

$a_{\mu}^{\text{HVP,LO}}$ with 2+1+1 HISQ fermions:
Status and Prospects from FNAL-HPOCD-MILC

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University of Illinois

DWQ@25 Workshop
hosted by BNL & RBRC (virtual)
13 - 17 December 2021

Fermilab Lattice, HPQCD, MILC g-2 group

Subgroup of members of the three collaborations actively engaged in the g-2 projects:

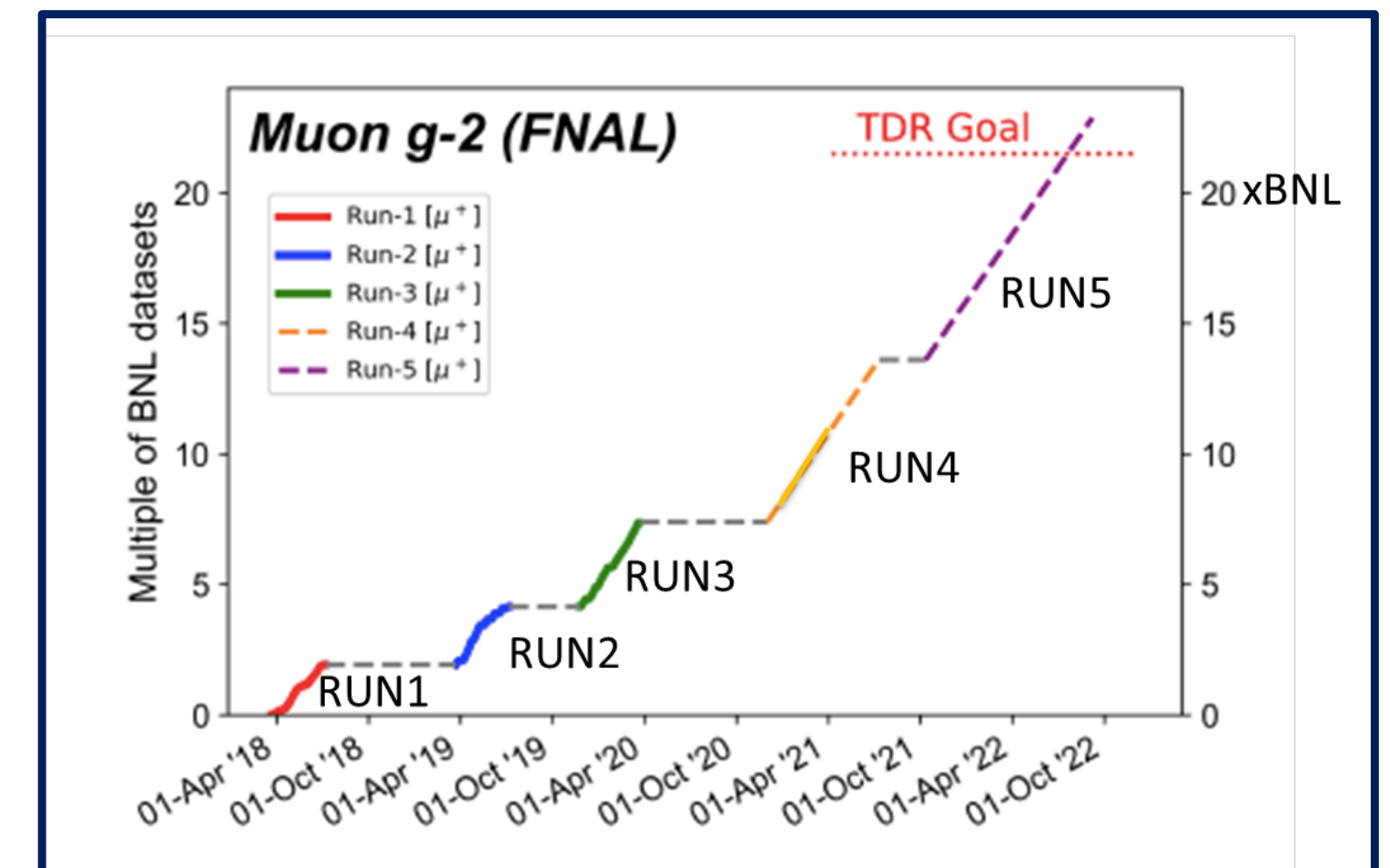
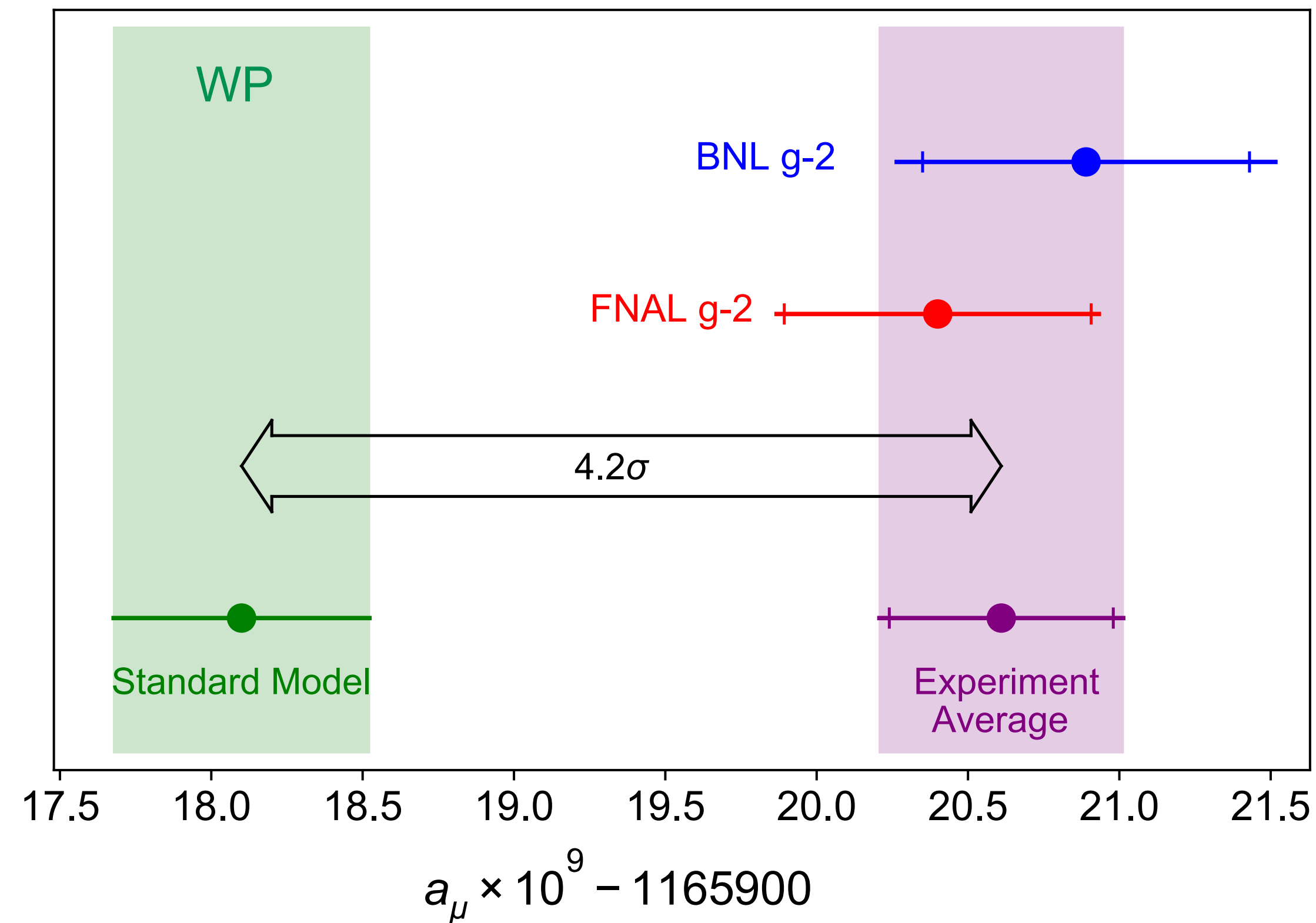
- ❖ Alexei Bazavov
- ❖ Carleton DeTar
- ❖ A. El-Khadra
- ❖ Elvira Gámiz
- ❖ Steve Gottlieb
- ❖ Andreas Kronfeld
- ❖ [Shaun Lahert](#)
- ❖ [Michael Lynch](#)
- ❖ Yin Lin (Ω baryon mass)
- ❖ Ethan Neil
- ❖ [Curtis Peterson](#)
- ❖ Jim Simone
- ❖ Alex Vaquero
- ❖ Ruth Van de Water

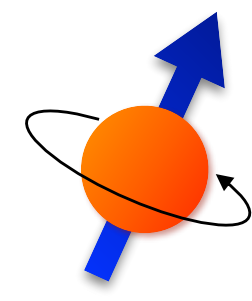


- ❖ Christine Davies
- ❖ Dan Hatton
- ❖ Peter Lepage
- ❖ Craig McNeile
- ❖ [Gaurav Ray](#)

Muon g-2: experiment

- The Fermilab experiment released the measurement result from their run 1 data on 7 April 2021. [B. Abi et al, *Phys. Rev. Lett.* 124, 141801 (2021)]
- Analysis of runs 2 and 3 is now underway.

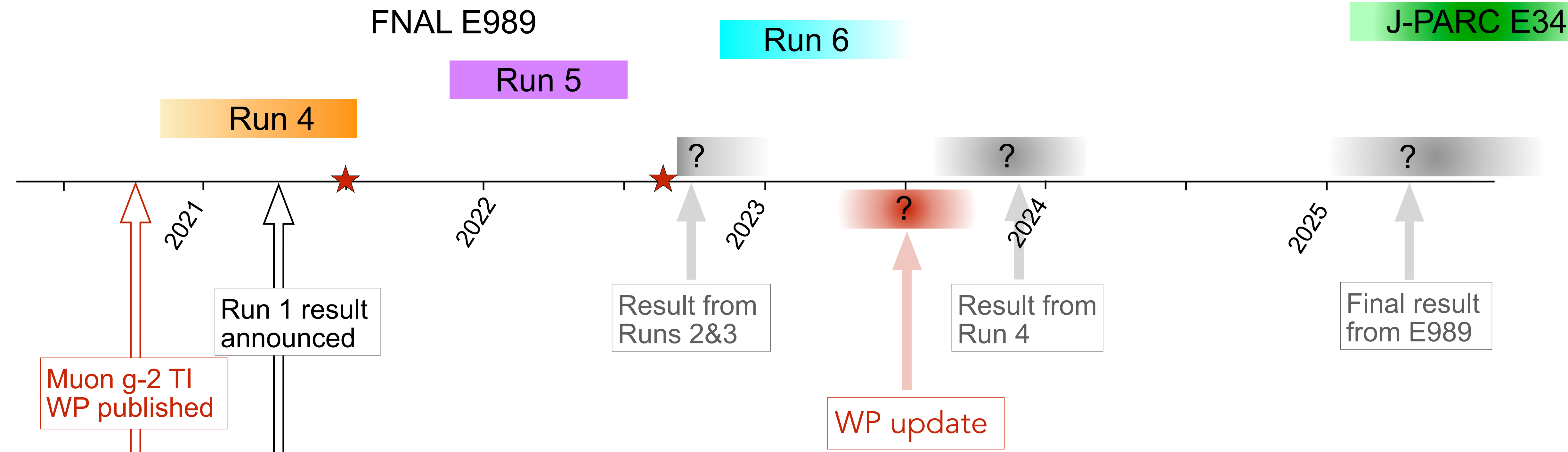




Muon $g-2$ Theory Initiative

- Maximize the impact of the Fermilab and J-PARC experiments
 - ▮ quantify and reduce the theoretical uncertainties on the hadronic corrections
- summarize the theory status and assess reliability of uncertainty estimates
- organize workshops to bring the different communities together:
 - [First plenary workshop @ Fermilab: 3-6 June 2017](#)
 - [HVP workshop @ KEK: 12-14 February 2018](#)
 - [HLbL workshop @ U Connecticut: 12-14 March 2018](#)
 - [Second plenary workshop @ HIM \(Mainz\): 18-22 June 2018](#)
 - [Third plenary workshop @ INT \(Seattle\): 9-13 September 2019](#)
 - [Lattice HVP at high precision workshop \(virtual\): 16-20 November 2020](#)
 - [Fourth plenary workshop @ KEK \(virtual\): 28 June - 02 July 2021](#)
 - [Fifth plenary workshop @ Higgs Centre \(Edinburgh\): 5-9 September 2022](#)
- 1st White Paper published in 2020 (132 authors, 82 institutions, 21 countries)
[T. Aoyama et al, [arXiv:2006.04822](#), Phys. Repts. 887 (2020) 1-166.]
- 2nd White Paper (~2023): First discussions @ KEK meeting in June 2021
expect to develop a concrete plan (outline, authors) @ Higgs Centre workshop

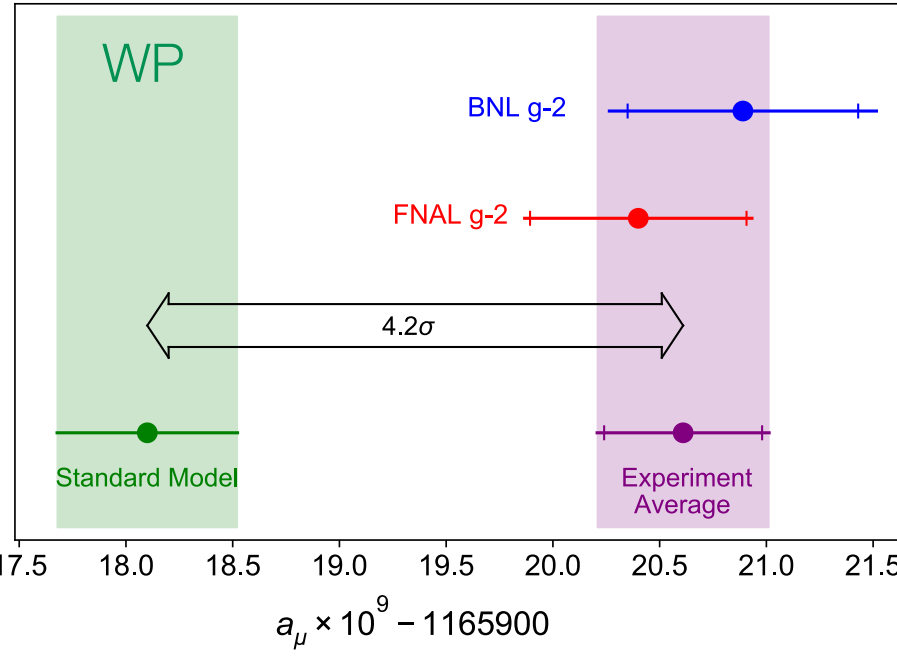
Timeline



Physics Reports 887 (2021) 1–106
 Contents lists available at ScienceDirect
 Physics Reports
 journal homepage: www.elsevier.com/locate/physrep

The anomalous magnetic moment of the muon in the Standard Model

T. Aoyama^{1,2,3}, N. Asmussen⁴, M. Benayoun⁵, J. Bijm⁶, S. Blum^{7,8}, M. Bruno⁹, I. Caprini¹⁰, C.M. Cariani Calame¹¹, M. Cè^{12,13}, G. Colangelo¹⁴, F. Crotty^{15,16}, H. Czyż¹⁷, J. Danilkin¹⁸, M. Davier¹⁹, C.H. Davies²⁰, M. Della Morte²¹, S.I. Eidelman²², A.K. El-Khadra²³, A. Gérardin²⁴, D. Giusti²⁵, M. Golterman²⁶, Steven Gottlieb²⁷, V. Gulpers²⁸, F. Hagelstein²⁹, M. Hayakawa³⁰, G. Herdoíza³¹, D.W. Hertzog³², A. Hoecker³³, M. Hofrichter³⁴, B.-L. Hoid³⁵, R.J. Hudspeth³⁶, F. Ignotov³⁷, T. Izubuchi³⁸, F. Jegerlehner³⁹, L. Jin⁴⁰, A. Keshavarzi⁴¹, T. Kinoshita⁴², B. Kubis⁴³, A. Kupchik⁴⁴, A. Kuznetsov⁴⁵, I. Laudi⁴⁶, C. Lehner⁴⁷, I. Leifschuch⁴⁸, I. Logashenko⁴⁹, B. Malaescu⁵⁰, K. Maltman⁵¹, M.K. Marinković⁵², P. Masjuan⁵³, A.S. Meyer⁵⁴, H.B. Meyer⁵⁵, T. Mibe⁵⁶, K. Mura⁵⁷, S.E. Müller⁵⁸, M. Nio⁵⁹, D. Nomura⁶⁰, A. Nyfeler⁶¹, V. Pascalutsa⁶², M. Passera⁶³, E. Perez del Rio⁶⁴, S. Peris⁶⁵, A. Portelli⁶⁶, M. Procura⁶⁷, C.F. Redmer⁶⁸, B.L. Roberts⁶⁹, J. Sánchez-Puertas⁷⁰, S. Seidenfaden⁷¹, B. Schwartz⁷², S. Simula⁷³, D. Stöckinger⁷⁴, H. Stöckinger-Kim⁷⁵, P. Stoffer⁷⁶, T. Teubner⁷⁷, R. Van de Water⁷⁸, M. Vanderhaeghe⁷⁹, G. Venanzoni⁸⁰, G. von Hippel⁸¹, H. Wittig⁸², Z. Zhang⁸³, M.N. Acharya⁸⁴, A. Bashir⁸⁵, N. Cardoso⁸⁶, B. Chakraborty⁸⁷, E.-H. Cho⁸⁸, J. Charles⁸⁹, A. Crivellin⁹⁰, O. Deineka⁹¹, A. Denig⁹², C. DeTar⁹³, C.A. Dominguez⁹⁴, A.E. Dorokhov⁹⁵, V.P. Druzhinin⁹⁶, G. Eichmann⁹⁷, M. Fael⁹⁸, C.S. Fischer⁹⁹, E. Gámiz¹⁰⁰, Z. Geiser¹⁰¹, J.R. Green¹⁰², S. Guellati-Khelifa¹⁰³, D. Hatton¹⁰⁴, R. Herrmann-Trojanowski¹⁰⁵, S. Holz¹⁰⁶, B. Hörz¹⁰⁷, M. Kiechl¹⁰⁸, J. Koponen¹⁰⁹, A.S. Kronfeld¹¹⁰, J. Laiho¹¹¹, S. Leupold¹¹², P.B. Mackenzie¹¹³, W.J. Marciano¹¹⁴, C. McNeile¹¹⁵, D. Mohler¹¹⁶, J. Monnard¹¹⁷, E.T. Neil¹¹⁸, A.V. Nesterenko¹¹⁹, K. Ottnaad¹²⁰, V. Pauk¹²¹, A.E. Radhababu¹²², E. de Rafael¹²³, K. Raya¹²⁴, A. Rich¹²⁵, A. Rodríguez-Sánchez¹²⁶, P. Roig¹²⁷, T. San José¹²⁸, E.P. Solodov¹²⁹, R. Sugar¹³⁰, K. Yu. Todyshin¹³¹, A. Vainshtein¹³², A. Vagstad Avilés-Casco¹³³, E. Weil¹³⁴, J. Wilhelm¹³⁵, R. Williams¹³⁶, A.S. Zhevlakov¹³⁷



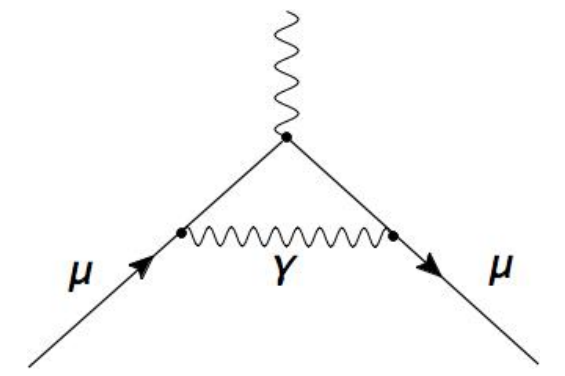
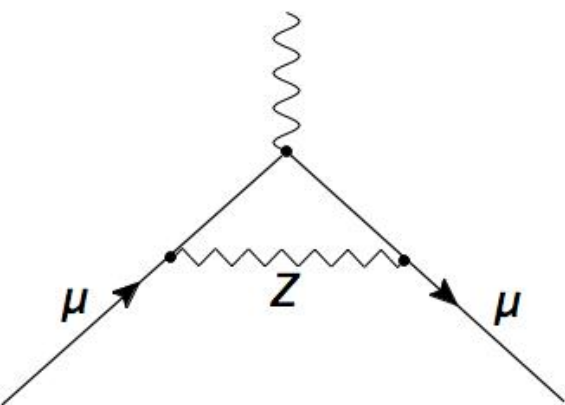
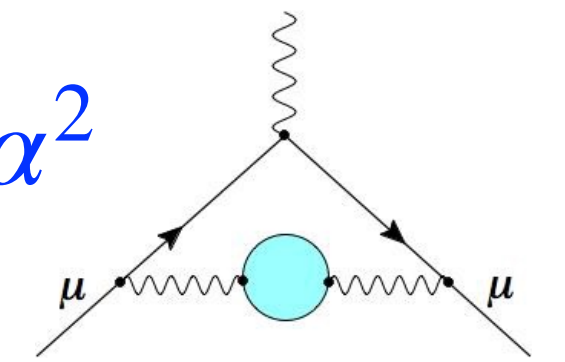
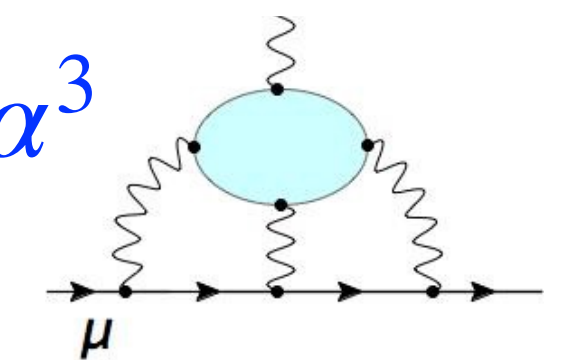
Theory Initiative:

- ★ ongoing activities: develop method average for Lattice HVP
- ★ plan to update WP with new SM predictions (~ 2023)

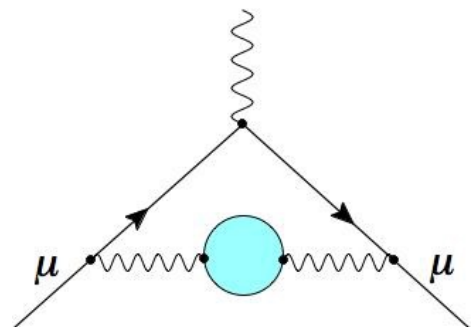
★ TI workshops: Jun 2021 @ KEK (virtual)
 Sep 2022 @ Higgscentre

Muon g-2: SM contributions

$$a_\mu = a_\mu(\text{QED}) + a_\mu(\text{EW}) + a_\mu(\text{hadronic})$$

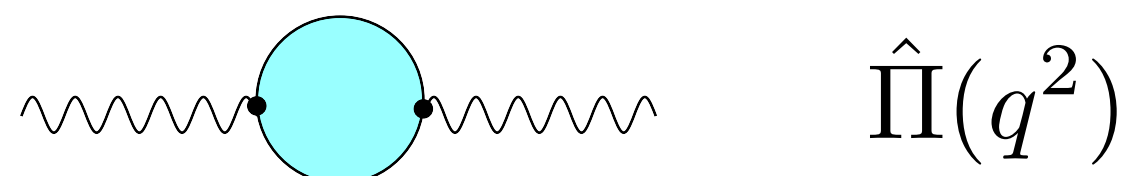
QED		+ ...	$116\,584\,718.9(1) \times 10^{-11}$	0.001 ppm
EW		+ ...	$153.6(1.0) \times 10^{-11}$	0.01 ppm
HVP	α^2 	+ ...	$6845(40) \times 10^{-11}$ [0.6%]	0.34 ppm
HLbL	α^3 	+ ...	$92(18) \times 10^{-11}$ [20%]	0.15 ppm

Hadronic corrections



Lattice HVP: Introduction

[B. Lautrup, A. Peterman, E. de Rafael, Phys. Rep 1972;
E. de Rafael, Phys. Let. B 1994; T. Blum, PRL 2002]



$$\hat{\Pi}(q^2)$$

Leading order HVP correction:

$$a_{\mu}^{\text{HVP,LO}} = \left(\frac{\alpha}{\pi}\right)^2 \int dq^2 \omega(q^2) \hat{\Pi}(q^2)$$

- Calculate $a_{\mu}^{\text{HVP,LO}}$ in Lattice QCD

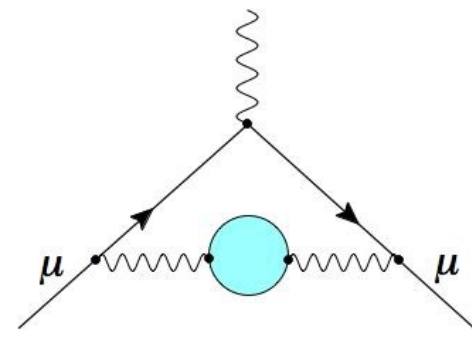
Compute correlation function: $C(t) = \frac{1}{3} \sum_{i,x} \langle j_i(x,t) j_i(0,0) \rangle$

and
$$\hat{\Pi}(Q^2) = 4\pi^2 \int_0^{\infty} dt C(t) \left[t^2 - \frac{4}{Q^2} \sin^2 \left(\frac{Qt}{2} \right) \right]$$

[D. Bernecker and H. Meyer, arXiv:1107.4388,
EPJA 2011]

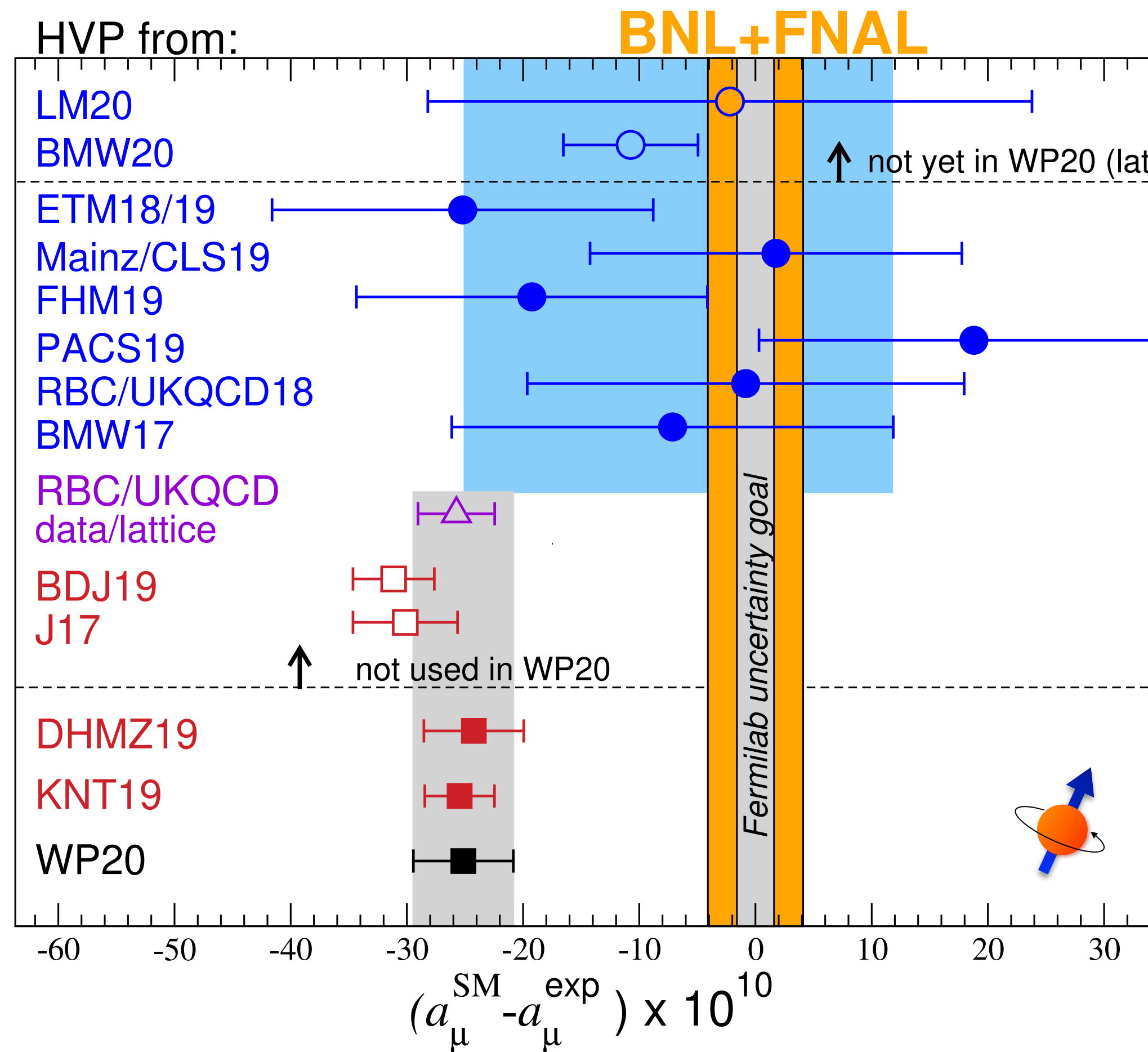
Obtain $a_{\mu}^{\text{HVP,LO}}$ from an integral over Euclidean time:

$$a_{\mu}^{\text{HVP,LO}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^{\infty} dt \tilde{w}(t) C(t)$$



HVP: Comparison

$$a_{\mu}^{\text{SM}} = a_{\mu}^{\text{HVP}} + [a_{\mu}^{\text{QED}} + a_{\mu}^{\text{Weak}} + a_{\mu}^{\text{HLbL}}]$$

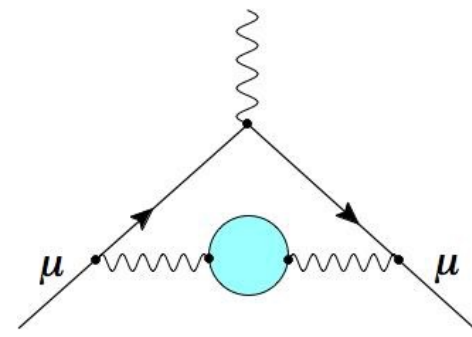


Lattice QCD + QED

hybrid: combine data & lattice

data driven

+ unitarity/analyticity constraints



HVP: Comparison

$$a_{\mu}^{\text{SM}} = a_{\mu}^{\text{HVP}} + [a_{\mu}^{\text{QED}} + a_{\mu}^{\text{Weak}} + a_{\mu}^{\text{HLbL}}]$$

FNAL-HPQCD-MILC:

[Chakraborty et al, arXiv:1710.11212, 2018 PRL]

[Davies et al, arXiv:1902.04223, 2020 PRD]

Updates in:

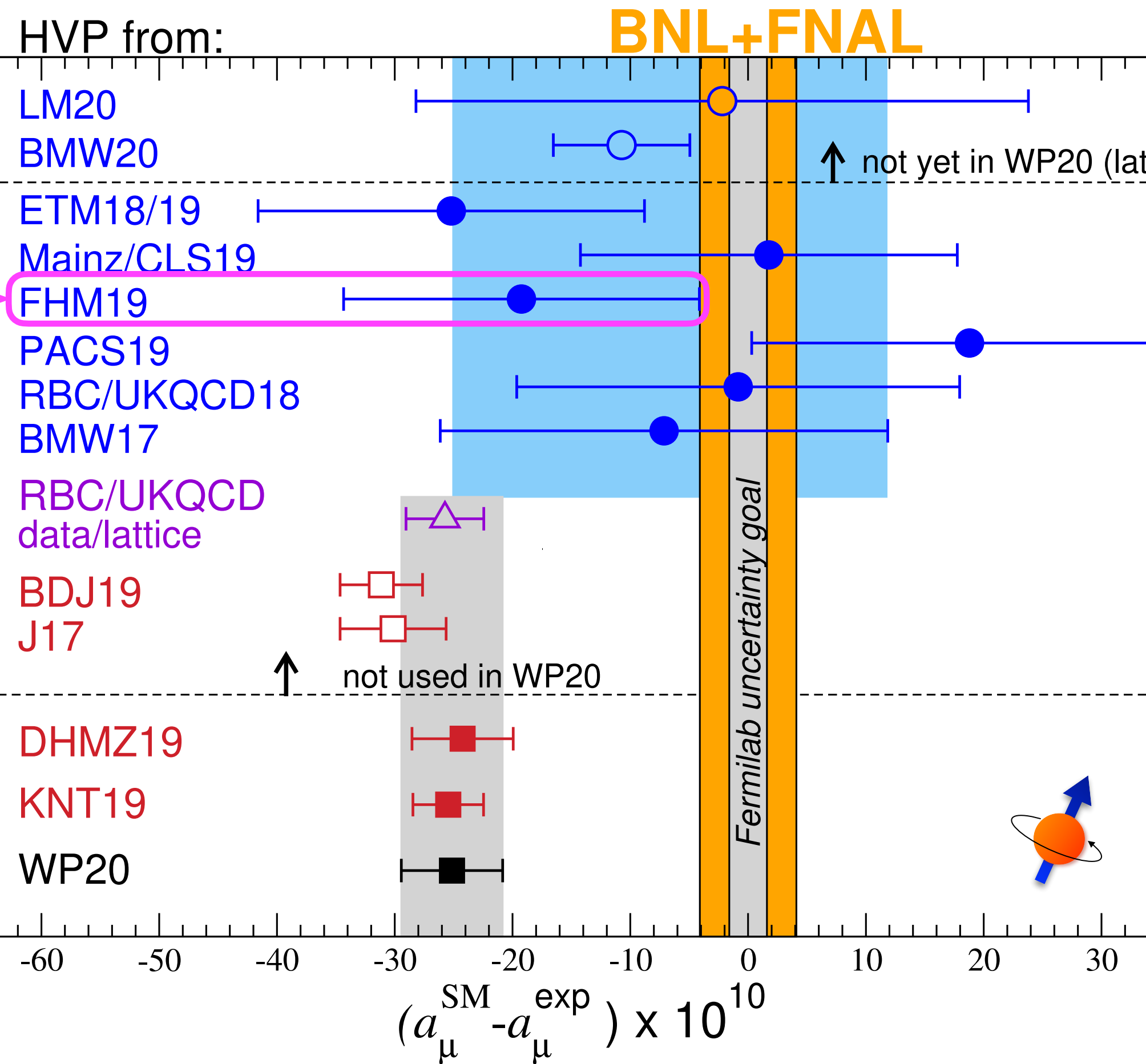
[C. DeTar et al, arXiv:1912.04382, Lattice 2019]

[S. Lahert @ Lattice 2021]

[C. McNeile @ Lattice 2021]

Ongoing program:

- Reduce uncertainties on connected light quark contribution
- Disconnected contribution
- Strong IB corrections
- QED corrections

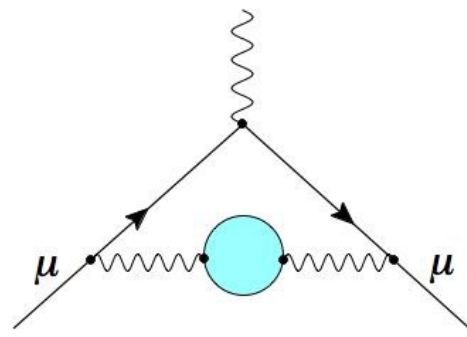


Lattice QCD + QED

hybrid: combine data & lattice

data driven

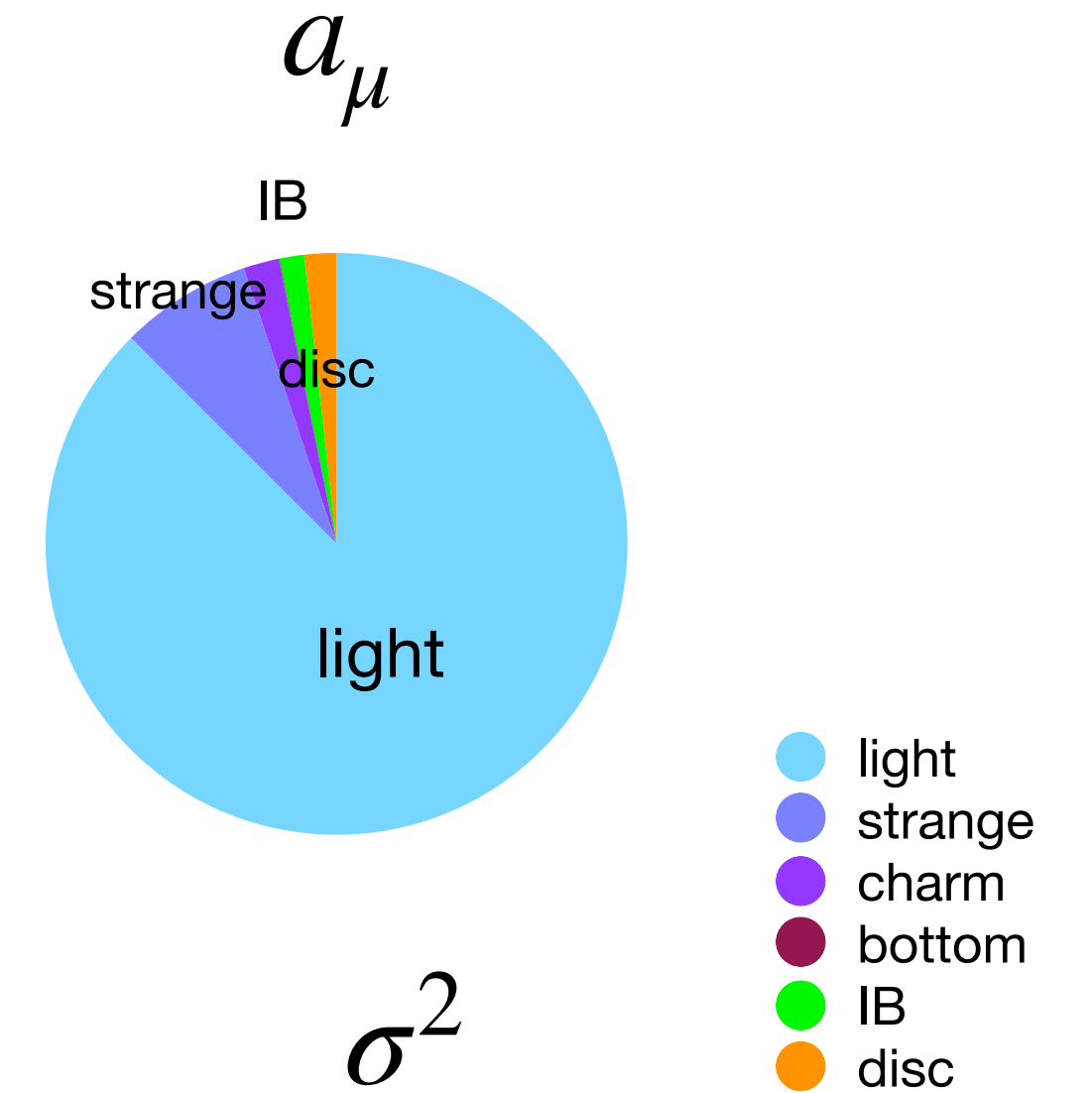
+ unitarity/analyticity constraints



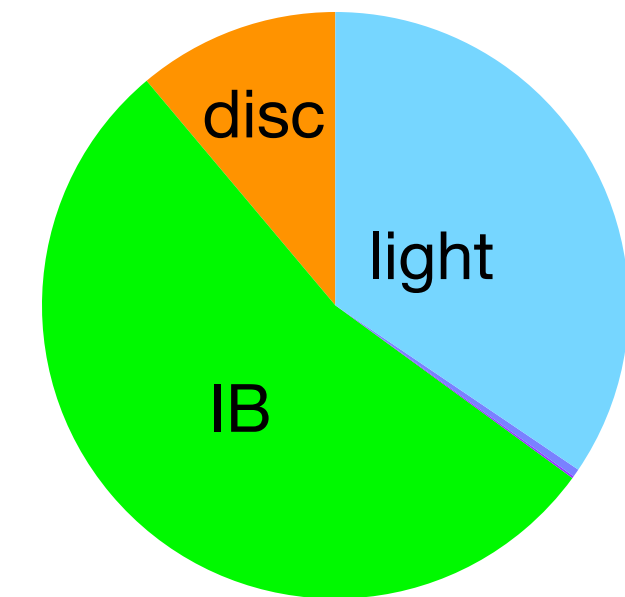
FNAL-HPQCD-MILC summary

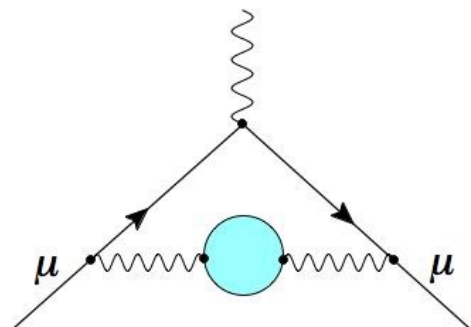
[Davies et al, arXiv:1902.04223, 2020 PRD]

- light-quark connected contribution:
 - $a_\mu^{\text{HVP,LO}}(ud) \sim 90\%$ of total
 - s, c, b -quark contributions
 - $a_\mu^{\text{HVP,LO}}(s, c, b) \sim 8\%, 2\%, 0.05\%$ of total
- disconnected contribution:
 - $a_{\mu, \text{disc}}^{\text{HVP,LO}} \sim 2\%$ of total
- Isospinbreaking (QED + $m_u \neq m_d$) corrections:
 - $\delta a_\mu^{\text{HVP,LO}} \sim 1\%$ of total



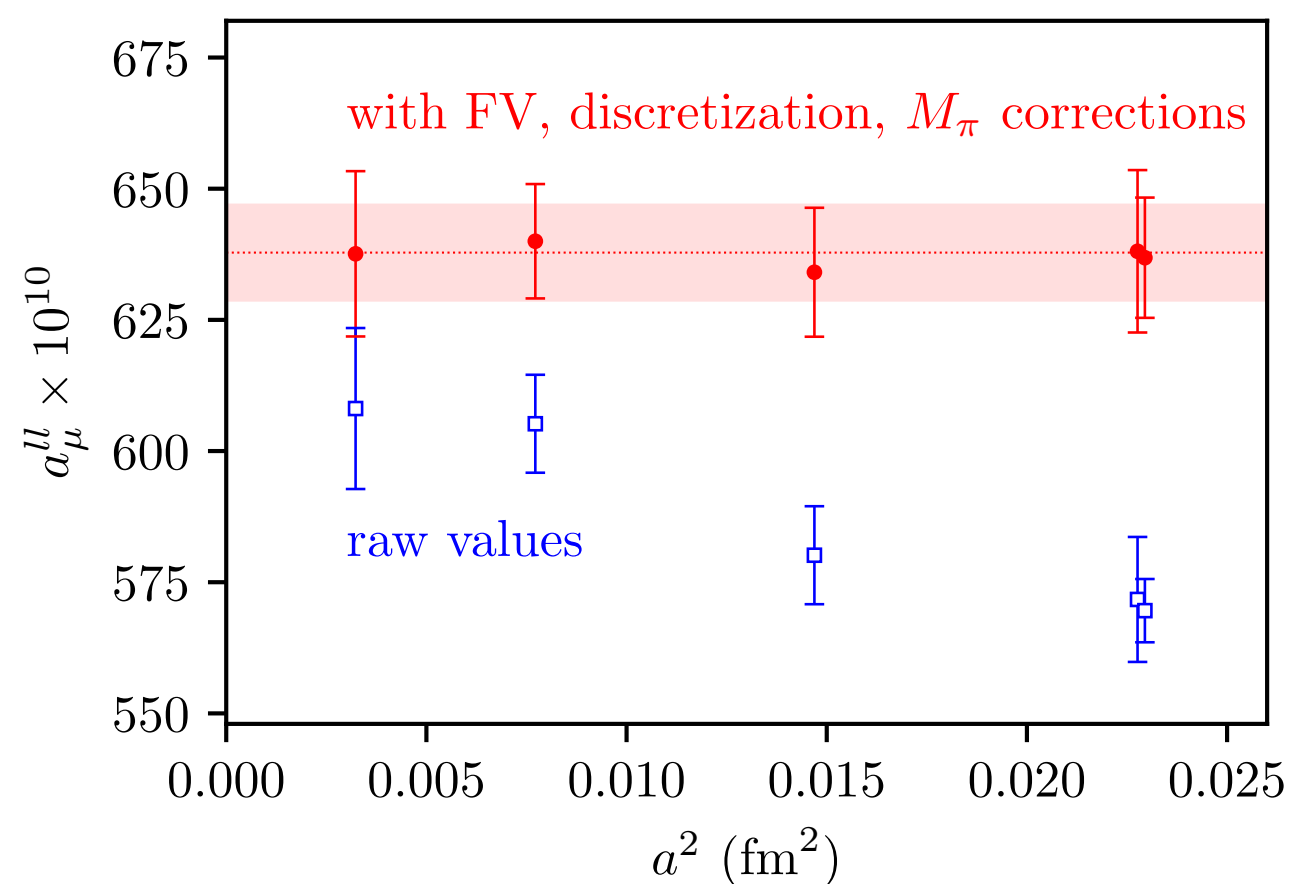
$$a_\mu^{\text{HVP,LO}} = a_\mu^{\text{HVP,LO}}(ud) + a_\mu^{\text{HVP,LO}}(s) + a_\mu^{\text{HVP,LO}}(c) + a_{\mu, \text{disc}}^{\text{HVP,LO}} + \delta a_\mu^{\text{HVP,LO}}$$





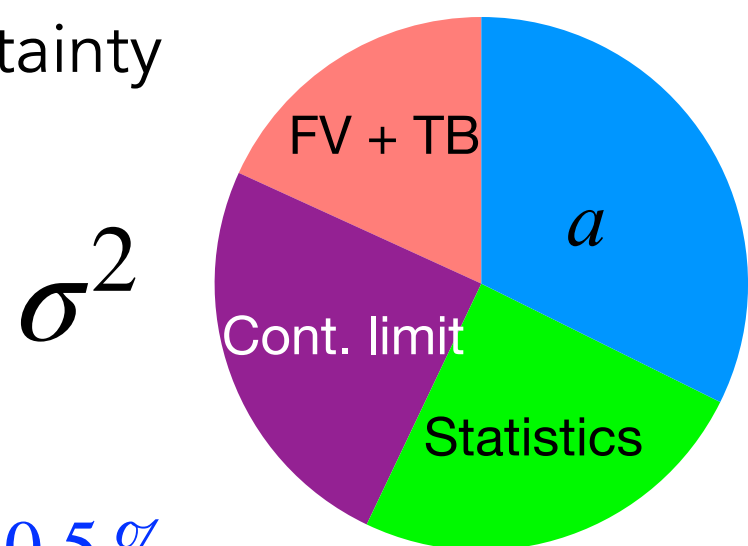
Light-quark connected contribution

[Davies et al, arXiv:1902.04223, 2020 PRD]



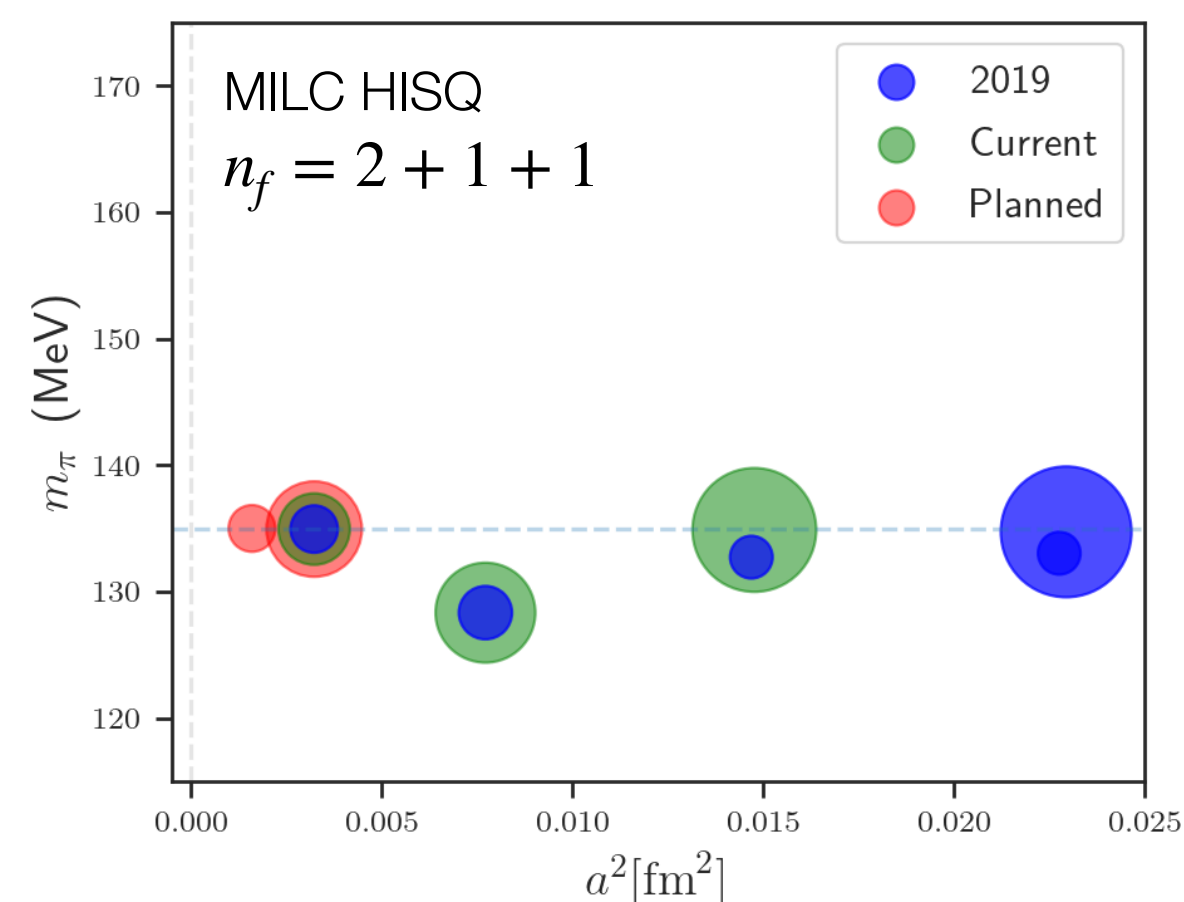
raw \rightarrow corrected:

- FV + taste-breaking corrections using chiral model
- adjust for small differences between simulation M_π and M_{π^0}
- 1.4 % total uncertainty
- error breakdown:



• ultimate goal: $\lesssim 0.5\%$

Statistics, continuum limit:



- MILC HISQ ensembles with (near) physical mass light quarks
- increasing statistics at 3 finest ensembles
- plan to add data at a 5th lattice spacing, $a \approx 0.042$ fm

FV corrections:

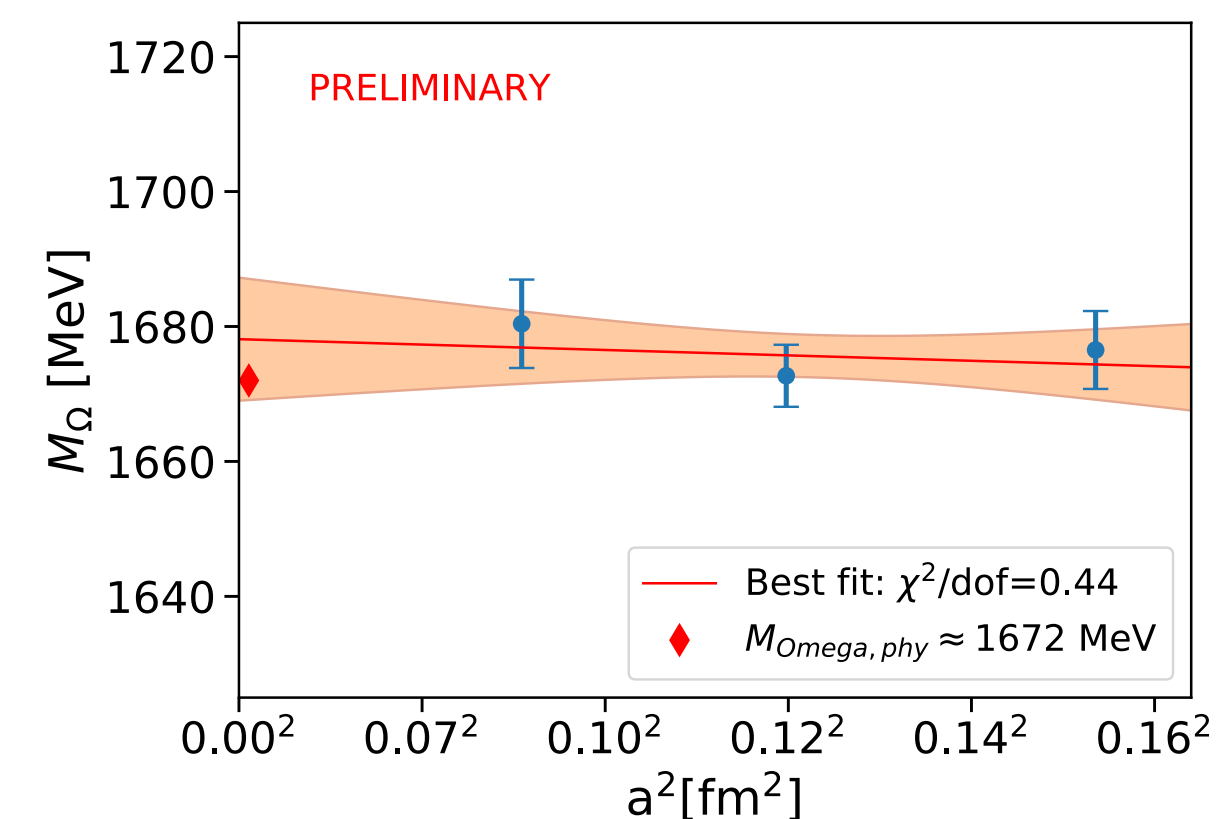
- Compare FV corrections using different approaches (chiral model, NNLO ChPT, HP, GS)

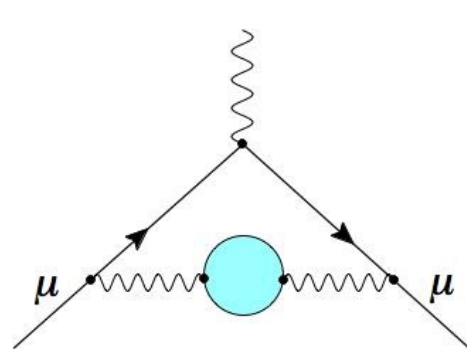
Lattice spacing uncertainty w_0 (fm):

- current $\sim 0.5\%$ error based on f_π

Ongoing work:

- Refining calculation of f_π on MILC ensembles (Claude Bernard)
- Refining calculation of relative scale (Alexei Bazavov)
- Computing Ω baryon mass (Yin Lin)
 - extending pilot project [Yin Lin, arXiv:1912.00028]
 - adding data at $a \approx 0.06$ fm
 - increasing statistics





Windows in Euclidean time

$$a_{\mu}^{\text{HVP,LO}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^{\infty} dt \tilde{w}(t) C(t)$$

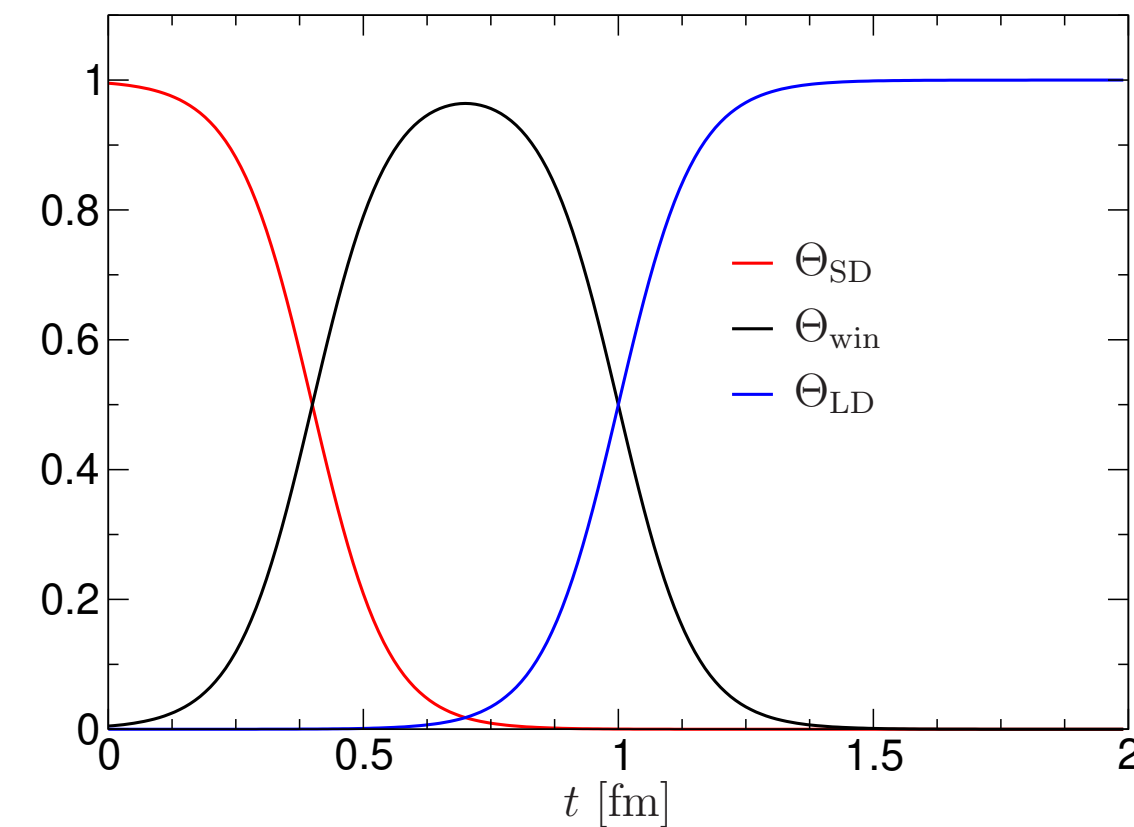
- Use windows in Euclidean time to consider the different time regions separately [T. Blum et al, arXiv:1801.07224, 2018 PRL]

Short Distance (SD) $t : 0 \rightarrow t_0$

Intermediate (W) $t : t_0 \rightarrow t_1$

Long Distance (LD) $t : t_1 \rightarrow \infty$

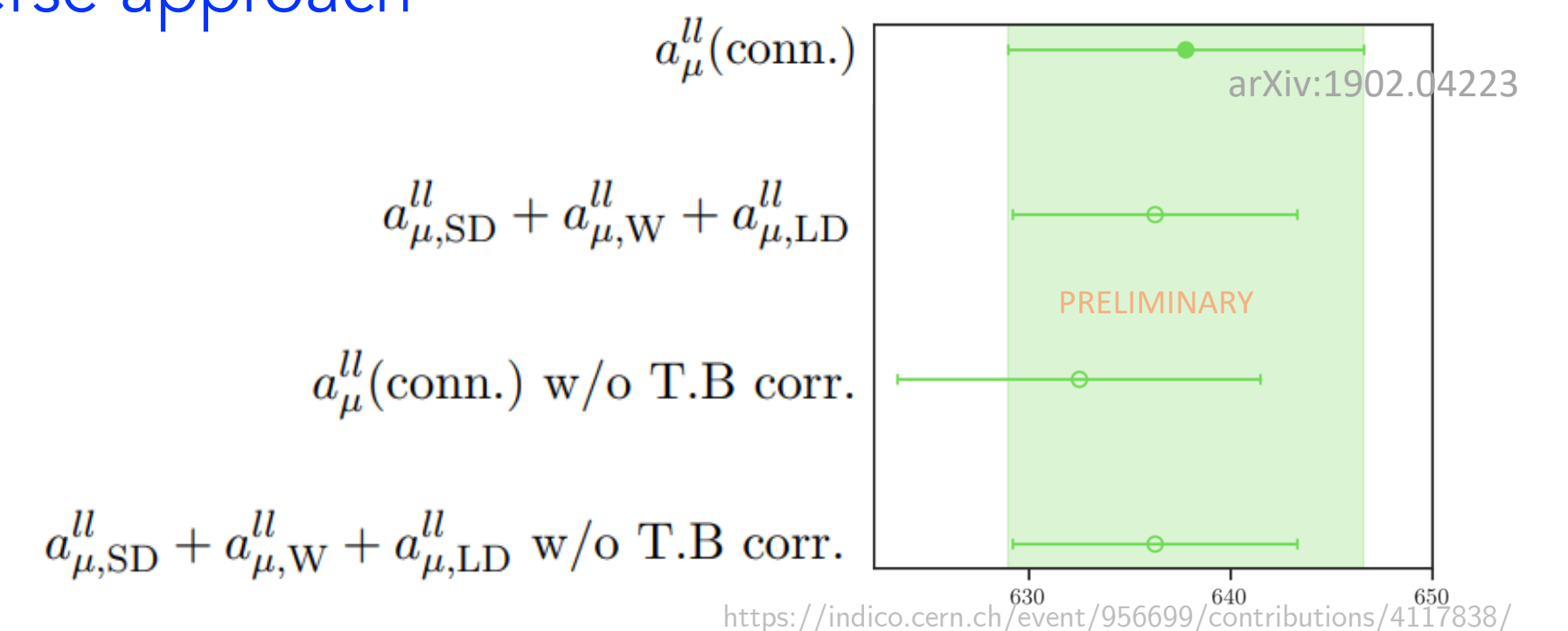
$$t_0 = 0.4 \text{ fm}, t_1 = 1.0 \text{ fm}$$



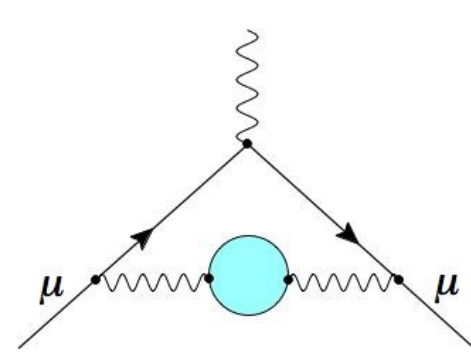
- disentangle systematics/statistics from long distance/FV and discretization effects
- intermediate window: easy to compute in lattice QCD & compare to disperse approach
- Internal cross check: compute each window separately (in continuum, infinite volume limits,...) and combine:

$$a_{\mu} = a_{\mu}^{\text{SD}} + a_{\mu}^{\text{W}} + a_{\mu}^{\text{LD}}$$

Shaun Lahert @ HVP 2020 workshop

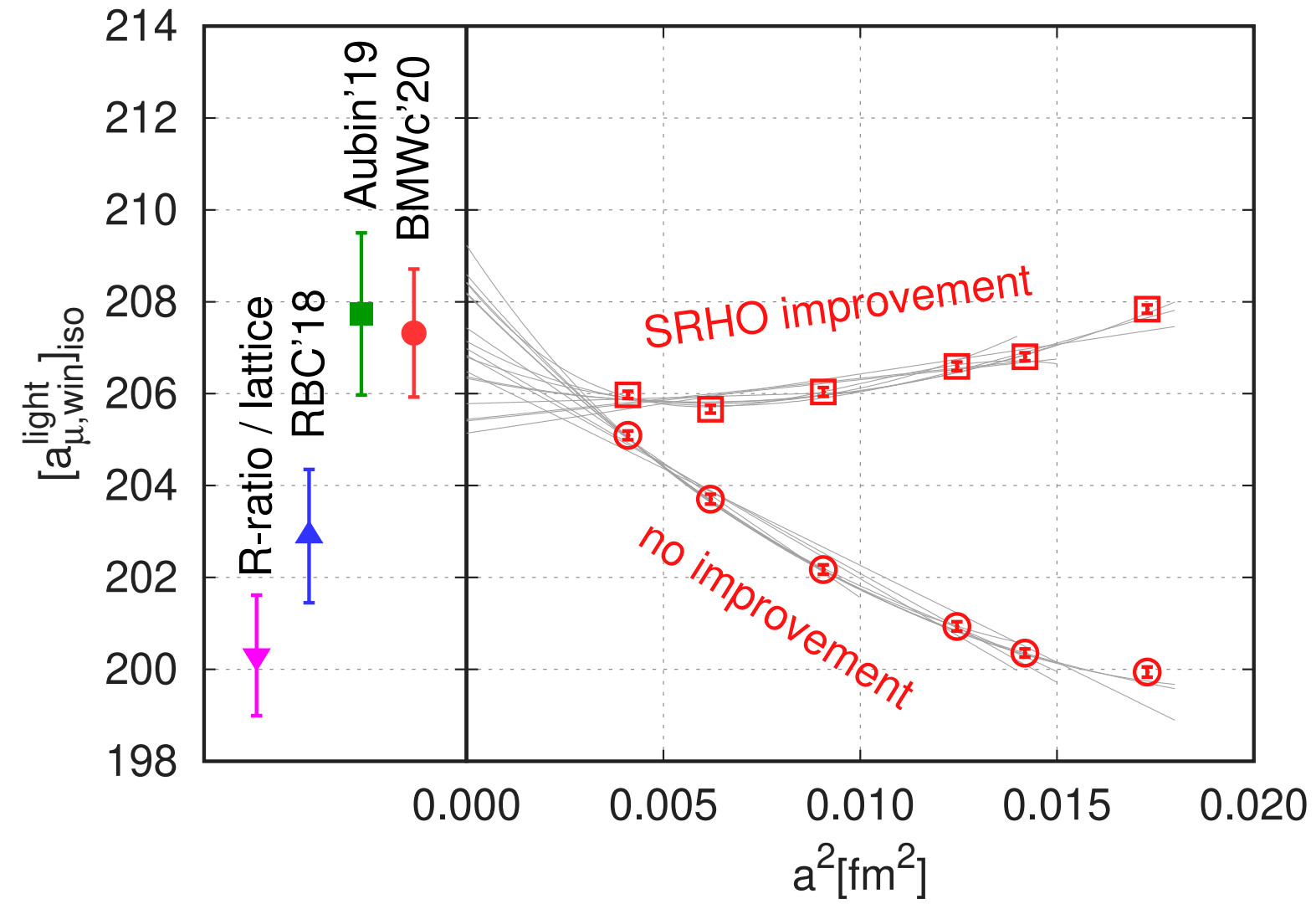


<https://indico.cern.ch/event/956699/contributions/4117838/>



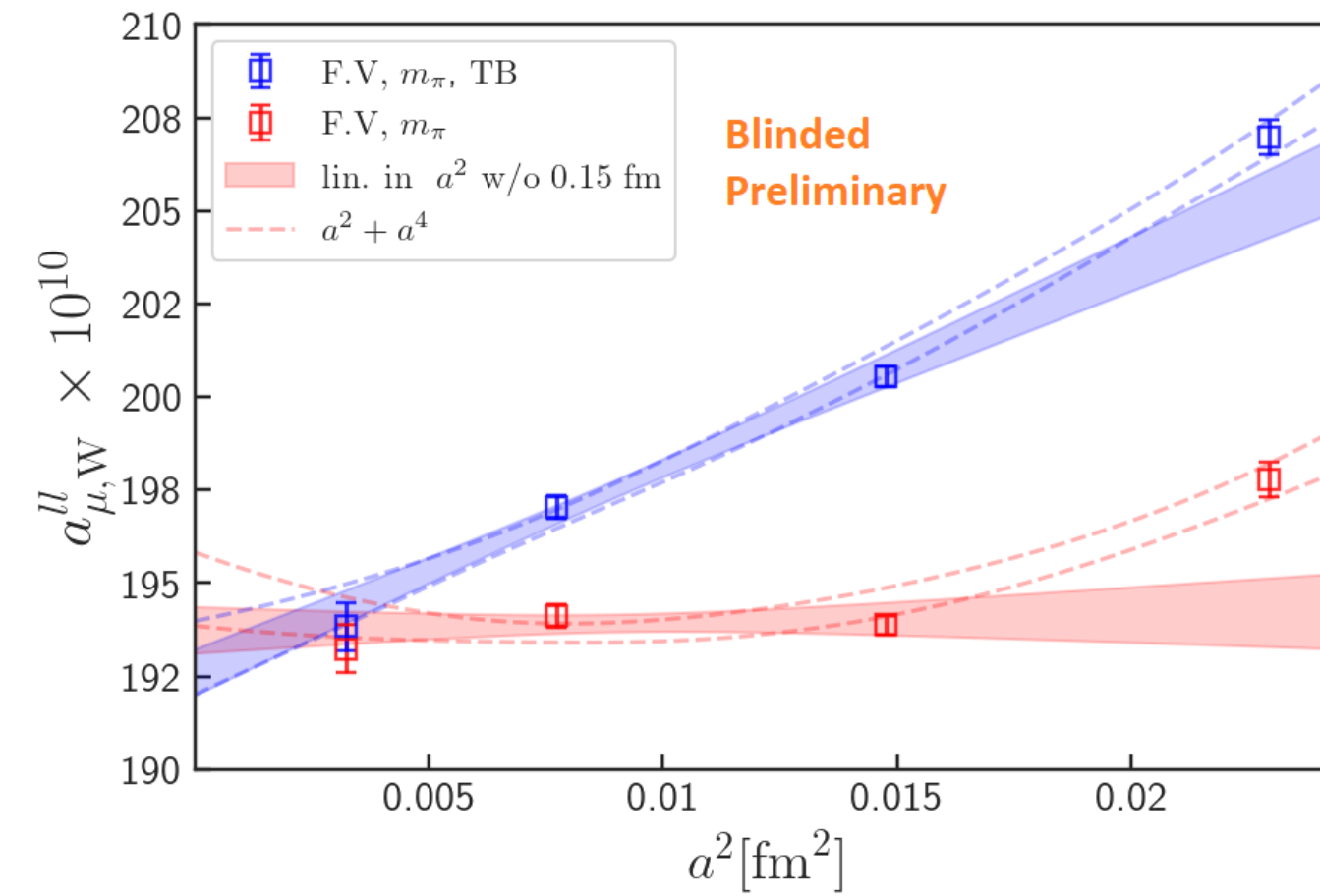
Intermediate window (ud)

BMWc [S. Borsanyi et al, arXiv:2002.12347, Nature 2021]



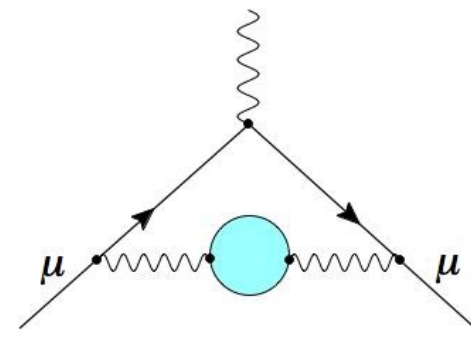
-3.7 σ tension with data-driven evaluation
 -2.2 σ tension with RBC/UKQCD18

Shaun Lahert @ Lattice 2021 (updated)



Ongoing work:

- continuum limit with & without taste-breaking corrections
- varying functional forms for continuum extrapolations
- FV corrections using different approaches

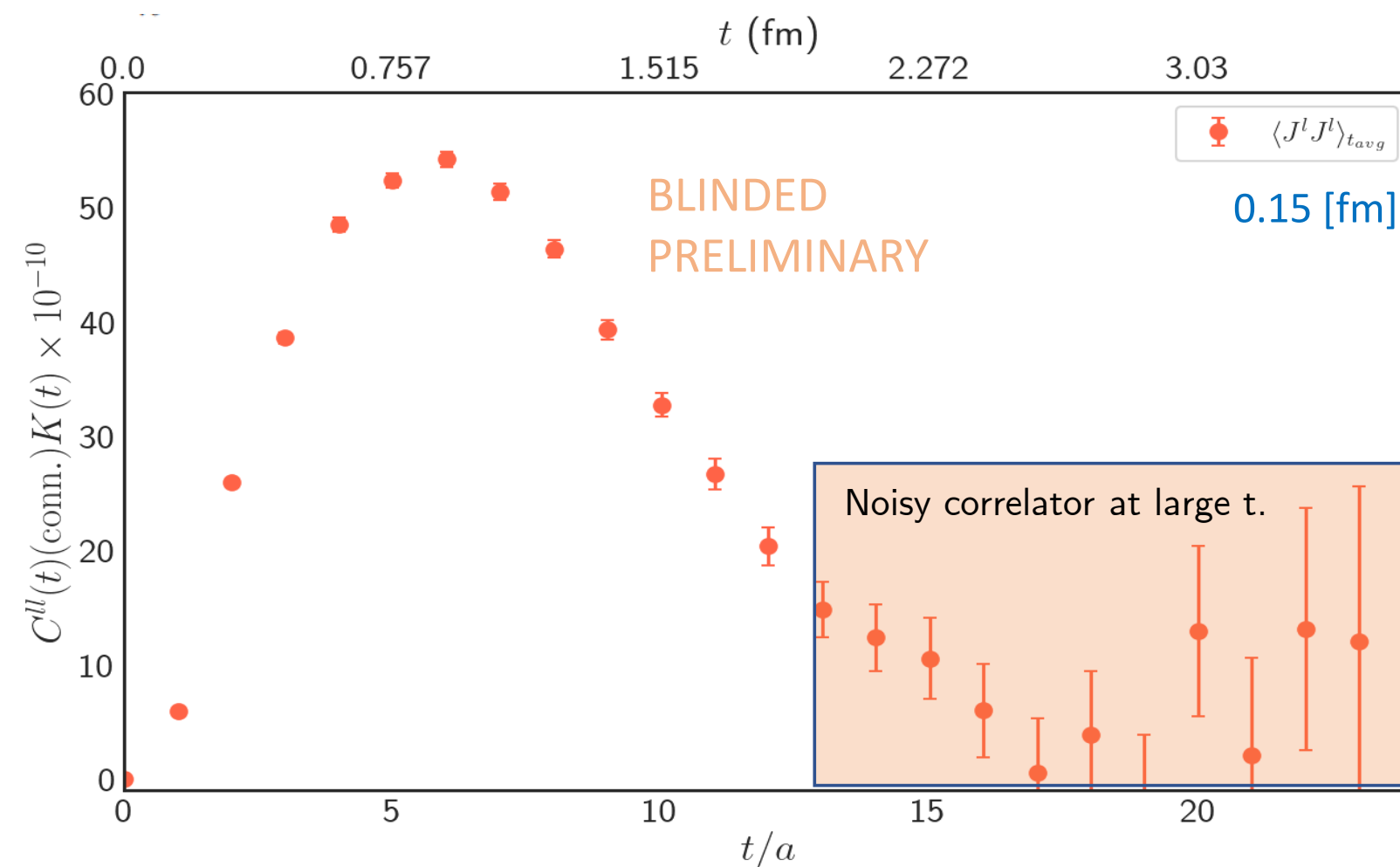


Long-distance tail (ud)

Shaun Lahert @ Lattice 2021

$$G(t) = \frac{1}{3} \sum_{i,x} \langle j_i(x,t) j_i(0,0) \rangle$$

- Growth of statistical noise at large t:

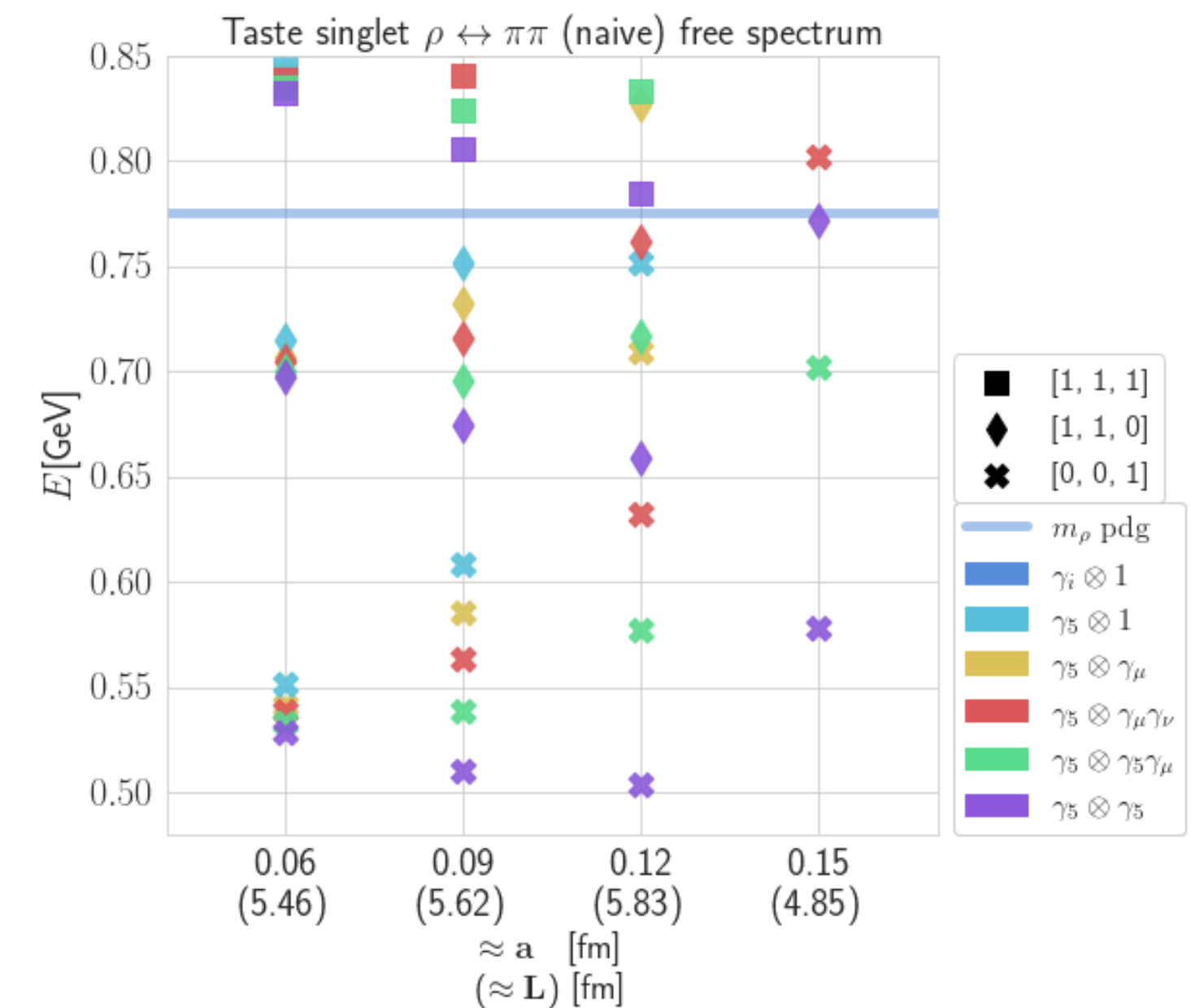


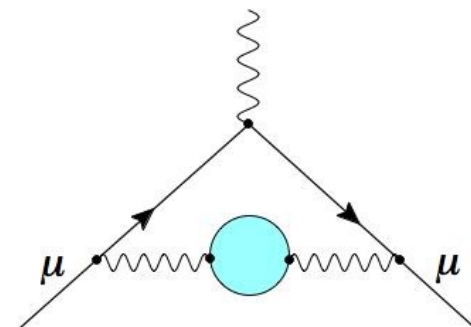
- Spectral reconstruction:

- ♦ obtain low-lying finite-volume spectrum (E_n, A_n) in dedicated study using additional operators that couple to two-pion states
- ♦ First LQCD calculation with staggered multi-pion operators
- ♦ Construct matrix of correlators (2,3,4-point functions):

$$\mathbf{C}(t) = \begin{pmatrix} C(t)_{J,\bar{J} \rightarrow J,\bar{J}} & C(t)_{J,\bar{J} \rightarrow \pi\pi} \\ C(t)_{\pi\pi \rightarrow J,\bar{J}} & C(t)_{\pi\pi \rightarrow \pi\pi} \end{pmatrix}$$

- ♦ Use GEVP to obtain energies and amplitudes for $\pi\pi$ states
- ♦ Reconstruct vector-current correlator





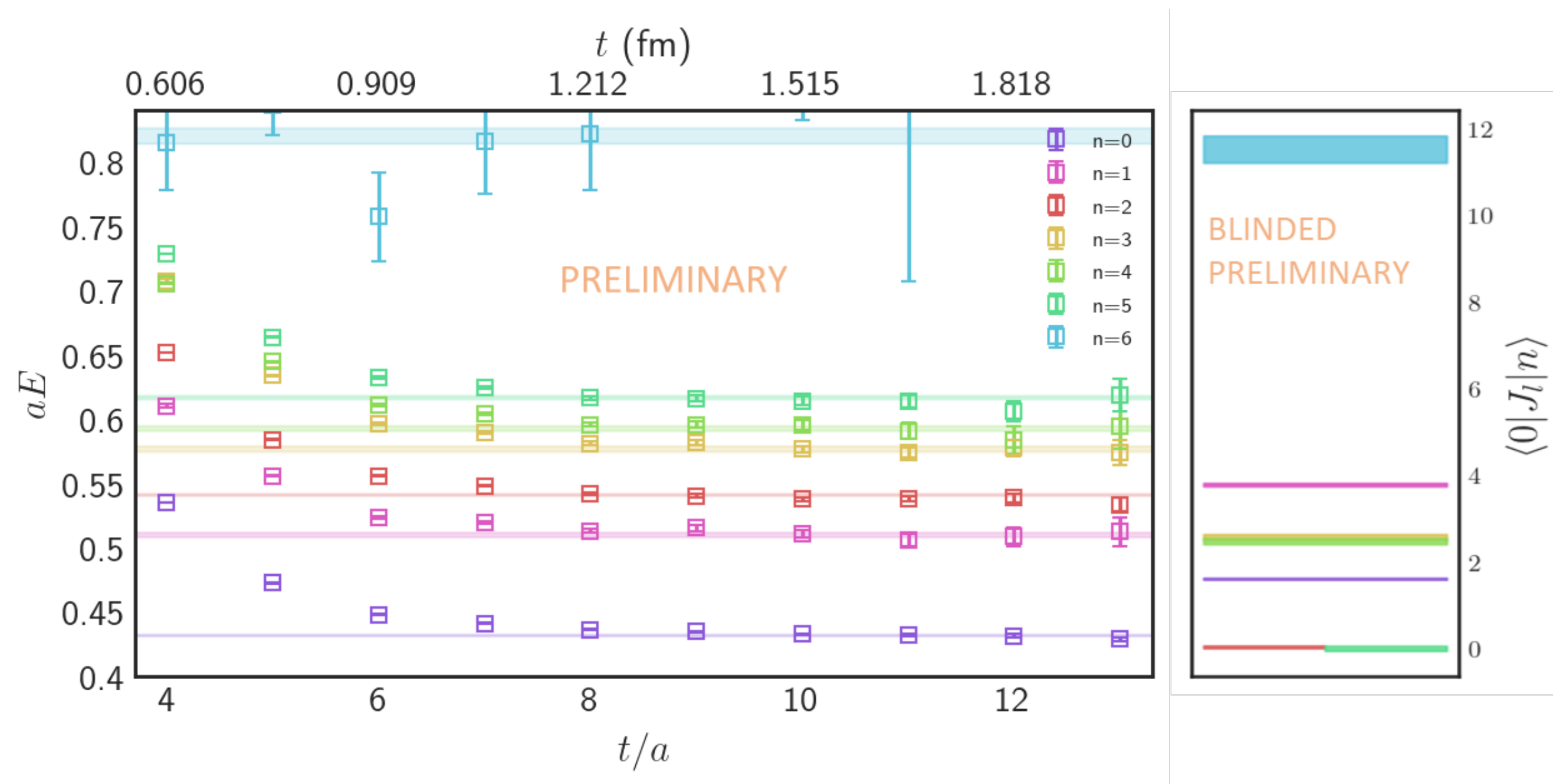
Long-distance tail (ud)

Shaun Lahert @ Lattice 2021

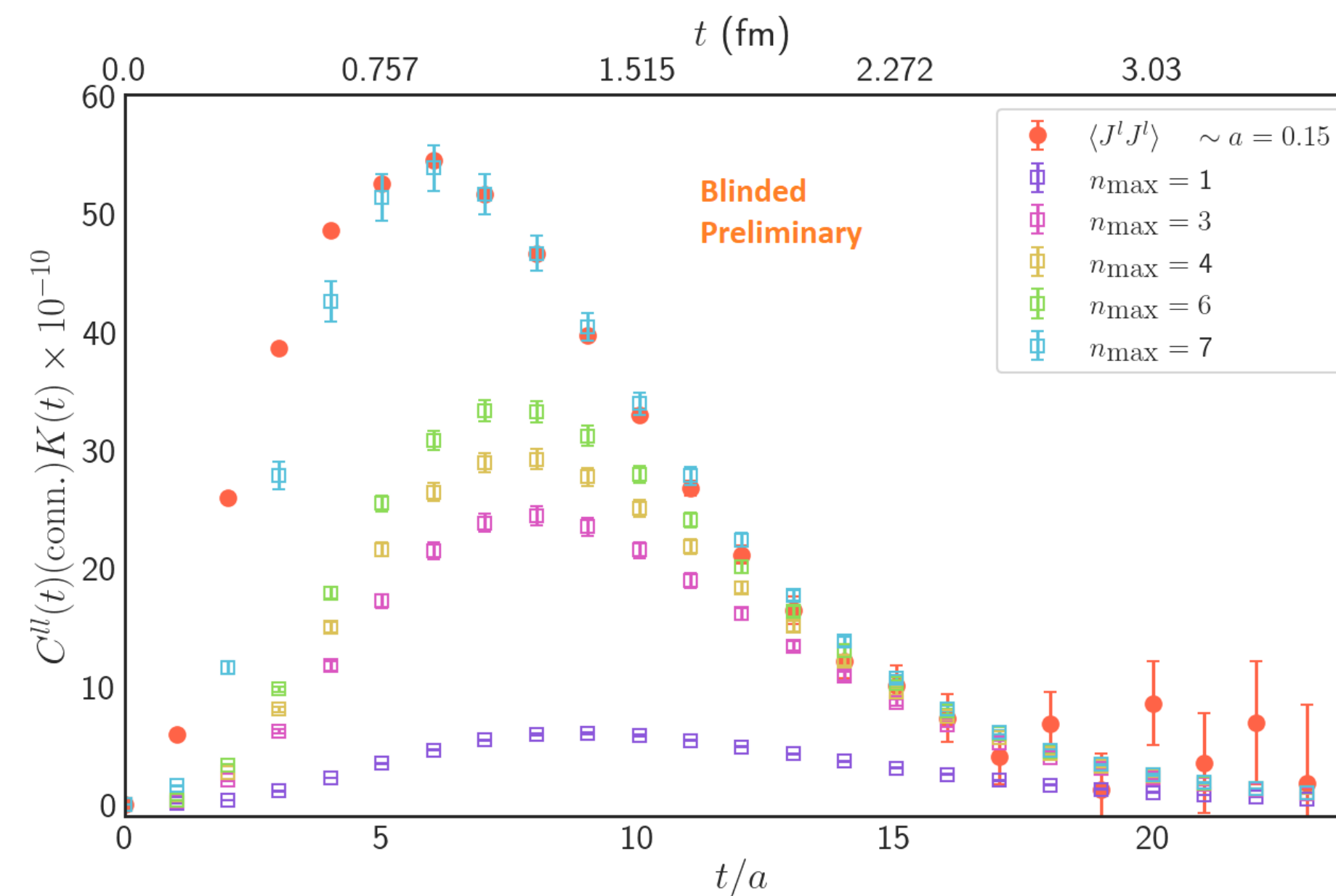
$$G(t) = \frac{1}{3} \sum_{i,x} \langle j_i(x,t) j_i(0,0) \rangle$$

• Spectral reconstruction:

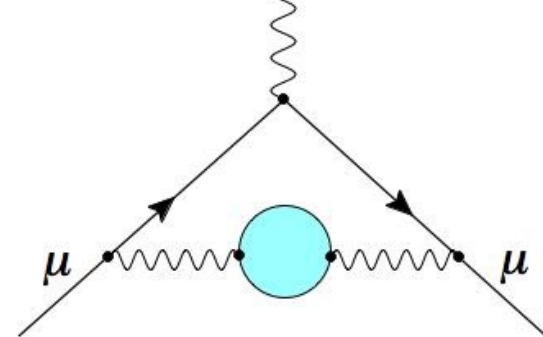
- ◆ obtain low-lying finite-volume spectrum (E_n, A_n) in dedicated study using additional operators that couple to two-pion states



- ◆ Reconstruct vector current correlator at large t



~four-fold reduction in statistical uncertainty compared to bounding method



Disconnected contribution

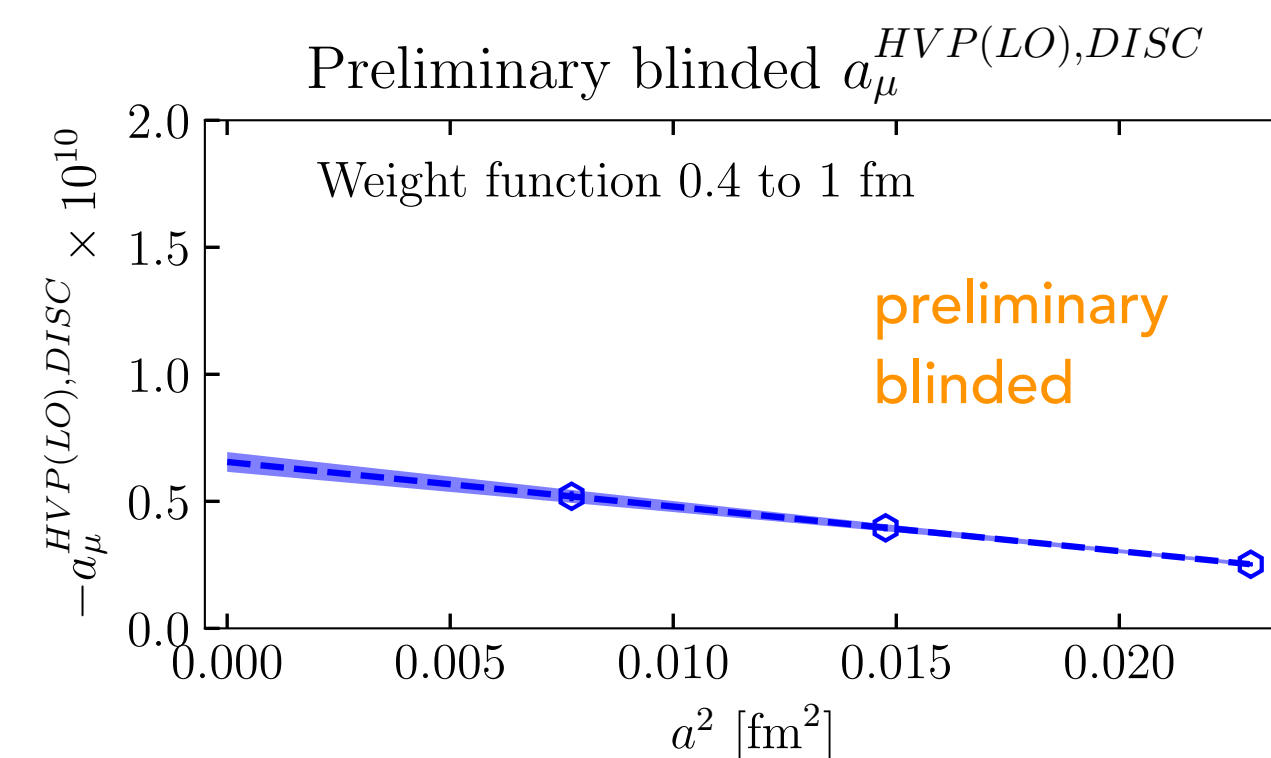
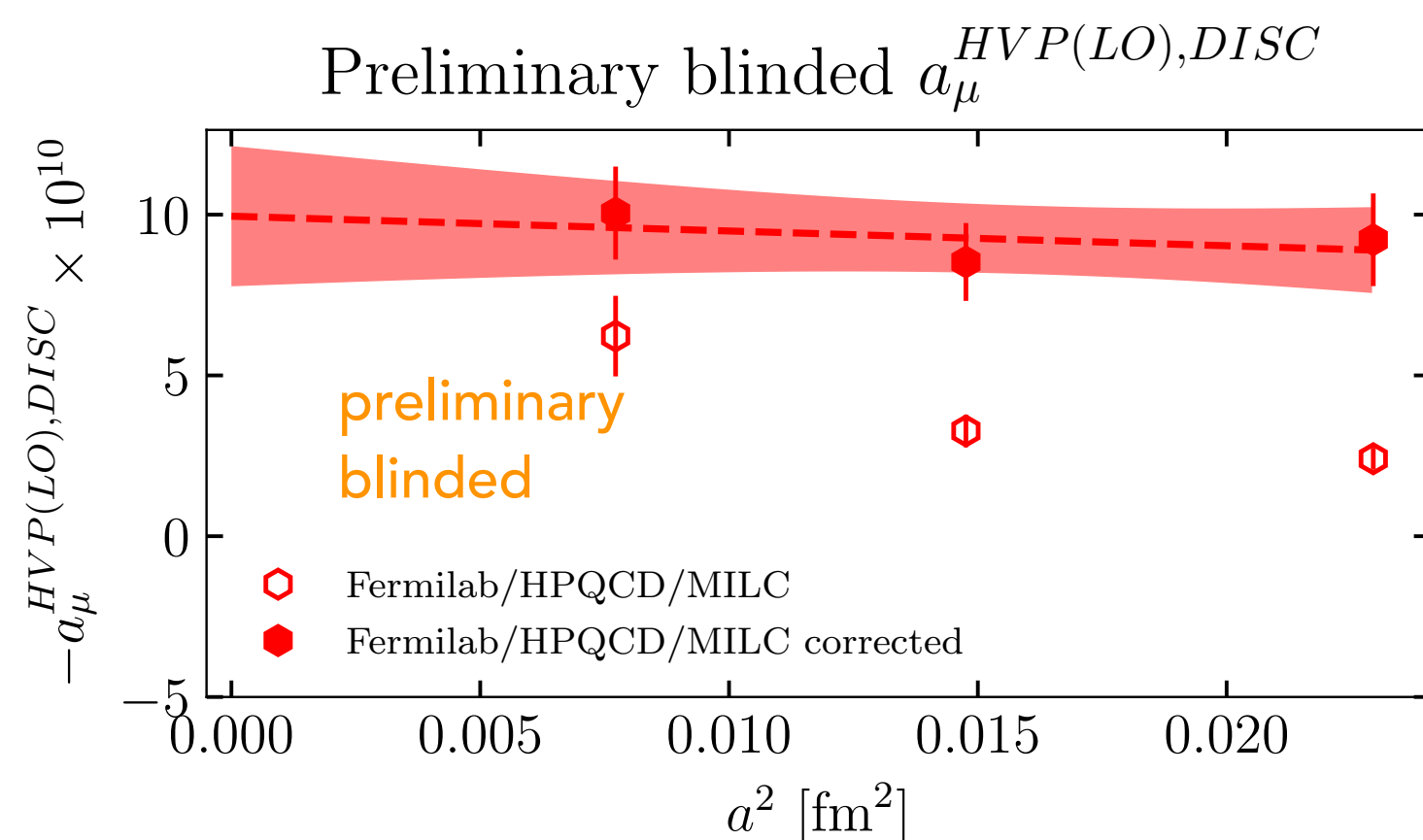
Craig McNeile @ Lattice 2021



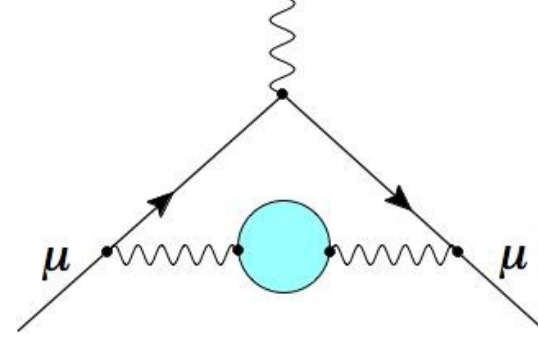
a fm	m_π MeV	L fm	Eigenmodes	N_{meas}
0.15	134.7	4.8	300	1692
0.12	134.9	5.8	-	787
0.09	128.3	5.8	1000	271

Ongoing work:

- computing SIB correction of disconnected contribution
- increasing statistics
- continuum limit with & without taste-breaking corrections
- varying functional forms for continuum extrapolations
- FV corrections using different approaches



No taste-breaking corrections



SIB + QED corrections

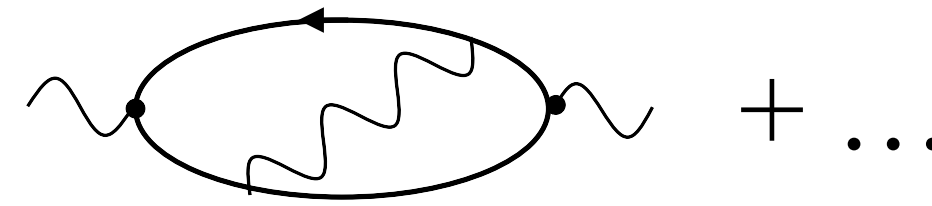
Craig McNeile @ Lattice 2021

Set-up for QED corrections: (Gaurav Ray)

- Quenched QED_L study
- Physical mass ensembles at 3 lattice spacings (0.15, 0.12, 0.09 fm)
- Range of valence quark masses: light → strange

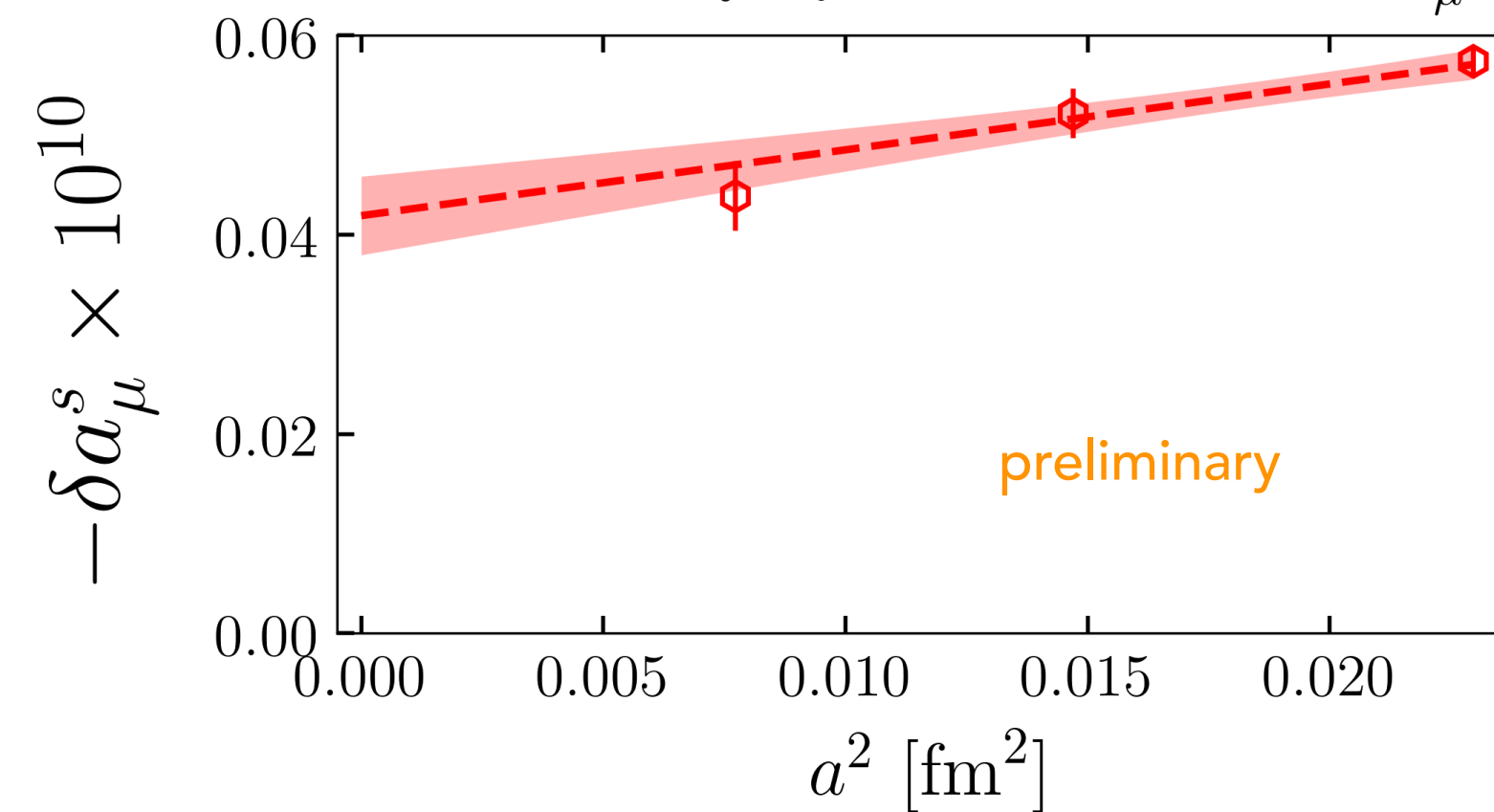
Ongoing work:

- Increasing statistics
- retune quark masses
- varying functional forms for continuum extrapolations
- dynamical QED+QCD ensemble @ 0.15 fm being generated
- disc. QED corrections



$$\delta a_\mu^s = a_\mu^s [QCD + qQED] - a_\mu^s [QCD]$$

Preliminary QED corrections to a_μ^s



Ongoing work SIB corrections: (Curtis Peterson)

- follow-up to previous work [Chakraborty et al, [arXiv:1710.11212](https://arxiv.org/abs/1710.11212), 2018 PRL]:
- analyzing data @ 0.12fm
- add data at finer lattice spacings
- use results to study M_π dependence

Summary and Outlook

- ★ Ultimate goal: $a_\mu^{\text{HVP,LO}}$ at $\lesssim 0.5\%$
- ★ Blind analysis of dominant contributions
- ★ light-quark connected contribution is needed with commensurate precision:
 - 📍 Increasing statistics at finest lattice spacings
 - 📍 Reduce scale setting uncertainty (Ω baryon mass, refined f_π)
 - 📍 Refine estimates of FV and other corrections
 - 📍 Reconstruction of $\pi\pi$ tail
 - 📍 Add data at a fifth lattice spacing
- ★ Intermediate window: analysis in progress (~early 2022)
- ★ Disconnected contribution (includes SIB): goal $< 10\%$
 - 📍 Increasing statistics, refining analysis
- ★ QED corrections:
 - 📍 Quenched QED_L study at three lattice spacings (phys. mass ensembles)

A watercolor drawing on a light-colored background. The central focus is a starburst or sunburst pattern. It consists of several radiating lines or streaks. The top and bottom streaks are composed of small, repeating circles in shades of red and pink. The left and right streaks are primarily blue with some green and yellow accents. In the center, there are larger, more varied circles in colors like red, yellow, green, and purple. The overall effect is a vibrant, multi-colored burst of light or energy.

Thank you!

Appendix

Staggered two-pion operators

Shaun Lahert @ Lattice 2021

Staggered two-pion operators

Staggered pion operator

$$\pi^-(\vec{p})_\xi = \sum_x e^{i\vec{p}\cdot\vec{x}} \bar{u}(x) \gamma_5 \otimes \gamma_\xi d(x)$$

Zero momentum taste irreps:

Scalar & Pseudo-scalar (1D)
Vector & Pseudo-vector (3D)
Tensor (3D+3D)

Non-zero momentum taste irreps:

1D irreps are Wilson like.
3D irreps can split (1D+2D) based on momentum & taste direction alignment.

Two-pion operators

$$\mathcal{O}_{\pi\pi}(0) = \sum_{\substack{\xi_1, \xi_2 \\ \vec{p} \in \{p\}^*}} CG_{\text{stag, iso.}} \pi(\vec{p})_{\xi_1} \pi(-\vec{p})_{\xi_2}$$

Clebsch-Gordon's from irreducible representations of staggered lattice symmetry group.

$$SU_I(2) \times \left(T_N^3 \rtimes \{ \Xi_\mu, C_0 \} \rtimes \{ \tilde{R}_{ij}, I_S \} \right) = SU(2) \times (Z_N^3 \rtimes \Gamma_{4,1} \rtimes W_3)$$

Wigner's method twice ~ little groups of little groups.

$$CG = \text{unitarity} : \sum_{g \in G} \left[D^{(\pi_1)}(g) \times D^{(\pi_2)}(g) \right] AD^{(J)}(g)^\dagger$$

-Sakata, J.Math.Phys. 15 (1974) 1702-1709

Taste scalar vector current "one-link" -> taste diagonal two pion states.

Operator	Momentum (back-to-back)
J_l, \tilde{J}_l	
$\mathcal{O}_{\pi\pi}^{\otimes \gamma_5}$	[0, 0, 1], [1, 1, 0]
$\mathcal{O}_{\pi\pi}^{\otimes \gamma_{5x/y}}, \mathcal{O}_{\pi\pi}^{\otimes \gamma_{5z}}$	[0, 0, 1]
$\mathcal{O}_{\pi\pi}^{\otimes \gamma_{xt/yt}}, \mathcal{O}_{\pi\pi}^{\otimes \gamma_{zt}}$	[0, 0, 1]

$$\mathbf{C}(t) = \begin{pmatrix} C_{J_l, \tilde{J}_l \rightarrow J_l, \tilde{J}_l}(t) & C_{J_l, \tilde{J}_l \rightarrow \pi\pi}(t) \\ C_{\pi\pi \rightarrow J_l, \tilde{J}_l}(t) & C_{\pi\pi \rightarrow \pi\pi}(t) \end{pmatrix}$$

$$\mathbf{C}(t)v = \lambda \mathbf{C}(t_0)v.$$

Solve GEVP:

$\chi_n = (v_n)_i \mathcal{O}_i$ with best overlap with state n .

$$\langle \chi_n \chi_n^\dagger \rangle = \sum_n Z_n^2 e^{-E_n t}$$

$$\langle \chi_n J_l^\dagger \rangle = \sum_n Z_n \langle 0 | J_l | n \rangle e^{-E_n t}$$