Lattice calculation of the pion mass splitting using the infinite volume reconstruction method

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in collaboration with

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arXiv:2108.05311 (lattice calculation), arXiv:1812.09817 (method)

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BNL-HET & RBRC Joint Workshop "DWQ@25":

The event marks the passage of twenty-five years since the first numerical simulations with Domain Wall Quarks (DWQ)

The RBC & UKQCD collaborations

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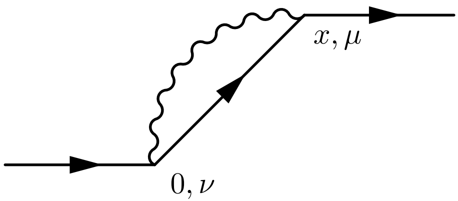
Outline

- **Introduction to the finite volume effects in lattice QCD + QED**
- QED correction to hadron masses & the infinite volume reconstruction method
  Feng and Jin, PRD [arXiv:1812.09817]
- Lattice calculation of the pion mass splitting
  Feng, Jin, and Riberdy [arXiv:1812.09817]
- Summary and outlook

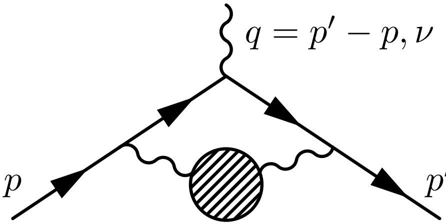
- No massless particles in QCD $\rightarrow$ Finite volume effects for many observables are **exponentially suppressed** by the spatial lattice size L.
  - Mass of a stable particle M. Lüscher, Commun.Math.Phys. 104, 177-206 (1986)
- QED include massless photon $\rightarrow$ Use treatments similar to QCD for QED leads to **power-law suppressed** finite volume effects.
  - Mass of a stable particle in QED$_L$ M. Hayakawa and S. Uno, Prog.Theor.Phys. (2008).



$$\Delta M(L) = \Delta M(\infty) - \frac{q^2}{4\pi}\frac{\kappa}{2L}\left(1 + \frac{2}{mL}\right) + \mathcal{O}\left(\frac{1}{L^3}\right) \tag{1}$$

where $\kappa = 2.8372997\cdots$. S. Borsanyi et al., Science 347, 1452 (2015).

- One may analytically (and perturbatively) treat the QED part of the diagram in the **infinite volume** (and in the continuum) to eliminate the power-law suppressed finite volume effects due to the massless photons.
  - Hadronic vacuum polarization (HVP) contribution to muon $g-2$:

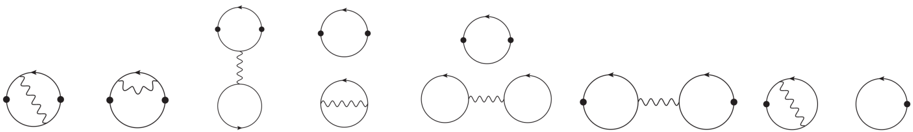


$$a_\mu^{\text{HVP LO}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dK^2 f(K^2) \hat{\Pi}(K^2) = \sum_{t=0}^{+\infty} w(t) C(t) \quad (2)$$

$$C(t) = \frac{1}{3} \sum_{\vec{x}} \sum_{j=0,1,2} \langle J_j^{em}(\vec{x}, t) J_j^{em}(0) \rangle_{\text{QCD}} \quad (3)$$

T. Blum (2003) D. Bernecker, H. Meyer (2011)

- One may analytically (and perturbatively) treat the QED part of the diagram in the **infinite volume** (and in the continuum) to eliminate the power-law suppressed finite volume effects due to the massless photons.
  - QED corrections to the hadronic vacuum polarization (HVP):



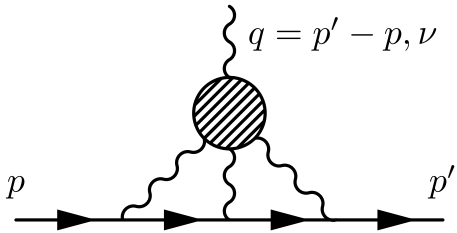
(a) V (b) S (c) T (d) D1 (e) D2 (f) F (g) D3

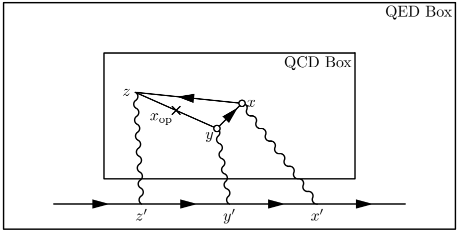
$$S^{\gamma}_{\mu,\nu}(x) = \frac{\delta_{\mu,\nu}}{4\pi^2 x^2} \tag{4}$$

T. Blum (2018)

Lattice QCD + QED_∞ and finite volume effects

- One may analytically (and perturbatively) treat the QED part of the diagram in the **infinite volume** (and in the continuum) to eliminate the power-law suppressed finite volume effects due to the massless photons.
  - Hadronic light-by-light (HLbL) contribution to muon $g-2$:

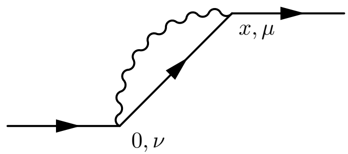




N. Asmussen et al (2016) T. Blum et al (2017)

- One may analytically (and perturbatively) treat the QED part of the diagram in the **infinite volume** (and in the continuum) to eliminate the power-law suppressed finite volume effects due to the massless photons.

- Does **NOT** work for calculating the QED correction to the mass of a stable hadron.



$$\Delta M = \mathcal{I} = \frac{1}{2}\int d^4x\, \mathcal{H}_{\mu,\nu}(x) S^{\gamma}_{\mu,\nu}(x), \tag{5}$$

$$\mathcal{H}_{\mu,\nu}(x) = \frac{1}{2M}\langle N|T J_\mu(x) J_\nu(0)|N\rangle, \quad S^{\gamma}_{\mu,\nu}(x) = \frac{\delta_{\mu,\nu}}{4\pi^2 x^2} \tag{6}$$

  - The hadronic function does not always fall exponentially in the long distance region. When $t \gg |\vec{x}|$:

$$\mathcal{H}_{\mu,\nu}(t,\vec{x}) \sim e^{-M\left(\sqrt{t^2+\vec{x}^2}-t\right)} \sim e^{-M\frac{\vec{x}^2}{2t}} \sim O(1) \tag{7}$$

  - Truncate the integral: $\int d^4x \to \int_{-L/2}^{L/2} d^4x$ & Approx the $\mathcal{H}(x)$: $\mathcal{H}(x) \to \mathcal{H}^L(x)$ $\to$ Power-law suppressed finite volume errors.

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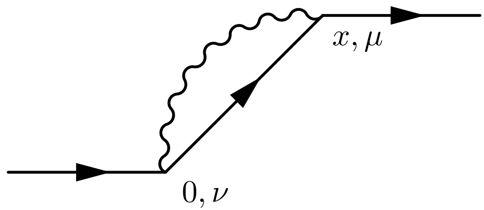
- Summary and outlook

QED correction to hadron masses

$$\Delta M = \mathcal{I} = \frac{1}{2}\int d^4x\, \mathcal{H}_{\mu,\nu}(x) S^{\gamma}_{\mu,\nu}(x)$$

$$\mathcal{H}_{\mu,\nu}(x) = \frac{1}{2M}\langle N|T J_{\mu}(x) J_{\nu}(0)|N\rangle$$

$$S^{\gamma}_{\mu,\nu}(x) = \frac{\delta_{\mu,\nu}}{4\pi^2 x^2}$$



- Evaluate the QED part, the photon propagator, in infinite volume.

- The hadronic function does not always fall exponentially in the long distance region
  $\rightarrow$ Separate the integral into two parts ($t_s \lesssim L$):

$$\Delta M = \mathcal{I} = \mathcal{I}^{(s)} + \mathcal{I}^{(l)}$$

$$\mathcal{I}^{(s)} = \frac{1}{2}\int_{-t_s}^{t_s} dt \int d^3x\, \mathcal{H}_{\mu,\nu}(x) S^{\gamma}_{\mu,\nu}(x) \quad (8)$$

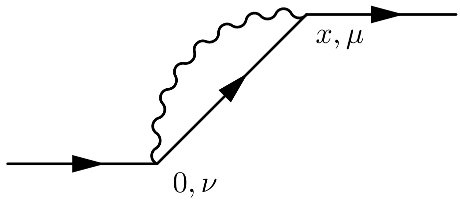
$$\mathcal{I}^{(l)} = \int_{t_s}^{\infty} dt \int d^3x\, \mathcal{H}_{\mu,\nu}(x) S^{\gamma}_{\mu,\nu}(x) \quad (9)$$

- For the short distance part, $\mathcal{I}^{(s)}$ can be directly calculated on a finite volume lattice:

$$\mathcal{I}^{(s)} \approx \mathcal{I}^{(s,L)} = \frac{1}{2}\int_{-t_s}^{t_s} dt \int_{L/2}^{L/2} d^3x\, \mathcal{H}^{L}_{\mu,\nu}(x) S^{\gamma}_{\mu,\nu}(x)$$

- For the **long distance part**, $\mathcal{I}^{(l)}$, a different treatment is required.

The infinite volume reconstruction (IVR) method



- For the long distance part, we can evaluate $\mathcal{H}_{\mu,\nu}(x)$ **indirectly** in the **infinite volume**.

$$\mathcal{I}^{(l)} = \int_{t_s}^{\infty} dt \int d^3x \, \mathcal{H}_{\mu,\nu}(x) S^{\gamma}_{\mu,\nu}(x) \quad (10)$$

- Note that when t is large ($t > t_s$), the intermediate states between the two currents are dominated by the single particle states (possibly with small momentum). Therefore:

$$\mathcal{H}_{\mu,\nu}(x) \approx \int \frac{d^3p}{(2\pi)^3} \left[\frac{1}{2E_{\vec{p}}} \frac{1}{2M} \langle N | J_\mu(0) | N(\vec{p}) \rangle \langle N(\vec{p}) | J_\nu(0) | N \rangle \right] e^{i\vec{p}\cdot\vec{x} - (E_{\vec{p}} - M)t} \quad (11)$$

  - We only need to calculate the form factors: $\langle N(\vec{p}) | J_\nu(0) | N \rangle$!
  - Values for all $\vec{p}$ are needed. Inversely Fourier transform the above relation **at** t_s!

$$\int d^3x \, \mathcal{H}_{\mu,\nu}(t_s, \vec{x}) e^{-i\vec{p}\cdot\vec{x} + (E_{\vec{p}} - M)t_s} \approx \frac{1}{2E_{\vec{p}}} \frac{1}{2M} \langle N | J_\mu(0) | N(\vec{p}) \rangle \langle N(\vec{p}) | J_\nu(0) | N \rangle \quad (12)$$

Master formula for QED correction to hadron masses

- The final expression for QED correction to hadron mass is split into two parts:

$$\Delta M = \mathcal{I} = \mathcal{I}^{(s)} + \mathcal{I}^{(l)}$$

- For the short distance part: $\mathcal{I}^{(s)} \approx \mathcal{I}^{(s,L)} = \frac{1}{2}\int_{-t_s}^{t_s} dt \int_{L/2}^{L/2} d^3x\, \mathcal{H}^L_{\mu,\nu}(x) S^{\gamma}_{\mu,\nu}(x)$ (13)

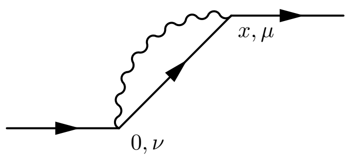
- For the long distance part: $\mathcal{I}^{(l)} \approx \mathcal{I}^{(l,L)} = \int_{-L/2}^{L/2} d^3x\, \mathcal{H}^L_{\mu,\nu}(t_s, \vec{x}) L_{\mu,\nu}(t_s, \vec{x})$

- For Feynman gauge:

$$S^{\gamma}_{\mu,\nu}(x) = \frac{\delta_{\mu,\nu}}{4\pi^2 x^2} \qquad L_{\mu,\nu}(t_s, \vec{x}) = \frac{\delta_{\mu,\nu}}{2\pi^2}\int_0^{\infty} dp \frac{\sin(p|\vec{x}|)}{2(p + E_p - M)|\vec{x}|} e^{-pt_s}$$

- Only use $\mathcal{H}^L_{\mu,\nu}(t, \vec{x})$ within $-t_s \leq t \leq t_s$.

- Choose $t_s = L/2$, **finite volume errors and the ignored excited states contribution to $\mathcal{I}^{(l)}$ are both exponentially suppressed by the spatial lattice size L.**



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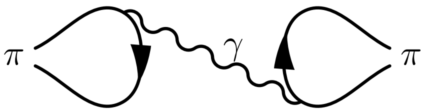
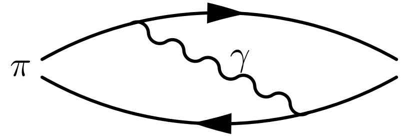
Pion mass splitting: $\Delta m_\pi = m_{\pi^\pm} - m_{\pi^0}$



Disconnected diagram



Connected diagram

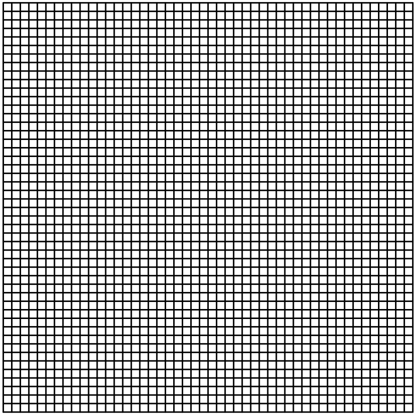
- Coulomb gauge fixed wall sources are used to interpolate the pion interpolating operators.
- Fixed time separation between the vector current operator and the closest pion interpolating operators: $t_{\text{sep}} \approx 1.5\text{fm}$.

$$\mathcal{H}^L_{\mu,\nu}(t,\vec{x}) = L^3 \frac{\langle \pi(t+t_{\text{sep}}) J_\mu(t,\vec{x}) J_\nu(0) \pi^\dagger(-t_{\text{sep}})\rangle_L}{\langle \pi(t+t_{\text{sep}}) \pi^\dagger(-t_{\text{sep}})\rangle_L^{[*]}} \tag{14}$$

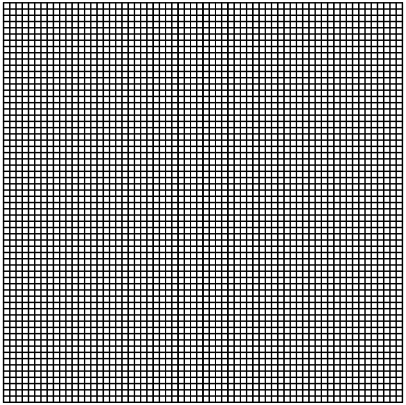
- Diagrams are similar to the $\pi^- \to \pi^+ ee$ neutrinoless double beta $(0\nu 2\beta)$ decay. D. Murphy and W. Detmold (2018), Tuo, Feng, and Jin (2019)
- At $\mathcal{O}(\alpha_{\text{QED}}, (m_u - m_d)/\Lambda_{\text{QCD}})$, all UV divergence are canceled. The two diagrams are the only diagrams contributing to $m_{\pi^\pm} - m_{\pi^0}$. RM123 (2013)
- In particular, the pion mass splitting at leading order does not depend on $m_u - m_d$.

[*]: Around the world effects of the pion two point function are already corrected.

48I



64I

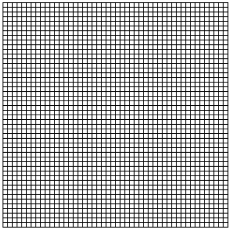


- Domain wall fermion action (preserves Chiral symmetry, no $\mathcal{O}(a)$ lattice artifacts).
- Iwasaki gauge action.
- $M_\pi = 135$ MeV *, $L = 5.5$ fm box, $1/a_{48I} = 1.73$ GeV, $1/a_{64I} = 2.359$ GeV.

*: Valence pion mass. Slightly different from the 139 MeV unitary pion mass used in the ensemble generation.

RBC/UKQCD, PRD [arXiv:1411.7017]

48I



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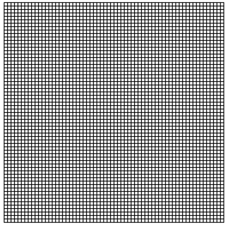
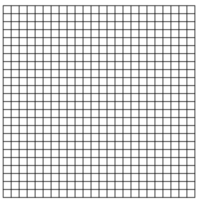
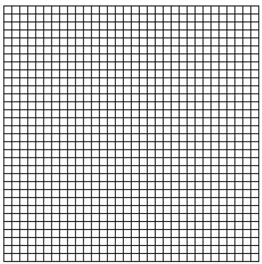


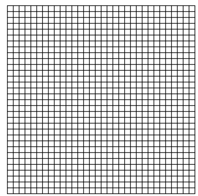
24D



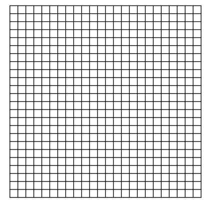
32D



32Dfine



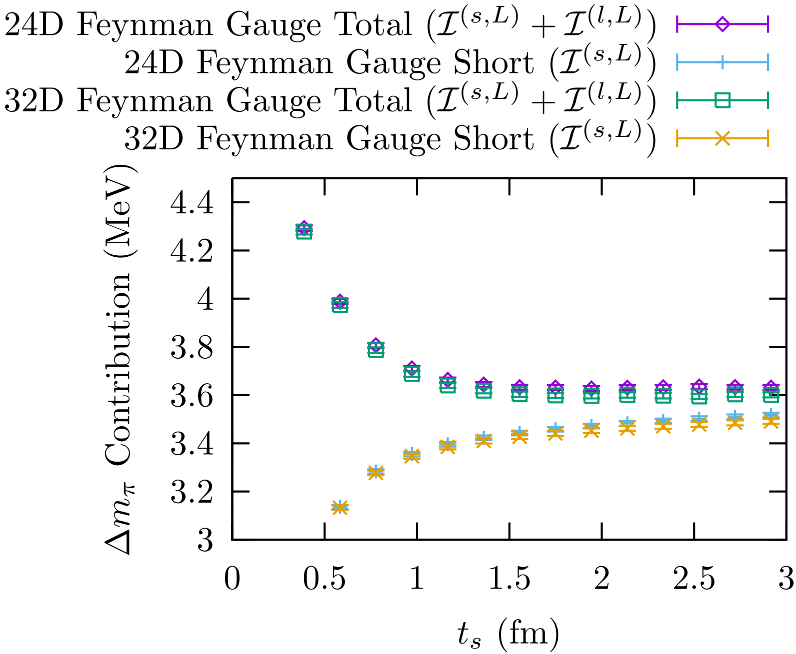
24DH



- For 24D, 32D, 32Dfine, $M_\pi \approx 140$ MeV
- For 24DH, $M_\pi \approx 340$ MeV

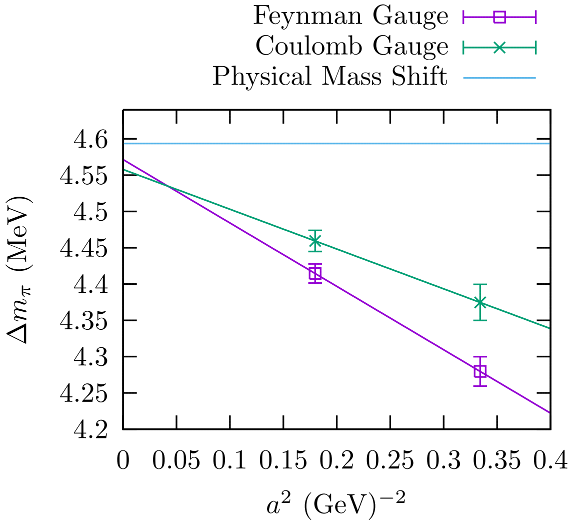
RBC/UKQCD, PRD [arXiv:1411.7017]

Finite volume effects and t_s dependence of Δm_π



- The difference between 32D and 24D is $-0.035(16)$MeV. This is consistent with a scalar QED calculation, which yields -0.022MeV.

Continuum extrapolation of Δm_π



	Disc (MeV)	Conn (MeV)	Total (MeV)
Feyn	0.051(9)(22)	4.483(40)(28)	4.534(42)(43)
Coul	0.052(2)(13)	4.508(46)(42)	4.560(46)(41)
Coul-t	0.018(1)(4)	1.840(22)(39)	1.858(22)(41)

Finite volume corrections calculated with the difference of the 32D and 24D ensembles are already included.

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- **Summary and outlook**

Summary and outlook

- We invent the infinite volume reconstruction (IVR) method, eliminates all power-law suppressed finite volume errors in QED self-energy calculations.
  Feng and Jin, PRD [arXiv:1812.09817]

- We have used this method to calculate the pion mass splitting $m_{\pi^\pm} - m_{\pi^0}$. In Feynman gauge, we obtained 4.534(42)(43)MeV, in good agreement with the experimental value 4.5936(5)MeV. Feng, Jin, and Riberdy [arXiv:1812.09817]

Reference	$m_{\pi^\pm} - m_{\pi^0}$(MeV)
RM123 2013	$5.33(48)_{\rm stat}(59)_{\rm sys}$
R. Horsley et al. 2015	$4.60(20)_{\rm stat}$
RM123 2017	$4.21(23)_{\rm stat}(13)_{\rm sys}$
This work	$4.534(42)_{\rm stat}(43)_{\rm sys}$

  - Pion mass splitting with a percent-level uncertainty, which is about 5-10 times smaller than previous lattice QCD calculations.
  - This is the first lattice calculation of pion mass splitting at the physical pion mass.
  - For the first time in literature, we have clearly resolved and included the contribution from the quark disconnected diagram.

Summary and outlook

- The IVR method and the 4-point hadronic function have more applications:
  - Talk by Xu Feng today at 9:00 AM "Two-photon Exchange Contribution to the Muonic-hydrogen Lamb Shift from Lattice QCD".
  - $\pi^- \to \pi^+ e^- e^-$ neutrinoless double beta ($0\nu 2\beta$) decay.

    $$g_\nu^{\pi\pi}(\mu)\Big|_{\mu=m_\rho} = -10.89(28)_{\mathrm{stat}}(33)_L(66)_a$$

    Tuo, Feng, and Jin, PRL [arXiv:1909.13525]

    Also $-10.78(12)_{\mathrm{stat}}(51)_{\mathrm{sys}}$ W. Detmold, D.J. Murphy [arXiv:2004.07404]
  - Electroweak box diagrams in $\pi^- \to \pi^0 e^- \bar{\nu}_e$.

    Feng, Gorchtein, Jin, Ma, and Seng, PRL [arXiv:2003.09798]

    Ma, Feng, Gorchtein, Jin, and Seng, PRD [arXiv:2102.12048]
  - Rare kaon decay Christ, Feng, Jin, and Sachrajda, PRD [arXiv:2009.08287]
  - $K \to \ell \nu_\ell \ell'^+ \ell'^-$ Tuo, Feng, Jin, Wang, [arXiv:2103.11331]
  - QED correction to the meson leptonic decay. Work in progress.

- Calculation performed by reusing propagators generated for the lattice HLbL calculation at MIRA.

Thank You!