

A VARIANCE REDUCTION TECHNIQUE FOR HADRONIC CORRELATORS WITH PARTIALLY TWISTED BOUNDARY CONDITIONS

N. Asmussen, A. Barone, A. Jüttner

University of Southampton

DWQ@25 BNL 13-17 Dec 2021

UNIVERSITY OF
Southampton

The RBC & UKQCD collaborations

[UC Berkeley/LBNL](#)

Aaron Meyer

[BNL and BNL/RBRC](#)

Yasumichi Aoki (KEK)

Peter Boyle (Edinburgh)

Taku Izubuchi

Yong-Chull Jang

Chulwoo Jung

Christopher Kelly

Meifeng Lin

Hiroshi Ohki

Shigemi Ohta (KEK)

Amarjit Soni

[CERN](#)

Andreas Jüttner (Southampton)

[Columbia University](#)

Norman Christ

Duo Guo

Yikai Huo

Yong-Chull Jang

Joseph Karpie

Bob Mawhinney

Ahmed Sheta

Bigeng Wang

Tianle Wang

Yidi Zhao

[University of Connecticut](#)

Tom Blum

Luchang Jin (RBRC)

Michael Riberdy

Masaaki Tomii

[Edinburgh University](#)

Matteo Di Carlo

Luigi Del Debbio

Felix Erben

Vera Gülpers

Tim Harris

Raoul Hodgson

Nelson Lachini

Michael Marshall

Fionn Ó hÓgáin

Antonin Portelli

James Richings

Azusa Yamaguchi

Andrew Z.N. Yong

[KEK](#)

Julien Frison

[University of Liverpool](#)

Nicolas Garron

[Michigan State University](#)

Dan Hoying

[Milano Bicocca](#)

Mattia Bruno

[Peking University](#)

Xu Feng

[University of Regensburg](#)

Davide Giusti

Christoph Lehner (BNL)

[University of Siegen](#)

Matthew Black

Oliver Witzel

[University of Southampton](#)

Nils Asmussen

Alessandro Barone

Jonathan Flynn

Ryan Hill

Rajnandini Mukherjee

Chris Sachrajda

[University of Southern Denmark](#)

Tobias Tsang

[Stony Brook University](#)

Jun-Sik Yoo

Sergey Syritsyn (RBRC)

GOAL: we want to induce momentum into hadrons through **twisted boundary conditions**^a with less computational effort but still with good statistical precision

^a[Sachrajda, Villadoro, PLB 609 (2005) 73-85], [de Divitiis et al., PLB 595 (2004) 408-413]

IDEA: exploit the known **untwisted** correlators, which are in general computed with good precision

We construct the (θ) twisted correlator (2-point and 3-point) using

$$C_{\theta}^{\alpha}(t) = \tilde{C}_{\theta}(t) + \alpha(t) \underbrace{(C_0(t) - \tilde{C}_0(t))}_{\text{noisy zero}}$$

$$\begin{cases} C = \frac{1}{N} \sum_{i=1}^N C^i \\ \tilde{C} = \frac{1}{n} \sum_{i=1}^n C^i \end{cases}$$

$$N > n,$$

N =full number of source planes
 (\mathbb{Z}_2) per configuration

Computing the error

$$(\sigma_\theta^\alpha)^2 = \langle (\Delta C_\theta^\alpha)^2 \rangle$$

we can optimize with respect to α imposing $\frac{\partial}{\partial \alpha} (\sigma_\theta^\alpha)^2 = 0$:

$$\alpha_{min} = - \frac{\text{Cov}(C_0, \tilde{C}_\theta) - \text{Cov}(\tilde{C}_0, \tilde{C}_\theta)}{\sigma_0^2 + \tilde{\sigma}_0^2 - 2\text{Cov}(C_0, \tilde{C}_0)}$$

$$(\sigma_\theta^\alpha)^2 = \tilde{\sigma}_\theta^2 - \frac{[\text{Cov}(C_0, \tilde{C}_\theta) - \text{Cov}(\tilde{C}_0, \tilde{C}_\theta)]^2}{\sigma_0^2 + \tilde{\sigma}_0^2 - 2\text{Cov}(C_0, \tilde{C}_0)}$$

Test on Kaon/Pion on $24^3 \times 64$ RBC/UKQCD DWF ensembles

[Allton et al, Phys. Rev. D (2008) 114509]

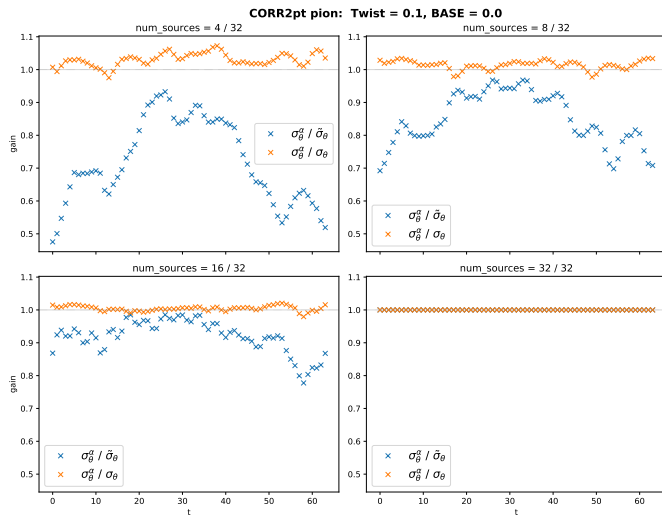
32 total sources / 39 configurations

Twist on light quark ($m_\pi \simeq 330$ MeV) on 3 spatial directions

$$\left\{ \begin{array}{l} \theta_1 = 0.1 \times (2\pi, 2\pi, 2\pi, 0) \\ \theta_2 = 0.2 \times (2\pi, 2\pi, 2\pi, 0) \\ \theta_3 = 0.3 \times (2\pi, 2\pi, 2\pi, 0) \\ \theta_4 = 0.4 \times (2\pi, 2\pi, 2\pi, 0) \end{array} \right. \quad \left\{ \begin{array}{l} p_1 = 0.1\sqrt{3} \cdot \frac{2\pi}{L} \simeq 0.17 \cdot \frac{2\pi}{L} \\ p_2 = 0.2\sqrt{3} \cdot \frac{2\pi}{L} \simeq 0.35 \cdot \frac{2\pi}{L} \\ p_3 = 0.3\sqrt{3} \cdot \frac{2\pi}{L} \simeq 0.52 \cdot \frac{2\pi}{L} \\ p_4 = 0.4\sqrt{3} \cdot \frac{2\pi}{L} \simeq 0.69 \cdot \frac{2\pi}{L} \end{array} \right.$$

Simulations carried out on the DiRAC Extreme Scaling service
at the University of Edinburgh using the Hadrons and Grid
software packages

RESULTS FOR 2PT $C_\pi(t, \rho_\theta)$ WITH $\theta = 0.1$

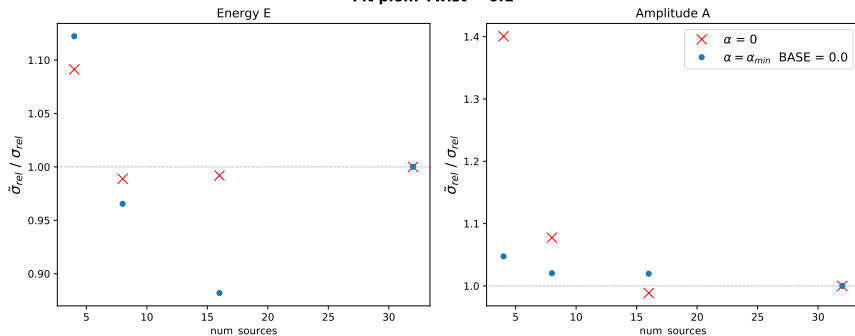


With 8 sources (1/4 of the cost) we get a good error improvement, $\sigma_\theta^\alpha / \sigma_\theta \simeq 1$ (orange data).

2PT CORRELATOR FIT

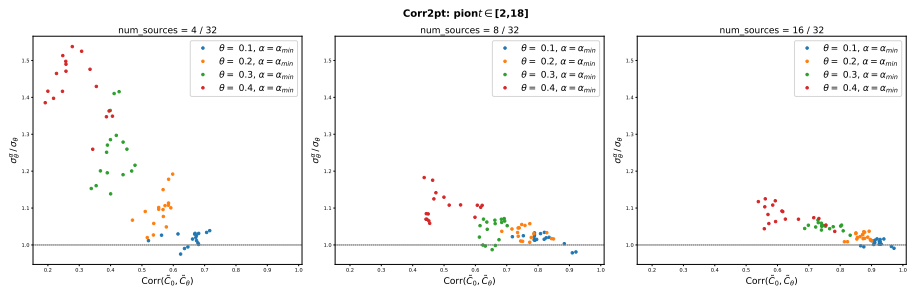
$$C_{\pi}(t) = 2Ae^{-E\frac{T}{2}} \cosh \left[E \left(\frac{T}{2} - t \right) \right]$$

Fit pion: Twist = 0.1



The blue points ($\alpha = \alpha_{min}$) show that the relative error with the new approach is smaller than the standard reduced case. In this case, 8 sources out of 32 gives a good improvement with only 1/4 of the total cost.

2PT IMPROVEMENT AS FUNCTION OF CORRELATION



The improvement $\sigma_{\theta}^{\alpha} / \sigma_{\theta}$ (y-axis) is controlled by the correlation $\text{Corr}(\tilde{C}_0, \tilde{C}_{\theta})$ between twisted and untwisted correlator. The smaller the twist, the higher the correlation and the better is the improvement $\sigma_{\theta}^{\alpha} / \sigma_{\theta}$.

The matrix element is

$$M_\mu = \langle \pi(p_\pi) | V_\mu | K(p_K) \rangle = f_+(q^2)(p_K + p_\pi)_\mu + f_-(q^2)(p_K - p_\pi)_\mu,$$

$$M_\mu = 4\sqrt{E_K E_\pi} \sqrt{\frac{C_{\mu, K\pi}^{\alpha_{min}}(t) C_{\mu, \pi K}^{\alpha_{min}}(t)}{C_K(t) C_\pi^{\alpha_{min}}(t)}}$$

The form factors are

$$\begin{cases} f_+(q^2) = \frac{1}{2E_K} \left(M_0 + \frac{E_K - E_\pi}{p_\pi} M_S \right) \\ f_-(q^2) = \frac{1}{2E_K} \left(M_0 - \frac{E_K + E_\pi}{p_\pi} M_S \right) \end{cases}$$

and the twist is on the π .

OPTIMIZATION OF THE FORM FACTORS ERROR

So far we have tuned α in order to reduce the error on the **correlators** (α_{min}).

We could tune α in order to reduce the error on the **form factors** (α_{opt}).

We consider the matrix elements as functions of α s

$$M_{\mu}(\alpha_1, \alpha_2, \alpha_3) \propto \sqrt{\frac{C_{\mu, K\pi}^{\alpha_1}(t) C_{\mu, \pi K}^{\alpha_2}(t)}{C_K(t) C_{\pi}^{\alpha_3}(t)}}$$

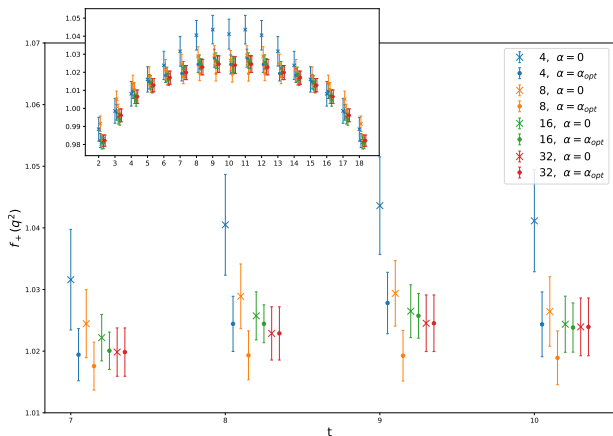
such that the error on the form factors is

$$\sigma_f \rightarrow \sigma_f(\alpha) \quad \text{with} \quad \nabla_{\alpha} \sigma_f(\alpha) = 0$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, with n up to 9.

FORM FACTOR f_+ OPTIMIZED

Form factor, $\theta = 0.1$



num_sources = 4 / 32		
t	$\tilde{\sigma}_\theta / \sigma_\theta$	$\tilde{\sigma}_\theta^{\alpha_{opt}} / \sigma_\theta$
7.0	2.0813	1.0789
8.0	1.8931	1.0364
9.0	1.7226	1.0858
10.0	1.7604	1.1208

num_sources = 8 / 32		
t	$\tilde{\sigma}_\theta / \sigma_\theta$	$\tilde{\sigma}_\theta^{\alpha_{opt}} / \sigma_\theta$
7.0	1.4084	0.9922
8.0	1.216	0.9183
9.0	1.1569	0.8947
10.0	1.201	0.9299

num_sources = 16 / 32		
t	$\tilde{\sigma}_\theta / \sigma_\theta$	$\tilde{\sigma}_\theta^{\alpha_{opt}} / \sigma_\theta$
7.0	0.9626	0.774
8.0	0.9015	0.711
9.0	0.9333	0.7911
10.0	0.9711	0.8512

The numerical optimization on the form factors $f_+(q^2)$ (in bold) improves significantly the error ($\sigma_\theta^\alpha \lesssim \sigma_\theta$) for at least $\sim 1/4$ of the total number of sources. The fits show a similar behaviour!

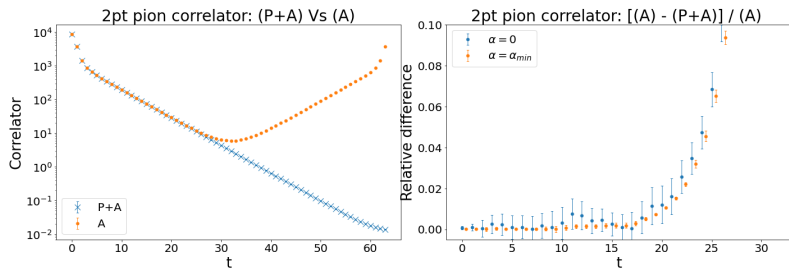
P±A BOUNDARY CONDITIONS

We could consider a similar procedure on a **pion** for the case

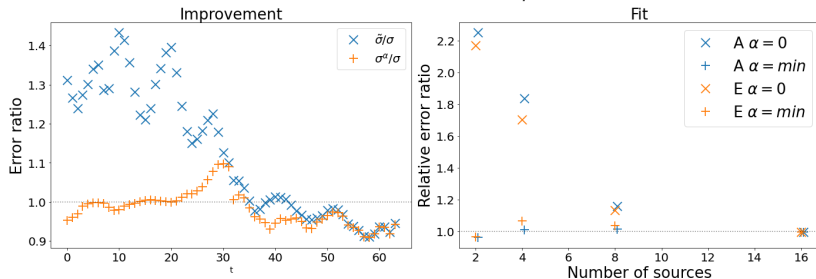
Periodic ± Anti-periodic Boundary Conditions in **time direction**

which have the advantage of **eliminating the around-the-world effects** related to the finite volume formulation. As before we can build

$$C_{P\pm A}^\alpha = \tilde{C}_{P\pm A} + \alpha(t)(C_A - \tilde{C}_A)$$



2pt pion correlator with P+A Number of sources = 4/16



With 1/4 of the cost the improvement is already very good in the first half of the lattice, $\tilde{\sigma}_{P+A}^\alpha \simeq \sigma_{P+A}$. The fit parameters are basically always equivalent to the case with the full number of sources.

SUMMARY

On correlators (twisting/ $P\pm A$):

- error is reduced, OK with $\sim \frac{1}{4}$ of total cost
- α_{min} has analytical expression
- central value (BIAS) must be taken into account

On generic observables (twisting):

- error is reduced, OK with $\sim \frac{1}{4}$ of total cost
- α_{opt} is optimized numerically

NEXT STEP

- BIAS correction
- Testing $P\pm A$ to more complex analysis (e.g. $K\pi$ scattering)