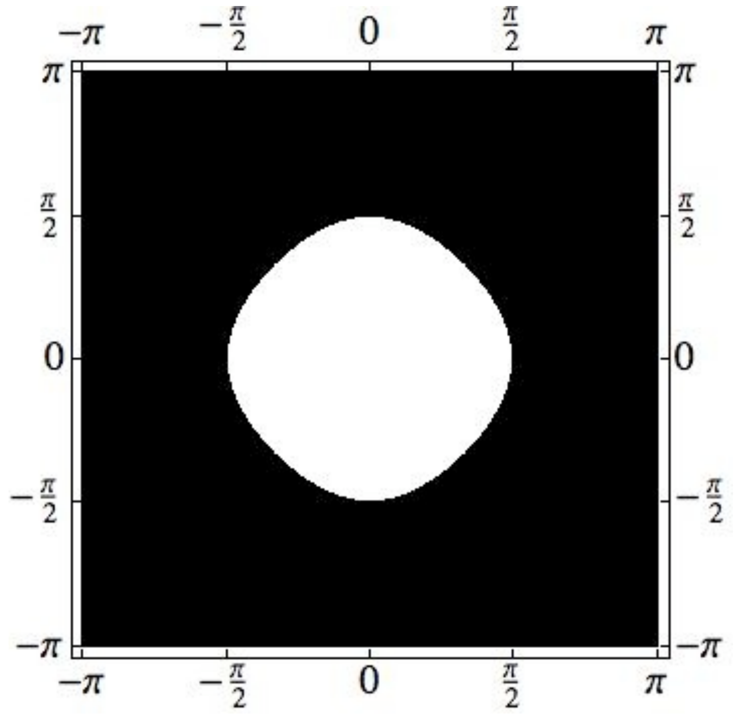
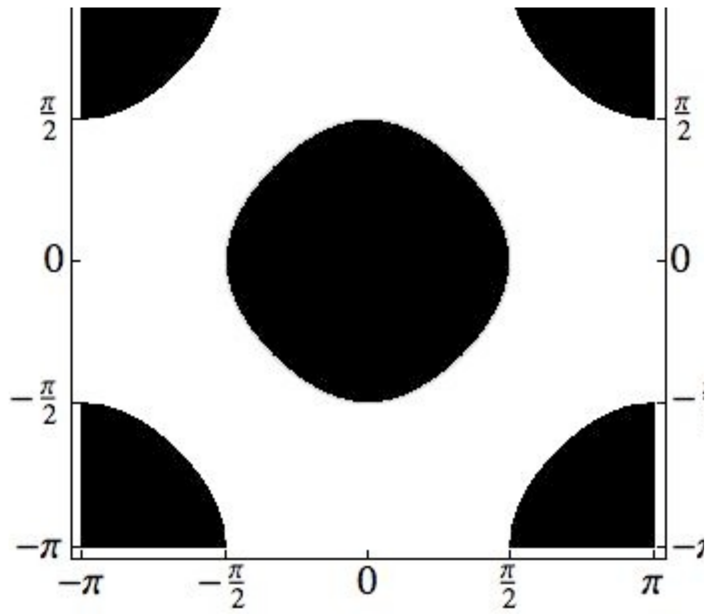
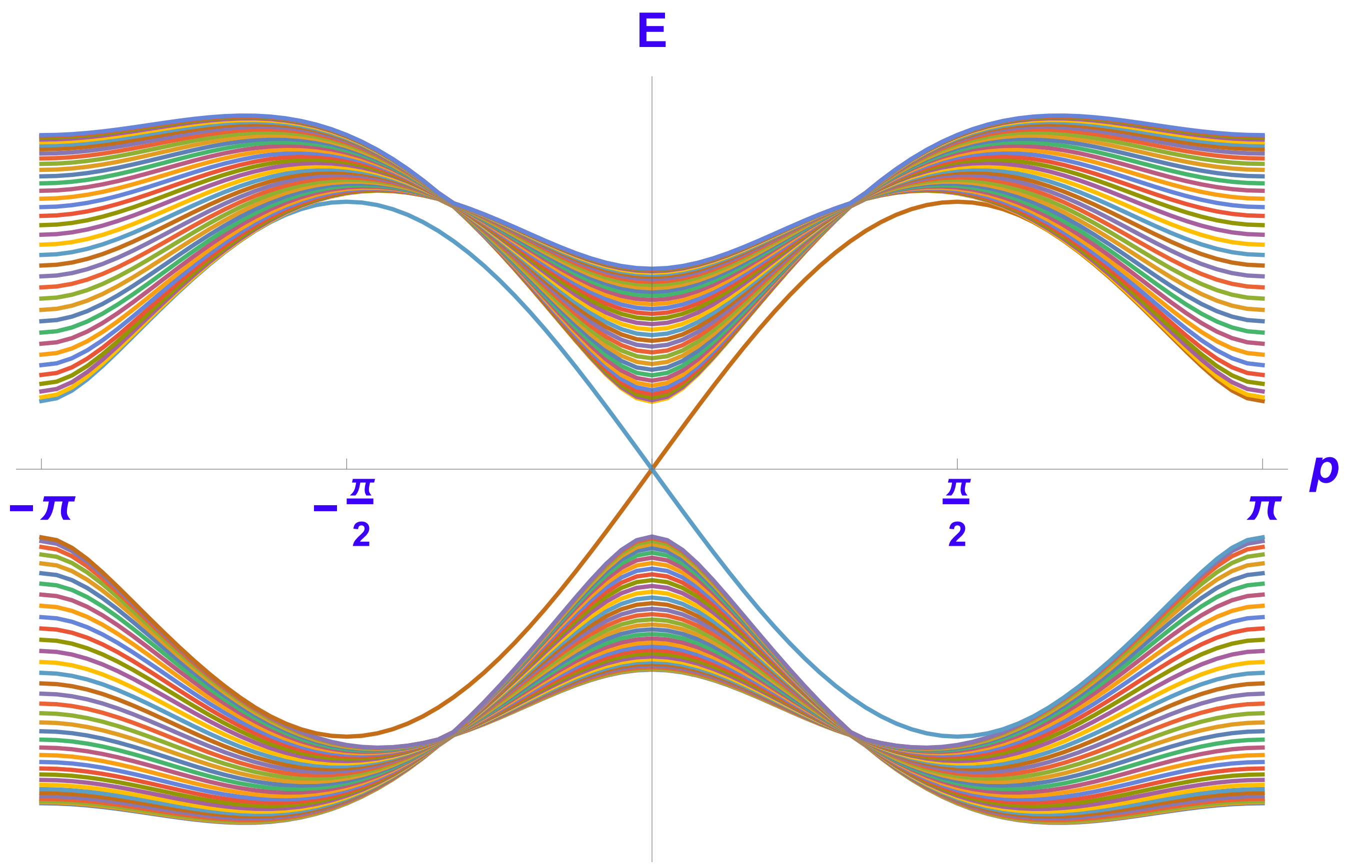
Index Theorems, Generalized Hall Currents, and Topology for Domain Wall Fermions

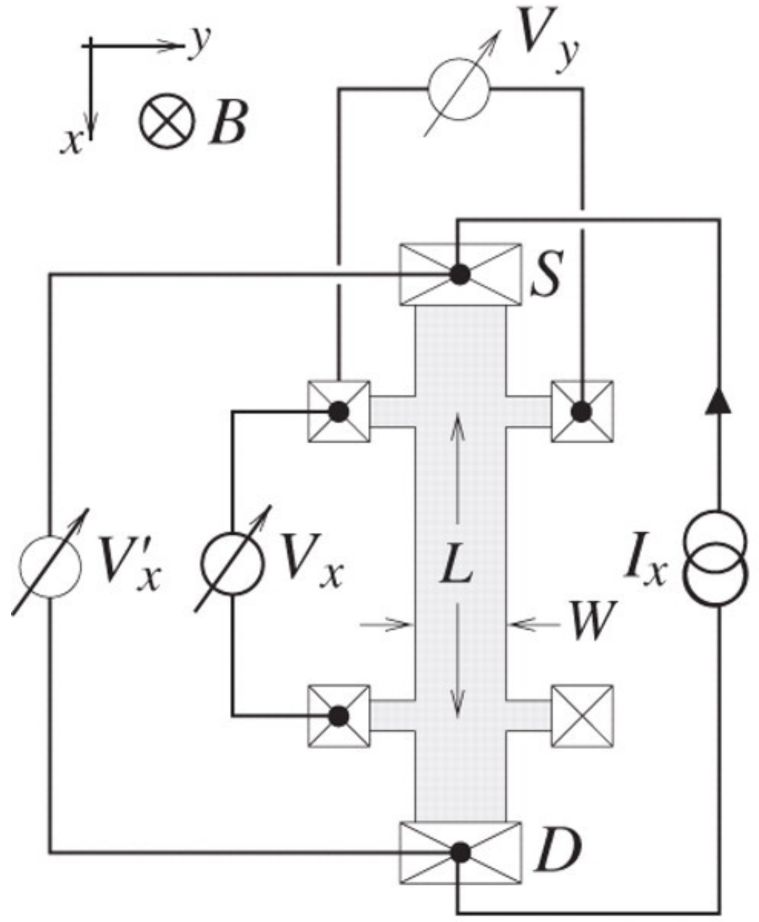
in collaboration with: Srimoyee Sen, Iowa State University

arXiv on Wednesday

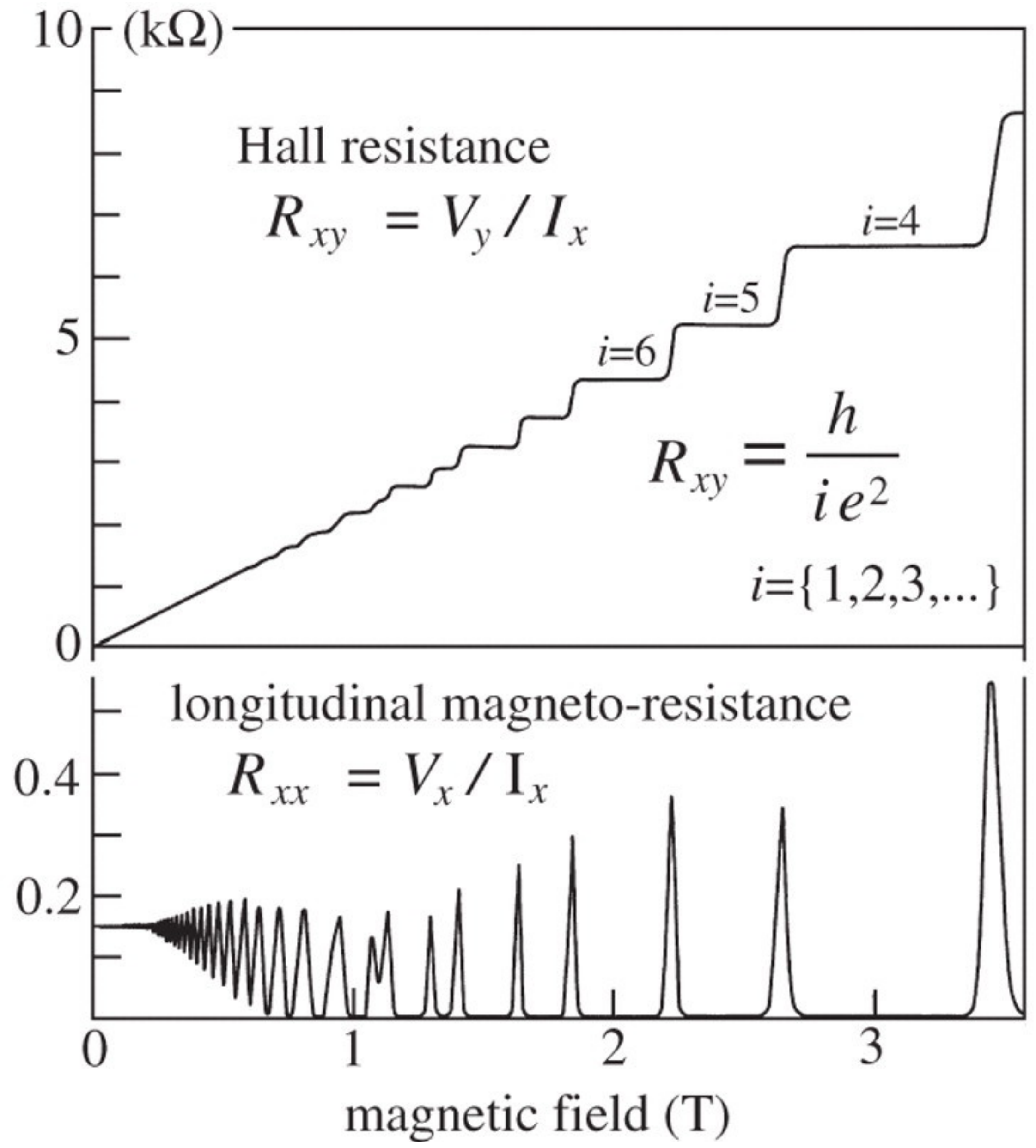




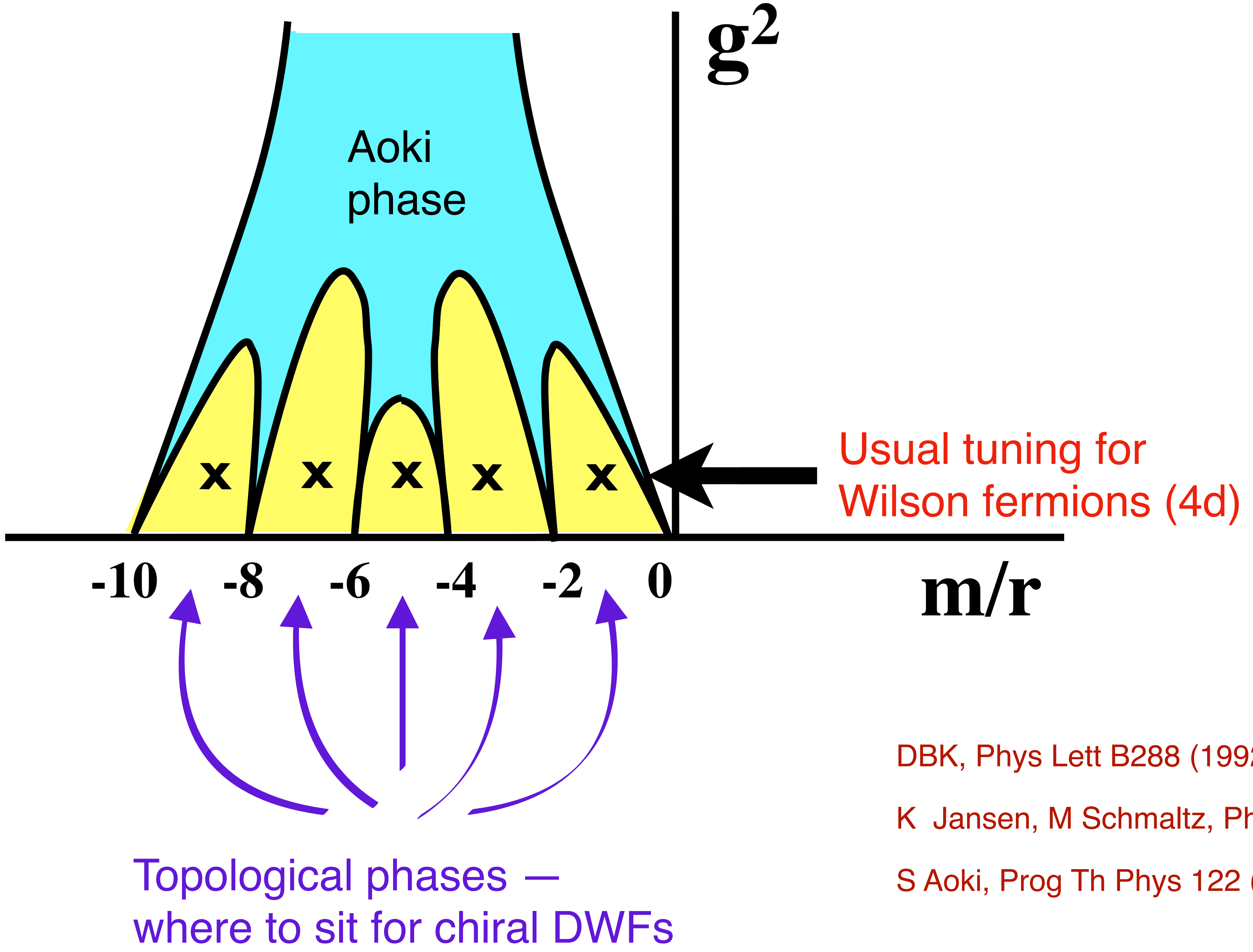




Credit: J. Chakhalian, https://www.physics.rutgers.edu/grad/601/CM601/



In principle, the phase diagram for Wilson fermions could have been found in the 70s:



DBK, Phys Lett B288 (1992) 342

K Jansen, M Schmaltz, Phys Lett B296 (1992) 374

S Aoki, Prog Th Phys 122 (1996) 179

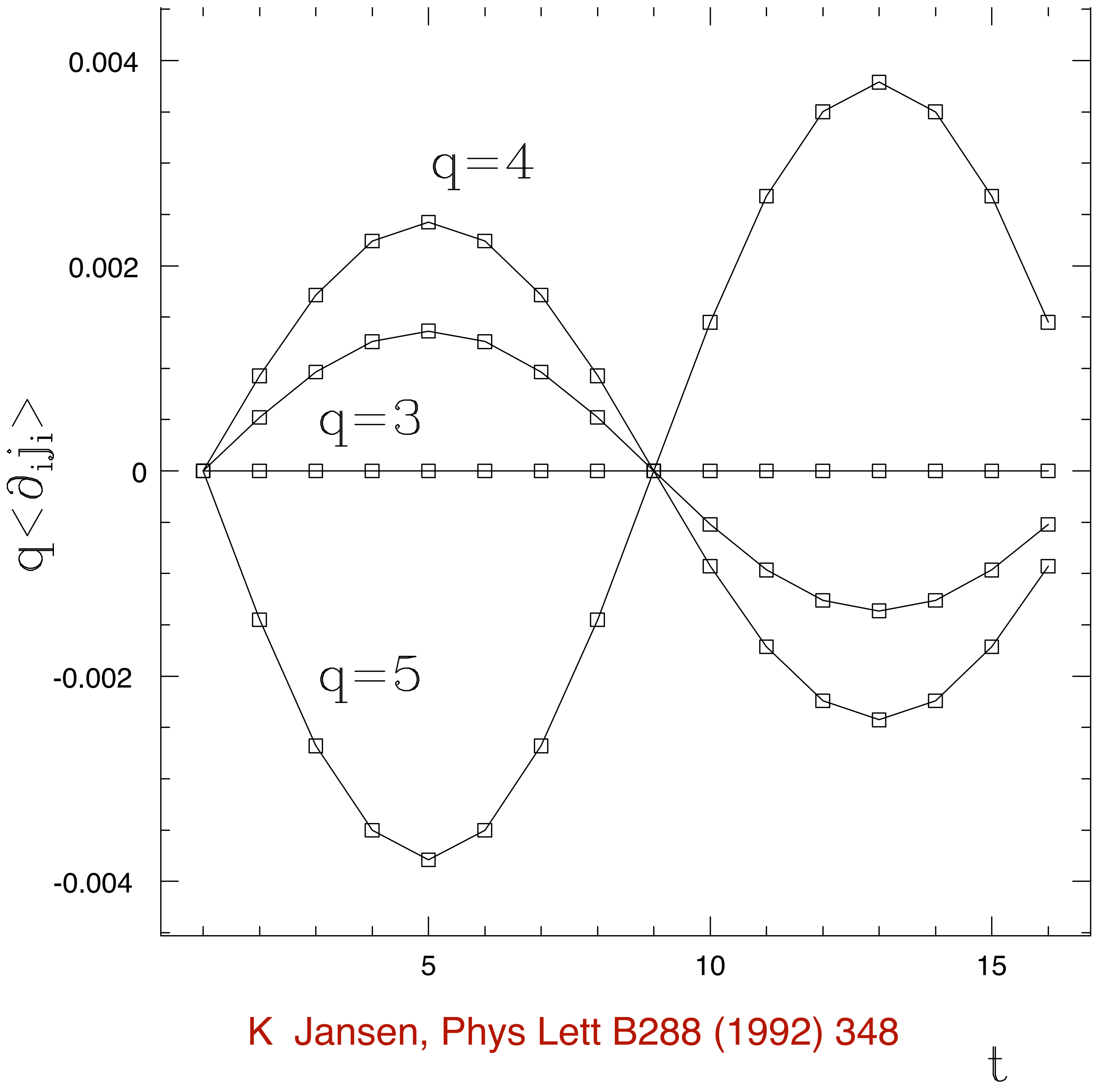
The topological phases are examples of the Integer Quantum Hall Effect w/o a magnetic field & Landau levels

Nobel prizes for von Klitzing, Thouless, Haldane — Kane & Mele

publication: 1980 — 1982 — 1988 — 2005

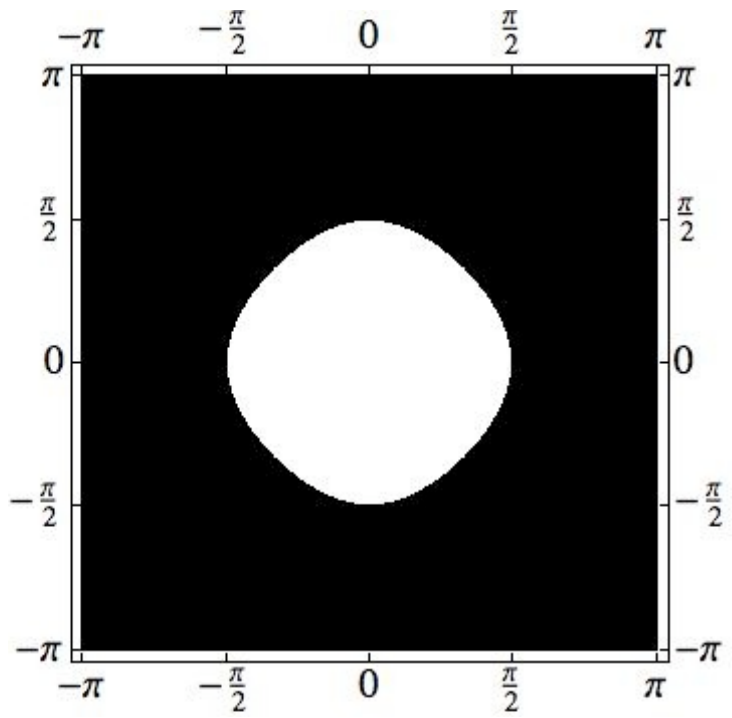
To recognize the topological phases one must look at:

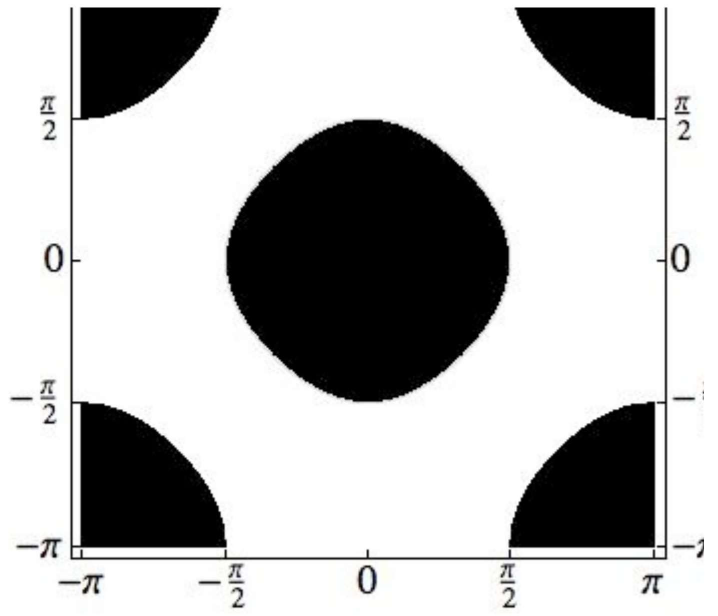
- Massless chiral edge states (DWF)
- Quantized currents in the bulk



K Jansen, Phys Lett B288 (1992) 348

m/r=1.2

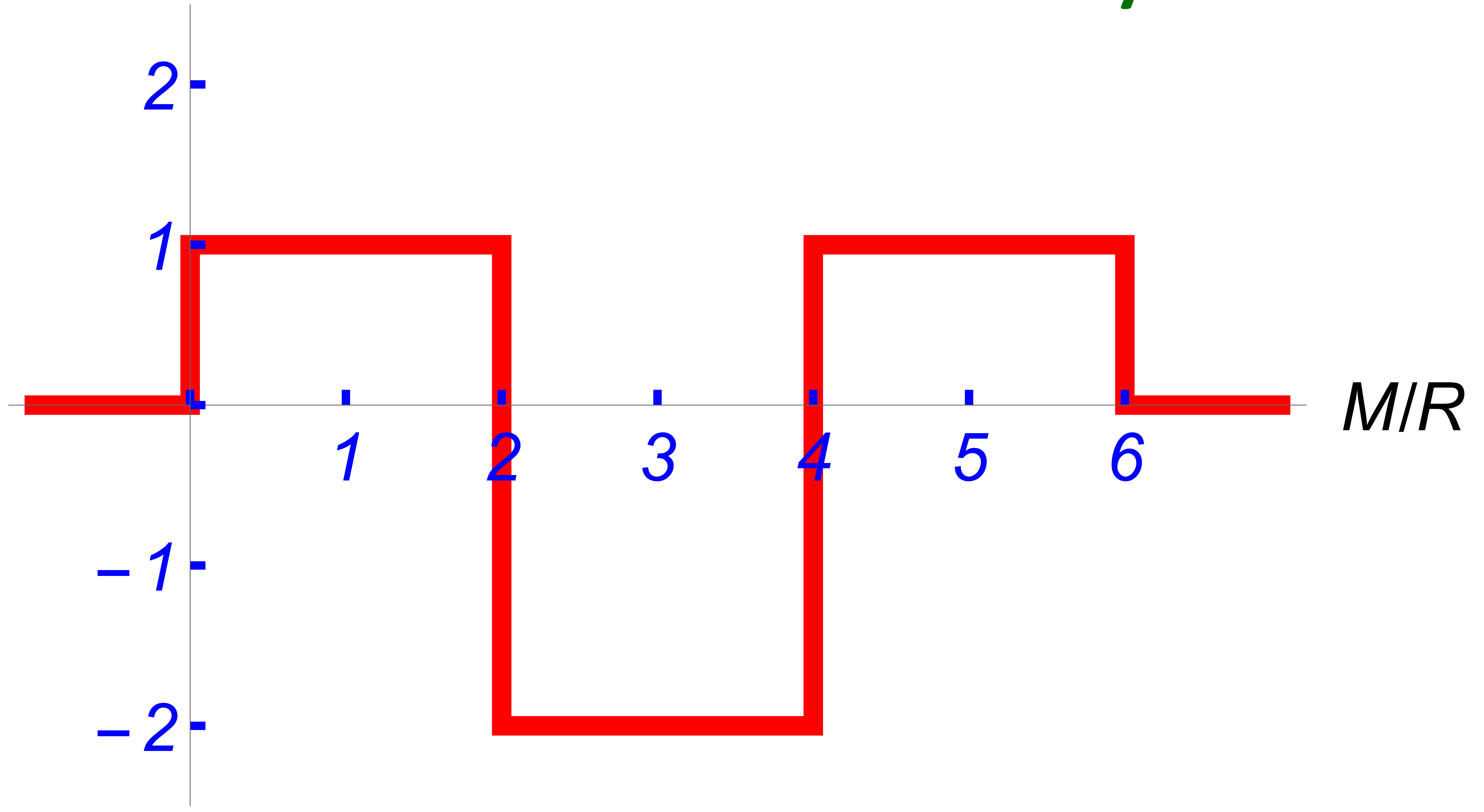




m/r=3.0

$$\frac{J}{\left(\frac{q^2 E}{2\pi}\right)} = \frac{\sigma_{xy}}{q{\wedge}2/h}$$

von Klitzing's quantum of conductivity



current in d=3

K Jansen, M Schmaltz, Phys Lett B296 (1992) 374

The role of topology in the Integer Quantum Hall Effect was recognized in TKNN

Thouless, Kohomoto Nightingale, den Nijs, 1982

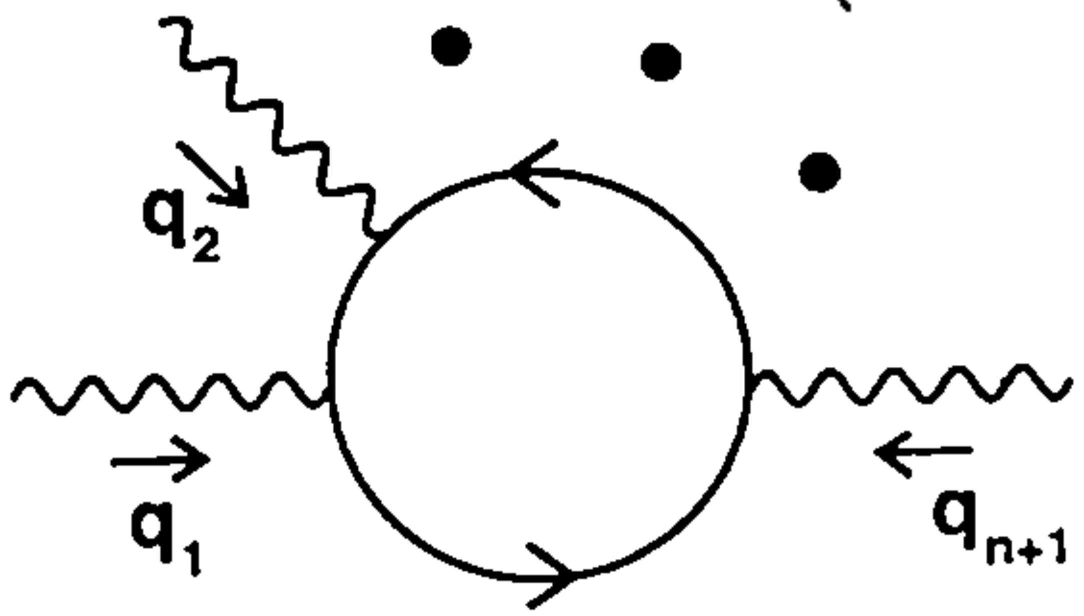
The QFT analog: quantization of the Chern-Simons coefficient

(H. So, 1985) Jansen, Golterman, DBK, 1992

- Integrating out the massive bulk fermions gives a Chern-Simons operator with quantized coefficient

- The fermion propagator is a map from momentum space (T^d) to "Dirac space" (S^d), and the 1-loop Feynman diagram computes the winding number

$$c_n = \frac{(-i)^n \epsilon_{\mu_1 \ \mu_{2n+1}}}{(n+1)(2n+1)!} \int \frac{d^{2n+1}p}{(2\pi)^{2n+1}} \mathrm{Tr}\{[S(p)\,\partial_{\mu_1} S(p)^{-1}] \ldots [S(p)\,\partial_{\mu_{2n+1}} S(p)^{-1}]\}$$



d=2n+1

$S(p)$ = fermion propagator

CM theorists have identified a plethora of topological phases in various dimensions

Periodic table for topological insulators and superconductors

Alexei Kitaev

California Institute of Technology, Pasadena, CA 91125, U.S.A.

Abstract. Gapped phases of noninteracting fermions, with and without charge conservation and time-reversal symmetry, are classified using Bott periodicity. The symmetry and spatial dimension determines a general universality class, which

Symmetry Class	Classifying Space	π_0 of Classifying Space
A	$\bigcup_n U(N)/(U(N-n)\times U(n))$	$\mathbb{Z}$
AIII	$U(N)$	0
AI	$\bigcup_n O(N)/(O(N-n)\times O(n))$	$\mathbb{Z}$
BDI	$O(N)$	$\mathbb{Z}_2$
D	$O(2N)/U(N)$	$\mathbb{Z}_2$
DIII	$U(N)/Sp(N)$	0
AII	$\bigcup_n Sp(N)/(Sp(N-n)\times Sp(n))$	$\mathbb{Z}$
CII	$Sp(N)$	0
C	$Sp(2N)/U(N)$	0
CI	$U(N)/O(N)$	0

Symmetry Class	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

Complicated!

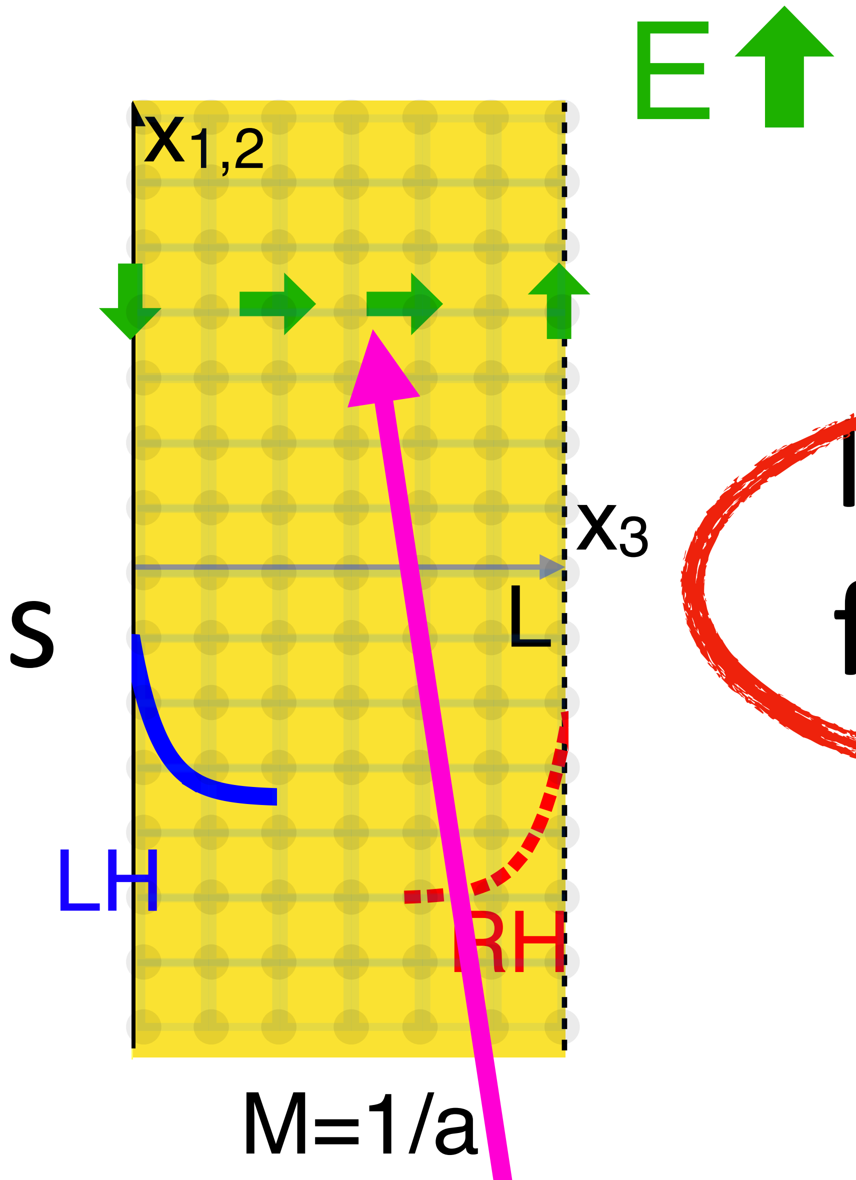
But what is simple:
massless DWF occur in all dimensions, with and without Chern-Simons currents, chiral anomalies… so there must be different sorts of topological phases…

Goal of this project:

- develop a "test" for topological phases to reveal whether or not it has massless DWF ⬅
- understand what happens when interactions are added
- perhaps apply to chiral gauge theories

Application to chiral gauge theories?

SM fermions



Mirror fermions

Can these be gapped w/o breaking gauge symmetry?

Eichten, Preskill (1986)

Anomaly cancellation
= only flavor Hall currents
= Quantum Spin Hall effect

The effects of interactions on the topological classification of free fermion systems

Lukasz Fidkowski and Alexei Kitaev
California Institute of Technology, Pasadena, CA 91125, U.S.A.

2009

In 1+1 d, yes. Can this be extended to higher dimensions?
Any more clues besides anomaly cancellation?

Proposal: compute the index of the Euclidian fermion operator for the system

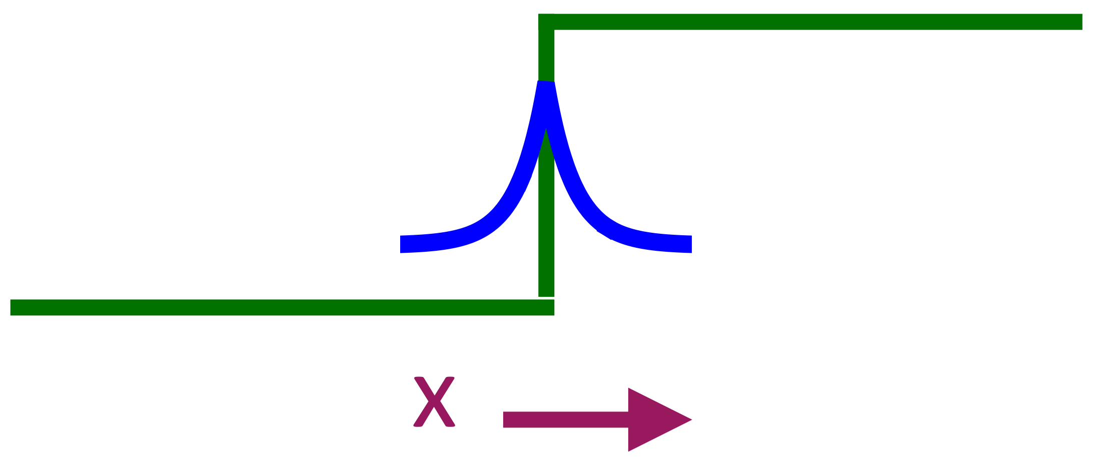
Suppose a Minkowski theory has a massless edge state, and the Euclidian theory is: $\mathcal{L}_E = \bar{\psi}\mathcal{D}\psi$

By adding background fields to localize the edge state in spacetime, *D* will have a nonzero index if the Minkowski theory has a massless edge state.

Example: Majorana fermion in 1+1 dimensions with a domain wall mass: this will have massless a Majorana state localized at the kink. (Kitaev's proposal for quantum computing)

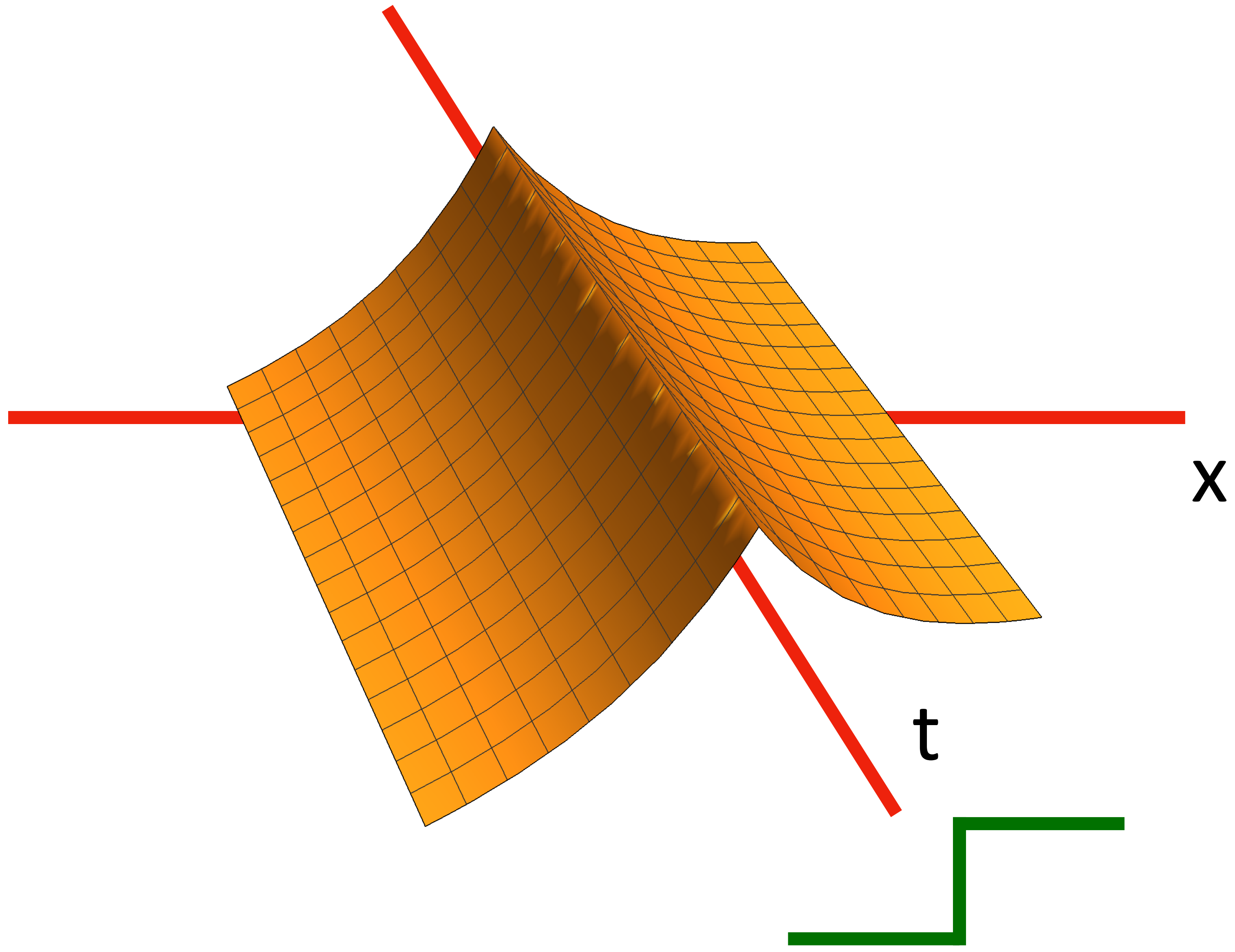
$$\mathcal{L}_M = \frac{1}{2}\psi^T C \left(i\partial\!\!\!/ - m\right)\psi$$

domain wall profile

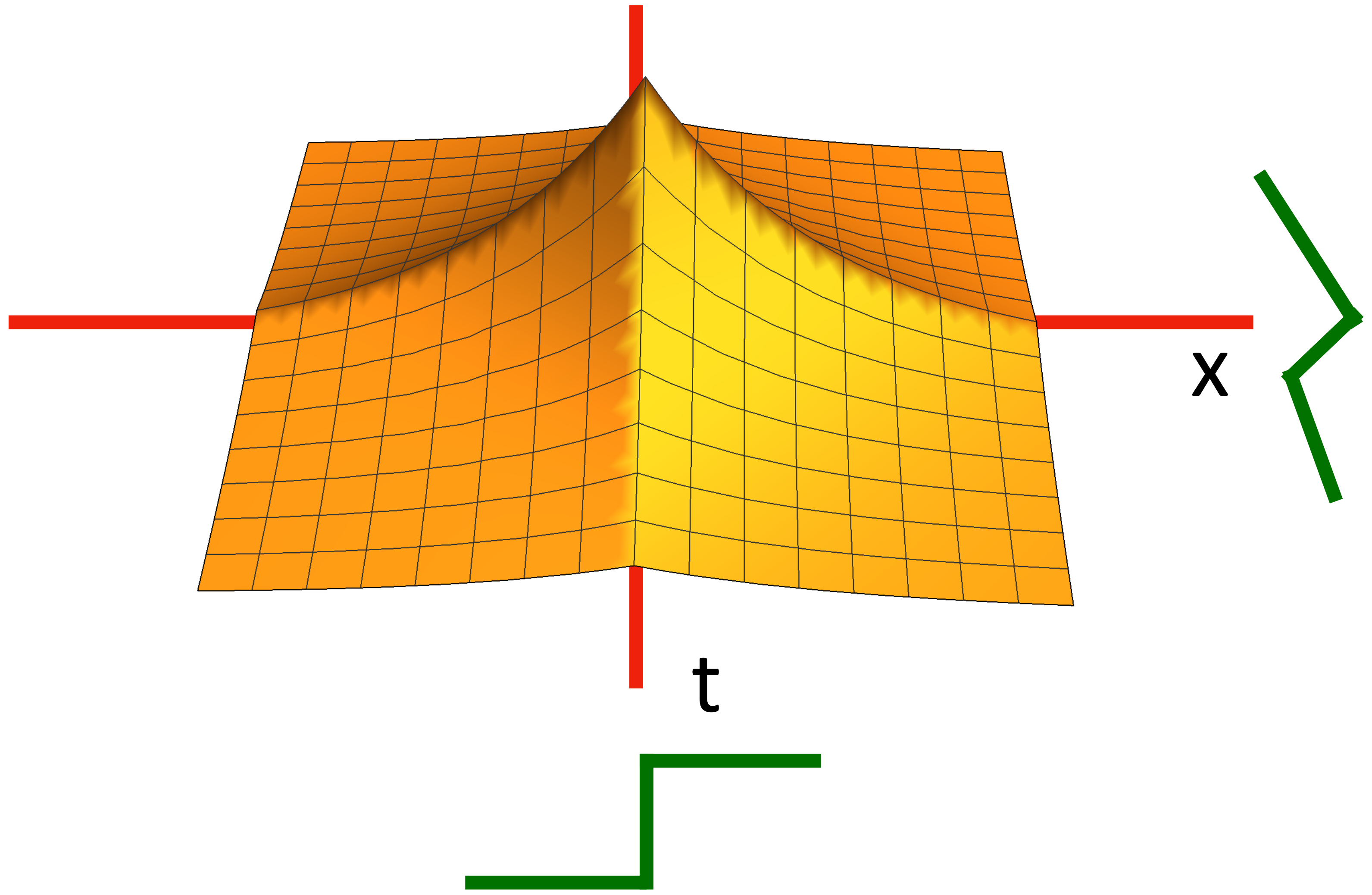


No currents, no anomaly

No localized mode in Euclidian spacetime…



But there is if you add a domain wall in Euclidian time direction:



Massless edge state of original theory will show up as localized zeromode of

$$\mathcal{D} = (\not{\partial} + \phi_1 + i\phi_2\gamma_\chi)$$

with proper choice for ϕ_1 , ϕ_2 …but not for $D^\dagger$ *… so index of D = 1.*

How to compute the (Callias) index?

$$\mathcal{I}(M) = \mathrm{Tr}\left[\left(\frac{M^2}{\mathcal{D}^\dagger\mathcal{D}+M^2} - \frac{M^2}{\mathcal{D}\mathcal{D}^\dagger+M^2}\right)\right]$$

index(D) = $\lim_{M\to 0}$ $\mathcal{I}$(M)

Can write as:

$$\mathcal{I}(M) = \mathrm{Tr}\left[\left(\frac{M^2}{\mathcal{D}^\dagger\mathcal{D}+M^2} - \frac{M^2}{\mathcal{D}\mathcal{D}^\dagger+M^2}\right)\right]$$

$$= -\mathrm{Tr}\,\Gamma_\chi \frac{M}{K+M}\ ,$$

where:

$$K = \begin{pmatrix} 0 & -\mathcal{D}^\dagger \\ \mathcal{D} & 0 \end{pmatrix},\ \Gamma_\chi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- K looks like a Dirac operator for a fermion with **twice** as many components as the original fermion
- Γ_χ looks like the γ_5 for that doubled theory

Recipe for computing index(D):

1. Construct $K = \begin{pmatrix} 0 & -\mathcal{D}^\dagger \\ \mathcal{D} & 0 \end{pmatrix}$

2. Consider the Euclidian QFT with doubled fermions: $S = \int d^{d+1}x\, \overline{\Psi}(K+M)\Psi$

3. $\text{index}(\mathcal{D}) = -\lim_{M\to 0} \text{Tr}\,\Gamma_\chi \frac{M}{K+M} = \lim_{M\to 0} M \int d^{d+1}x\, \langle \bar{\Psi}\Gamma_\chi \Psi \rangle$

4. This can be computed as the divergence of the axial current for this theory:

   index(D) = $\lim_{M\to 0}$ · $\frac{1}{2}\int d^{d+1}x\, \partial_\mu \langle \overline{\Psi}\Gamma_\mu \Gamma_\chi \Psi \rangle$

5. …which can be computed from a 1-loop Feynman diagram

Bac k to the example of 1+1 dimensional Majorana fermion:

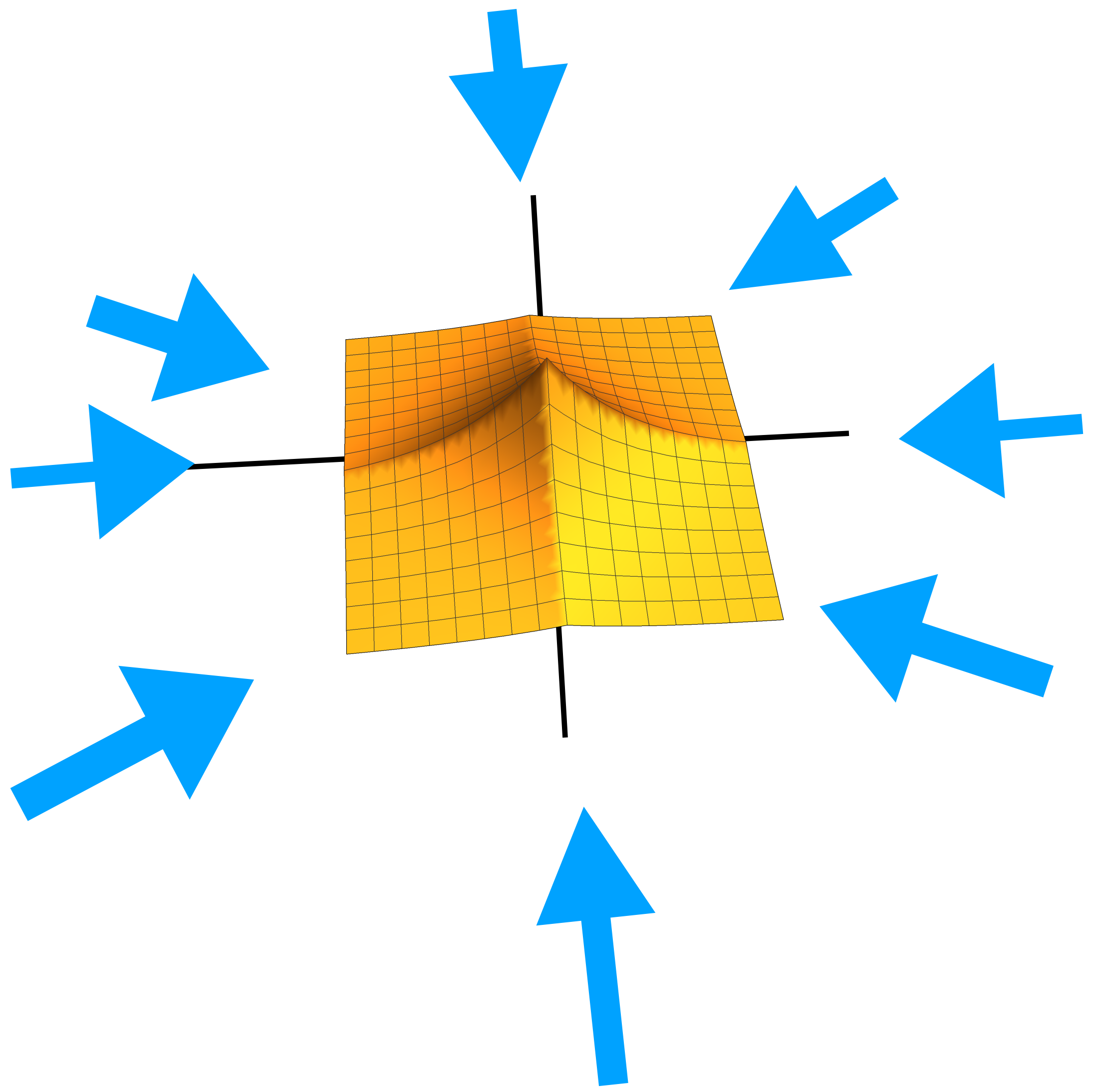
$$\mathcal{L} = \psi^T C \mathcal{D} \psi \ , \qquad \mathcal{D} = \not{\partial} + \phi_1 + i\phi_2\gamma_\chi$$

$$\phi_1 + i\phi_2 = \rho e^{i\theta}$$

In doubled theory compute the axial current: $\Gamma_\mu \gamma_\chi$

$$\frac{1}{K+M}$$

Find that when $\phi_1 + i\phi_2 = \rho e^{i\theta}$ has nonzero vorticity, this generalized Hall current flows to the defect with zeromode



Furthermore, the divergence of the current is recognized as being proportional to an integer winding number in momentum space

index(*D*) = $\lim_{M\to 0}$ $\frac{1}{2}\int d^{d+1}x\,\partial_\mu\langle\overline{\Psi}\Gamma_\mu\Gamma_\chi\Psi\rangle$

result: $\mathrm{index}(\mathcal{D}) = \nu_q \nu_\phi$

winding number of fermion propagator in **momentum space**

winding number of background scalar field in **position space** result: $\mathrm{index}(\mathcal{D}) = \nu_q \nu_\phi$

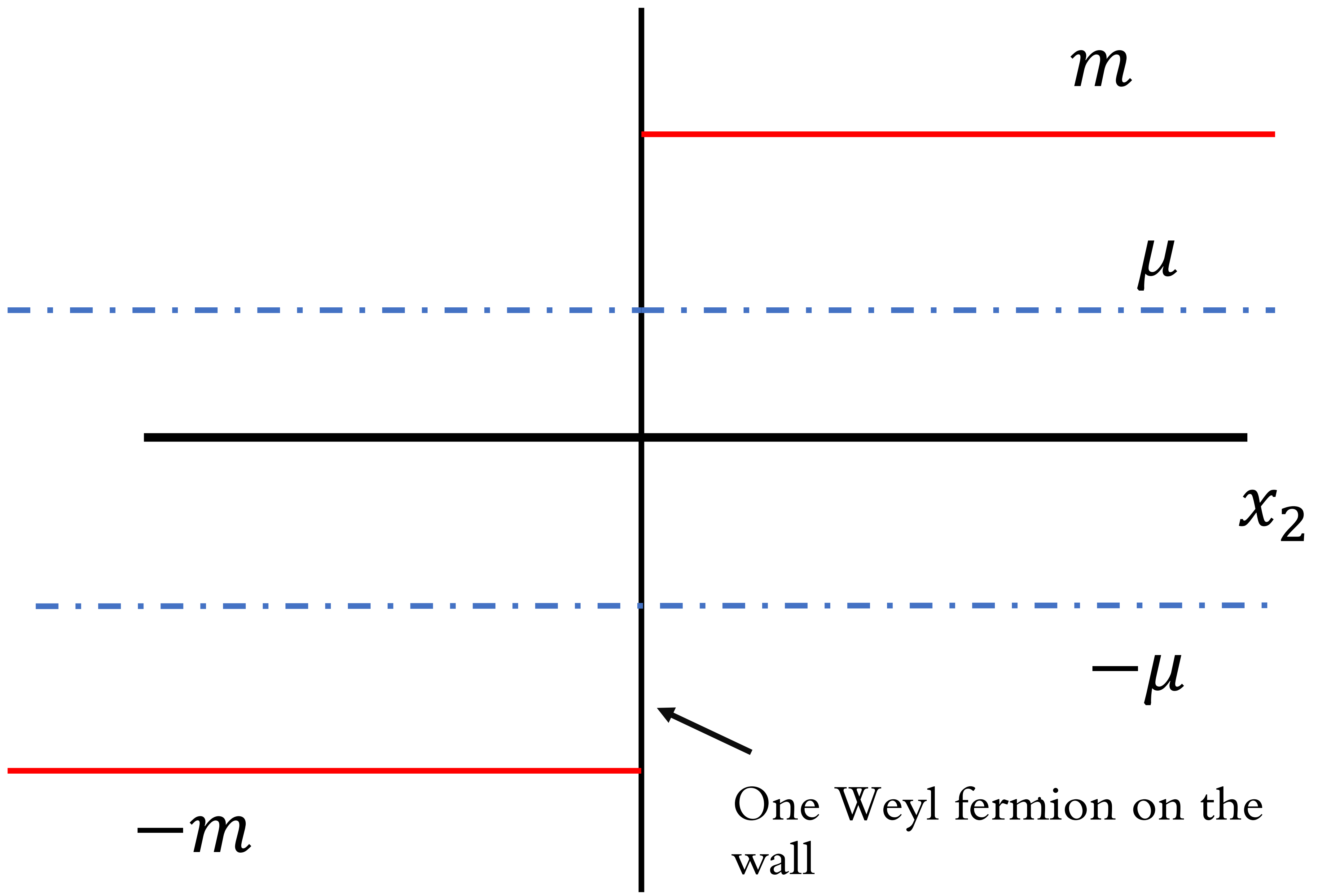
winding number of fermion propagator in momentum space

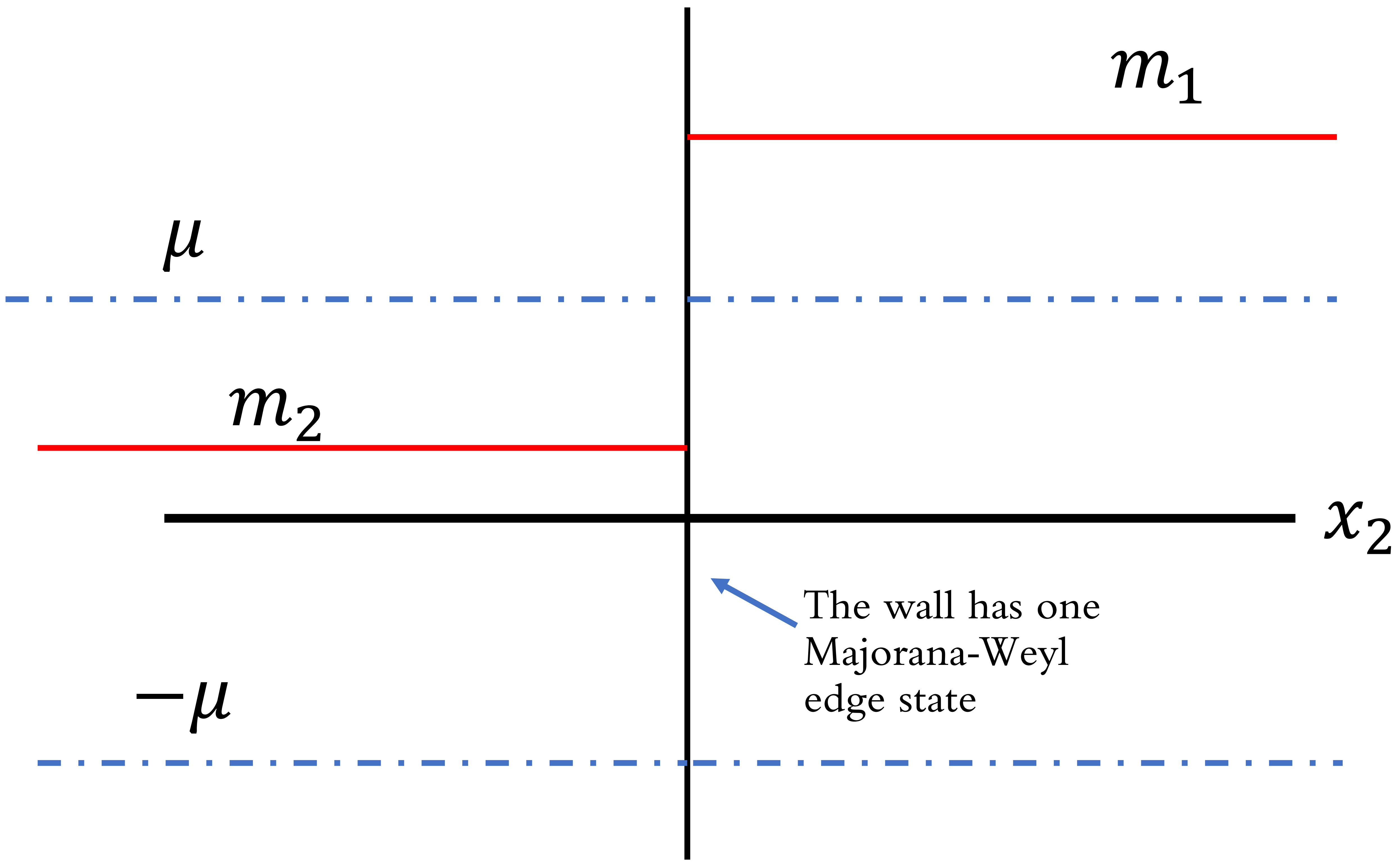
winding number of background scalar field in position space

- Nonzero result indicates that the Minkowski theory has a massless state
- The index is computed as the divergence of an inflowing generalized Hall currents in the doubled theory…even though no Hall currents in original theory
- The computation clearly reveals the underlying topology in *phase* space.

We have applied this to more complicated cases, such as IQH system with superconductivity
= fermion in 2+1 with both constant Majorana mass μ + Dirac mass m with domain wall profile

Depending on m & μ we find (i) no massless edge state, (ii) one Majorana-Weyl edge state, (iii) one Weyl edge state





Conclusions

- Topological phases are ubiquitous in theories of Dirac and Majorana fermions in various dimensions (and can be related to CM systems by symmetries)
- Both the existence of massless edge states and theory topological origin can be computed by calculating the divergence of a “generalized Hall current” in Euclidian spacetime via 1-loop Feynman diagram in background fields.
- Currently exploring whether the “periodic table of topological insulators and superconductors” (Kitaev, 2009) can be derived in this manner…
- …whether the effect of interactions be incorporated?
- …and whether the doubled theories have any direct counterpart in Minkowski spacetime that could be explored experimentally…