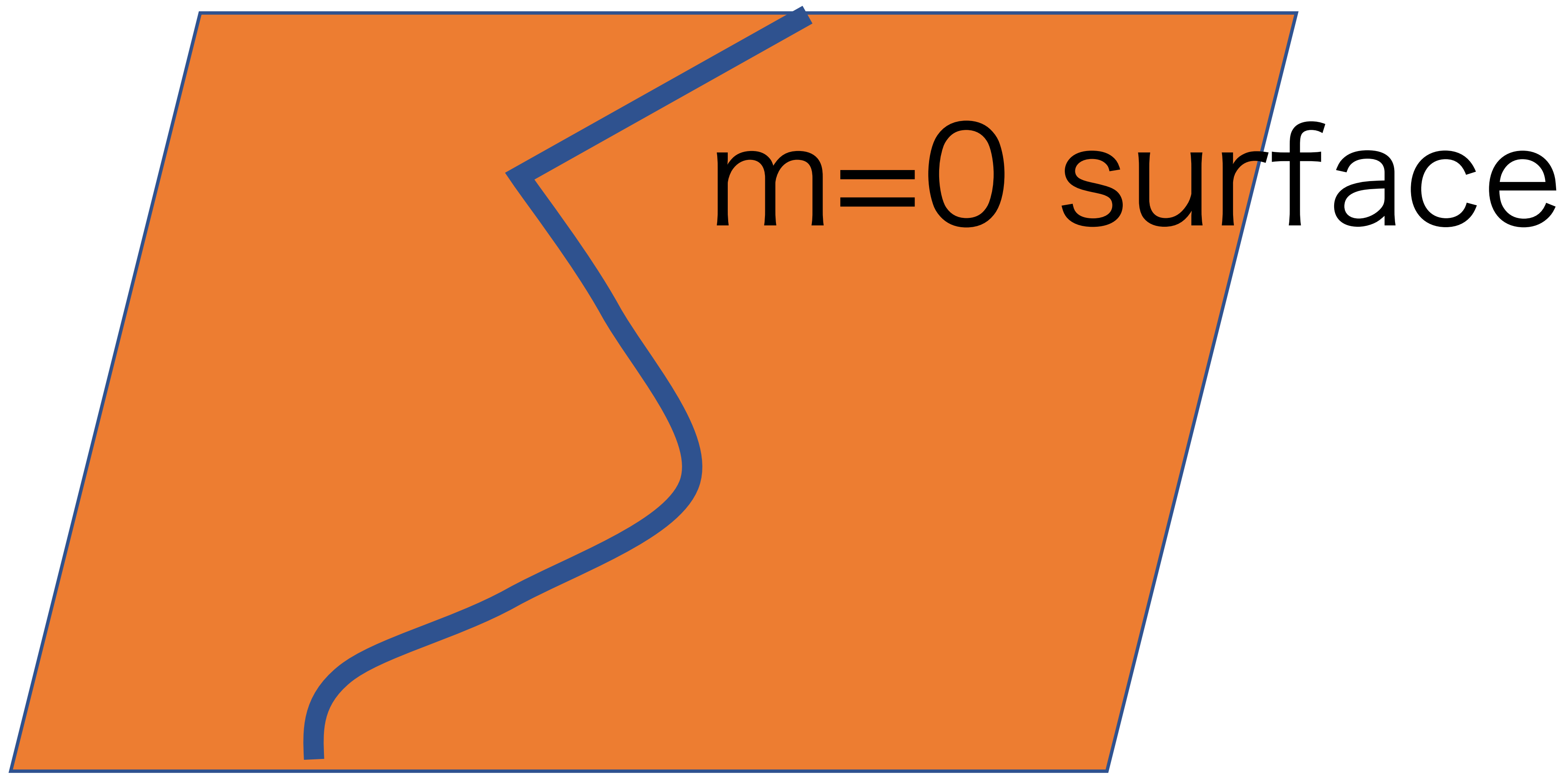
Index w/ APS b.c. = Index with infinite cylinder attached to the original boundary (w.r.t. square integrable modes).

* On cylinder, gauge fields are constant in the extra-direction.

Theorem 2:
Localization (& product formula)

By giving position-dependent “mass”, we can localize the zero modes to “massless” lower-dimensional surface and the index is given by the product:

$$Ind(\gamma_s(D^d + \partial_s + i\gamma_s M(s))) =$$
$$Ind(D^d) \times Ind(\gamma_s\partial_s + M(s))$$



Theorem 3:
In odd-dim, APS index = boundary eta-invariant

$$\int F \wedge F \wedge \cdots$$

exists only in even-dim.

$$Ind(D_{\rm APS}^{odd-dim}) = \frac{1}{2}\left[\eta(D^{\rm boundary1}) - \eta(D^{\rm boundary2})\right]$$

5-dimensional Dirac operator

we consider

$$D^{5\mathrm{D}} = \begin{pmatrix} 0 & \partial_5 + \gamma_5(D^{4\mathrm{D}} + m(x_4, x_5)) \\ -\partial_5 + \gamma_5(D^{4\mathrm{D}} + m(x_4, x_5)) & 0 \end{pmatrix}$$

where

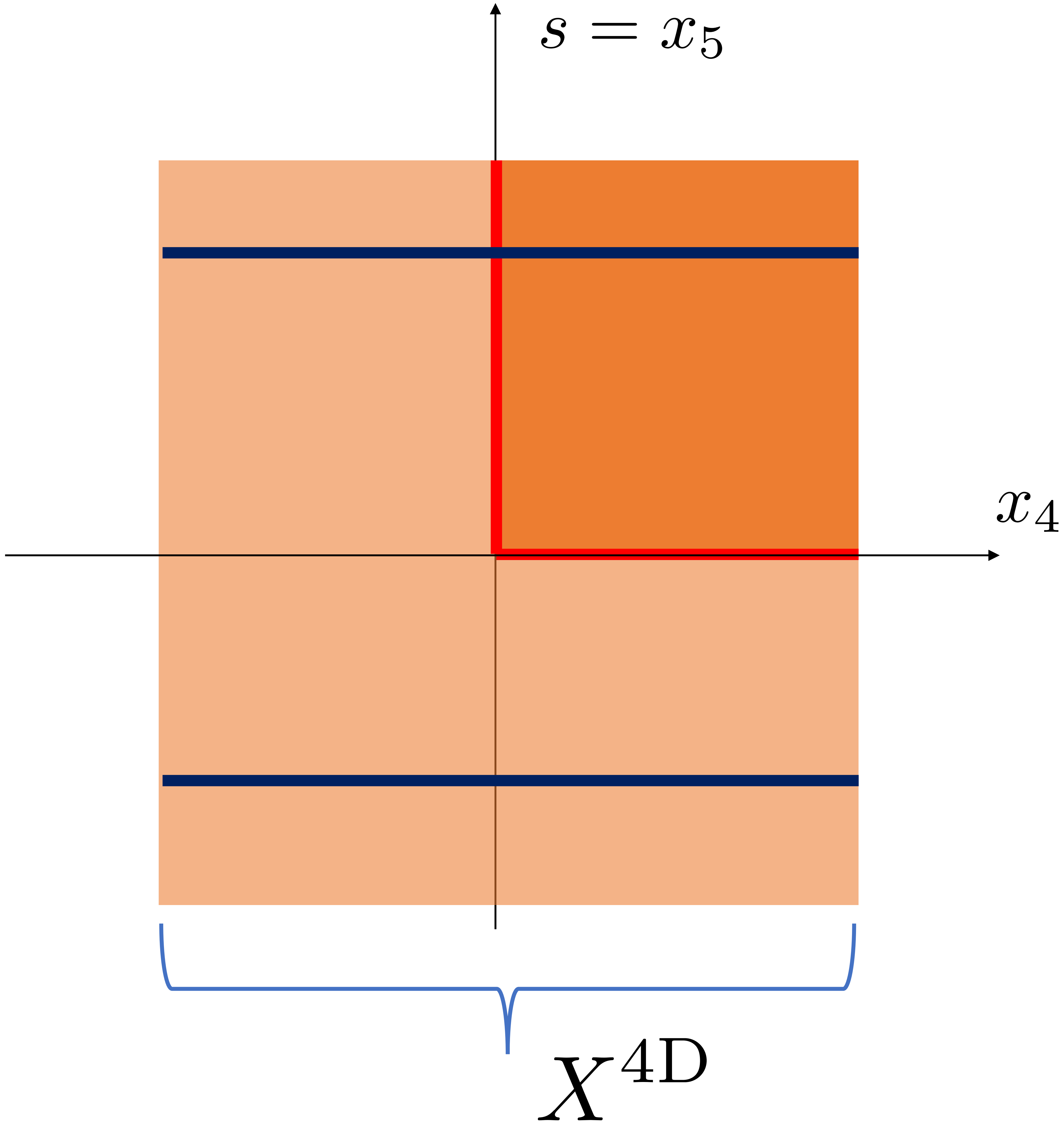
$$m(x_4, x_5) = \begin{cases} M & \text{for } x_4 > 0 \ \& \ x_5 > 0 \\ 0 & \text{for } x_4 = 0 \ \& \ x_5 = 0 \\ -M_2 & \text{otherwise} \end{cases}$$

and A_μ is

independent of x_5.

* Application is straightforward to any 2n+1 dimensions.

On X^{4D} x R,



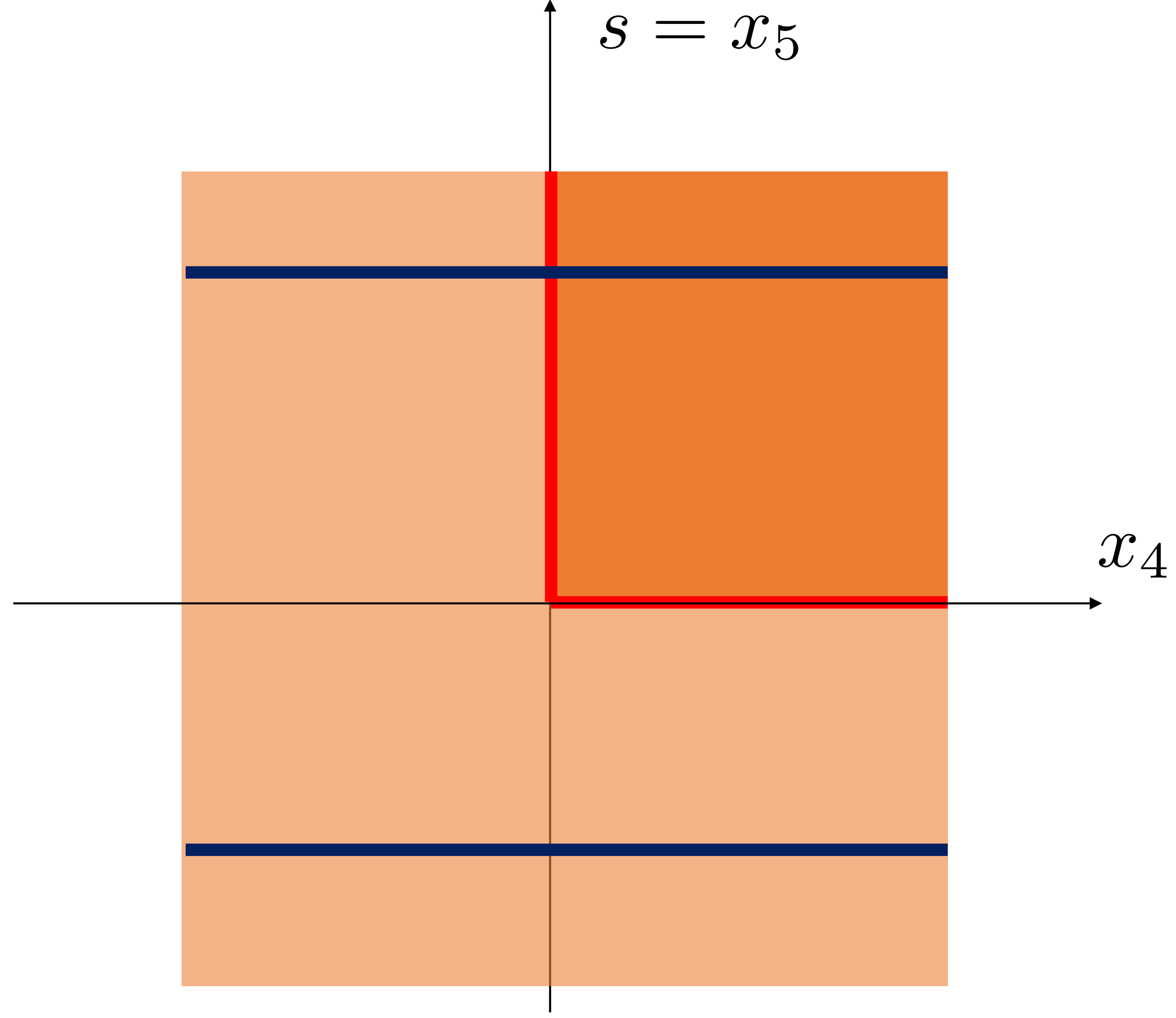
we compute

$$Ind(D^{5\mathrm{D}})$$

in two different ways:

1. localization
2. eta-inv. at $x_5 = \pm 1.$

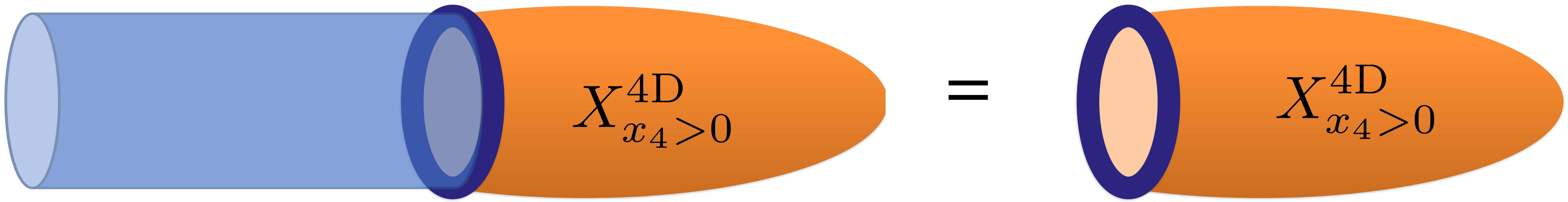
Localization



Theorem 2 tells us

$$Ind(D^{5D})|_{M,M_2\to\infty} = Ind(D^{4D}_{m=0\text{surface}}) \times \underbrace{Ind D^{1D}_{normal}}_{=1}$$

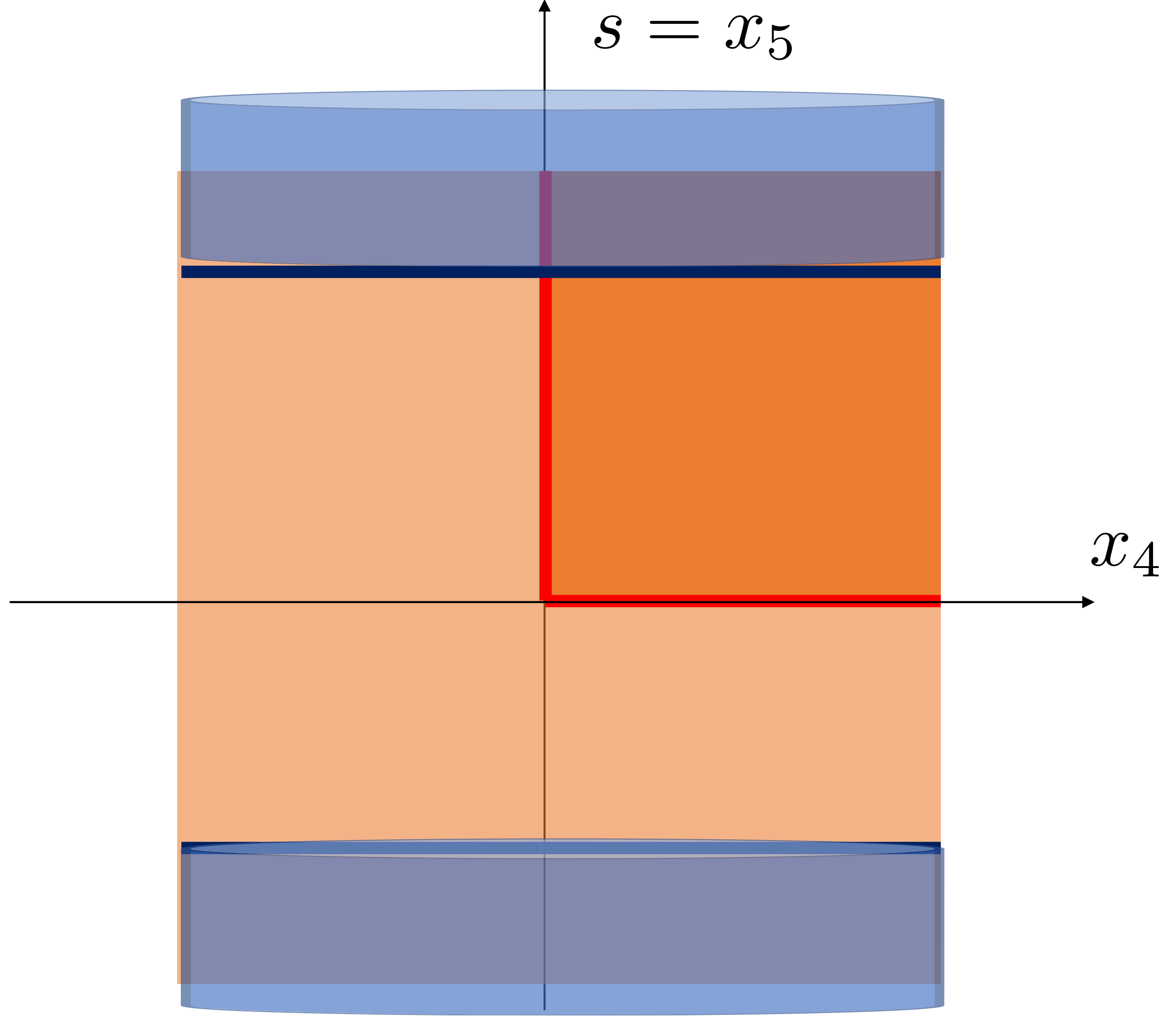
and on the massless surface



theorem 1 indicates

$$Ind(D^{4D}_{m=0\text{surface}}) = Ind(D^{X^{4D}_{x_4>0}}_{\text{APS}})$$

Boundary eta invariants



Theorem 1 tells us

$$Ind(D^{5\mathrm{D}}) = Ind(D^{5\mathrm{D}}_{\mathrm{APS\ b.c.}ats=\pm 1})$$

and from theorem 3, we obtain

$$Ind(D^{5\mathrm{D}}_{\mathrm{APS\ b.c.}ats=\pm 1}) = \frac{1}{2}\left[\eta(D^{4\mathrm{D}}_{s=1}) - \eta(D^{4\mathrm{D}}_{s=-1})\right]$$

$$= \frac{1}{2}\left[\eta(\gamma_5(D^{4\mathrm{D}} + M\epsilon(x_4)) - \eta(\gamma_5(D^{4\mathrm{D}} - M_2)\right] = \frac{1}{2}\eta^{PVreg.}(\gamma_5(D^{4\mathrm{D}} + M\epsilon(x_4))$$

therefore,

$$Ind(D^{5D}) = Ind(D_{\mathrm{APS}}) = \frac{1}{2}\eta(H_{DW})$$

Q.E.D.

Revisiting lattice index theorems with mathematicians

Yamashita, “A lattice version of the Atiyah-Singer index theorem,” arXiv:2007.06239

F, Furuta, Matsuo, Onogi, Yamaguchi, Yamashita, “On analytic indices in lattice gauge theory and their continuum limits,”

in preparation.

Different explanation why APS appears [Witten Yonekura 2019]

They rotate the x4 to the “time” direction and introduced the APS boundary condition as intermediate “states”. The unphysical property of APS is canceled between the bra/ket states.

(In our work, we try to remove it.)

Eta invariant = Chern Simons term + integer (non-local effect)

$$\frac{\eta(iD^{\mathrm{3D}})}{2} = \frac{CS}{2\pi} + \mathrm{integer}$$

$$CS \equiv \frac{1}{4\pi}\int_Y d^3x\ \mathrm{tr}_c\left[\epsilon_{\nu\rho\sigma}\left(A^\nu\partial^\rho A^\sigma + \frac{2i}{3}A^\nu A^\rho A^\sigma\right)\right],$$

= surface term.

$$\mathfrak{I} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{\mathrm{3D}})}{2}$$