The E&M Corrections to $K \to \pi\pi$

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Introduction

- $K \to \pi\pi$ decay and direct CP violation *[R. Abbott et al, arXiv:2004.09440]*

$$\epsilon' = \frac{ie^{\delta_2-\delta_0}}{\sqrt{2}} \frac{\mathrm{Re}A_2}{\mathrm{Re}A_0} \left(\frac{\mathrm{Im}A_2}{\mathrm{Re}A_2} - \frac{\mathrm{Im}A_0}{\mathrm{Re}A_0} \right) \qquad (1)$$

- $\Delta I = 1/2$ rule, where $\frac{\mathrm{Re}A_0}{\mathrm{Re}A_2} \sim 20$, amplifies possible corrections
- Naïvely 1% isospin breaking, such as QED and light quark mass difference, can be $\sim$ 20% corrections
- In order to study how to include QED in a finite box with two hadrons, begin with the simpler system of $\pi - \pi$ scattering
- For $K \to \pi\pi$, see C. Kelly's and M. Tomii's talks from this morning
- For $\pi\pi$ scattering, see T. Wang's talk this afternoon

QED_L in a finite box

- Lüscher quantization works for interactions with a finite range, such as QCD unlike QED
- QED_L is a popular way of including QED into a finite box by removing the zero modes. $V_{\mathrm{QED}_L}(x) = \sum_{|k|\neq 0} \frac{e^{ik\cdot x}}{k^2}$
  - Adds new $1/L$ power law corrections to be determined
- Relation between finite volume energies with QED_L and scattering phase shift, in non-relativistic regime, derived by Beane and Savage *[S. Beane and M. Savage, arXiv:1407.4846]* and implemented by NPLQCD and QCDSF *[S. Beane et al, arXiv:2003.12130]* at heavy pion masses for understanding nucleon scattering
- Correction for Lellouch-Lüscher for $K \to \pi\pi$ discussed in proceedings *[Y. Cai and Z. Davoudi, arXiv:1812.11015]*

Splitting up QED

- First, choose the Coulomb gauge
  - Separates transverse radiation and instantaneous Coulomb interaction
  - The non-Lorentz covariance of the gauge is not issue. Use rest frame of the Kaon
- Second, truncate the Coulomb interaction to finite range R
  - The truncated interaction works perfectly with Lüscher quantization in a finite volume
  - Gives residual power law corrections in R^{-1}
  - The remainder of the Coulomb interaction, and the power law corrections in R, can be fixed after the fact
- Three pieces
  - Truncated Coulomb potential $V_{TC} = \frac{e^2}{r}\theta(R - r)$ [Numerically]
  - Complement of the truncated Coulomb potential $V_{\overline{TC}} = \frac{e^2}{r}\theta(r - R)$ [Analytically]
  - Transverse Radiation [Neglected for now]

Truncated Coulomb potential

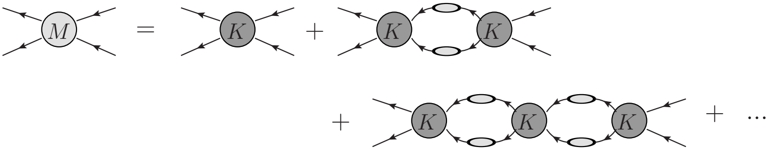
- Truncated Coulomb potential works perfectly well with Lüscher quantization given
  - $R < L/2$ so that the potential remains within the finite volume
  - $R > \sim \Lambda_{\mathrm{QCD}}^{-1}$ so that $V_{\overline{TC}}$ can be corrected with out needing quark dynamics
  - Limits need to be studied realistic calculations
- Unlike QED_L, the only $1/L$ power law corrections are those from neglected higher partial waves, instead have $1/R$ power law corrections
  - $1/R$ can be studied without changing the ensemble!
  - $1/R$ can also be corrected analytically
- All that remains are exponentially suppressed L and R corrections

Long Distance Calculation

[N. Christ, X. Feng, JK, T. Nguyen, arXiv:2111.04668]

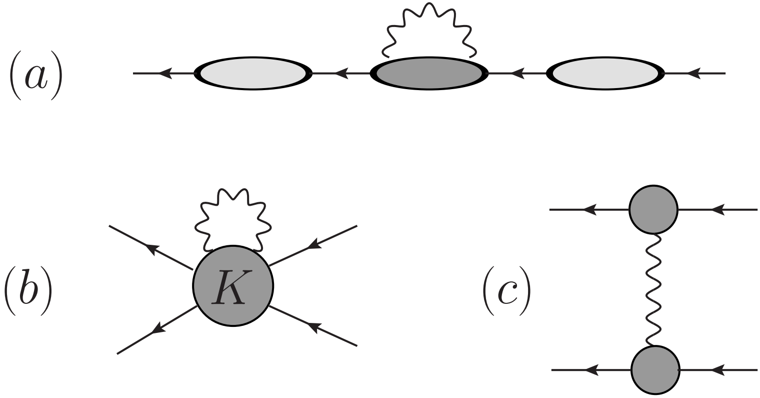
Calculating Corrections

- Want to remove all power corrections in R from the truncation, neglecting exponentially suppressed terms
- With sufficiently large R, the interacting pions can be treated as elementary particles in infinite volume
- Include $V_{\overline{TC}}$ into the Lippmann-Schwinger series



Calculating Corrections

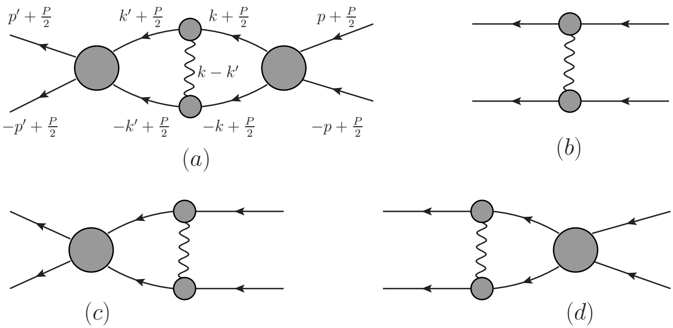
- (a) Self Energy Correction [Suppressed]
- (b) 2-PI Kernel Correction [Suppressed]
- (c) Exchange Diagrams Correction [Dominant]



Exponentially Suppressed Corrections

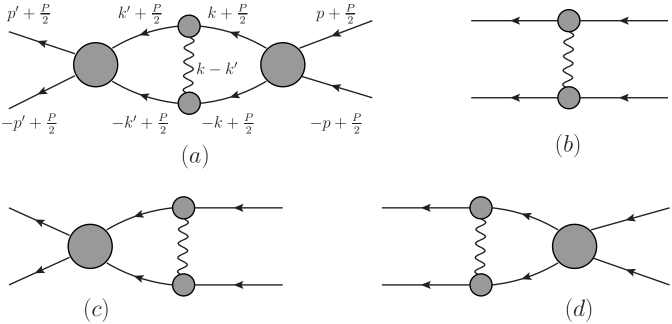
- When the two pion energy is below the 4 pion threshold, the self energy and 2-PI Kernel Correction can be calculated in Euclidean space and analytically continued to Minkowski space
- These diagrams will be dominated by short range interactions due to exponential decay of the propagators
- When $V_{\overline{TC}}$ is included, at least 2 propagators must travel the large distance R giving exponential suppression of the diagram
- *[N. Christ, X. Feng, JK, T. Nguyen, arXiv:2111.04668]*

The Exchange Diagrams Correction



- Generally diagrams involve off-shell scattering kernel and off-shell pion EM vertex
- $V_{\overline{TC}}$ restricts to long distance regions where they go on shell
- On shell quantities are obtainable from LQCD or combination of experiment and phenomenology

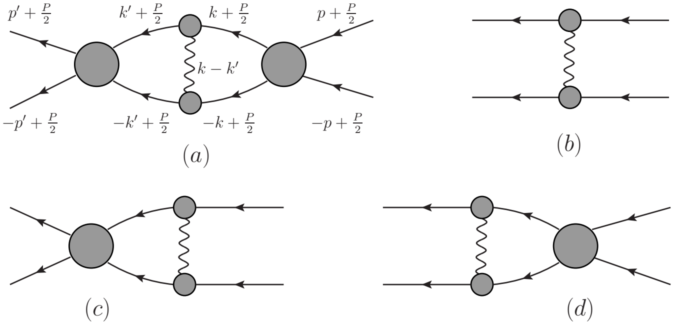
The Exchange Diagrams Correction



$$M_{\overline{TC},l} = \frac{1}{2l+1} \sum_{m=-l}^{l} \int \int d^4k' d^4k \Psi^{\text{out}}_{lm}(k', P)^* K_{\overline{TC}}(k', k, P) \Psi^{\text{in}}_{lm}(k, P) \quad (2)$$

$$\Psi^{\text{in/out}}_{lm}(k, P) = \psi^0_{lm}(k, P) + \psi^{\text{in/out}}_{lm}(k, P)$$

The Exchange Diagrams Correction



$$\delta_l^{\overline{TC}} = \frac{\omega_p}{2p} \frac{1}{2l+1} \sum_{m=-l}^{l} \sum_{l'=0}^{\infty} \sum_{m'=-l'}^{l'} \int_0^{\infty} dz \iint d\Omega_{\hat{p}'} d\Omega_{\hat{p}} V_{\overline{TC}}(|\vec{z}|)$$

$$\cdot \left[Y_{lm}(\hat{p}')^* Y_{l'm'}(\hat{p}')^* Y_{lm}(\hat{p}) Y_{l'm'}(\hat{p}) \right] F(|\vec{p} - \vec{p}\,'|)^2 \sin^2\left(p|\vec{z}| - \frac{\pi l}{2} + \delta_l \right)$$

Short Distance Calculation

Reminder of Finite Volume Quantization

- Relationship between finite volume energy levels and infinite volume phase shifts
  Valid when interaction is exponentially suppressed at distance smaller than volume
  Truncated Coulomb potential naturally works in finite volume for $R < L/2$
- Quantization condition ($q = Lp/2\pi$)

$$\delta_0(p) + \phi(q) = n\pi \tag{3}$$

$$\tan\phi(q) = -\frac{\pi^{3/2} q}{Z_{00}(1, q)} \tag{4}$$

Perturbation Theory

- Energy and phase shifts can be expanded in the coupling
  $$E = E^{(0)} + \alpha E^{(1)} + \dots$$
  $$\delta_0 = \delta_0^{(0)} + \alpha \delta_0^{(1)} + \dots$$
- Perturbative corrections to the energy calculated by

$$E^{(1)} = \frac{\langle O_{\pi\pi}(t_f) \frac{1}{2} \int d^3 r_1 d^3 r_2 \rho(r_2, t_V) V_{\mathrm{TC}}(|r_2 - r_1|_L) \rho(r_1, t_V) O_{\pi\pi}(t_i) \rangle}{\langle O_{\pi\pi}(t_f) O_{\pi\pi}(t_i) \rangle} \tag{5}$$

- Perturbative corrections to Lüscher quantization equation

$$\delta_0^{(1)}(p^{(0)}) = -\left\{ \frac{d\delta_0(p)}{dp} + \frac{d\phi(q)}{dq} \frac{L}{2\pi} \right\}_{p=p^{(0)}} \frac{E^{(0)}}{4p^{(0)}} E^{(1)}, \tag{6}$$

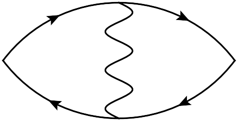
$$p^{(0)} = \sqrt{(E^{(0)}/2)^2 - m^2}$$

Renormalization of the quark mass

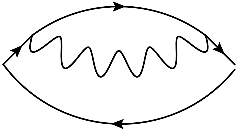
- Ensemble used was generated at physical pion mass with isospin symmetry
  Wish to maintain pion mass in isospin breaking calculation
- The quark mass must be adjusted perturbatively to keep the physical pion mass fixed when QED is added
  Tune $\alpha m_q^{(1)}$ correction to counteract QED pion mass shift
- Uses same matrix element as the light quark mass splitting
- Apply same quark mass shift from single pion to the scattering pions case

Lattice Setup

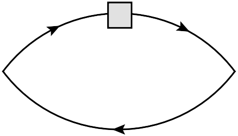
- Diagrams for single pion
- Currents meant at fixed time for Coulomb interaction
- Analogous diagrams needed for two pions



Exchange Diagram



Self Energy Diagram



Scalar Current

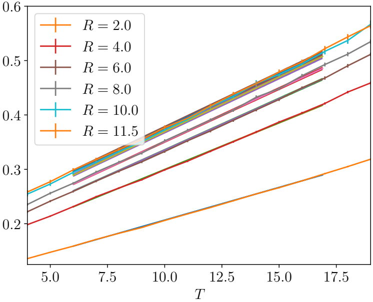
Lattice Setup

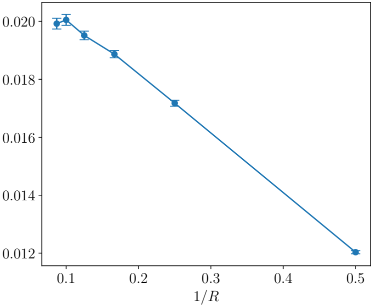
- RBC/UKQCD's 24ID ensemble, $a^{-1} = 1$ GeV, $m_\pi = 140$ MeV, $24^3 \times 64$
- Form correlation functions with Wall source and Wall sink interpolators
- Coulomb photons inserted stochastically
- Self-Energy diagram created with a sequential propagator (Still being calculated)
- Other diagrams by tying together source and sink propagators
- Use $R/a = 2.0, 4.0, 6.0, 8.0, 10.0, 11.5$ to test dependence
- Matrix elements obtained through summation method

$$R(T) = \frac{\sum_t C_3(T,t)}{C_2(T)} = MT + b \tag{7}$$

Exchange diagram for single pion

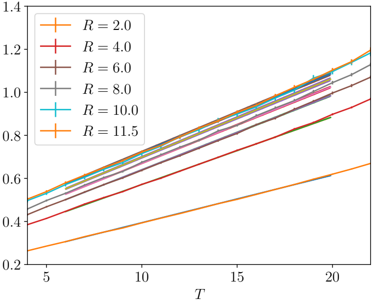
- Linear behavior in $1/R$ from truncating the interaction which is $1/r$
- $R = 11.5$, which has interaction filling almost entire volume, may be seeing edge effects from finite volume
- Preliminary to study in detail without self energy diagram

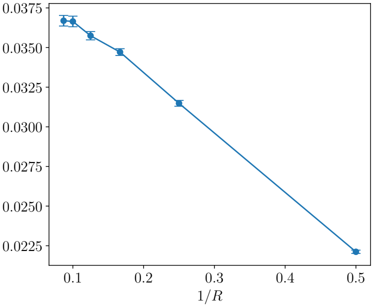




Exchange diagram for two pions

- Linear behavior in $1/R$ from truncating the interaction which is $1/r$
- $R = 11.5$, which has interaction filling almost entire volume, may be seeing edge effects from finite volume
- Preliminary to study in detail without self energy diagram





Scalar current insertion

- Scalar current insertion for light quark mass splitting
- Appears noisier than the insertion of photons for exchange diagram

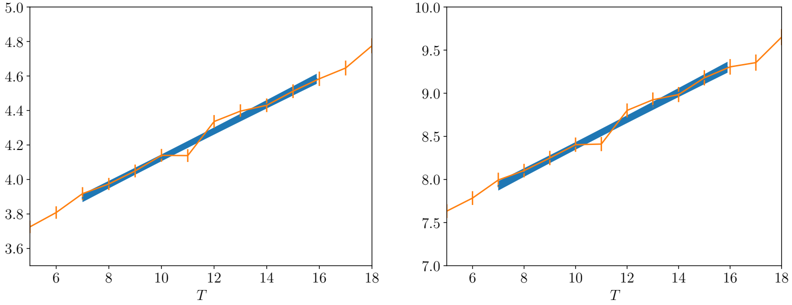


Figure: The scalar current diagram for single pion with $M = 0.077(4)$ (Left) and two pions with $M = 0.154(9)$(Right)

What needs doing to get $K \to \pi\pi$

- Finish diagram for $\pi^+\pi^+$ scattering and study R dependence in phase shifts
- Study with twisted boundary conditions to move away from threshold
- Formalism for transverse radiation in $\pi\pi$ scattering and $K \to \pi\pi$
- Generalize truncated Coulomb interaction formalism to neutral $\pi\pi$ scattering with two channels
- Expand truncated Coulomb interaction formalism to $K \to \pi\pi$ decays
- Work on methods for efficient calculation of QED insertions to $K \to \pi\pi$ decay amplitudes

Conclusions

- Can add Coulomb interactions to Lüscher finite volume quantization without new $1/L$ power law corrections, at cost of new $1/R$ corrections
- If R is within appropriate limits, $1/R$ corrections can be calculated analytically in the infinite volume
- Need to study R dependence in realistic lattice calculation to study the limits and R independence of final phase shifts
- Need to add back in the transverse radiation to solve full relativistic problem