

Chiral symmetry, the topological charge and the Yang–Mills gradient flow

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Based in part on work done with Peter Weisz

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Topological charge beyond the semi-classical level

Conceptually challenging, since

- Fields are typically nowhere continuous
- $\langle q(x)q(0) \rangle \underset{x \rightarrow 0}{\sim} |x|^{-8} \Rightarrow \langle Q^n \rangle$ not obviously well defined
- Field space in LQCD is connected

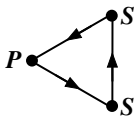
Often considered to be a “lattice problem” ...

Density-chain formulae

Consider 4d LQCD with exact chiral symmetry

Topological charge = chiral anomaly

$$Q = \frac{a}{2} \text{Tr}\{\gamma_5 D\} = m_1 m_2 m_3 \sum_{x,y,z} [P_{12}(x) S_{23}(y) S_{31}(z)]_{\text{Wick}}$$



Giusti, Rossi & Testa '04
ML '04

Ginsparg & Wilson '82
Kaplan '92, . . .

For example

Assuming 5 or more quark flavours

$$\langle Q^2 \rangle = m_1 \dots m_5 \times \left\langle \begin{array}{c} \bullet S \\ \nearrow \\ \bullet P \\ \searrow \\ \bullet S \end{array} \quad \begin{array}{c} \bullet P \\ \curvearrowright \\ \bullet S \end{array} \right\rangle$$

- Short-distance singularities are integrable
- Does not require renormalization

Well-defined universal formula for the topological susceptibility!

Flowed topological charge

ML '10

Weisz & ML '11

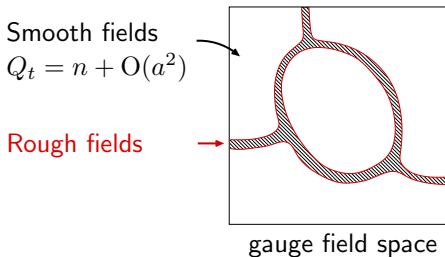
Hieda, Makino & Suzuki '17

At gradient-flow time $t > 0$

- No short-distance singularities
- The topological charge Q_t does not require renormalization
- In the continuum limit, $\partial_t \langle Q_t \dots \rangle = 0$

In particular, the moments $\langle Q_t^n \rangle$ are well defined and independent of t

Moreover, as $a \rightarrow 0$ the gauge field becomes “smooth” with probability $1 - O(a^2)$



⇒ Topological sectors emerge dynamically!

Gradient flow vs density chains

Does the equality

$$\langle Q_t^2 \rangle = m_1 \dots m_5 \times \left\langle \begin{array}{c} \bullet S \\ \nearrow \\ \bullet P \\ \searrow \\ \bullet S \end{array} \quad \begin{array}{c} \bullet P \quad \bullet S \\ \curvearrowright \\ \bullet P \quad \bullet S \end{array} \right\rangle$$

hold beyond the semi-classical level?

Topology of field space \leftrightarrow chiral anomaly

Cè, Consonni, Engel & Giusti '15 (YM theory)

ML '21 (full QCD)

Small flow-time expansion

As $t \rightarrow 0$

$$q_t(x) \sim c_1(t)\phi_1(x) + c_2(t)\phi_2(x) + \mathcal{O}(t)$$

where

$$\phi_1(x) = q(x) + X_A \partial_\mu A_\mu(x)$$

$$\phi_2(x) = Z_A \partial_\mu A_\mu(x)$$

Moreover, using perturbation theory and the RG

$$\lim_{t \rightarrow 0} c_1(t) = 1, \quad \lim_{t \rightarrow 0} c_2(t) = 0$$

Hieda & Suzuki '16; ML & Weisz '21

As a consequence

$$\langle Q_t^2 \rangle = \langle Q_t Q_s \rangle = \langle Q_t Q \rangle = m_4 m_5 \times \langle Q_t \text{P} \circlearrowright \text{S} \rangle$$

The correlation function

$$\langle Q_t \text{P} \circlearrowright \text{S} \rangle = \int_{x,y,z} \langle q_t(x) P_{12}(y) S_{21}(z) \rangle$$

however develops additional short-distance singularities as $t \rightarrow 0$

Must control these to be able to establish the identity

$$\langle Q_t^2 \rangle = m_1 \dots m_5 \times \left\langle \begin{array}{c} \text{S} \\ \nearrow \\ \text{P} \\ \searrow \\ \text{S} \end{array} \text{P} \circlearrowright \text{S} \right\rangle$$

Generating function for density chains

Consider a complex mass matrix M

$$S_F = \int_x \bar{\psi}(x) \{ \mathcal{D} + MP_- + M^\dagger P_+ \} \psi(x), \quad P_\pm = \frac{1}{2}(1 \pm \gamma_5)$$

$$\partial_{rs}^S S_F = \int_x S_{rs}(x), \quad \partial_{rs}^P S_F = \int_x P_{rs}(x)$$

⇒ The free energy

$$F(M) = -\ln\{Z(M)\}$$

generates density chains, e.g.

$$\partial_{12}^P \partial_{23}^S \partial_{31}^S F(M) = \int_{x,y,z} \langle P_{12}(x) S_{23}(y) S_{31}(z) \rangle_c$$

Exact flavour symmetries

For any anti-Hermitian flavour matrix λ

$$\delta_\lambda^V M = [\lambda, M] \quad \Rightarrow \quad \delta_\lambda^V F(M) = 0$$

$$\delta_\lambda^A M = \{\lambda, M\} \quad \Rightarrow \quad \delta_\lambda^A F(M) = -2 \operatorname{tr}\{\lambda\} \langle Q \rangle$$

Renormalization ($N_f \geq 5$)

$F(M)$ requires additive renormalization

$$\Delta F(M) = V \left[\frac{z_0}{a^4} + \frac{z_1}{a^2} \operatorname{tr}\{M^\dagger M\} + z_2 \operatorname{tr}\{M^\dagger M\}^2 + z_3 \operatorname{tr}\{(M^\dagger M)^2\} \right]$$

$\Rightarrow F + \Delta F$ has the same symmetry properties as F

Final steps

Free energy \leftrightarrow density-chain formula at $a > 0$

$$[\delta_\lambda^A \delta_\eta^A \{F(M) + \Delta F(M)\}]_{\text{diagonal } M} = 4 \text{tr}\{\lambda\} \text{tr}\{\eta\} \langle Q^2 \rangle$$

$$= 4 \text{tr}\{\lambda\} \text{tr}\{\eta\} m_1 \dots m_5 \times \left\langle \begin{array}{c} \bullet S \\ \nearrow \\ P \bullet \\ \searrow \\ \bullet S \end{array} \quad \begin{array}{c} \bullet P \\ \rightarrow \\ \bullet S \\ \leftarrow \\ \bullet P \end{array} \right\rangle$$

Final steps

Free energy \leftrightarrow density-chain formula at ~~$a \neq 0$~~ $a = 0$

$$[\delta_\lambda^A \delta_\eta^A \{F(M) + \Delta F(M)\}]_{\text{diagonal } M} \equiv \cancel{4 \text{tr}\{\lambda\} \text{tr}\{\eta\} \langle Q^2 \rangle}$$

$$= 4 \text{tr}\{\lambda\} \text{tr}\{\eta\} m_1 \dots m_5 \times \left\langle \begin{array}{c} \bullet S \\ \nearrow \\ P \bullet \\ \searrow \\ \bullet S \end{array} \quad \begin{array}{c} \bullet P \\ \rightarrow \\ \bullet S \\ \rightarrow \\ \bullet P \end{array} \right\rangle$$

Conclusions

The equality of the density-chain and gradient-flow formulae

- ★ *Relates the chiral anomaly to the topology of field space at the fully non-perturbative level*
- ★ *Provides the definitive justification for using the flow-formula for the topological susceptibility*

Would be difficult to show w/o formulations of LQCD preserving chiral symmetry!