Application of Tensor Renormalization Group to Quantum Field Theories

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Plan of Talk

- Introduction
- Tensor Renormalization Group (TRG)
- TRG Approaches to Quantum Field Theories
- Application of TRG method to
  - 2D, 4D Complex ϕ^4 at Finite Density
  - 4D Nambu–Jona-Lasinio (NJL) Model at Finite Density
- Summary

Tensor Network Scheme

What is Tensor Network (TN) Scheme?

Theoretical and numerical methods for high precision analyses of many body problems with tensor network formalism

Advantages of Tensor Renormalization Group (TRG)

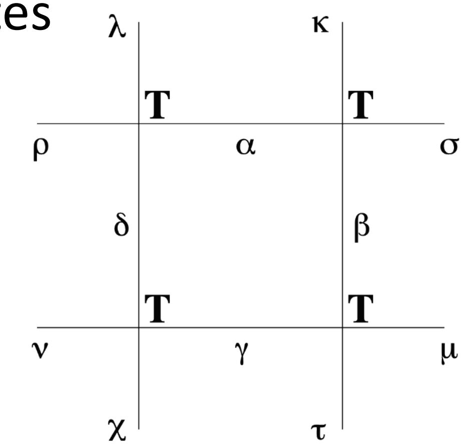
Free from sign problem and complex action problem in Monte Carlo method
Computational cost for L^D system size $\propto D \times \log(L)$
Direct treatment of Grassmann numbers
Direct evaluation of partition function Z itself

Applications in particle physics :
Finite density QCD, QFTs w/ θ-term, Lattice SUSY etc.
Also, in condensed matter physics
Hubbard model (Mott transition, High T_c superconductivity) etc.

Tensor Renormalization Group (TRG)

Levin-Nave
PRL99(2007)120601

Explain the algorithm with 2D Ising model with N sites



Hamiltonian $H = \sum_{\langle i,j \rangle} s_i s_j \qquad s_i \pm 1$

Partition Function $Z = \sum_{\{s_i\}} \exp\left(-\beta H\right)$

$$= \sum_{\alpha,\beta,\gamma,\delta,\cdots=1}^{2} T_{\alpha,\lambda,\rho,\delta} T_{\sigma,\kappa,\alpha,\beta} T_{\mu,\beta,\gamma,\tau} T_{\gamma,\delta,\nu,\chi} \cdots$$

Tensor Network representation

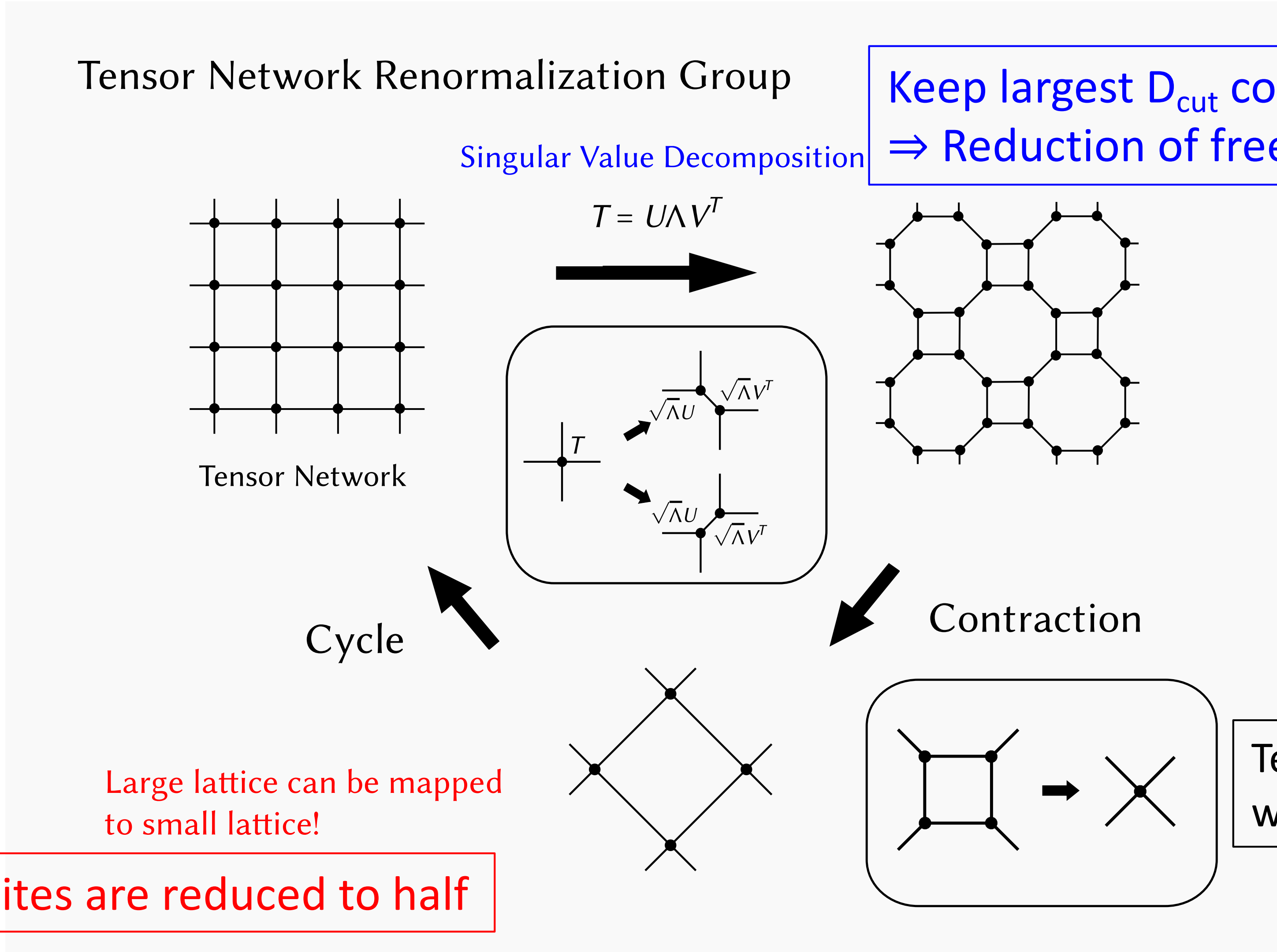
Details of model are specified in initial tensor
The algorithmic procedure is independent of models

Of course, direct contraction is impossible for large N even with current fastest supercomputer

⇒ How to evaluate the partition function?

Schematic View of TRG Algorithm

1. Singular Value Decomposition of local tensor T
2. Contraction of old tensor indices (coarse-graining)
3. Repeat the iteration



Keep largest D_{cut} components
⇒ Reduction of freedom

Tensors
w/ new indices

#sites are reduced to half

Development of TRG method

Original TRG algorithm (Levin-Nave) is applicable to only 2D models

Applicable to ≧3D models

- Higher Order TRG (HOTRG) : Xie+, PRB86(2012)045139
- Anisotropic TRG (ATRG) : Adachi-Okubo-Todo, PRB102(2020)054432
- Triad TRG (TTRG) : Kadoh-Nakayama, arXiv:1912.02414

Grassmann versions

- Grassmann TRG (GTRG) : Shimizu-YK, PRD90(2014)014508
- Grassmann HOTRG (GHOTRG) : Sakai-Takeda-Yoshimura,PTEP2017(2017)063B07
- Grassmann ATRG (GATRG) : Akiyama-YK-Yamashita-Yoshimura, JHEP01(2021)121

TRG method/approach refers to all the above algorithms
(not restricted to the original one)

Collaborators

Y. Kuramashi, S. Akiyama	U. Tsukuba
Y. Nakamura, (Y. Shimizu)	R-CCS
S. Takeda, Y. Yoshimura R. Sakai(→Syracuse U.)	Kanazawa U.
D. Kadoh	Doshisha U.

Collaborations are dynamically changed depending on the research topics

TRG Approaches to QFTs (1)

2D models

Real ϕ^4 theory :

Shimizu, Mod.Phys.Lett.A27(2012)1250035

Kadoh-YK-Nakamura-Sakai-Takeda-Yoshimura, JHEP05(2019)184

Complex ϕ^4 theory at finite density :

Kadoh-YK-Nakamura-Sakai-Takeda-Yoshimura, JHEP02(2020)161

U(1) gauge theory+θ :

YK-Yoshimura, JHEP04(2020)089

Schwinger, Schwinger+θ :

Shimizu-YK, PRD90(2014)014508, PRD90(2014)074503,
PRD97(2018)034502

Gross-Neveu model at finite density :

Takeda-Yoshimura, PTEP2015(2015)043B01

N=1 Wess-Zumino model :

Kadoh-YK-Nakamura-Sakai-Takeda-Yoshimura, JHEP03(2018)141

・Free from sign/complex action problems
・Development of numerical algorithms for scalar, fermion, gauge theories

TRG Approaches to QFTs (2)

3D models

Free Wilson fermion :

Sakai-Takeda-Yoshimura, PTEP2017(2017)063B07,

Yoshimura-YK-Nakamura-Takeda-Sakai, PRD97(2018)054511

Z_2 gauge theory at finite temperature :

YK-Yoshimura, JHEP1908(2019)023

4D models

Ising : Akiyama-YK-Yamashita-Yoshimura, PRD100(2019)054510

Complex ϕ^4 theory at finite density :

Akiyama-Kadoh-YK-Yamashita-Yoshimura, JHEP09(2020)177

NJL model at finite density :

Akiyama-YK-Yamashita-Yoshimura, JHEP01(2021)121

Real ϕ^4 theory :

Akiyama-YK-Yoshimura, PRD104(2021)034507

⇒ Now our research target is moving from 2D models to 4D ones

TRG Approaches to QFTs (3)

Condensed matter physics

Similarity btw NJL model and Hubbard model

Action consists of hopping term and four-fermi interaction

$$S=\sum_{n\in\Lambda_{d+1}}\epsilon\left\{\bar{\psi}(n)\left(\frac{\psi(n+\hat{\tau})-\psi(n)}{\epsilon}\right)-t\sum_{\sigma=1}^{d}\left(\bar{\psi}(n+\hat{\sigma})\psi(n)+\bar{\psi}(n)\psi(n+\hat{\sigma})\right)+\frac{U}{2}\left(\bar{\psi}(n)\psi(n)\right)^2-\mu\bar{\psi}(n)\psi(n)\right\}$$

First principle calculation at finite chemical potential

(1+1)D Hubbard model : Akiyama-YK, PRD104(2021)014504

(2+1)D Hubbard model : Akiyama-YK-Yamashita, arXiv:2109.14149

Two examples in this talk

2D, 4D Complex ϕ4 at Finite Density

4D Nambu−Jona-Lasinio (NJL) Model at Finite Density

2D Complex ϕ^4 Theory at Finite Density

Kadoh+, JHEP02(2020)161

Continuum action of 2D complex ϕ^4 theory at finite μ

$$S_{\text{cont}} = \int \mathrm{d}^2x \left\{ |\partial_\rho \phi|^2 + (m^2 - \mu^2)|\phi|^2 + \mu(\phi^* \partial_2 \phi - \partial_2 \phi^* \phi) + \lambda |\phi|^4 \right\}$$

Introduction of finite chemical potential ⇒ complex action

Lattice action

$$Z(\text{original}) = \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \exp(-S)$$

$$S = \sum_n \left[(4 + m^2)|\phi_n|^2 + \lambda |\phi_n|^4 - \sum_{\rho=1}^{2} \left(e^{\mu \delta_{\rho,2}} \phi_n^* \phi_{n+\hat{\rho}} + e^{-\mu \delta_{\rho,2}} \phi_{n+\hat{\rho}}^* \phi_n \right) \right]$$

First step is to construct TN representation

Tensor Network Representation

Kadoh+, JHEP02(2020)161

Boltzmann weight is expressed as

$$e^{-S} = \prod_{n\in\Gamma}\prod_{\nu=1}^{2} f_\nu\left(\phi_n, \phi_{n+\hat{\nu}}\right)$$

$$f_\nu\left(z, z'\right)$$
$$= \exp\left\{-\frac{1}{4}\left(4+m^2\right)\left(|z|^2+|z'|^2\right) - \frac{\lambda}{4}\left(|z|^4+|z'|^4\right) + e^{\mu\delta_{\nu,2}} z^* z' + e^{-\mu\delta_{\nu,2}} z z'^*\right\}$$

⇒ Need to discretize the continuous d. o. f.

Use of Gauss-Hermite quadrature

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}z_1 \mathrm{d}z_2\ e^{-z_1^2-z_2^2} h\left(z\right) \approx \sum_{\alpha,\beta=1}^{K} w_\alpha w_\beta h\left(\frac{y_\alpha + i y_\beta}{\sqrt{2}}\right)$$

Discretized version of partition function

$$Z \approx Z\left(K\right) = \sum_{\{\alpha,\beta\}}\prod_{n\in\Gamma} w_{\alpha_n} w_{\beta_n} \exp\left(y_{\alpha_n}^2 + y_{\beta_n}^2\right)\prod_{\nu=1}^{2} f_\nu\left(\frac{y_{\alpha_n} + i y_{\beta_n}}{\sqrt{2}}, \frac{y_{\alpha_{n+\hat{\nu}}} + i y_{\beta_{n+\hat{\nu}}}}{\sqrt{2}}\right)$$

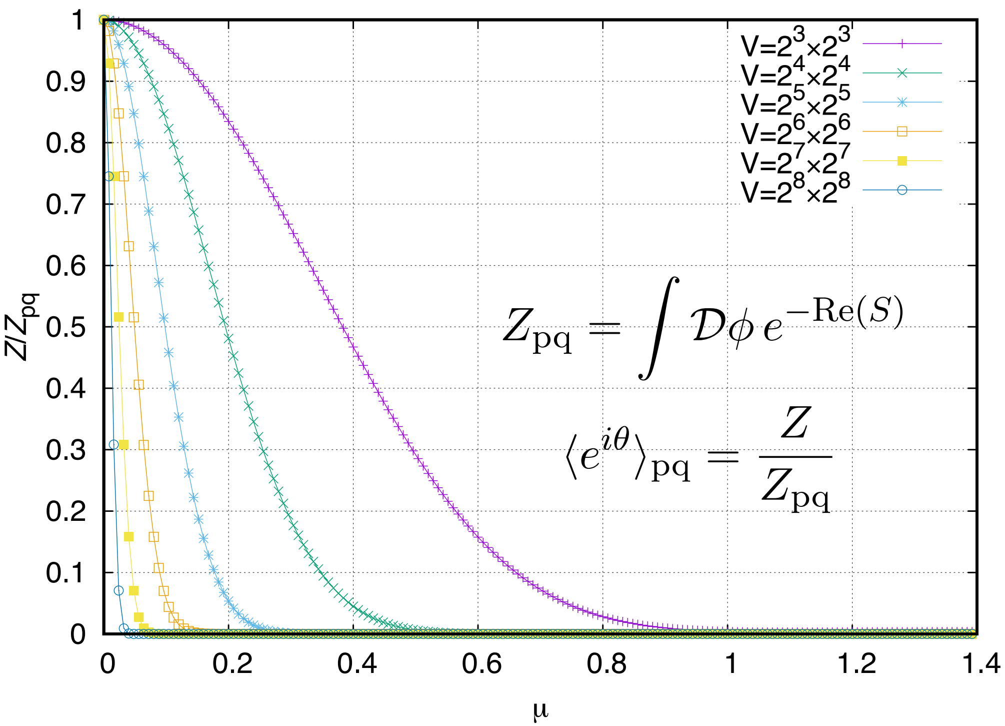
Results for Z(original) with TRG

Kadoh+, JHEP02(2020)161

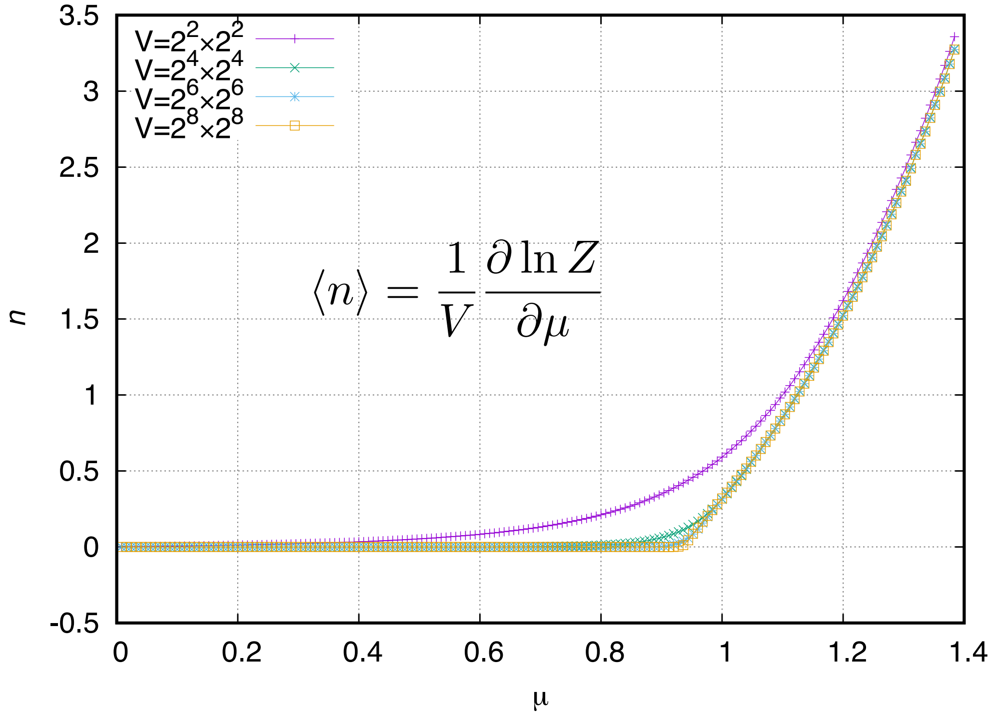
Parameters: m^2=0.01, λ=1, K=64, D_{cut}=64

V=L × L is changed from 4 × 4 to 256 × 256

Average phase factor



Number density



Sign problem: $<e^{i\theta}>_{pq} \ll 1$

Silver Blaze feature

Bose condensation@μ≳0.95

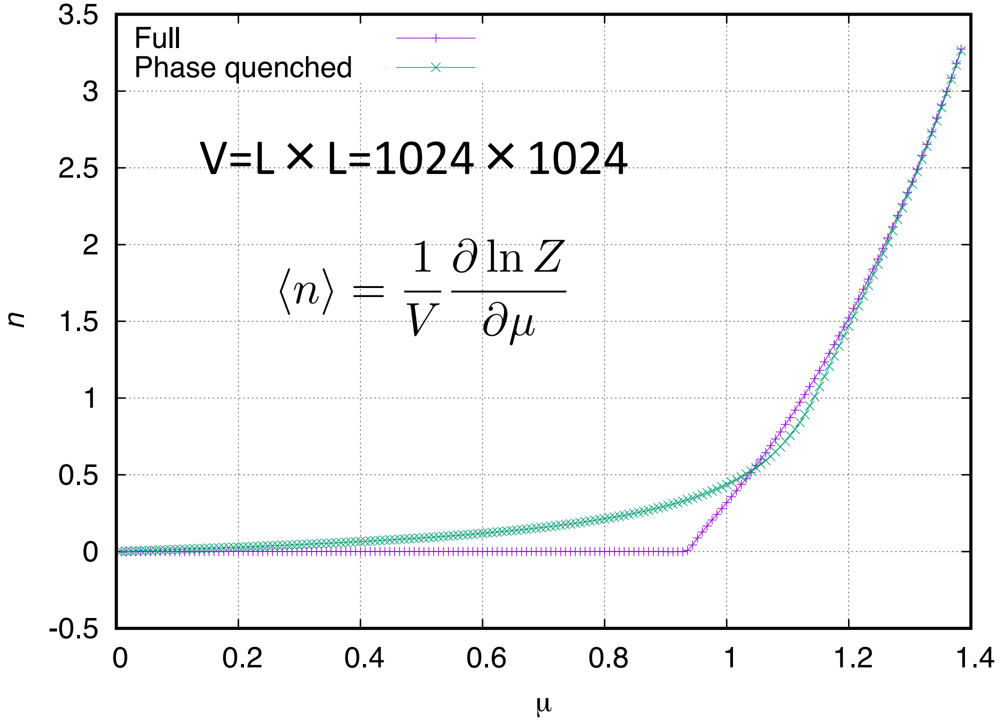
Effects of Phase

Kadoh+, JHEP02(2020)161

Parameters: m^2=0.01, λ=1, K=64, D_{cut}=64

$$Z_{\rm pq} = \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \exp(-\mathrm{Re}(S))$$

<n> w/ and w/o phase



No Silver Blaze feature in phase quenched case

⟹ Clear phase effect

Sign-Problem-Free Representation

Endres, PRD75(2007)065012

Mathematical tools

- Polar coordinate

$$\phi_n = (\phi_{n,1}, \phi_{n,2}) \to (r_n \cos\theta_n, r_n \sin\theta_n)$$

- Character expansion

$$\exp(x\cos z) = \sum_{k=-\infty}^{\infty} \mathrm{I}_k(x)\exp(ikz) \quad x \in R,\ z \in C$$

Partition function can be expressed in a sign-problem-free form

$$Z(\text{positive})$$
$$= \left(\prod_n \sum_{k_{n,1},k_{n,2}=-\infty}^{\infty}\right)\left(\prod_n \int_0^\infty \mathrm{d}r_n\right)\prod_n 2\pi r_n \prod_{\rho=1}^{2} e^{-\frac{1}{4}(4+m^2)(r_n^2+r_{n+\hat{\rho}}^2)-\frac{\lambda}{4}(r_n^4+r_{n+\hat{\rho}}^4)}$$
$$\cdot \mathrm{I}_{k_{n,\rho}}(2r_n r_{n+\hat{\rho}})e^{k_{n,\rho}\mu\delta_{\rho,2}}\delta_{(k_{n,1}+k_{n,2}-k_{n-\hat{1},1}-k_{n-\hat{2},2}),0}$$

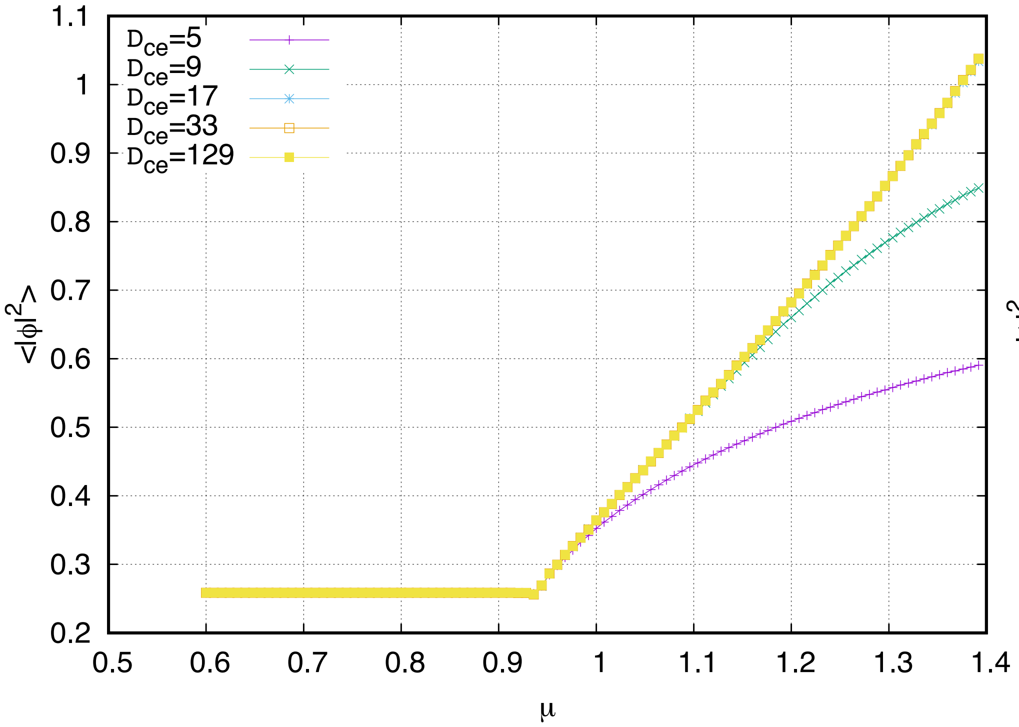
Apply TRG to Z(positive)

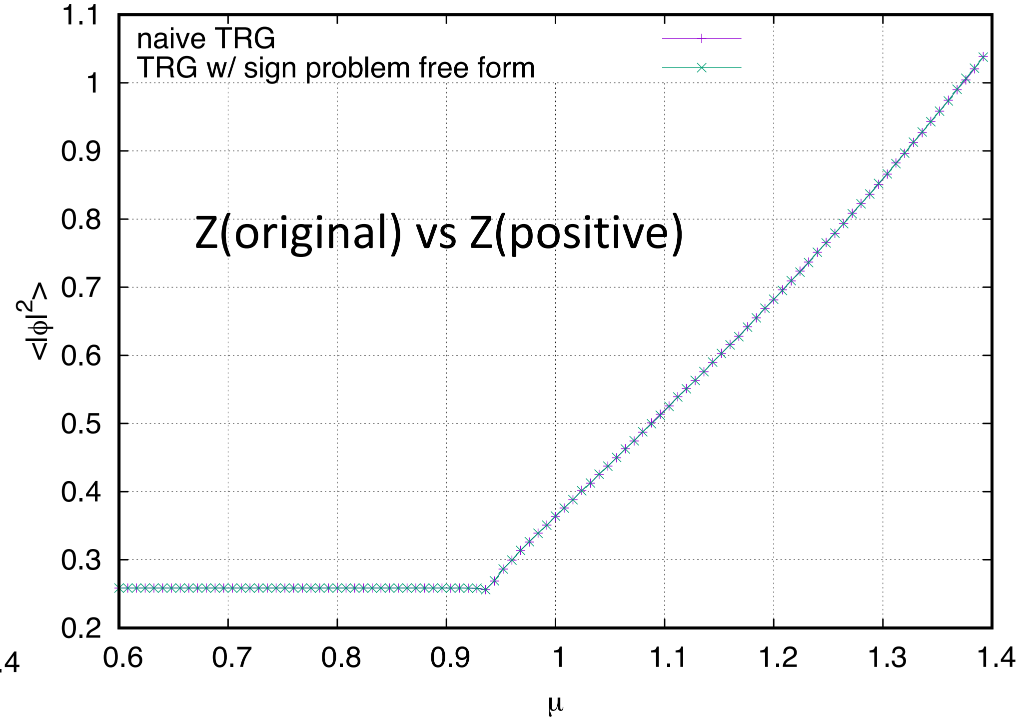
Consistency check btw the results for Z(original) and Z(positive)

Comparison btw Z(original) and Z(positive)

Kadoh+, JHEP02(2020)161

Parameters: m^2=0.01, V=1024x1024, λ=1, K=256, D_{cut}=64





Good convergence
in character expansion

Good agreement (degenerate)
⇒ Free from sign problem

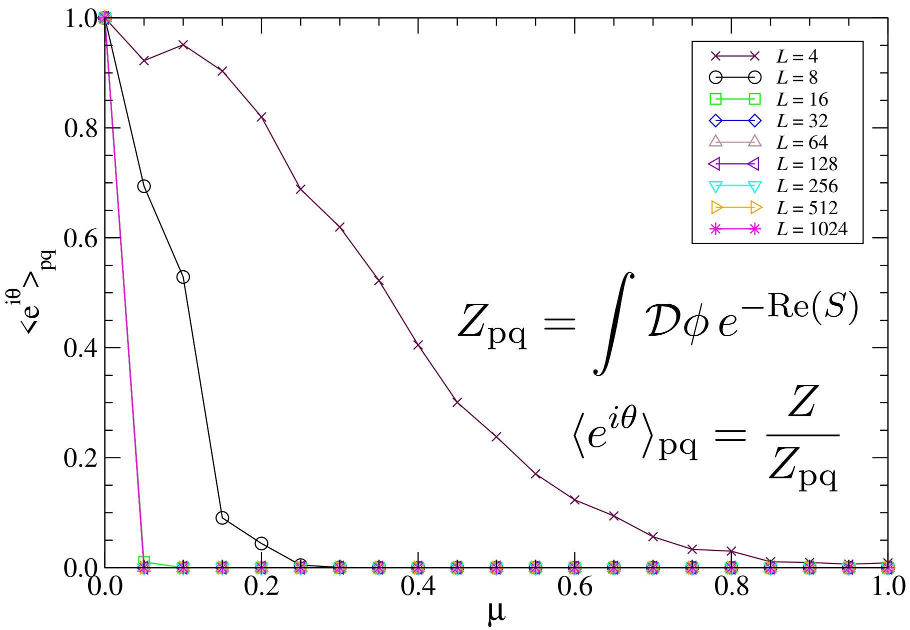
Extension to 4D

Akiyama+, JHEP09(2020)177

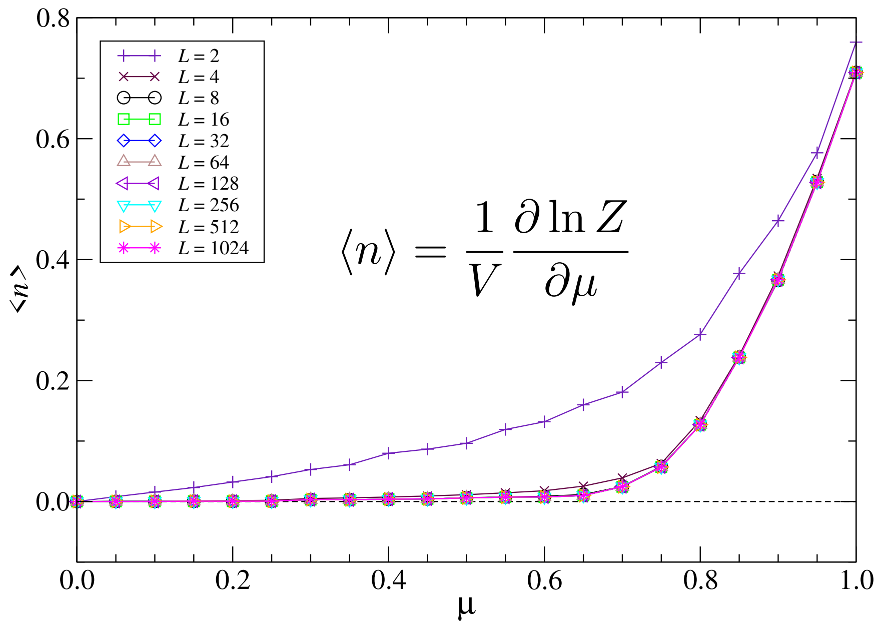
Parameters: m^2=0.01, λ=1, K=64, D_{cut}=45

V=L^4 is changed from 2^4 to 1024^4

Average phase factor



Number density



Sign problem: $\langle e^{i\theta}\rangle \ll 1$

Silver Blaze feature

Bose condensation@μ≳0.65

(μ_c ≈0.7 by mean field theory)

NJL Model at Finite Density

Akiyama+, JHEP01(2021)121

NJL model at zero density in the continuum

$$\mathcal{L} = \bar{\psi}(x)\gamma_\nu \partial_\nu \psi(x) - g_0 \left\{ \left(\bar{\psi}(x)\psi(x)\right)^2 + \left(\bar{\psi}(x)\mathrm{i}\gamma_5\psi(x)\right)^2 \right\}$$

NJL model at finite density on the lattice w/ Kogut-Susskind fermion

$$S = \frac{1}{2}a^3 \sum_{n\in\Lambda} \sum_{\nu=1}^{4} \eta_\nu(n) \left[\mathrm{e}^{\mu a \delta_{\nu,4}} \bar{\chi}(n)\chi(n+\hat{\nu}) - \mathrm{e}^{-\mu a \delta_{\nu,4}} \bar{\chi}(n+\hat{\nu})\chi(n) \right]$$
$$+ m a^4 \sum_{n\in\Lambda} \bar{\chi}(n)\chi(n) - g_0 a^4 \sum_{n\in\Lambda} \sum_{\nu=1}^{4} \bar{\chi}(n)\chi(n)\bar{\chi}(n+\hat{\nu})\chi(n+\hat{\nu})$$

μ : chemical potential

m : fermion mass

g_0 : coupling constant of four-fermi interaction

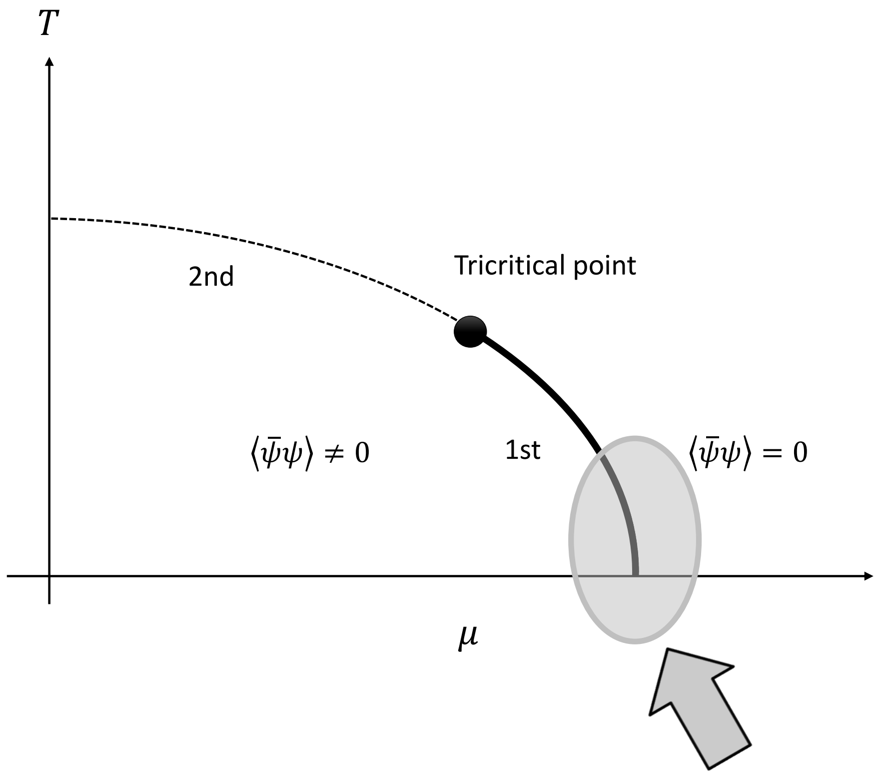
a : lattice spacing

Phase Diagram of NJL Model at Finite Density

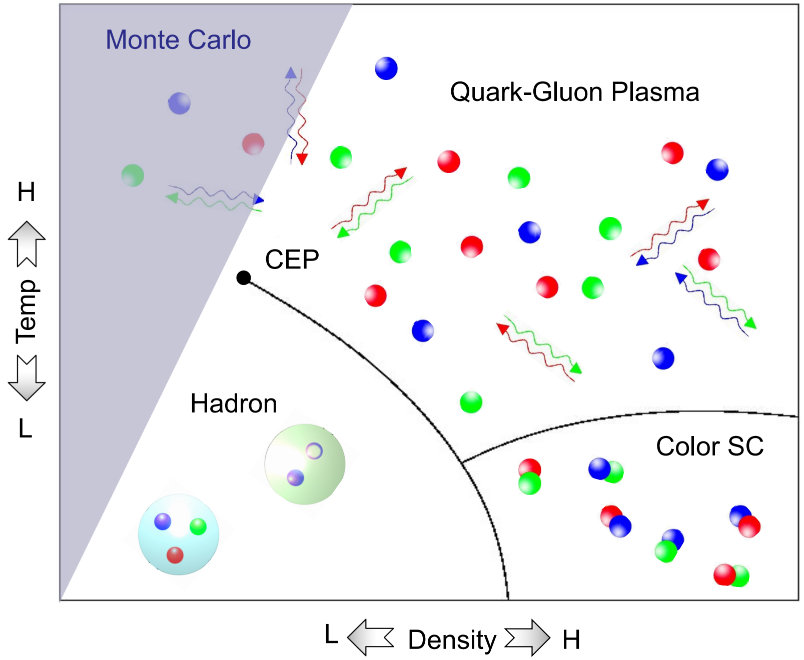
Akiyama+, JHEP01(2021)121

NJL model as a prototype of QCD

Expected phase diagram for NJL model



Expected phase diagram for QCD



Check the first-order phase transition at cold and dense region

Algorithm and Parameters

Akiyama+, JHEP01(2021)121

TN representation

$$\mathcal{T}_{n;i_4(n)i_1(n)i_2(n)i_3(n)i_4(n-\hat{4})i_1(n-\hat{1})i_2(n-\hat{2})i_3(n-\hat{3})}$$

$$= \int \mathrm{d}\chi \mathrm{d}\bar{\chi}\ \mathrm{e}^{-m\bar{\chi}\chi} \prod_{\nu=1}^{4} \left(\frac{\mathrm{e}^{\frac{\mu}{2}\delta_{\nu,4}}}{\sqrt{2}} \eta_\nu(n)\bar{\chi}\mathrm{d}\Phi_\nu(n) \right)^{i_{\nu,1}(n)} \left(\frac{\mathrm{e}^{\frac{\mu}{2}\delta_{\nu,4}}}{\sqrt{2}} \chi \mathrm{d}\bar{\Phi}_\nu(n) \right)^{i_{\nu,1}(n-\hat{\nu})}$$

$$\times \left(\frac{\mathrm{e}^{-\frac{\mu}{2}\delta_{\nu,4}}}{\sqrt{2}} \eta_\nu(n)\chi\mathrm{d}\Psi_\nu(n) \right)^{i_{\nu,2}(n)} \left(\frac{\mathrm{e}^{-\frac{\mu}{2}\delta_{\nu,4}}}{\sqrt{2}} \bar{\chi}\mathrm{d}\bar{\Psi}_\nu(n) \right)^{i_{\nu,2}(n-\hat{\nu})} (\sqrt{g_0}\bar{\chi}\chi)^{i_{\nu,3}(n)}$$

$$\times (\sqrt{g_0}\bar{\chi}\chi)^{i_{\nu,3}(n-\hat{\nu})} \left(\bar{\Phi}_\nu(n+\hat{\nu})\Phi_\nu(n) \right)^{i_{\nu,1}(n)} \left(\bar{\Psi}_\nu(n+\hat{\nu})\Psi_\nu(n) \right)^{i_{\nu,2}(n)}.$$

$$Z = \sum_{\{t,x,y,z\}} \int \prod_{n\in\Lambda} \mathcal{T}_{n;txyzt'x'y'z'} \qquad (x = i_1, y = i_2, z = i_3, t = i_4)$$

Coarse-graining procedure: GATRG w/ D_{cut}=55

Parameters

$$V = L\times\beta = (aN_\sigma)\times(aN_\tau) = 2^4, \cdots, (1024)^4 \qquad \left(\beta = \frac{1}{T}\right)$$

Periodic BC for spatial direction and anti-periodic BC for temporal direction

a is fixed at finite value (a=1)

$g_0 = 32$ for coupling constant of four-fermi interaction

Heavy Dense Limit as a Benchmark

Akiyama+, JHEP01(2021)121

Heavy dense limit: $m, \mu \to \infty$ while e^{μ}/m kept fixed

Chiral condensate and number density in the heavy dense limit

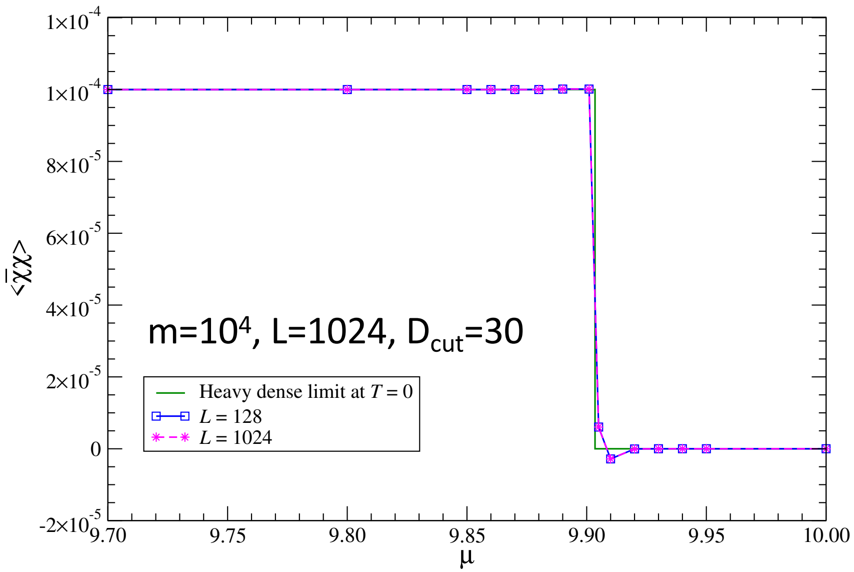
$$\langle \bar{\chi}(n)\chi(n) \rangle = \frac{1}{m}\Theta(\mu_c - \mu) \qquad \langle n \rangle = \Theta(\mu - \mu_c)$$

Pawlowski-Zielinski, PRD87(2013)094509

Analytical solutions are step function at $\mu_c = \ln(2m)$

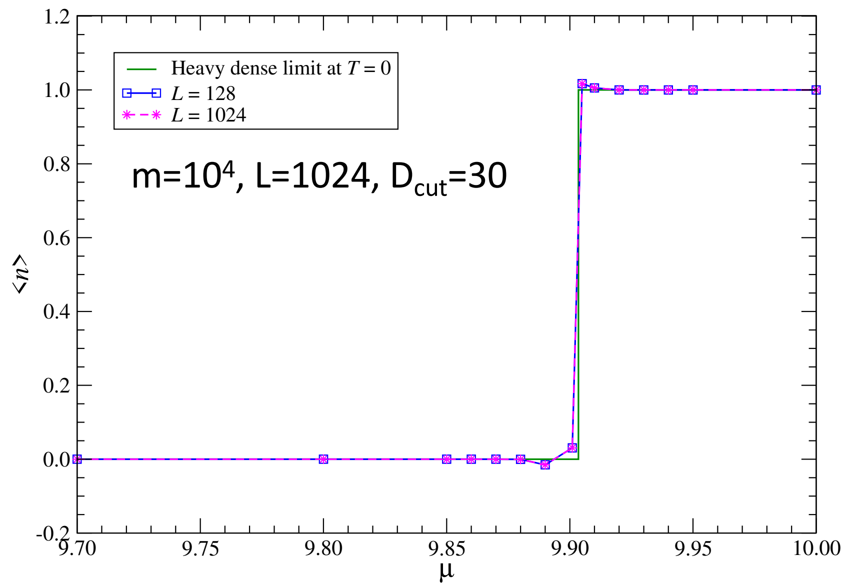
Chiral condensate

$$\langle \bar{\chi}(n)\chi(n) \rangle|_{m=10^4} = \frac{1}{V}\frac{\ln Z(m+\Delta m) - \ln Z(m)}{\Delta m}\bigg|_{m=10^4}$$



Number density

$$\langle n \rangle = \frac{1}{V}\frac{\partial \ln Z(\mu)}{\partial \mu} \approx \frac{1}{V}\frac{\ln Z(\mu+\Delta\mu) - \ln Z(\mu)}{\Delta \mu}$$



Good consistency with analytical solutions

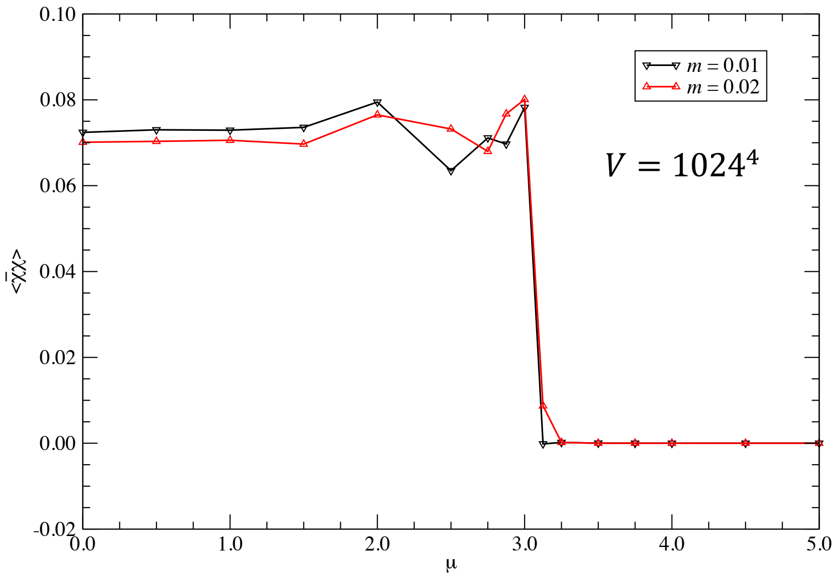
μ Dependence of Chiral Condensate

Akiyama+, JHEP01(2021)121

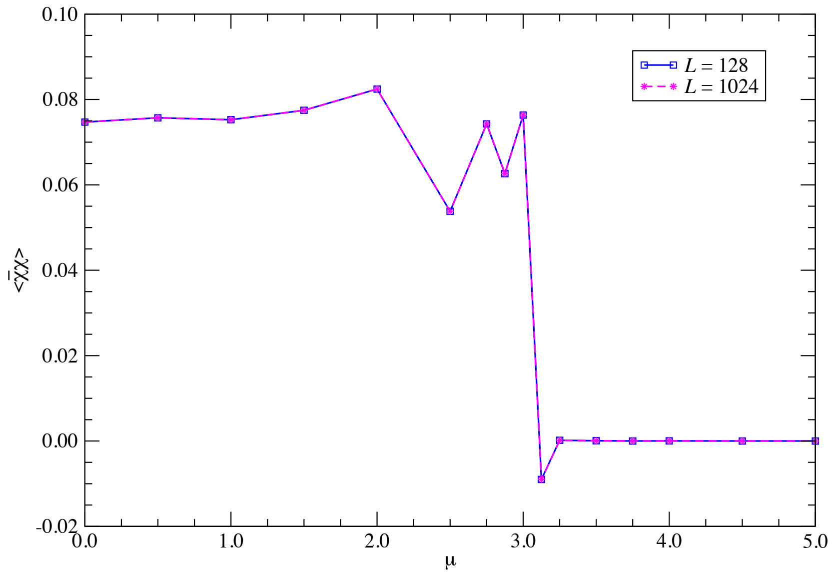
Order parameter of chiral phase transition

$$\langle \bar{\chi}(n)\chi(n) \rangle = \lim_{m\to 0} \lim_{V\to\infty} \frac{1}{V} \frac{\partial}{\partial m} \ln Z$$

Mass dependence@L=1024



Volume dependence@m=0



Jump around $\mu \approx 3.0 \Rightarrow$ First-order phase transition

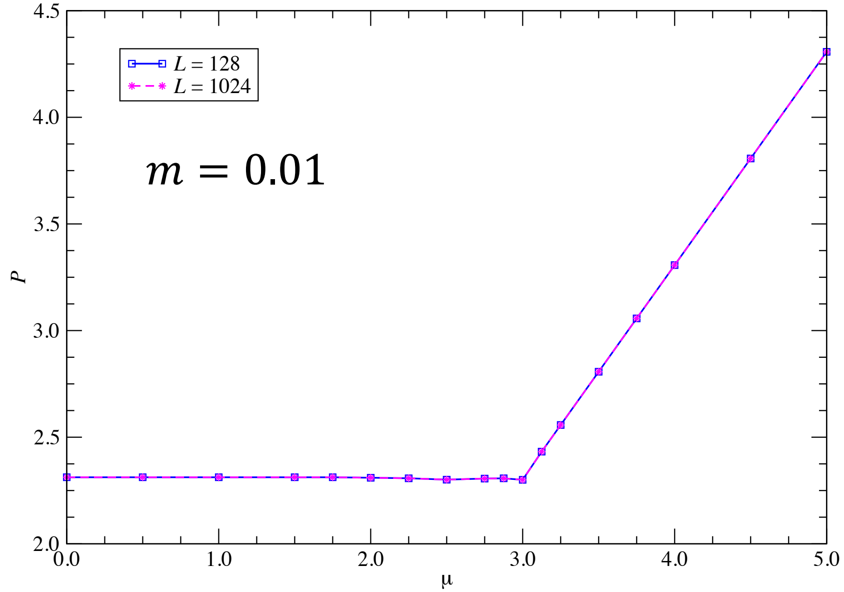
μ Dependence of Number Density

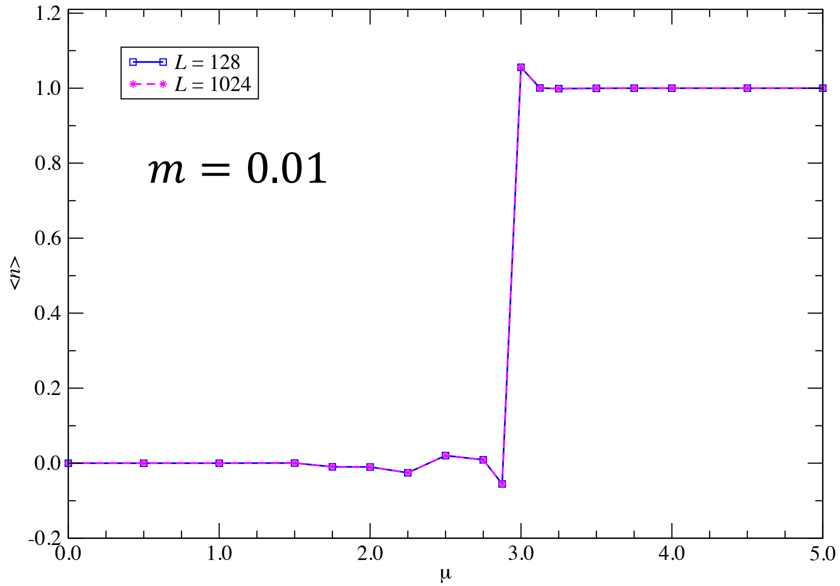
Akiyama+, JHEP01(2021)121

Pressure and number density (EOS)

$$P = \frac{\ln Z}{V}$$

$$\langle n(\mu) \rangle = \frac{\partial P(\mu)}{\partial \mu} \approx \frac{P(\mu + \Delta\mu) - P(\mu)}{\Delta\mu}$$





Jump around $\mu \approx 3.0$ ⇒ Another evidence of first-order phase transition

Summary

<u>What we have achieved so far</u>

- Studies of various 2D models

− Show that the TRG method is free from sign problems

− Development of algorithms for scalar, fermion, gauge theories

- Studies of 4D models

− Ising model

− Complex ϕ^4 theory at finite density

− NJL model at finite density

− Real ϕ^4 theory

<u>Current status</u>

- Research topics is moving from 2D models to 4D ones
- A new research direction: Hubbard model in condensed matter physics