Opportunities and problems with inclusive processes

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New challenges for lattice, in flavor physics

What can we compute other than the form factors? (choices of my interest)

$B \to X_c \ell\nu$ (inclusive)

$B \to X_u \ell\nu$ (inclusive)

$B \to X_s \ell^+\ell^-$ (inclusive)

hadronic decays: $B \to \pi\pi$ etc

D^0–$\bar{D}^0$ mixing

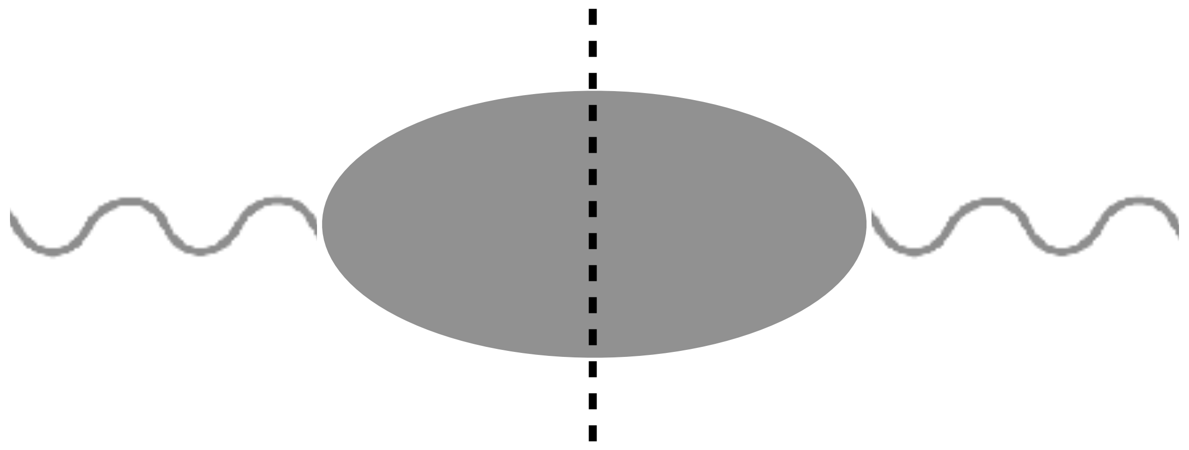
$\tau(D^+)/\tau(D^0)$ (inclusive hadronic)

$B \to K^{(*)}\ell^+\ell^-$ (charm loop)

→ View from a different angle:
reweighting of the spectral function
(or a "smeared spectrum")

Spectral function at work

Muon g-2: a well-known story



spectral function

$$a_\mu^{\rm HVP} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \frac{ds}{s} \overbrace{\frac{1}{\pi}{\rm Im}\Pi(s)}\, K(s)$$

"smeared spectrum"

$$a_\mu^{\rm HVP} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dQ^2\, f(Q^2)\hat{\Pi}(Q^2)$$

Bernecker-Meyer (2011)

$$\hat{\Pi}(Q^2) = 4\pi^2 \int_0^\infty dt\, C(t) \left[t^2 - \frac{4}{Q^2}\sin^2\frac{Qt}{2}\right]$$

$$a_\mu^{\rm HVP} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, C(t)\tilde{f}(t)$$

Lesson: "smeared spectrum" can be written using Euclidean correlator.

Spectrum → Physics

see also, Hansen, Meyer, Robaina, arXiv:1704.08993 (idea to go through approx spectrum)

Spectral function: $\rho(\omega) \propto \sum_X \delta(\omega - E_X) |\langle X|J|0\rangle|^2 \quad \sim \langle 0|J\,\delta(\omega - \hat{H})\,J|0\rangle$

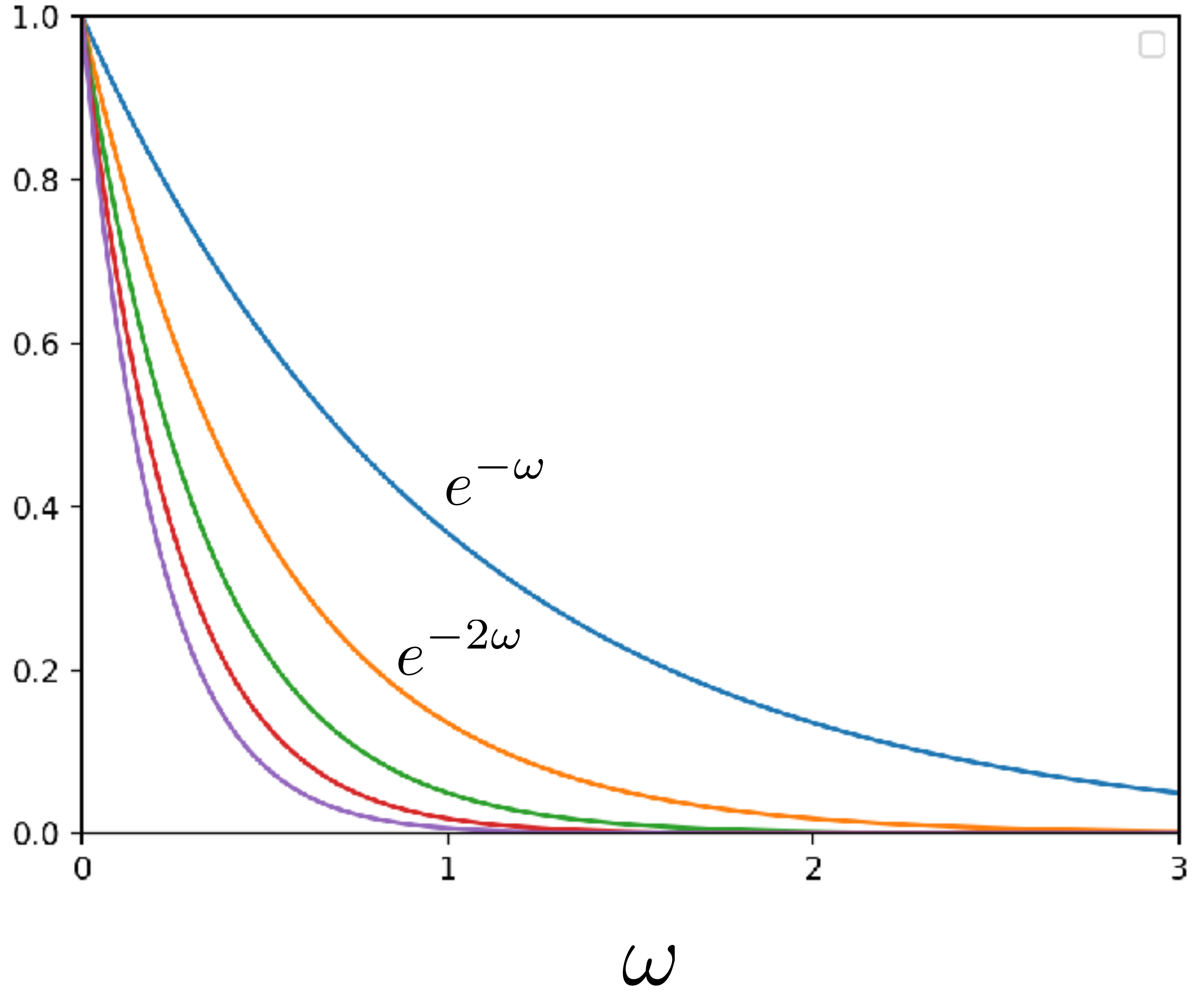
What you want: $\bar{K} = \int_0^\infty d\omega\, K(\omega)\rho(\omega) \quad \sim \langle 0|J K(\hat{H}) J|0\rangle$

approx: $K(\hat{H})$ by $e^{-\hat{H}t}$?

What you have: $C(t) = \int_0^\infty d\omega\, \rho(\omega) e^{-\omega t} \quad \sim \langle 0|J\, e^{-\hat{H}t}\, J|0\rangle$

Approximation

$$K(\hat{H}) \simeq k_0 + k_1 e^{-\hat{H}} + k_2 e^{-2\hat{H}} + \cdots + k_N e^{-N\hat{H}}$$



- Not always possible; when the function varies rapidly, in particular.
- Some methods developed recently.

  - Modified Backus-Gilbert

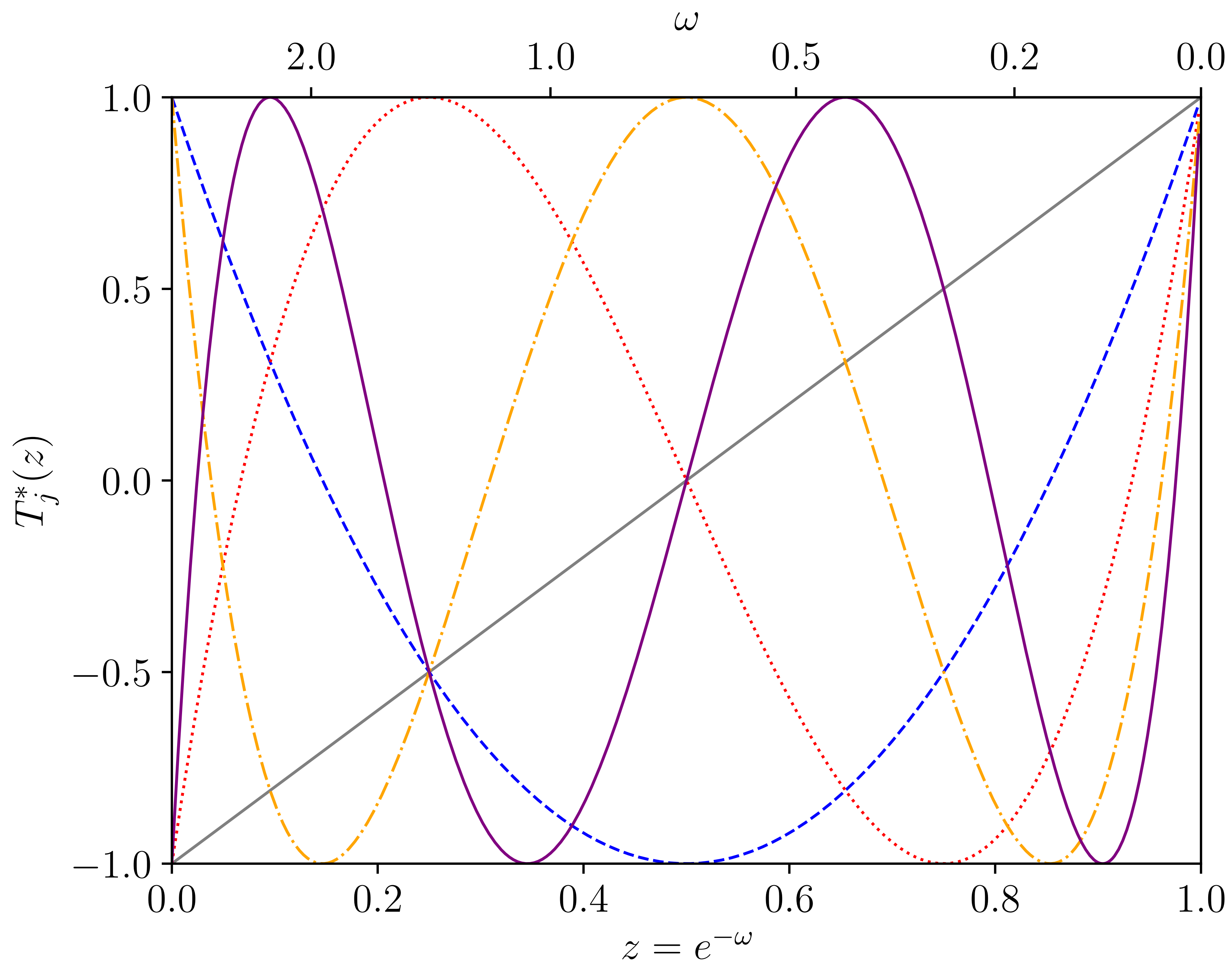
    Hansen, Lupo, Tantalo, arXiv:1903.06476
  - Or, Chebyshev polynomial

    Bailas, Ishikawa, SH, arXiv:2001.11779

Chebyshev approx:

Bailas, Ishikawa, SH, arXiv:2001.11779

$$K(\omega) \simeq \frac{c_0}{2} + \sum_{j=1}^{N} c_j^* T_j^*(e^{-\omega})$$



(shifted) Chebyshev polynomials

$$T_0^*(x) = 1$$
$$T_1^*(x) = 2x - 1$$
$$T_2^*(x) = 8x^2 - 8x + 1$$

$x^t \to C(t)$

$$T_{j+1}^*(x) = 2(2x-1)T_j^*(x) - T_{j-1}^*(x)$$

"Best" approximation can be obtained with

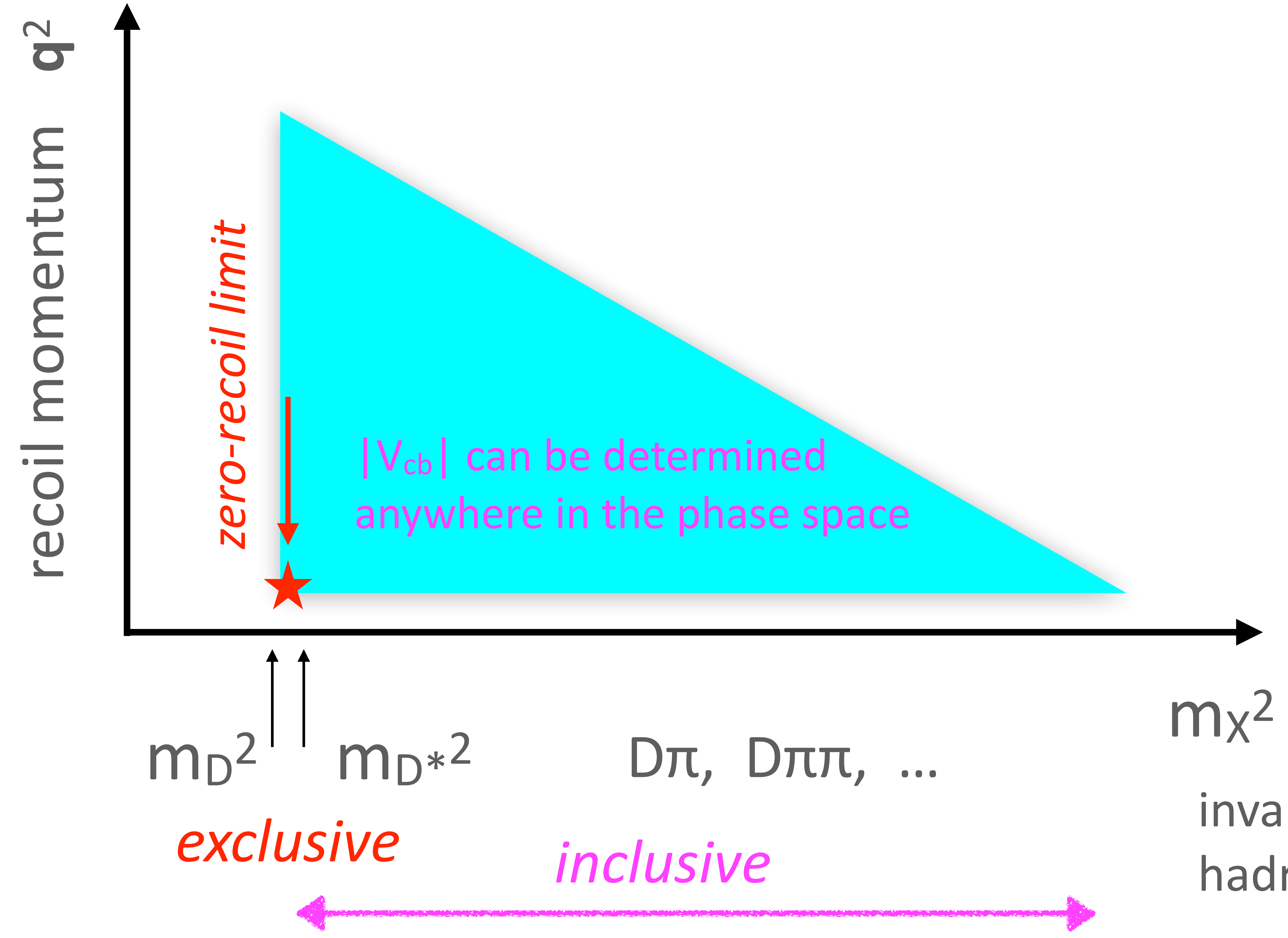
$$c_j^* = \frac{2}{\pi}\int_0^\pi d\theta\, S\left(-\ln\frac{1+\cos\theta}{2}\right)\cos(j\theta)$$

- Constraint $|T_j^*(e^{-\omega})| \leq 1$ stabilizes the expansion.
- Higher orders are suppressed when the coefficients are. It is the case for smooth function $K(\omega)$

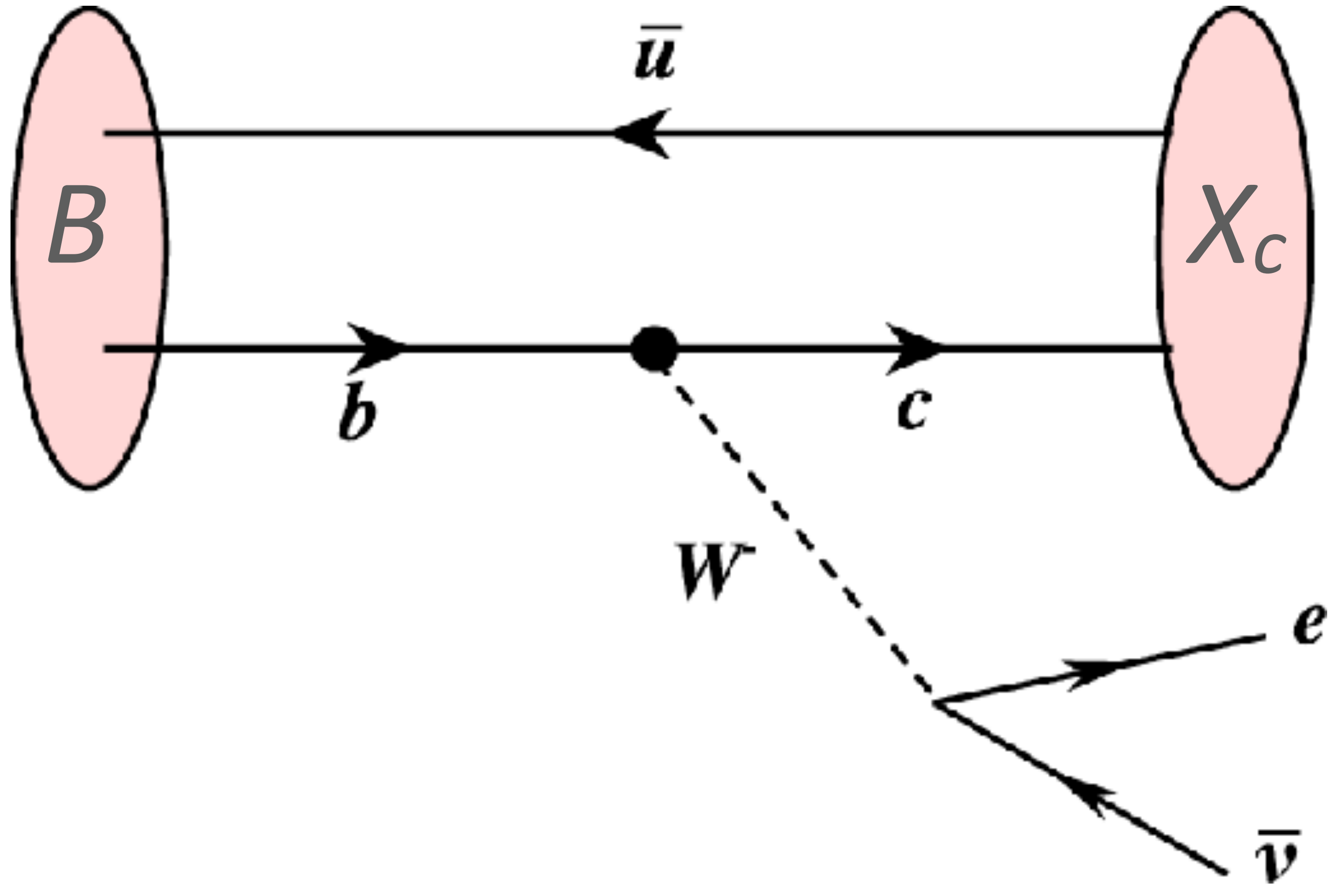
Inclusive semileptonic decay

Inclusive versus exclusive

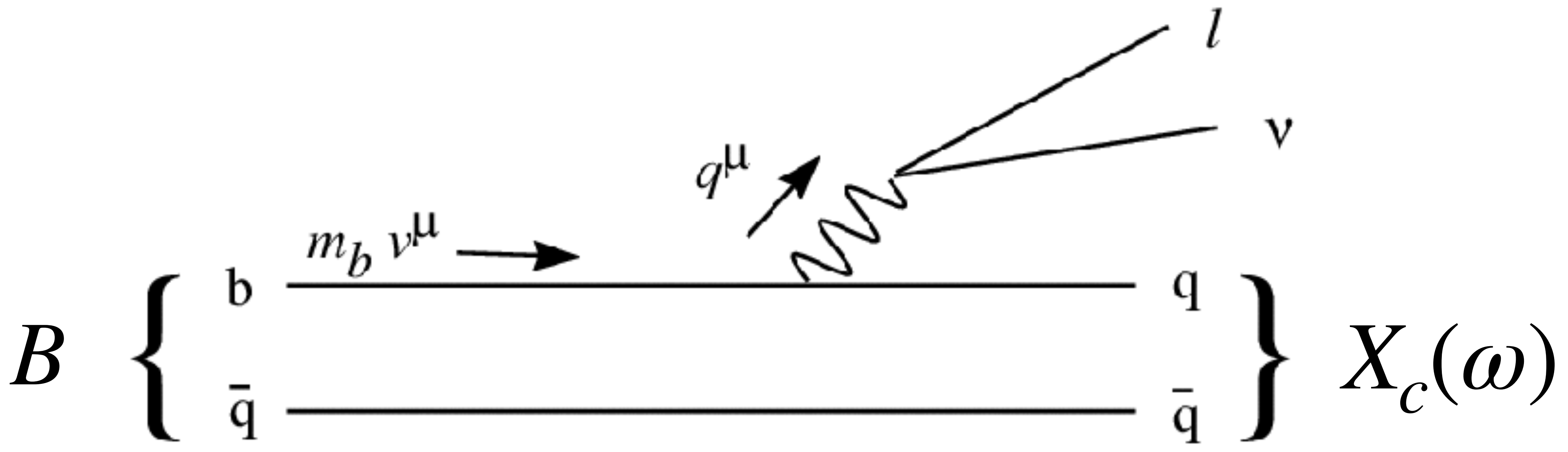
Can we treat the both on the lattice?



invariant mass of the
hadronic system



Inclusive rate



Differential decay rate:

$$d\Gamma \sim |V_{cb}|^2 l^{\mu\nu} W_{\mu\nu}$$

Structure function:

$$W_{\mu\nu} = \sum_X (2\pi)^2 \delta^4(p_B - q - p_X) \frac{1}{2M_B} \langle B(p_B)|J_\mu^\dagger(0)|X\rangle\langle X|J_\nu(0)|B(p_B)\rangle$$

$$\rightarrow \langle B(\mathbf{0})|\tilde{J}_\mu^\dagger(-\boldsymbol{q};t)\ \delta(\omega - \hat{H})\ \tilde{J}_\nu(\boldsymbol{q};0)|B(\mathbf{0})\rangle$$

"spectral function"

Decay rate:

$$\Gamma \propto \int_0^{\boldsymbol{q}^2_{\max}} d\boldsymbol{q} \int_{\sqrt{m_D^2+\boldsymbol{q}^2}}^{m_B-\sqrt{\boldsymbol{q}^2}} d\omega\, K(\omega;\boldsymbol{q}^2)\langle B(\mathbf{0})|\tilde{J}^\dagger(-\boldsymbol{q})\delta(\omega-\hat{H})\tilde{J}(\boldsymbol{q})|B(\mathbf{0})\rangle$$

known kinematical factor

Sum over states = energy integral

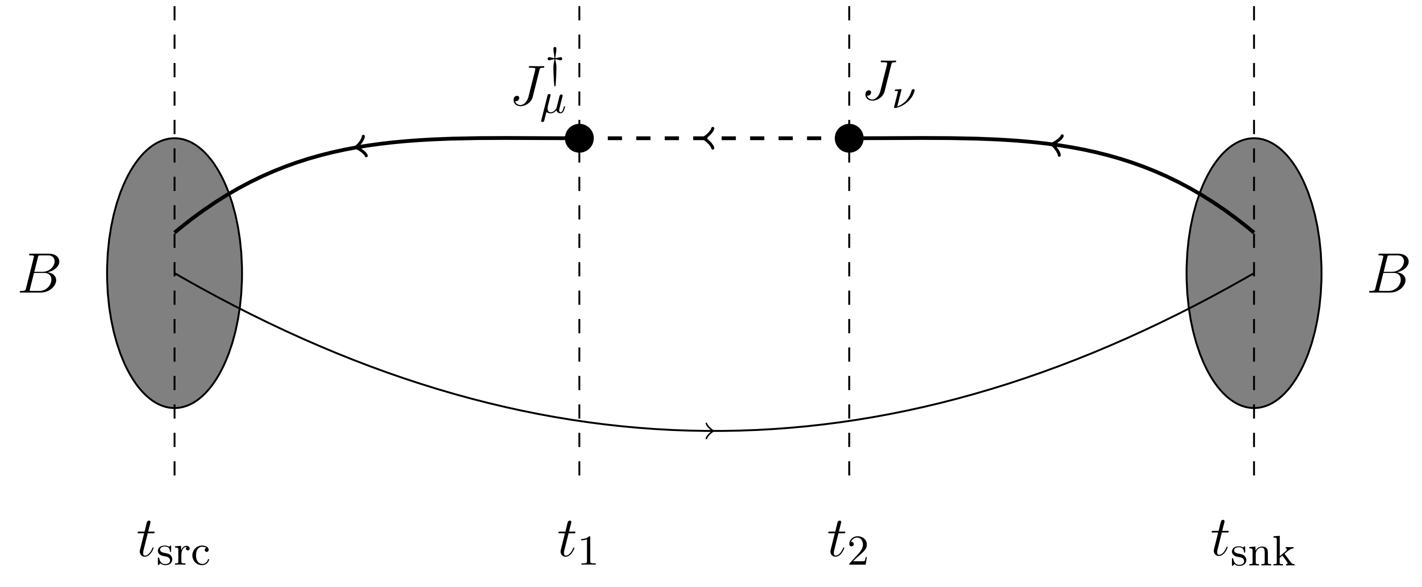
$$\Gamma \propto \int_0^{\boldsymbol{q}^2_{\max}} d\boldsymbol{q} \int_{\sqrt{m_D^2+\boldsymbol{q}^2}}^{m_B-\sqrt{\boldsymbol{q}^2}} d\omega\, K(\omega;\boldsymbol{q}^2) \langle B(\boldsymbol{0})|\tilde{J}^\dagger(-\boldsymbol{q})\delta(\omega-\hat{H})\tilde{J}(\boldsymbol{q})|B(\boldsymbol{0})\rangle$$

$$= \langle B(\boldsymbol{0})|\tilde{J}^\dagger(-\boldsymbol{q})K(\hat{H};\boldsymbol{q}^2)\tilde{J}(\boldsymbol{q})|B(\boldsymbol{0})\rangle$$

"smeared spectral function"

Lattice Compton amplitude:

$$\langle B(\boldsymbol{0})|\tilde{J}^\dagger_\mu(-\boldsymbol{q};t)\ \tilde{J}_\nu(\boldsymbol{q};0)|B(\boldsymbol{0})\rangle \longrightarrow \langle B(\boldsymbol{0})|\tilde{J}^\dagger(-\boldsymbol{q})e^{-\hat{H}t}\tilde{J}(\boldsymbol{q})|B(\boldsymbol{0})\rangle$$



approx: $K(\hat{H})$ by $e^{-\hat{H}t}$?

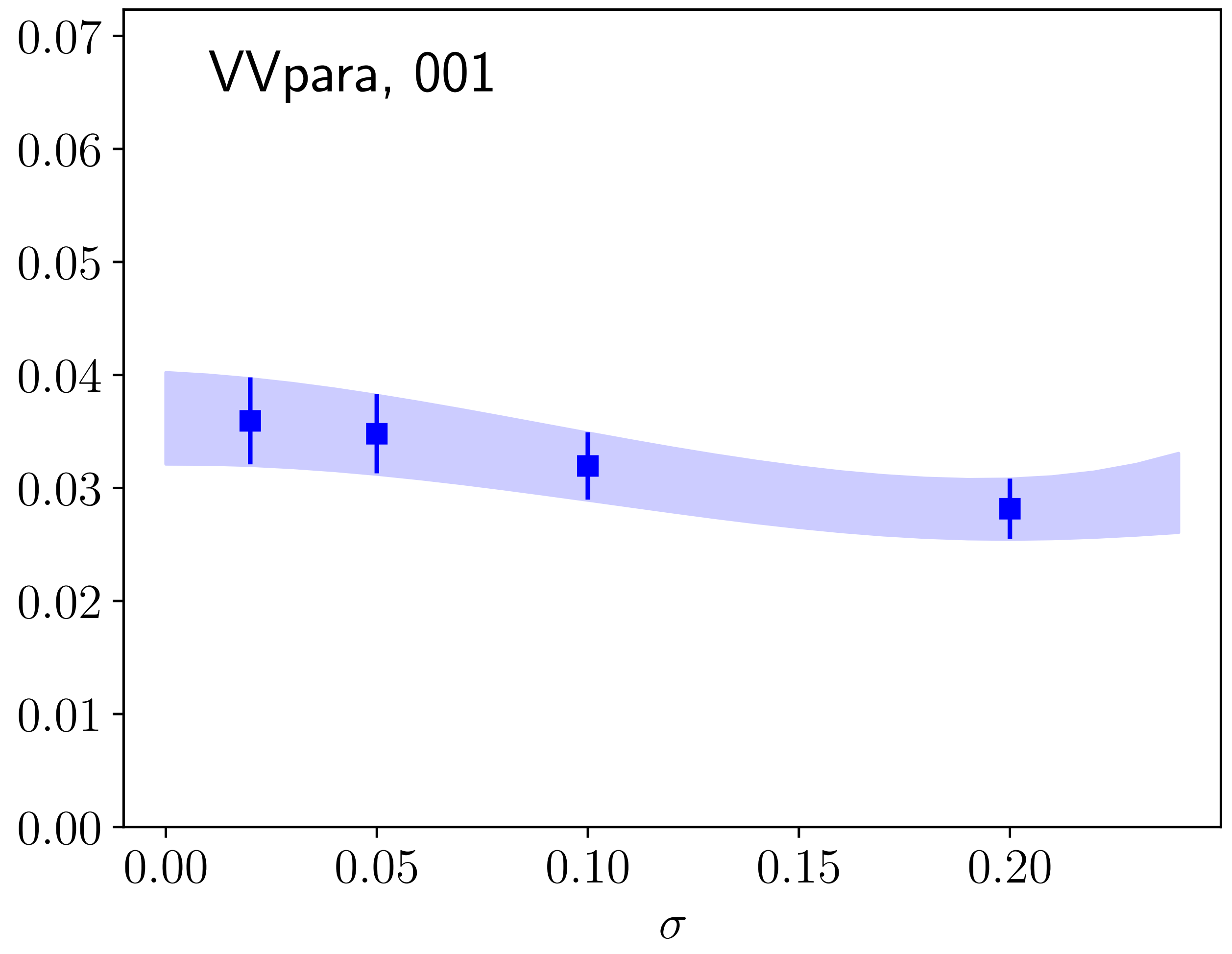
Kernel to approximate

To implement the upper limit of integ

$$K(\omega) \sim e^{2\omega t_0} (m_B - \omega)^l \theta(m_B - |\mathbf{q}| - \omega)$$

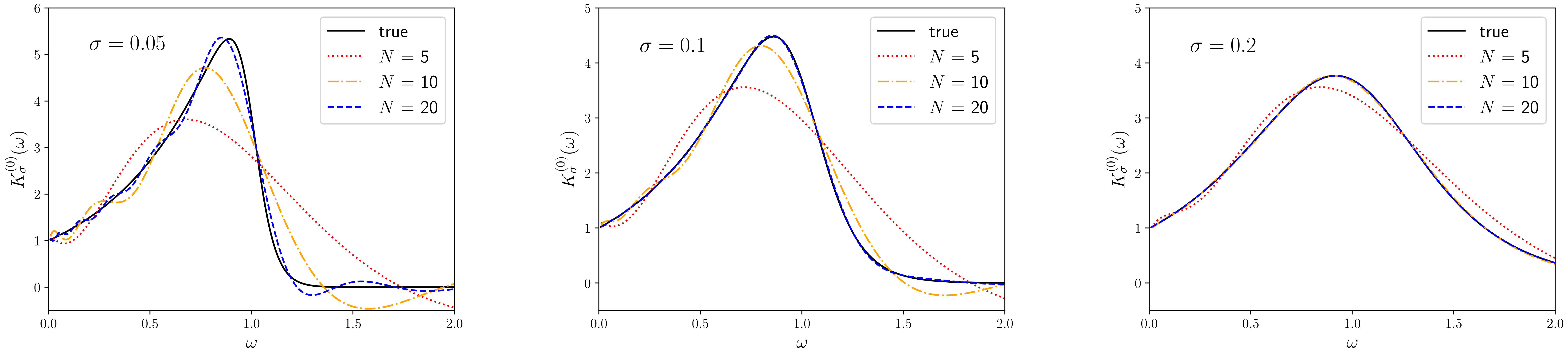
kinematical factor

Smear by "sigmoid" with a width σ
Need to take a limit of σ→ 0



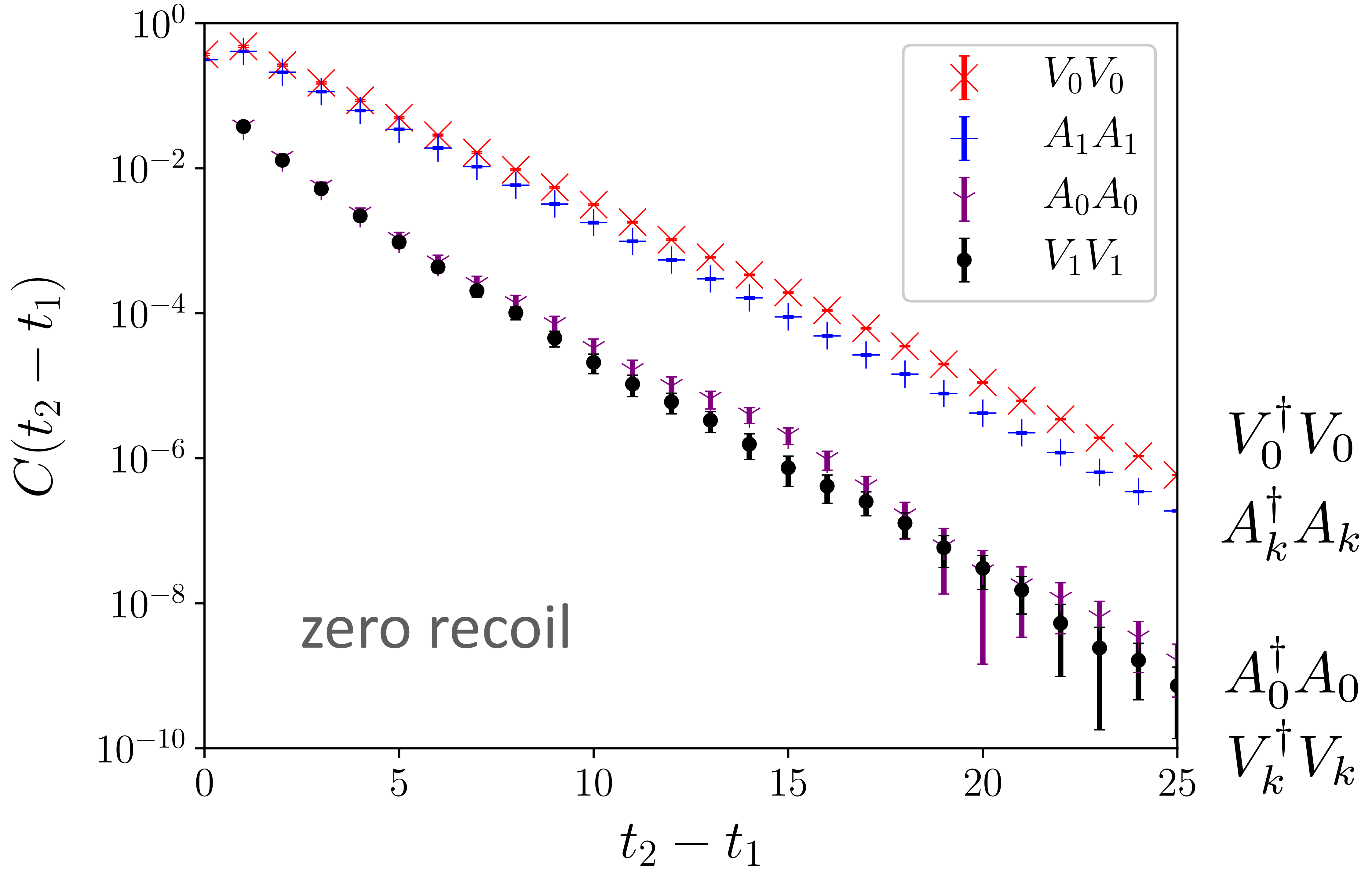
narrow

wide



Compton amplitude

$$\langle B(\mathbf{0})|\tilde{J}^{\dagger}_{\mu}(-\boldsymbol{q};t)\ \ \tilde{J}_{\nu}(\boldsymbol{q};0)|B(\mathbf{0})\rangle$$



Pilot lattice computation [JLQCD setup]

- On a lattice of 48^3x96 at $1/a$ = 3.6 GeV
- Strange spectator quark
- physical charm quark mass
- (unphysically) light b quark ~ 2.7 GeV
- 100 configs x 4 src

S-wave (D and D*)

- Very well approximated by a single-exp = no sign of excited state contrib.

P-wave (D**'s)

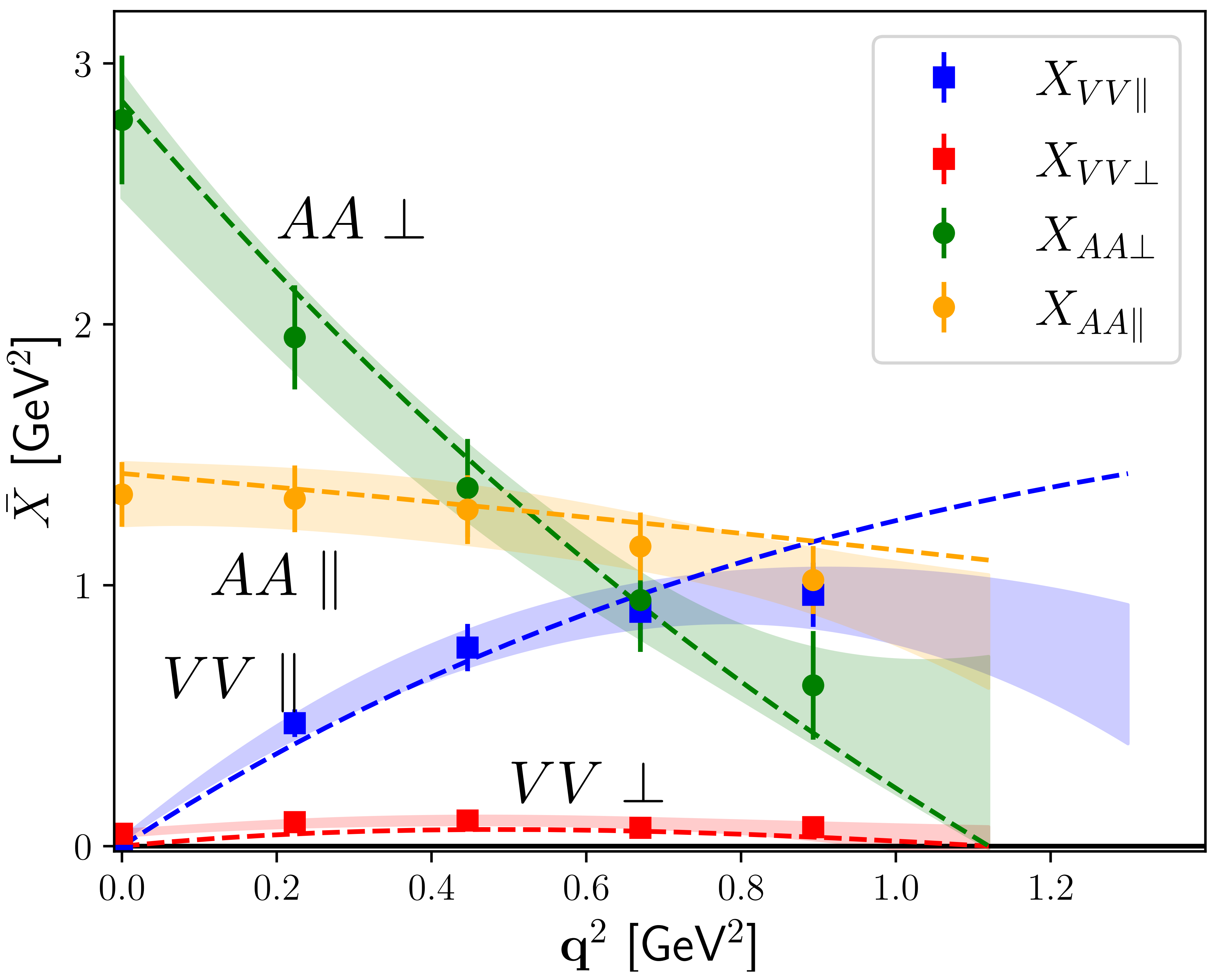
- Small : no wave function overlap of excited states when m_b=m_c and zero recoil

Inclusive decay rate

- Breakdown to individual channels: VV and AA; parallel and perp with respect to the recoil momentum
- Compared to exclusive contributions estimated from B→D(*) form factors (dashed line), that are separately calculated.

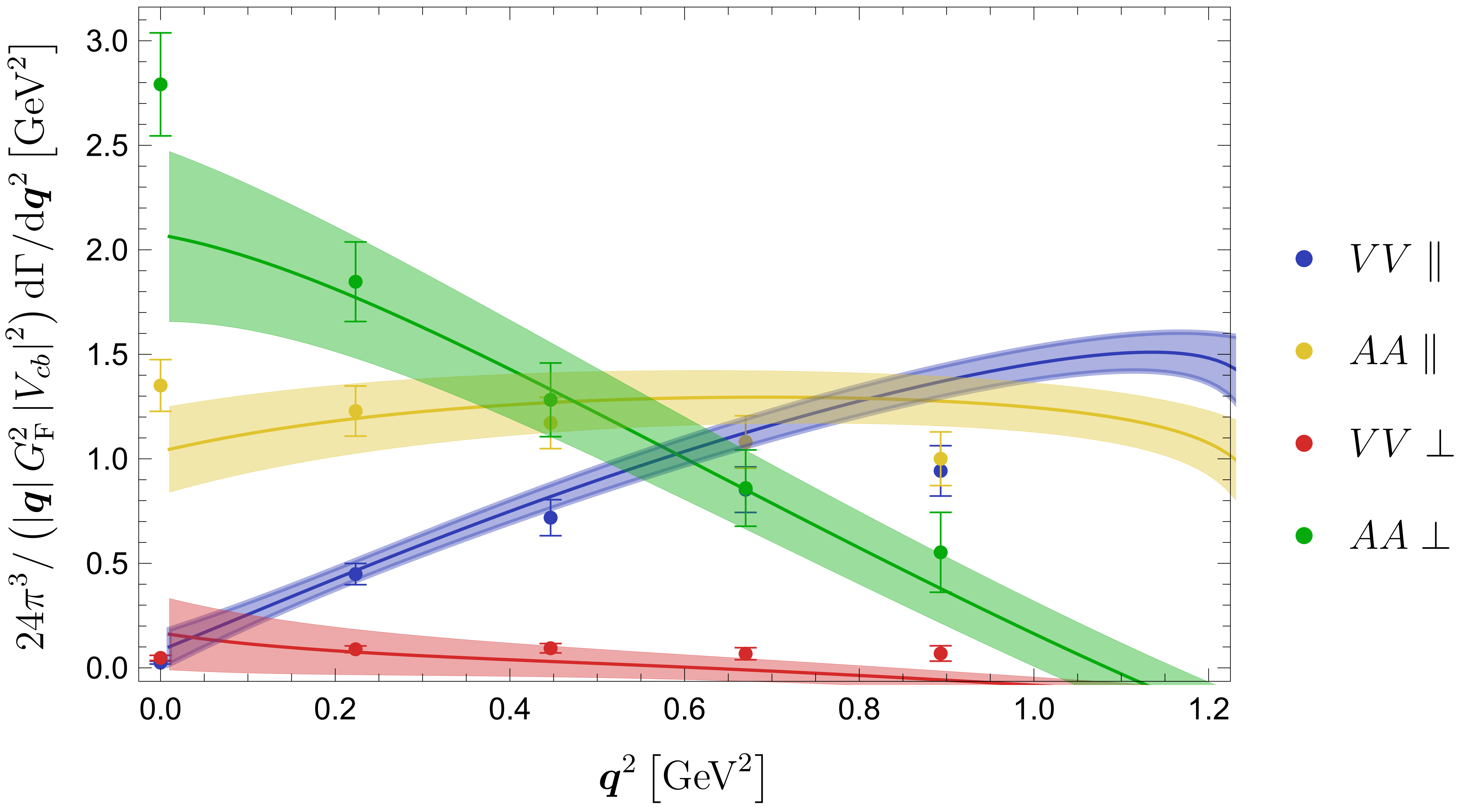
  - $VV_{||}$ dominated by B→D
  - All others by B→D*

differential decay rate / |**q**|



Comparison with OPE

Gambino, SH, Machler, arXiv:2111.02833



OPE at $O(\alpha_s)$, $O(1/m_b^3)$ with

- physical charm mass
- m_b to reproduce B_s mass

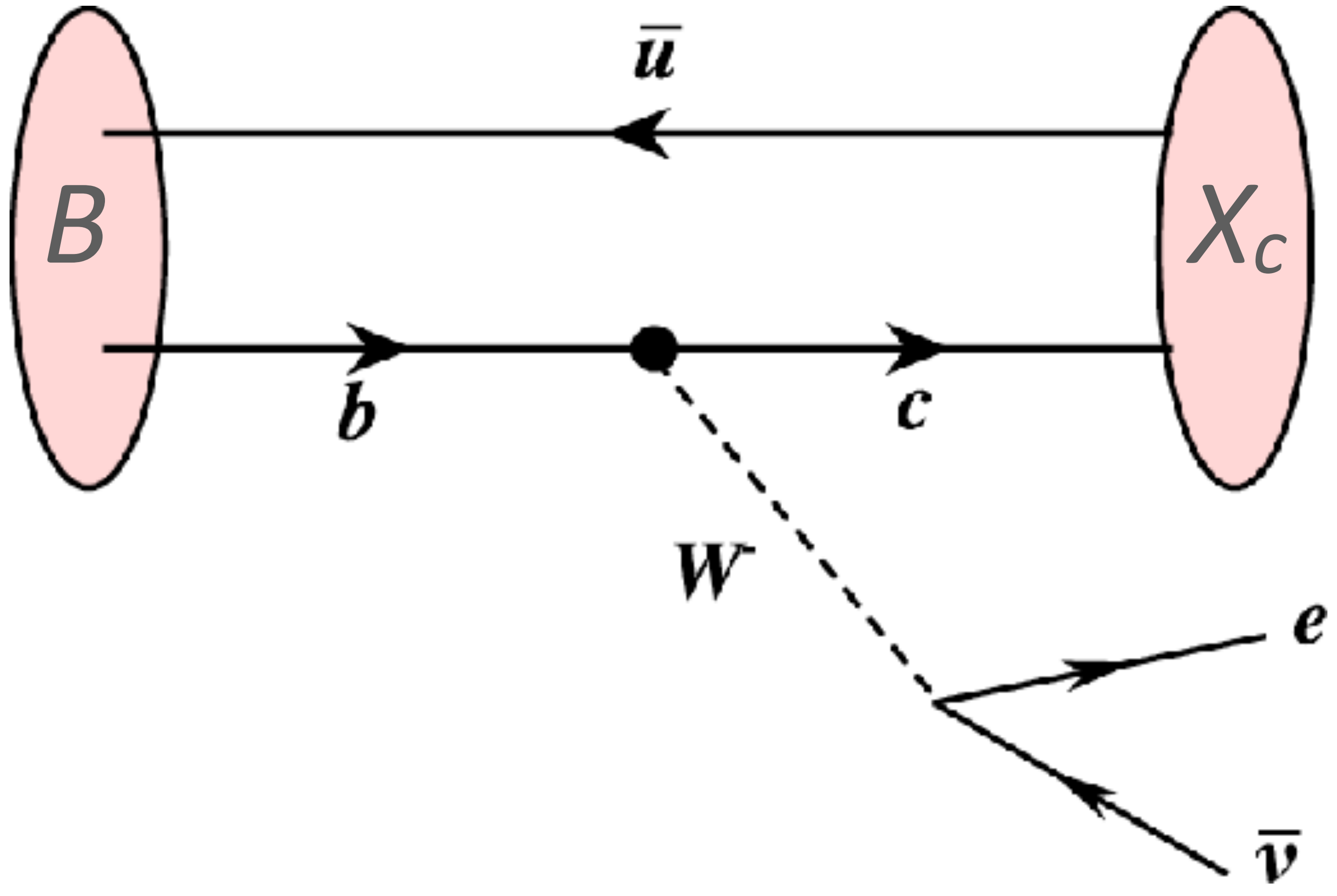
Gambino, Melis, Simula, arXiv:1704.06105

- MEs from fits of exp't; allowing 15% or 25% uncertainty (for those of $1/m_b^2$ and $1/m_b^3$)
- $\alpha_s = 0.32(1)$

Reasonable agreement observed. Further analysis to study the consistency between OPE and lattice.

(arbitrary) Moments

Gambino, SH, Machler, arXiv:2111.02833



Arbitrary moments/cuts can be implemented.

$$\int_0^{\boldsymbol{q}^2_{\max}} d\boldsymbol{q} \int_{\sqrt{m_D^2+\boldsymbol{q}^2}}^{m_B-\sqrt{\boldsymbol{q}^2}} d\omega \int_{E_\ell^{\min}}^{E_\ell^{\min}} dE_\ell\, K(\omega, E_\ell; \boldsymbol{q}^2)\langle B(\mathbf{0})|\tilde{J}^\dagger(-\boldsymbol{q})\delta(\omega-\hat{H})\tilde{J}(\boldsymbol{q})|B(\mathbf{0})\rangle$$

cuts by modifying the upper/lower limits

insert any function

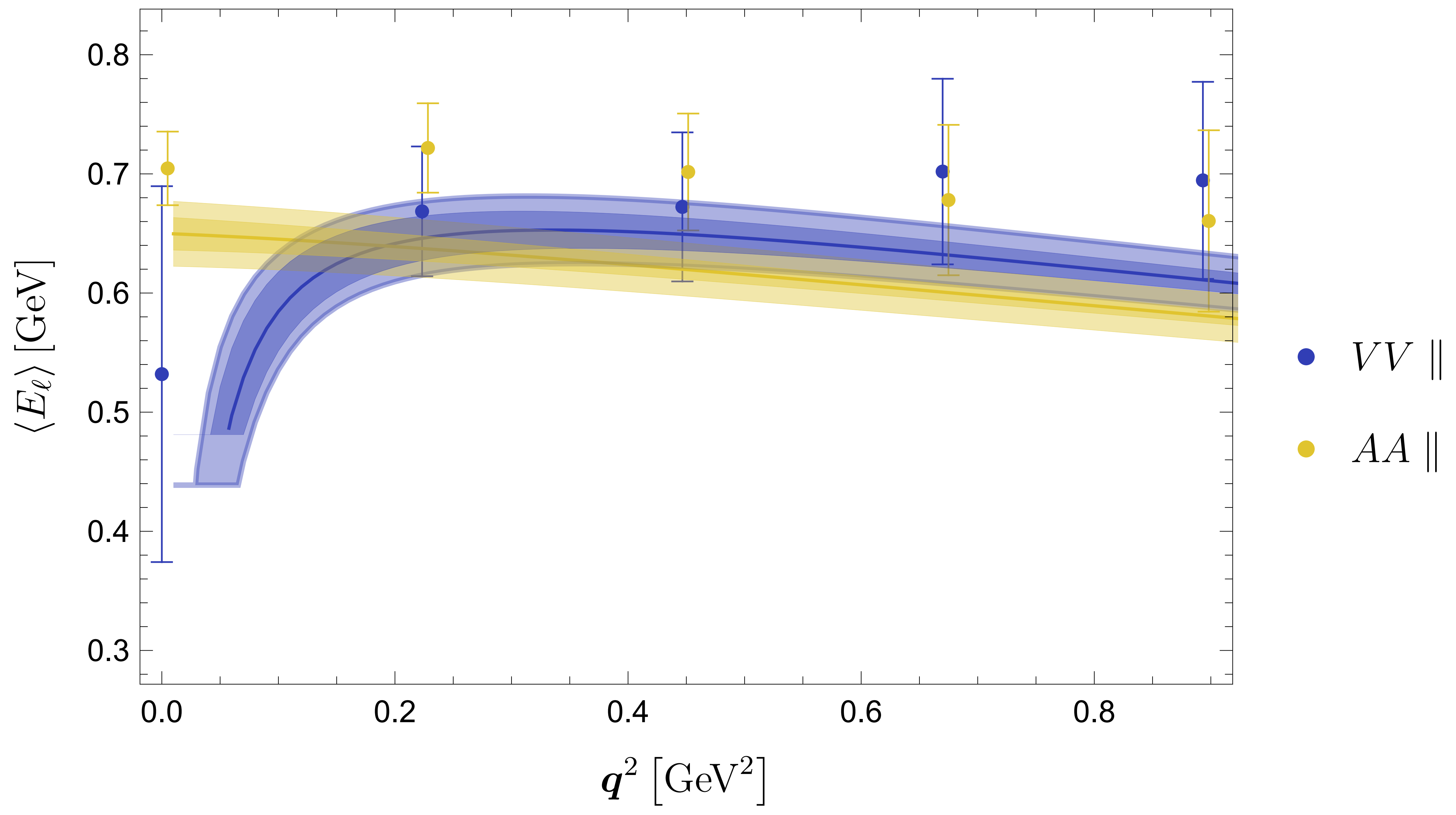
$X(\omega, E_\ell, \boldsymbol{q})$

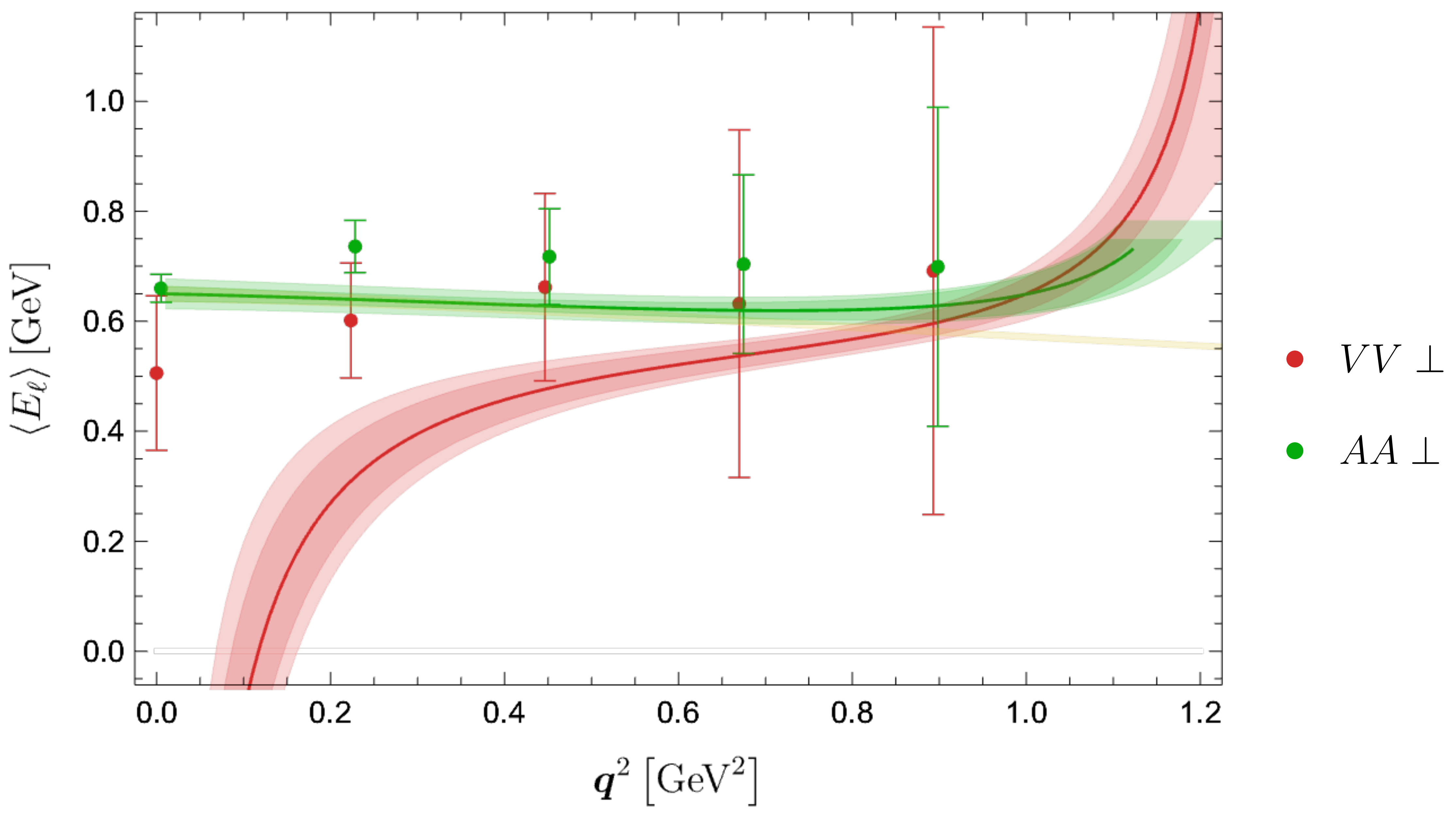
- Smooth function of ω can be approximated easier.
- Cuts would be crucial for b→u to avoid large background from b→c .

(arbitrary) Moments

e.g. Lepton energy moment $\langle E_l \rangle$

Gambino, SH, Machler, arXiv:2111.02833





Potential impacts

- Inclusive and Exclusive ($B \to D^{(*)}$) can be computed on the same lattice setup. Treat the both with similar systematics. Does the incl-vs-excl tension persist?
- Can be used to test the convergence of OPE.
- Can find the best moments to suppress errors for lattice, pQCD (or OPE), exp't.
- Can be applied for $b \to u$ and $b \to s$

Possible problems

- Does the approximation really control systematic errors? Non-trivial, especially on finite volumes where the spectrum is non-smooth.

More applications of the smeared spectrum

Scattering amplitude

Time-like pion form factor, 0→ππ, as an example: Bulava, Hansen, arXiv:1903.11735

LSZ reduction:

amplitude — at the pole

$$\int d^4x_1\, e^{-iq_1x_1}\theta(t_1)\langle\pi(\boldsymbol{p}_2)|\phi(x_1)J(0)|0\rangle = \frac{Z^{1/2}(\boldsymbol{p}_1)}{2E(\boldsymbol{p}_1)}\mathcal{M}(\pi(\boldsymbol{p}_2),\pi(\boldsymbol{p}_1);0)\frac{i}{-q_1^0 - E(\boldsymbol{p}_1)+i\epsilon} + \cdots$$

inserting complete set of states

$$\rho_{\boldsymbol{p}_2,0}(E_1,-\boldsymbol{p}_1) = \sum_\alpha \pi\delta(E_1 - E_\alpha)\langle\pi(\boldsymbol{p}_2)|\tilde{\phi}(q_1)|\alpha\rangle\langle\alpha|J(0)|0\rangle$$

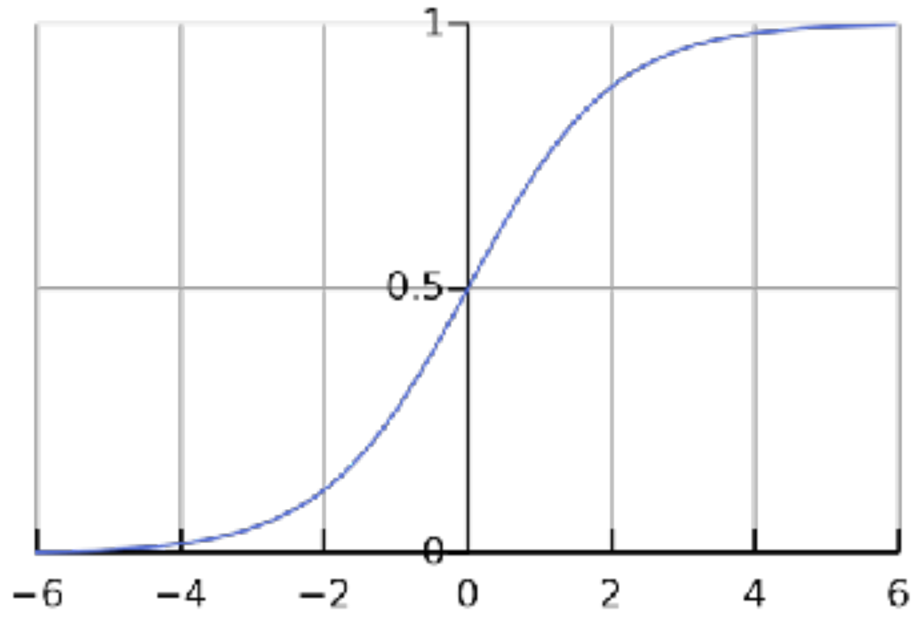
$$= \int_0^\infty \frac{dE_1}{\pi}\frac{i}{E(\boldsymbol{p}_2) - q_1^0 - E_1 + i\epsilon}\rho_{\boldsymbol{p}_2,0}(E_1,-\boldsymbol{p}_1)$$

➡ $$\mathcal{M}(\pi(\boldsymbol{p}_2),\pi(\boldsymbol{p}_1);0) = \frac{2E(\boldsymbol{p}_1)}{Z^{1/2}(\boldsymbol{p}_1)}\lim_{\epsilon\to 0}\epsilon\int_0^\infty \frac{dE_1}{\pi}\frac{i}{E(\boldsymbol{p}_2)+E(\boldsymbol{p}_1) - E_1 + i\epsilon}\rho_{\boldsymbol{p}_2,0}(E_1,-\boldsymbol{p}_1)$$

Looks like a smeared spectrum

Smeared spectrum as a filtering

M. Bruno and M. Hansen, arXiv:2012.11488



Maiani-Testa says that only the threshold amplitude ($\boldsymbol{q} = \mathbf{0}$) can be obtained from

$$\langle \pi(\boldsymbol{q}) | \phi_{-\boldsymbol{q}}(t) J(0) | 0 \rangle$$

Otherwise, the zero (relative) momentum states will dominate the correlator.

Consider, instead,

$$\langle \pi(\boldsymbol{q}) | \phi_{-\boldsymbol{q}}(t) \Theta(\hat{H} - \sqrt{s}, \Delta) J(0) | 0 \rangle$$

smoothed Heaviside function

- Look at the intermediate states only above $\sqrt{s}$
- Introduce the smearing so that it can be calculated more easily.

M. Bruno and M. Hansen, arXiv:2012.11488

$$\langle \pi(\boldsymbol{q}) | \phi_{-\boldsymbol{q}}(t) \Theta(\hat{H} - \sqrt{s}, \Delta) J(0) | 0 \rangle$$

$$\to e^{-E(\boldsymbol{q})t} \Big[\Theta(0, \Delta) \mathrm{Re}[f(s)] - 2\mathcal{J}^{(0)}(t, s, \Delta) \mathrm{Im}[f(s)] + \cdots \Big]_{s=(2E(\boldsymbol{q}))^2}$$

known functions

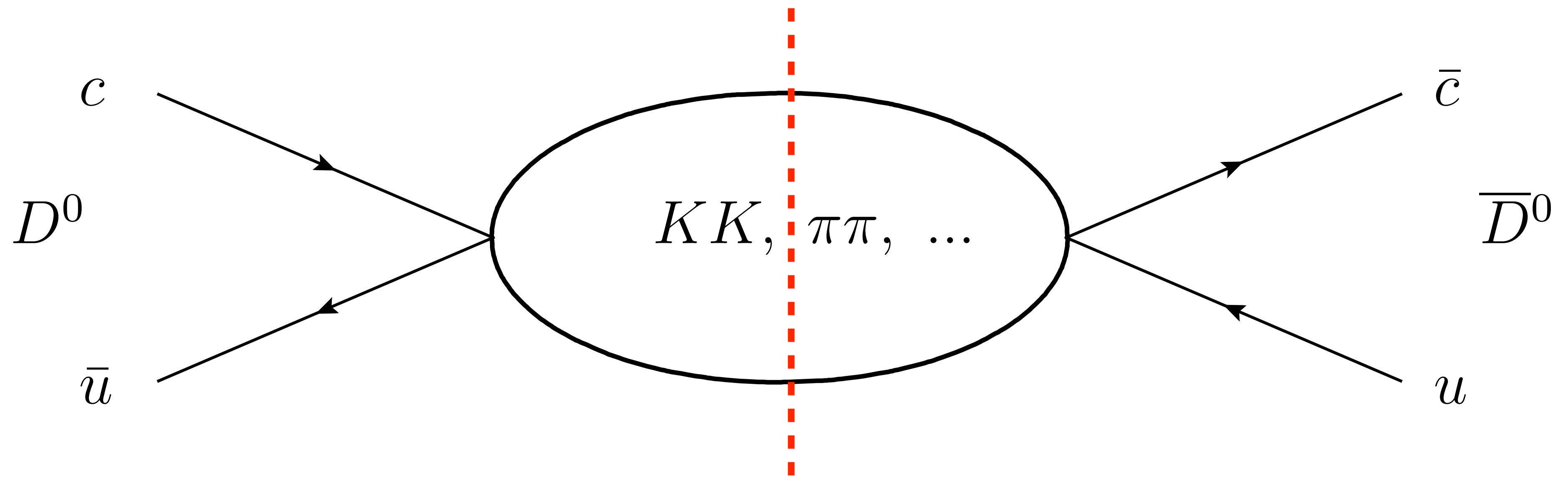
$f(s) = \langle \pi\pi; s | J(0) | 0 \rangle$: time-like form factor at any energy

Again, the question reduces to how well one can implement (matrix-valued) sigmoid function.

Potential application to: K→ππ, D→ππ, …, B→ππ

Even more challenging (towards DWQ@50)

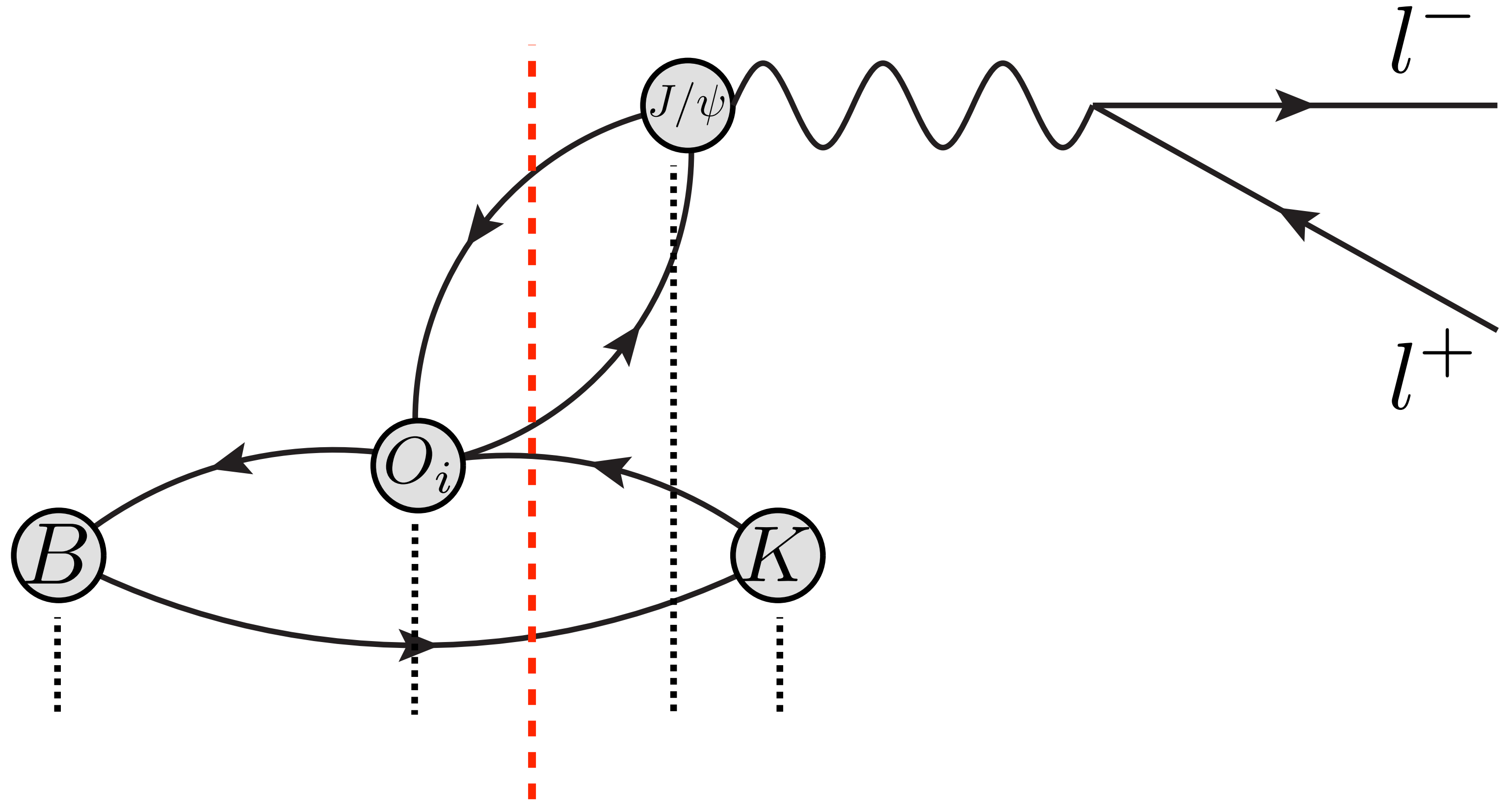
D^0–$\bar{D}^0$ mixing



all possible states with ω = m_D

$$\sim \langle D^0 | J\ \delta(\hat{H} - m_D)\ J | \bar{D}^0 \rangle$$

$B \to K(c\bar{c}) \to K\ell^+\ell^-$



all possible states with ω = m_B

$$\sim \langle B | O_i\ \delta(\hat{H} - m_B)\ J | K \rangle$$

To select the states that are otherwise inaccessible due to Maiani-Testa.

Towards DWQ@50

Be prepared; Not too distant future!

- Interesting new applications, even if only feasible with much larger lattices and statistics.
  - Inclusive B decays, …, inelastic νN scattering (see Yoo's talk on Thursday).
  - Experiments are there, or under construction. (LHC is accumulating more data; HL-LHC will be from 2027. Belle II still at the beginning, runs till 2031 at least; DUNE will run from 2027.)
  - (Challenging) Suggestions are welcome especially from exp colleagues.