# Two-photon Exchange Contribution to the Muonic-hydrogen Lamb Shift from Lattice QCD

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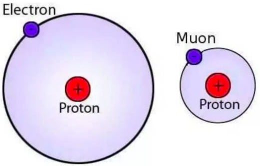
# Puzzles of proton size

## A decade puzzle since 2010

- Probe charge radius using electron
  - *e*-*p* scattering & hydrogen spectrum
  - consistent with uncertainty 0.7%
  - $r_p = 0.8751(61)$ fm

    [Rev. Mod. Phys. 88 (2016) 035009]
- Probe charge radius using muon
  - muonic hydrogen spectrum
  - high precision of uncertainty 0.05%
  - $r_p = 0.84087(39)$ fm

    [Nature 466 (2010) 213]
    [Science 339 (2013) 417]



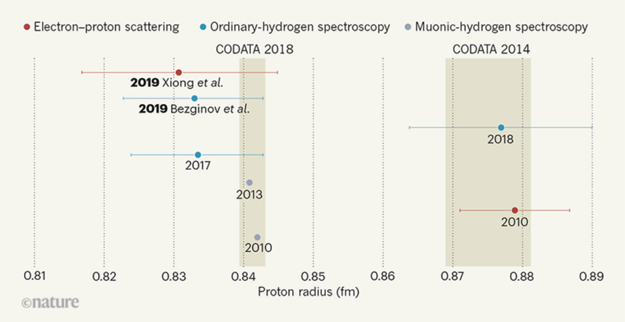


- Muon vs electron, diff by 4%, 5.6$\sigma$ smaller

# Recent progress from experiments

## Two very recent "electron" experiments favor smaller charge radius

- Hydrogen spectrum [N. Bezginov, et.al. Science 365 (2019) 1007]
- *e*-*p* scattering [W. Xiong, A. Gasparian, H. Gao, et.al. Nature 575 (2019) 147]



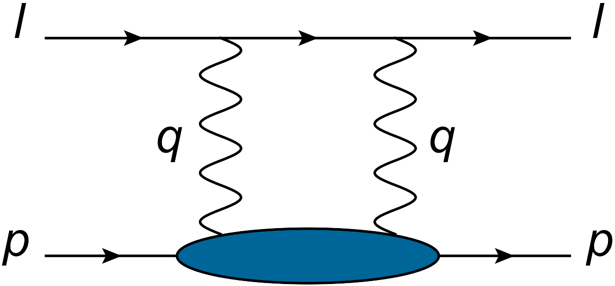

[J.-Ph. Karr & D. Marchand, Nature 575 (2019) 61-62]

> Discrepancy mainly arises from different experiments

## Theoretically, lattice QCD can provide the answer to the puzzle

$\Leftarrow$ if various systematic effects are under control

# Two-photon exchange



## eVP contribution to Lamb shift

- In non-relativistic limit, the potential of proton and lepton is

$$V(\vec{x}) = \int \frac{d^3\vec{q}}{(2\pi)^3} e^{i\vec{q}\cdot\vec{x}} \frac{-e^2}{\vec{q}^2\left[1 - \hat{\Pi}(-\vec{q}^2)\right]}$$

where $\hat{\Pi}(-\vec{q}^2)$ describes the vacuum polarization from the electron loop

- If $q^2 \ll m_e^2$, one can perform Taylor expansion for the integrand and obtain

$$V(\vec{x}) = -\frac{\alpha}{r} - \frac{4\alpha^2}{15 m_e^2}\delta^{(3)}(\vec{x})$$

- The $\delta^{(3)}(\vec{x})$ term contributes to the Lamb shift as

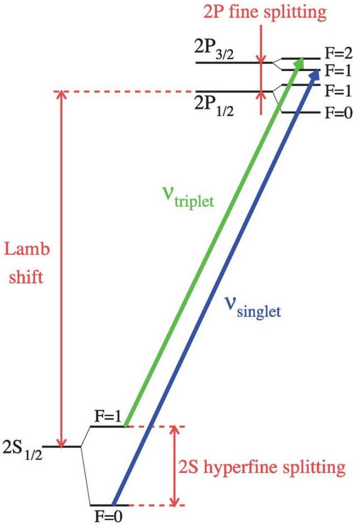
$$\Delta E_{\text{eVP}} = -\frac{4\alpha^2}{15 m_e^2}|\psi_\ell(0)|^2$$

where $|\psi_\ell(0)|^2 \propto m_\ell^3$ is the wave function at the origin point

  - In hydrogen atom, the eVP contribute about 1/40 of total Lamb shift
  - In $\mu$H atom, the condition of $q^2 \ll m_e^2$ does not hold

    but still a large enhancement can be expected from the ratio of $\frac{m_\mu^3}{m_e^3}$

# Experimental measurement of $\mu$H Lamb shift




- Lamb shift from hydrogen

$$E_{2P} - E_{2S} = 4.372\ \mu\text{eV}$$

- Lamb shift from $\mu$H [Science 2013]

$$E_{2P} - E_{2S} = 202.3706(23)\ \text{meV}$$

enhanced by a factor of $4.6 \times 10^4$

- $\mu$H EXP measured transitions $\nu_t$ and $\nu_s$
- Using 2P fine structure as inputs, one can determine both $\Delta E_{\text{Lamb}}$ and $\Delta E_{\text{HFS}}$
- Theoretical predic. [Ann. of Phy. 2013]

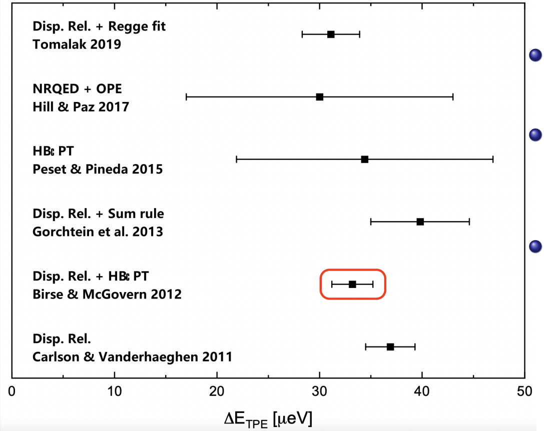
$$\begin{aligned} E_{2P} - E_{2S} &= \Delta E_{\text{QED}} + \Delta E_{\text{proton size}} + \Delta E_{\text{TPE}} \\ &= 206.0336(15) - 5.2275(10)\langle r_p^2 \rangle + 0.0332(20)\ \text{meV} \end{aligned}$$

TPE contributes the largest theoretical uncertainty

## Contributions to the 2S-2P $\mu$H Lamb shift

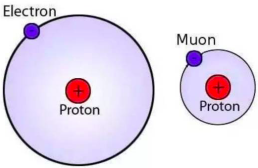
[Science 339 (2013) 417. Ann. of Phy. 331 (2013), 127]

$$E_{2P} - E_{2S} = \Delta E_{\text{QED}} + \Delta E_{\text{proton size}} + \Delta E_{\text{TPE}}$$
$$202.3706(23) = 206.0336(15) - 5.2275(10)\langle r_p^2\rangle + 0.0332(20)\ \text{meV}$$



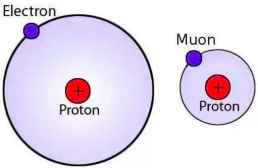

- It results in $r_p = 0.84087(39)$ fm
- The pheno. estimate of $\Delta E_{\text{TPE}}$ ranges from 20-50 $\mu$eV
- To resolve the discrepancy, one needs $\Delta E_{\text{TPE}} \sim 340\ \mu$eV, 10 times larger

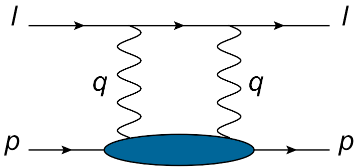
Understanding $\Delta E_{\text{TPE}}$ from first principles $\Rightarrow$ Lattice QCD




- Bound-state QED
- Treat proton as point-like particle + charge radius correction
- No IR divergence due to binding energy
- But more complicated strcuture dependence are not included yet

# Calculation of Lamb shift

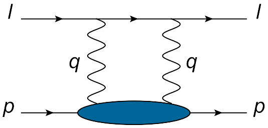



- Bound-state QED
- Treat proton as point-like particle + charge radius correction
- No IR divergence due to binding energy
- But more complicated strcuture dependence are not included yet

- QCD + QED loop
- Use free rather than bounded lepton
- IR divergence is treated by removing point-like + charge radius contribution
- At zero total momentum transfer, TPE correction is approximated by a $\alpha^2\delta^{(3)}(\vec{x})$ potential

# TPE from Compton tensor




- All external lines have zero three-momentum
- Lorentz invariant variables

$$q^2 = -Q^2, \quad \nu = p\cdot q/M = q_0$$

- Compton tensor

$$T_{\mu\nu}(P,Q) = \frac{1}{8\pi M}\int \mathrm{d}^4 x e^{iQ\cdot x}\langle p|\mathcal{T}[j_\mu(x) j_\nu(0)]|p\rangle$$

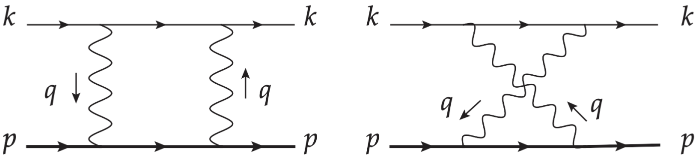
$$= \left(-\delta_{\mu\nu} + \frac{Q_\mu Q_\nu}{Q^2}\right) T_1(\nu, Q^2) - \left(P_\mu - \frac{P\cdot Q}{Q^2}Q_\mu\right)\left(P_\nu - \frac{P\cdot Q}{Q^2}Q_\nu\right)\frac{T_2(\nu,Q^2)}{M^2}$$

- Lorentz scalar function $T_1$ and $T_2$ contribute to $\Delta E$

$$\Delta E = \frac{8m\alpha^2}{\pi}|\phi_n(0)|^2 \int \mathrm{d}^4 Q \frac{(Q^2 + 2Q_0^2)\,T_1(iQ_0, Q^2) - (Q^2 - Q_0^2)\,T_2(iQ_0, Q^2)}{Q^4(Q^4 + 4m^2 Q_0^2)}$$

$|\phi_n(0)|$ is the $nS$-state wave function at the origin

How well do we know about the TPE contribution?




IR divergence is related to point-like particle + charge radius

- Point-like proton contribution (assuming form factor $F_1 = 1$ and $F_2 = 0$)

$$T_1^{pt}(\nu, Q^2) = \frac{M}{\pi}\frac{\nu^2}{Q^4 - 4M^2\nu^2}, \quad T_2^{pt}(\nu, Q^2) = \frac{M}{\pi}\frac{Q^2}{Q^4 - 4M^2\nu^2}$$

- Charge radius term from third Zemach moment contribution

$$\Delta E^{(\mathrm{sub})} = -\alpha^2|\phi_n(0)|^2 \int \frac{\mathrm{d}Q^2}{Q^2}\frac{16mM}{(M+m)Q}G_E'(0),$$

- $T_i$ can be reconstructed using dispersion relation

$$T_1(\nu, Q^2) = T_1(0, Q^2) + \frac{\nu^2}{\pi} \int_{\nu_{\text{el}}^2}^{\infty} \mathrm{d}\nu'^2 \frac{\text{Im } T_1(\nu, Q^2)}{\nu'^2(\nu'^2 - \nu^2)}$$

$$T_2(\nu, Q^2) = \frac{1}{\pi} \int_{\nu_{\text{el}}^2}^{\infty} \mathrm{d}\nu'^2 \frac{\text{Im } T_2(\nu, Q^2)}{\nu'^2 - \nu^2}$$

- Im $T_i$ known from experimental measurement of inelsatic scattering
- To avoid UV divergence $T_1$ requires a once subtracted dispersion relation
- Subtraction term $T_1(0, Q^2)$ cannot be extracted from experimental data
  $\Rightarrow$ Require a model description

Lattice QCD can determine the full TPE contribution

- On the lattice, we prefer to rewrite $\Delta E$ in terms of $T_{00}$ and $\sum_i T_{ii}$

$$\Delta E = \frac{8m\alpha^2}{\pi}|\phi_n(0)|^2 \int \mathrm{d}^4 Q \frac{-(Q^2+Q_0^2)T_{00} - Q_0^2 \sum_i T_{ii}}{Q^4(Q^4+4m^2Q_0^2)},$$

- Compton tensor in Euclidean space

$$T_{\mu\nu}(P,Q) = \frac{1}{8\pi M}\int \mathrm{d}^4 x e^{iQ\cdot x} \underbrace{\langle p|\mathcal{T}[j_\mu(x) j_\nu(0)]|p\rangle}_{H_{\mu\nu}(\vec{x},t)}$$

- We obtain

$$\Delta E = \frac{2m\alpha^2}{\pi M}|\phi_n(0)|^2 \sum_{i=1,2}\int \mathrm{d}^4 x\, \omega_i(\vec{x},t) H_i(\vec{x},t),$$

with $H_1 = H_{00}$ and $H_2 = \sum_i H_{ii}$ and known weight function $\omega_i$

# Signficant ground-state state contribution

$$
\begin{aligned}
H_{\mu\nu}(\vec{x}, t) &= \langle p|\mathcal{T}[j_\mu(x) j_\nu(0)]|p\rangle \\
&\sim \int \frac{d^3\vec{k}}{(2\pi)^3} \langle p|j_\mu(0)|p(\vec{k})\frac{1}{2E}\langle p(\vec{k})|j_\nu(0)]|p\rangle e^{i\vec{p}\cdot\vec{x}} e^{-(E-M)t}
\end{aligned}
$$

At small momnetum $\vec{k}$, $E - M \sim 0$

$\Rightarrow$ sufficiently large time integral is required to control the truncation effects

# Utilization of IVR method

Infinite-volume reconstruction (IVR) method [XF, L. Jin, PRD 100, 094509 (2019)]

- Divide the time integral into two regions

$$\Delta E \sim \int_{|t|<t_s} \mathrm{d}^4x\, \omega(\vec{x},t)H(\vec{x},t) + \int_{|t|>t_s} \mathrm{d}^4x\, \omega(\vec{x},t)H(\vec{x},t)$$

Ground-state dominance $\Rightarrow H(|t|>t_s)$ is reconstructed by $H(t=t_s) \Rightarrow$

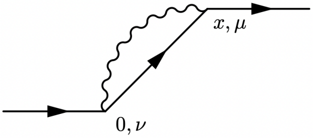
$$\Delta E(t_s) \sim \int_{|t|<t_s} \mathrm{d}^4x\, \omega(\vec{x},t)H(\vec{x},t) + \int \mathrm{d}^3\vec{x}\, L(\vec{x},t_s)H(\vec{x},t_s)$$

- Two roles played by $t_s$
  - $t_s$ sufficiently large to gaurantee the ground-state dominance
  - IR regulator: We hope the IR divergent part is contained in $L(\vec{x},t_s)$

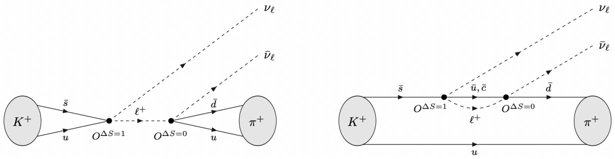
IR subtraction can be performing by modifying $L(\vec{x},t_s)$

# Related research topic using IVR

- Methodology of IVR developed [XF, L. Jin, PRD 100, 094509 (2019)]
  ⇒ Remove all the power-law FV effects in the QED self-energy




- Demonstrate method works for intermediate-state lighter than initial state
  [N. Christ, XF., L. Jin, C. Sachrajda, PRD 103, 014507 (2021)]




- Use the method to calculate meson leptonic decay process
  [N. Christ, XF, L. Jin, C. Sachrajda, M. Tomii, T. Wang]

- Apply the method to a realistic lattice calculation of pion mass splitting
  [XF, L. Jin, M. Riberdy, arXiv:2108.05311]

Luchang's talk at 14:30 - 15:00 today.

# Very singular IR structure

- The kinematic structure is very IR singular

$$\Delta E = \frac{8m\alpha^2}{\pi}|\phi_n(0)|^2 \int \mathrm{d}^4 Q \frac{-(Q^2+Q_0^2)T_{00} - Q_0^2 \sum_i T_{ii}}{Q^4(Q^4+4m^2Q_0^2)},$$

- As a result, both weight function $\omega(\vec{x}, t)$ and $L(\vec{x}, t_s)$ is IR divergent

$$\Delta E(t_s) \sim \sum_{i=1,2} \int_{|t|<t_s} \mathrm{d}^4 x \; \omega_i(\vec{x}, t) H(\vec{x}, t) + \int \mathrm{d}^3\vec{x} \; L_i(\vec{x}, t_s) H(\vec{x}, t_s)$$

with

$$\omega_i(\vec{x}, t) = -\int \frac{dQ^2}{Q^2} \int d\theta \frac{\alpha_i(\theta)}{Q^2 + 4m^2 \sin^2\theta} e^{iQx}, \quad \alpha_i = \begin{cases} 1 - \sin^4\theta, & i = 1 \\ \sin^2\theta(1-\sin^2\theta), & i = 2 \end{cases}$$

- Split $e^{iQx}$ into two parts: $e^{iQx} - 1$ and 1 so that

$$\omega_i = \hat{\omega}_i + \delta\omega_i, \quad L_i = \hat{L}_i + \delta L_i$$

$\delta\omega_i$ and $\delta L_i$ is IR divergent $\Rightarrow$ needs to introduce IR regulator

$$\delta\omega_i(\vec{x}, t) = -\int_{Q>\epsilon} \frac{dQ^2}{Q^2} \int d\theta \frac{\alpha_i(\theta)}{Q^2 + 4m^2\sin^2\theta}$$

Factor 1 is equivalent to $e^{iQx}$ at $Q = 0$ $\Rightarrow$ $\delta\omega_i(\vec{x}, t)$ is independent on $(\vec{x}, t)$

# Treatment of contributions from $\delta\omega_i$ and $\delta L_i$

For $\delta L_i$, it receives the long-distance contribution for $t > t_s$

- Only ground-intermediate state and no excited states contribute
- Zero momentum required by the factor of 1 $\Rightarrow$ charge conservation

To sum up, although singular, $\delta L_i$-contribution is known from point-like particle

For $\delta\omega_i$, it receives the short-distance contribution for $t \le t_s$

- Seems that it receives contributions from all excited states
- However, at $Q \to 0$, we have the hadronic scalar function

$$\lim_{Q\to 0} T_i(Q) - T_i^B(Q) = 0, \quad i = 1 \text{ for } T_{00} \text{ and } i = 2 \text{ for } \sum_i T_{ii}$$

As the structure of the Born term $T_i^B(Q)$ is known, we have

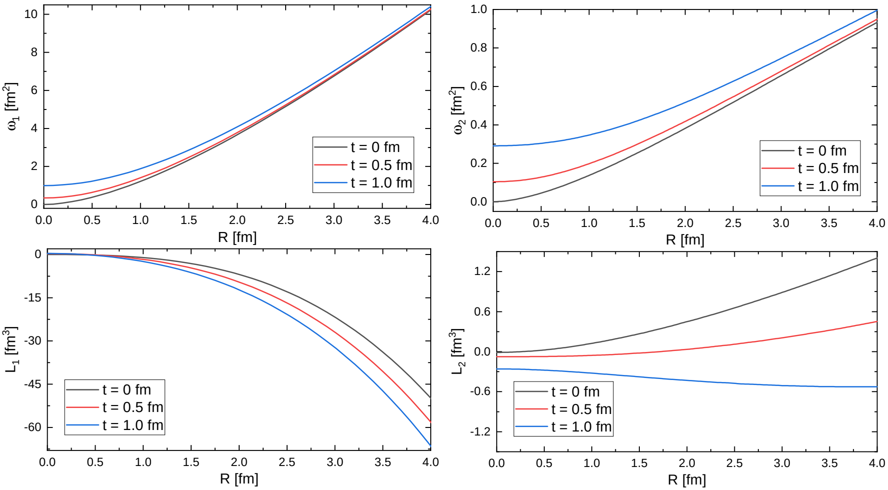
$$\frac{1}{2M}\int_{-t_s}^{t_s} dt \int d^3\vec{x}\, H_i(\vec{x},t) = \begin{cases} 2t_s, & i = 1 \\ \frac{3}{M}, & i = 2 \end{cases}$$

For $i = 1$, the condition simply originates from the charge conservation

To sum up, $\delta\omega_i$-contribution is known analytically

# Master formula

$$\Delta E(t_s) \sim \sum_{i=1,2} \left[ \int_{|t|<t_s} \mathrm{d}^4x \; \hat{\omega}(\vec{x}, t) H(\vec{x}, t) + \int \mathrm{d}^3\vec{x} \; \hat{L}(\vec{x}, t_s) H(\vec{x}, t_s) \right]$$




$\hat{L}_1$ dominate the contribution with large uncertainties

# Optimized subtraction scheme

- Realizing

$$G_E^2(0) = \int d^3\vec{x}\, L_0(\vec{x}, t_s) H_1(\vec{x}, t_s),$$
$$\langle r_p^2 \rangle = \int d^3\vec{x}\, L_r(\vec{x}, t_s) H_1(\vec{x}, t_s),$$

with

$$L_0(\vec{x}, t_s) = \frac{1}{2M}, \quad L_r(\vec{x}, t_s) = \frac{1}{4M}\left(\vec{x}^2 - \frac{3 + 6Mt_s}{2M^2}\right)$$

- Define a reduced weight function $L_1^{(r)}(\vec{x}, t_s)$ with a subtraction of

$$L_1^{(r)}(\vec{x}, t_s) = L_1(\vec{x}, t_s) - c_0 L_0(\vec{x}, t_s) - c_r L_r(\vec{x}, t_s)$$

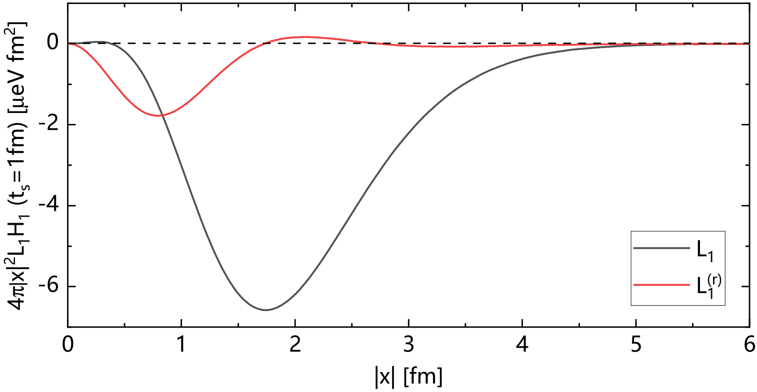
Choose the coefficients $c_0$ and $c_r$ by minimizing the following integral

$$I(c_0, c_1) = \int_{R_{\mathrm{min}}}^{R_{\mathrm{max}}} \mathrm{d}|\vec{x}|\, \left(4\pi|\vec{x}|^2\right) |L_1^{(r)}(\vec{x}, t_s)|^2$$

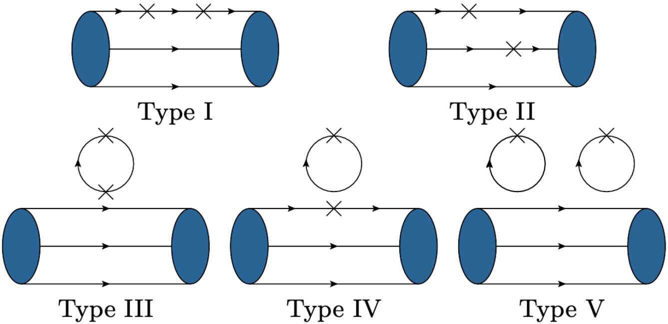
- $\Delta E_{\text{TPE}}$

$$\Delta E = -0.60\ \mu\text{eV} + c_0 + c_r\langle r_p^2\rangle + \Delta E^{(\text{lat})},$$

- Use $R_{\text{min}} = 1$ fm and $R_{\text{max}} = 3$ fm
- Minimization yields $c_0 = -0.17\ \mu$eV, $c_r = -93.72\ \mu$eV

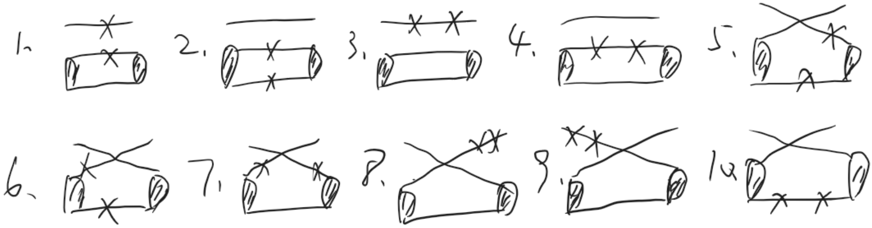


# Quark field contractions




- Include connected diagram Type I & II + disconnected diagram Type III
- Type IV & V vanishes at flavor SU(3) limit and thus neglected

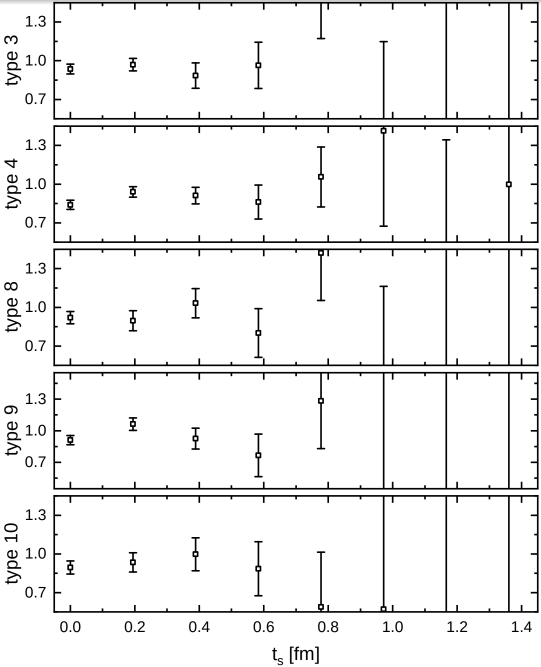
# Quark field contractions



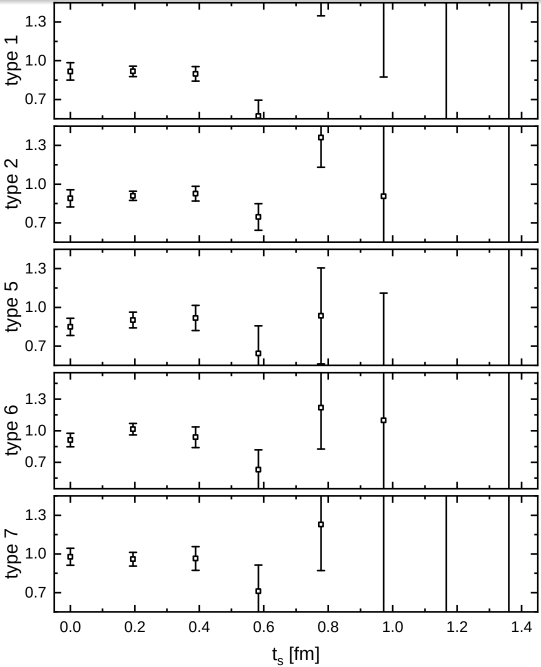
DWF fermion used in the calculation

| Ensemble | $m_\pi$[MeV] | $m_p$[MeV] | L/a | T/a | a[fm] | $N_{conf}$ |
|---|---|---|---|---|---|---|
| 24D | 141.7(2) | 935(5) | 24 | 64 | 0.1944 | 124 |

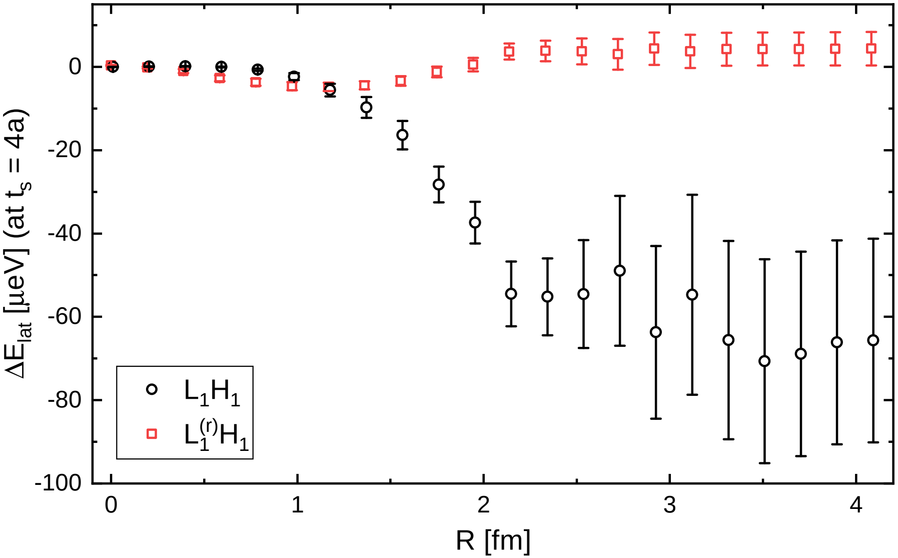
# Test of charge conservation

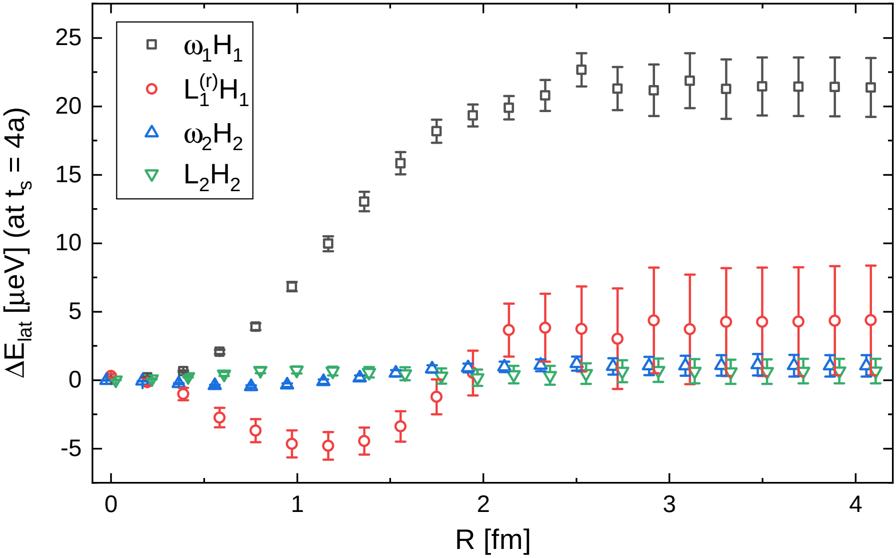


# Test of charge conservation



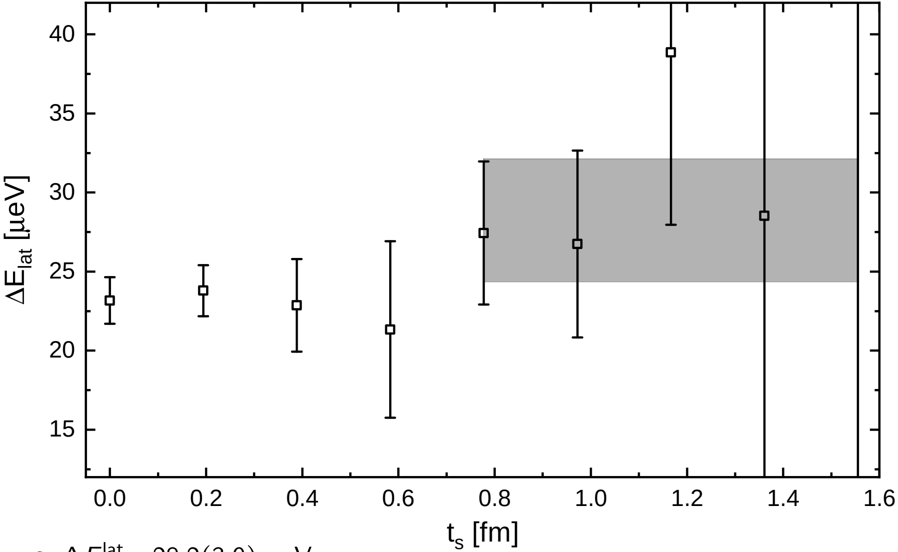
# Efficiency of optimized subtraction scheme





$\omega_1 H_1$
$L_1^{(r)} H_1$
$\omega_2 H_2$
$L_2 H_2$
$\Delta E_{lat}$ [μeV] (at $t_s$ = 4a)
R [fm]
25
20
15
10
5
0
-5
0
1
2
3
4

@ $m_\pi$ = 142 MeV, Connected contributions only




- $\Delta E^{\text{lat}} = 28.2(3.9)\ \mu\text{eV}$
- $\Delta E_{\text{TPE}} = -27.4(3.9) + 93.72\langle r_p^2 \rangle = 38.9(3.9)\ \mu\text{eV}$