Lattice Calculation of Δm_K

Bigeng Wang

Department of Physics and Astronomy
University of Kentucky

DWQ@25
December 13–17, 2021

The RBC & UKQCD collaborations

UC Berkeley/LBNL
Aaron Meyer

BNL and BNL/RBRC
Yasumichi Aoki (KEK)
Peter Boyle (Edinburgh)
Taku Izubuchi
Yong-Chull Jang
Chulwoo Jung
Christopher Kelly
Meifeng Lin
Hiroshi Ohki
Shigemi Ohta (KEK)
Amarjit Soni

CERN
Andreas Jüttner (Southampton)

Columbia University
Norman Christ
Duo Guo
Yikai Huo
Yong-Chull Jang
Joseph Karpie
Bob Mawhinney
Ahmed Sheta
Bigeng Wang
Tianle Wang
Yidi Zhao

University of Connecticut
Tom Blum
Luchang Jin (RBRC)
Michael Riberdy
Masaaki Tomii

Edinburgh University
Matteo Di Carlo
Luigi Del Debbio
Felix Erben
Vera Gülpers
Tim Harris
Raoul Hodgson
Nelson Lachini
Michael Marshall
Fionn Ó hÓgáin
Antonin Portelli
James Richings
Azusa Yamaguchi
Andrew Z.N. Yong

KEK
Julien Frison

University of Liverpool
Nicolas Garron

Michigan State University
Dan Hoying

Milano Bicocca
Mattia Bruno

Peking University
Xu Feng

University of Regensburg
Davide Giusti
Christoph Lehner (BNL)

University of Siegen
Matthew Black
Oliver Witzel

University of Southampton
Nils Asmussen
Alessandro Barone
Jonathan Flynn
Ryan Hill
Rajnandini Mukherjee
Chris Sachrajda

University of Southern Denmark
Tobias Tsang

Stony Brook University
Jun-Sik Yoo
Sergey Syritsyn (RBRC)

$K^0 - \overline{K^0}$ mixing and Δm_K

$K^0(S=-1)$ and $\overline{K^0}(S=+1)$, each having definite strangeness, which is conserved in the strong processes, mix through second order weak interactions.

$$i\frac{d}{dt}\begin{pmatrix} K^0(t) \\ \overline{K^0}(t) \end{pmatrix} = (M - \frac{i}{2}\Gamma)\begin{pmatrix} K^0(t) \\ \overline{K^0}(t) \end{pmatrix}, \quad (1)$$

where the matrix M is given by:

$$M_{ij} = m_K^{(0)}\delta_{ij} + \mathcal{P}\sum_n \frac{\langle K_i^0|H_W|n\rangle\langle n|H_W|K_j^0\rangle}{m_K - E_n}, \quad (2)$$

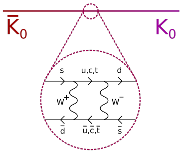


Figure: **from wikipedia**

If the small effects of CP violation are neglected, long-lived (K_L) and short-lived (K_S) are the two eigenstates:

$$K_S \approx \frac{K^0 - \overline{K^0}}{\sqrt{2}}, \quad K_L \approx \frac{K^0 + \overline{K^0}}{\sqrt{2}}. \quad (3)$$

$$\Delta m_K \equiv m_{K_L} - m_{K_S} = 2\mathrm{Re}M_{12}. \quad (4)$$

Different life times:

$K_S \xrightarrow[\text{even}]{\text{CP}} \pi\pi$,

$2m_\pi \approx 280\mathrm{MeV} < m_K$

$K_L \xrightarrow[\text{odd}]{\text{CP}} \pi\pi\pi$,

$3m_\pi \approx 420\mathrm{MeV} \lesssim m_K$

Physics motivation

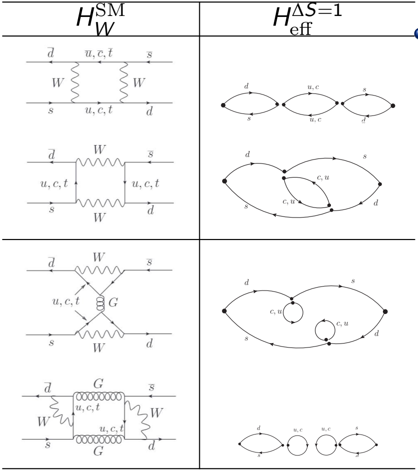
Δm_K is given by:

$$\Delta m_K \equiv m_{K_L} - m_{K_S} = 2\mathrm{Re}M_{12} = 2\mathcal{P}\sum_n \frac{\langle \overline{K^0}|H_W|n\rangle\langle n|H_W|K^0\rangle}{m_K - E_n}. \qquad (5)$$

- This quantity is:
  1. **Tiny** compared to the K^0 mass $\sim$ 498 MeV, and precisely measured
     $$\Delta m_{K,exp} = 3.483(6) \times 10^{-12} \text{ MeV}$$
  2. **Sensitive to new physics**: FCNC via 2nd order weak interaction
  3. Significant contribution from scale of m_c(GIM mechanism)
  4. **Difficult to compute by treating charm quark perturbatively**: strong coupling at m_c scale

J. Brod and M. Gorbahn, Phys. Rev. Lett. 108, 121801 (2012)

Non-perturbative calculation of Δm_K using a renormalization scale above the charm quark mass



- $\mathrm{Re}\,ⓧ_{GIM} \simeq \lambda_u^2\, ⓧ^{(u-c)(u-c)}$ [arXiv:1402.2577]
- The $\Delta S = 1$ effective weak Hamiltonian:

$$H_{\mathrm{eff}}^{\Delta S=1} = \frac{G_F}{\sqrt{2}} \sum_{q,q'=u,c} V_{qd} V^*_{q's} \left(C_1 Q_1^{qq'} + C_2 Q_2^{qq'} \right) \tag{6}$$

where the $Q_i^{qq'}{}_{i=1,2}$ are current-current opeartors, defined as:

$$Q_1^{qq'} = (\bar{s}_i \gamma^\mu (1-\gamma^5) d_i)(\bar{q}_j \gamma^\mu (1-\gamma^5) q'_j)$$

$$Q_2^{qq'} = (\bar{s}_i \gamma^\mu (1-\gamma^5) d_j)(\bar{q}_j \gamma^\mu (1-\gamma^5) q'_i)$$

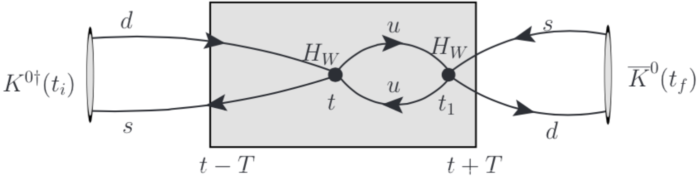
Extract Δm_K from single-integrated correlators

- The single-integrated correlator is defined as:

$$\mathcal{A}^s(t, T) \equiv \frac{1}{2!} \sum_{t_1=t-T}^{t+T} \langle 0 | T\{\overline{K^0}(t_f) H_W(t_1) H_W(t) K^0(t_i)\} | 0 \rangle \qquad (7)$$

- If we insert a complete set of intermediate states, we find:

$$\mathcal{A}^s = N_K^2 e^{-m_K(t_f - t_i)} \sum_n \frac{\langle \overline{K^0} | H_W | n \rangle \langle n | H_W | K^0 \rangle}{m_K - E_n} (-1 + e^{(m_K - E_n)(T+1)}) \quad (8)$$



Status of the calculation

- "$K_L - K_S$ **mass difference from Lattice QCD"**
  Z. Bai, N. H. Christ, T. Izubuchi, C. T. Sachrajda, A. Soni and J. Yu
  Phys. Rev. Lett. 113(2014), 112003
  All diagrams included on a $24^3 \times 64$ lattice with unphysical masses
- **"The** $K_L - K_S$ **mass difference"**
  Z. Bai, N. H. Christ, C. T. Sachrajda
  EPJ Web of Conferences 175(2018), 13017
  All diagrams included on a $64^3 \times 128$ lattice with **physical masses** on 59 configurations: $\Delta m_k = 5.5(1.7)_{stat} \times 10^{-12}$ MeV.
- **"Calculation of the** $K_L - K_S$ **mass difference for physical quark masses"**
  B. Wang
  PoS LATTICE2019 (2019) 093
  All diagrams included on a $64^3 \times 128$ lattice with **physical masses** on **152** configurations: $\Delta m_k = 6.7(0.6)_{stat}(1.7)_{sys} \times 10^{-12}$ MeV.
- In this talk I will present the most recent Δm_K results and address studies performed on smaller lattices to estimate the systematic errors in our result.

Δm_K calculation with physical quark masses

- $64^3 \times 128 \times 12$ lattice with Möbius DWF and the Iwasaki gauge action with physical pion mass (136 MeV).

Lattice ensemble	Action (F+G)	a^{-1} (GeV)	Lattice Volume	β	b+c	L_s	m_l	m_h	m_{res}
64I	MDWF+I	2.359(7)	$64^3 \times 128 \times 12$	2.25	2.0	12	0.000678	0.02661	0.000314

- Data analysis:
  - Sample AMA correction: [Phys. Rev. D88(9), 094503 (2013)]

data type	CG stop residual
Sloppy	$1e-4$
Exact	$1e-8$

Diagram types	sample AMA correction	# of Sloppy	# of Exact
Type-3&4	Y	116	36
Type-1&2	N	0	36

The super-jackknife method is used to estimate the statistical errors for the AMA corrected data.

  - Disconnected Type4 diagrams:
    save left- and right-pieces separately and use multiple source-sink separation for fitting.

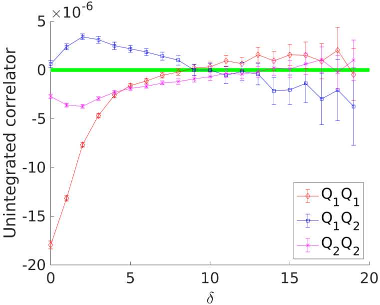
Calculation of Δm_K using single-integrated correlators

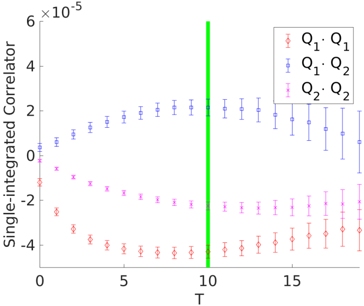
- Subtract light states from the averaged unintegrated correlator:

$$\widetilde{G}_{ij}^{\mathrm{sub}}(\delta) = \widetilde{G}_{ij}(\delta) - \sum_{n \in \{n_l\}} \langle \overline{K^0} | Q_i' | n \rangle \langle n | Q_j' | K^0 \rangle e^{(m_K - E_n)\delta} \tag{9}$$

- Perform a single-integration over δ for the subtracted correlator between $\delta = 0$ and $\delta = T$ to obtain:

$$\widetilde{\mathcal{A}}_{ij}^{S}(T) = \sum_{\delta=1}^{T} \widetilde{G}_{ij}^{\mathrm{sub}}(\delta) + \frac{1}{2} \widetilde{G}_{ij}^{\mathrm{sub}}(0) \tag{10}$$





Results for Δm_K preliminary

- We choose to use the results from the single-integration method:

Analysis method	$\Delta m_K/10^{-12}$MeV	Δm_K(type1&2)	Δm_K(type3&4)
Double-integration	6.31(0.98)	6.71(0.48)	-0.20(0.65)
Single-integration	**6.34(0.57)**	6.24(0.24)	0.33(0.50)

- Systematic errors:
  - **Finite-volume corrections**: small
    "Effects of finite volume on the $K_L - K_S$ mass difference"
    N.H. Christ, X. Feng, G. Martinelli and C.T. Sachrajda, arXiv:1504.01170
    $\delta(\Delta m_K)^{FV} = -0.54(18) \times 10^{-12}\mathrm{MeV}.$
  - **Discretization effects** are the **largest** source of systematic error:
    - effects from low-energy scale $\sim \Lambda_{\mathrm{QCD}}$
    - heavy charm quark, $\sim (m_c a)^2$ gives 25%
    - Another estimate based on HVP calculation is $\sim$ 15%

Finite lattice spacing effects: scaling tests preliminary

- Scaling tests: perform calculations of three- and four-point quantities on two lattices with different lattice spacings. We need a coarser lattice to be compared with a finer lattice.
    - **64I(2.4 GeV)** $\leftrightarrow$ 96I(2.8 GeV) Hard to do
    - **24I(1.8 GeV)** $\leftrightarrow$ 32I(2.4 GeV) ✓
- We would like to see how large the discrepancy is
    - at a relatively small $m_c a$, which shows the finite lattice spacing effects from low-energy scale $\sim \Lambda_{\mathrm{QCD}}$.
    - **at a relatively large $m_c a \sim 0.32$, which corresponds to the physical mass in our calculation on the 64I ensemble.**

Lattice name	Action (F+G)	a^{-1} (GeV)	Lattice Volume	β	b+c	L_s	m_l	m_h	m_{res}
24I	DWF+I	1.785(5)	$24^3 \times 64 \times 16$	2.13	1.0	16	0.0050	0.0400	0.00308
32I	DWF+I	2.383(9)	$32^3 \times 64 \times 16$	2.25	1.0	16	0.0040	0.0300	0.000664

Finite lattice spacing effects: scaling tests preliminary

- Set up input masses to keep physics consistent on the two ensembles:

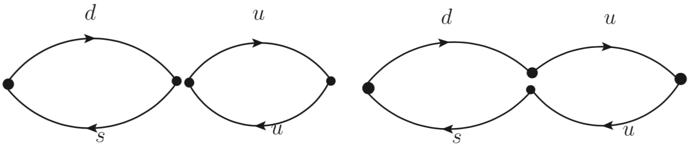
Lattice	m_x	m_y	m_π	m_K	m_c's
24I	0.00667	0.0321	0.2079	0.3125	0.15:0.05:0.35
32I	0.00649	0.0249	0.1557	0.2332	$(0.15{:}0.05{:}0.35)\frac{1.785}{2.383}$

- Quantities to be compared are:
  - 2-point:
    - m_π, m_K $\rightarrow$ confirm the valence masses yield consistent physical values.

Lattice	N_{conf}	m_π/MeV	m_K/MeV
24I	186	371.3(7)	556.2(7)
32I	222	371.4(6)	557.5(6)

    - m_D $\rightarrow$ to calculate $m_c(m_D)$ to match the physics.
  - 3-point figure-8 diagrams: light charm which has degenerate mass with m_u $\langle\pi|Z_{(84,1)}Q_+|K^0\rangle$, $\langle\pi|Z_{(20,1)}Q_-|K^0\rangle$,
  - 4-point single-integrated correlators(connected diagrams only) with m_c dependency:
    Integrated correlators: with operator combinations: Q_+Q_+ and Q_-Q_-

Scaling test - three-point figure-8 results preliminary



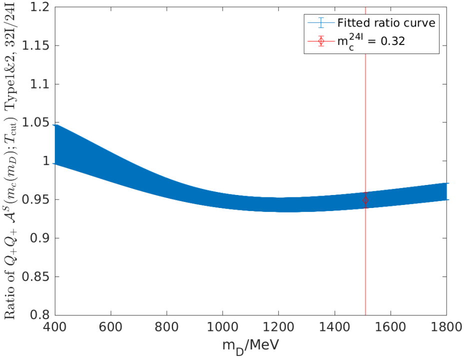
μ/GeV	Op.	Z factors 32I (a^{-1} =2.38GeV)	Z factors 24I (a^{-1} =1.78GeV)	Matrix elements in physical Unit 32I (a^{-1} =2.38GeV)	Matrix elements in physical Unit 24I (a^{-1} =1.78GeV)	Scaling violation
2.15	Q_+	0.52997(11)	0.47143(8)	0.003957(18)	0.004045(18)	-2.19 %
	Q_-	0.58755(14)	0.57493(26)	0.011949(65)	0.009936(59)	**18.39 %**
2.64	Q_+	0.52489(6)	0.46996(6)	0.003919(18)	0.004032(18)	-2.84 %
	Q_-	0.60358(11)	0.58239(11)	0.012275(67)	0.010065(60)	**19.78 %**

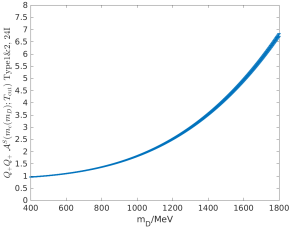
Table: The Z factors of NPR in (γ_μ, γ_μ) scheme and $\langle \pi | Q_\pm | K^0 \rangle$(figure-8 only) in physical units on the two lattice ensembles and different scale μ.

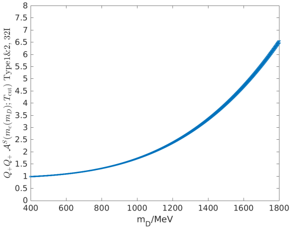
Comment: $K \to \pi\pi$ matrix elements, $O \in (8,8)$ irrep also show similarly large finite lattice spacing errors. [PhysRevD.91.07450]

Scaling test - four-point Q_+Q_+ single-integrated correlators

preliminary



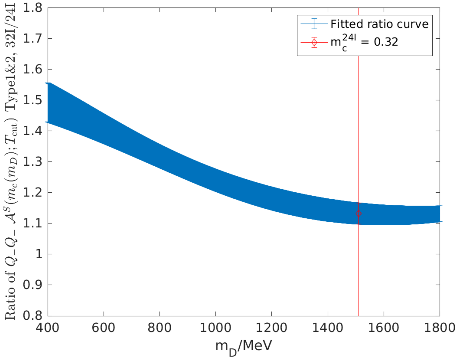


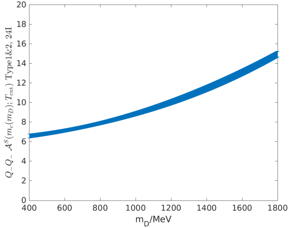


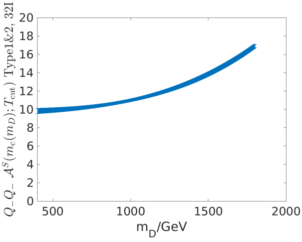
- at low m_D values, the ratio is consistent with 1.
- at $m_c a = 0.32$, which corresponds to the physical charm mass on 64I, the scaling violation is about 5%.

Scaling test - four-point Q_-Q_- single-integrated correlators

preliminary







- at low m_D values, the ratio is about 1.45. The scaling violation: $\sim 40\%$.
- at $m_c a = 0.32$, the scaling violation: $\sim 14\%$.

Summary of the scaling tests preliminary

Quantities compared between the two lattice spacings are:

- 2-point:
  - m_π, m_K ✓
  - m_D ✓
- 3-point: light charm which has degenerate mass with m_u:
  $\langle \pi | Z_{(84,1)} Q_+ | K^0 \rangle$ ✓ 3% difference
  $\langle \pi | Z_{(20,1)} Q_- | K^0 \rangle$! 20% difference
  Comment: $K \rightarrow \pi\pi$ matrix elements $\in$ (8,8) irrep also show similarly large finite lattice spacing errors. [PhysRevD.91.07450].
- 4-point(connected only) with m_c dependency:
  single-integrated correlators: with operators
  $Z^2_{(84,1)} Q_+ Q_+$ ✓ 5% difference
  $Z^2_{(20,1)} Q_- Q_-$! 40% difference

We estimate the finite lattice spacing error in our Δm_K calculation to be of order of 40%.

Conclusion and outlook

- Our **preliminary** result for Δm_K based on 152 configurations is:

$$\Delta m_K = 5.8(0.6)_{\text{stat}}(2.3)_{\text{sys}} \times 10^{-12}\text{MeV}, \quad (11)$$

to be compared to the experimental value:

$$(\Delta m_K)^{exp} = 3.483(6) \times 10^{-12}\text{MeV}. \quad (12)$$

We find reasonable agreement given the large finite lattice spacing errors.

- Outlook:
  Future calculations:
  - Δm_K: on $96^3 \times 192$ lattice with $a^{-1} = 2.8$ GeV
    - Better estimate of finite lattice spacing effect:
      64I(2.4 GeV) $\leftrightarrow$ 96I(2.8 GeV) continuum limit to be explored
    - Further improvement of the precision to $\sim 5\%$ level.
  - long-distance ϵ_K: Joe Karpie, improve the accuracy of ϵ_K to sub-percent level.

Thanks for your attention!

Backup slides

$K^0 - \overline{K^0}$ mixing and Δm_K

Two particles: K^0: $(d\bar{s}, S=-1)$ and $\overline{K^0}$ $(\bar{d}s, S=+1)$.

- Strong interactions: conserve strangeness
- Weak interactions: H_W changes strangeness, $\Delta S = 1$.

Second-order weak process: $K^0 \leftrightarrow \overline{K^0}$

$$i\frac{d}{dt}\begin{pmatrix} K^0(t) \\ \overline{K^0}(t) \end{pmatrix} = (M - \frac{i}{2}\Gamma)\begin{pmatrix} K^0(t) \\ \overline{K^0}(t) \end{pmatrix}. \tag{13}$$

If the small effects of CP violation are neglected, long-lived (K_L) and short-lived (K_S) are the two eigenstates:

$$K_S \approx \frac{K^0 - \overline{K^0}}{\sqrt{2}}, \quad K_L \approx \frac{K^0 + \overline{K^0}}{\sqrt{2}}. \tag{14}$$

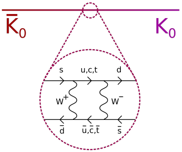


Figure: figure from wikipedia

Different life times:

$K_S \xrightarrow[\text{even}]{\text{CP}} \pi\pi$,

$2m_\pi \approx 280\text{MeV} < m_K$

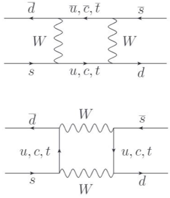
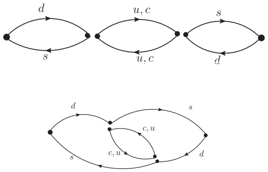
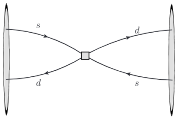
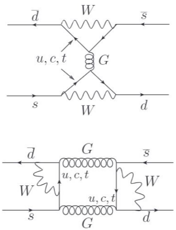
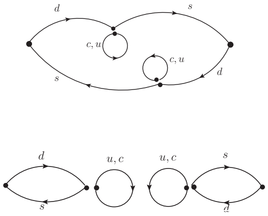
$K_L \xrightarrow[\text{odd}]{\text{CP}} \pi\pi\pi$,

$3m_\pi \approx 420\text{MeV} \lesssim m_K$

The operator product expansion(OPE) and Δm_K

OPE: full theory $H_W \xrightarrow[\text{heavy particles}]{\text{integrate out}} H_{\text{eff}} = \sum_j C_j(\mu) O_j(\mu)$, renormalized at scale μ

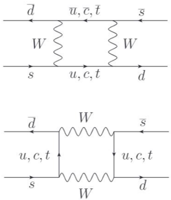
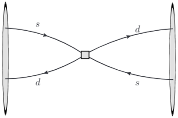
$C_j(\mu)$: short-distance, perturbative; $O_j(\mu)$: long-distance, non-perturbative

H_W^{SM}	$H_{\text{eff}}^{\Delta S=1}$	$H_{\text{eff}}^{\Delta S=2}$
		
		

Earlier calculations of Δm_K: charm quark is integrated out

The specific division $\mu < m_c$ in OPE where charm quark is integrated out.

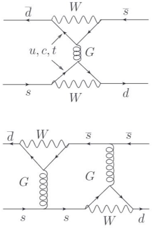
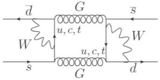
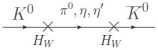
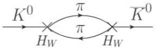
short-distance box only:

$H_W^{\rm SM} \rightarrow H_{\rm eff}^{\Delta S=2}$	
	

$$\mathcal{H}_{\rm eff}^{\Delta S=2} = C(\mu) O_{LL}(\mu), \qquad (15)$$

$$O_{LL} = (\bar{s}d)_{V-A}(\bar{s}d)_{V-A}, \qquad (16)$$

leaving out:

QCD penguin	disconnected
	
long-distance box	
	

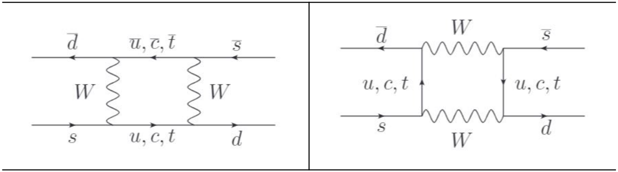
Only 36% accuracy in the next-to-next-to-leading-order(NNLO) calculation of the QCD correction factors using perturbation theory: slow convergence of the perturbative series

J. Brod and M. Gorbahn, Phys. Rev. Lett. 108, 121801 (2012)

→ **better to treat charm quark non-perturbatively on the lattice**

GIM mechanism and the short- and long-distance characteristics of Δm_K

GIM mechanism: flavor-changing neutral currents(FCNC) are suppressed in loop diagrams → charm quark → the CKM matrix



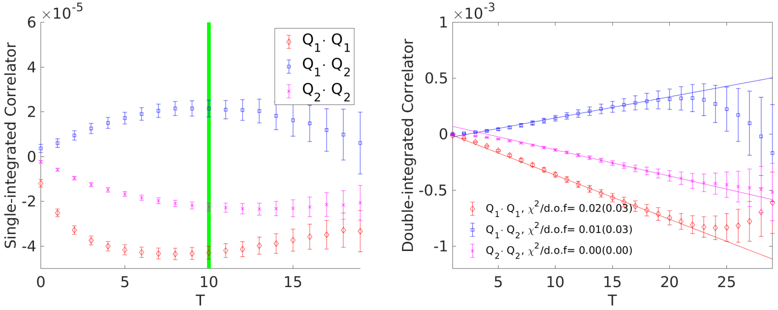
- Quark mixing: at each weak vertex
  → a product of CKM matrix elements $V_{qd}V^*_{q's}$, where $q, q' = u, c, t$.
- Define $\lambda_q = V_{q,d}V^*_{q,s}$, $q = u, c, t$,
  unitarity of the CKM matrix → $\lambda_u + \lambda_c + \lambda_t = 0 \rightarrow \lambda_c = -\lambda_u - \lambda_t$
- Specific diagram with GIM mechanism:
  $$\textcircled{X}_{GIM} = \lambda_u^2 \textcircled{X}^{(u-c)(u-c)} + \lambda_t^2 \textcircled{X}^{(t-c)(t-c)} + 2\lambda_u\lambda_t \textcircled{X}^{(u-c)(t-c)}$$
- For $\Delta m_K = 2\mathrm{Re}M_{12}$, the first term dominates.

Overview of the calculation of Δm_K

Quantities to be calculated are:

- two-point correlation functions:
  - meson masses: m_π, m_K, $m_{\pi\pi}$, m_η
  - normlization factors of meson interpolating operators: N_π, N_K, $N_{\pi\pi}$, N_η
- three-point correlation functions:
  - light state matrix elements to be subtracted: $\langle\pi|Q_i'|K^0\rangle = \langle\pi|Q_i|K^0\rangle - c_{\mathrm{s}i}\langle\pi|\bar{s}d|K^0\rangle$, and $\langle\pi\pi_{I=0}|Q_i - c_{\mathrm{p}i}\bar{s}\gamma_5 d|K^0\rangle$.
  - coefficients of the $\bar{s}d$ and $\bar{s}\gamma_5 d$ operators: $c_{\mathrm{s}i} = \frac{\langle\eta|Q_i|K^0\rangle}{\langle\eta|\bar{s}d|K^0\rangle}$, $\quad c_{\mathrm{p}i} = \frac{\langle 0|Q_i|K^0\rangle}{\langle 0|\bar{s}\gamma_5 d|K^0\rangle}$.
- four-point correlation functions:
  - unintegrated correlation functions calculated from diagrams having light state contribution subtracted: $\widetilde{G}^{\mathrm{sub}}(\delta) = \widetilde{G}(\delta) - \sum_{n\in\{n_l\}} \langle\bar{K}^0|H_W|n\rangle\langle n|H_W|K^0\rangle e^{(m_K-E_n)\delta}$
  - single-integrated correlation functions: $\widetilde{\mathscr{A}}^S(T) = \sum_{\delta=1}^{T} \widetilde{G}^{\mathrm{sub}}(\delta) + \frac{1}{2}\widetilde{G}^{\mathrm{sub}}(0) \to \Delta m_K$

Calculation of Δm_K using double-integrated correlators



Analysis method	Δm_K
Double-integration	6.31(0.98)
Single-integration	6.34(0.57)

Scaling of diagrams with a specific topology

- A collection of diagrams to be studied in isolation in a lattice calculation
  - Fermion propagators contracted with a fixed topology.
  - The path integral provides a sum over all possible gluon emissions, gluon self-interactions and closed fermion loop insertions.
- Conditions for a continuum limit:
  - The quark propagator topology DOES NOT introduce new divergent sub-diagrams not present in QCD. The renormalizability and chiral symmetry of DWF QCD will lead to a continuum limit with a ca^2 scaling behavior.
  - The quark propagator topology DOES introduce new divergent sub-diagrams not present in QCD: include these same diagrams when performing the NPR subtraction.

Non-perturbative renormalizations

- Renormalization of lattice operator $Q_{1,2}$ in 3 steps:

$$C_i^{lat} = C_a^{\overline{MS}} (1 + \Delta r)_{ab}^{RI \to \overline{MS}} Z_{bi}^{lat \to RI}$$

  - Non-perturbative Renormalization: from the lattice to the RI-SMOM

$$Z^{lat \to RI} = \begin{bmatrix} 0.5642 & -0.03934 \\ -0.03934 & 0.5642 \end{bmatrix} \quad (17)$$

  - Perturbation theory: from the RI-SMOM to the $\overline{MS}$

    C. Lehner, C. Sturm, Phys. Rev. D 84(2011), 014001

$$\Delta r^{RI \to \overline{MS}} = 10^{-3} \times \begin{bmatrix} -2.28 & 6.85 \\ 6.85 & -2.28 \end{bmatrix} \quad (18)$$

  - Use Wilson coefficients in the $\overline{MS}$ scheme

    G. Buchalla, A.J. Buras and M.E. Lautenbacher, arXiv:hep-ph/9512380

$$C^{\overline{MS}} = 10^{-3} \times \begin{bmatrix} -0.260 & 1.118 \end{bmatrix} \quad (19)$$

The jackknife and super-jackknife method

Raw data sets:

y_1	y_2	$\cdots$	y_{N_s-1}	y_{N_s}	Δy_1	Δy_2	$\cdots$	Δy_{N_c-1}	Δy_{N_c}

N_s elements N_c elements

Jackknife data sets:

Y_1	Y_2	$\cdots$	Y_{N_s-1}	Y_{N_s}	ΔY_1	ΔY_2	$\cdots$	ΔY_{N_c-1}	ΔY_{N_c}

N_s elements N_c elements

Super-jackknife data set:

$Y_1+\overline{\Delta Y}$	$Y_2+\overline{\Delta Y}$	$\cdots$	$Y_{N-1}+\overline{\Delta Y}$	$Y_N+\overline{\Delta Y}$	$\overline{Y}+\Delta Y_1$	$\overline{Y}+\Delta Y_2$	$\cdots$	$\overline{Y}+\Delta Y_{N_c-1}$	$\overline{Y}+\Delta Y_{N_c}$

(N_s+N_c) elements

The mean of the fitting parameter Θ is given by:

$$\overline{\Theta} = \frac{1}{N_s+N_c}\sum_{i=1}^{N_s+N_c}\Theta_i, \quad \sigma^2_{\overline{\Theta}} = \frac{N_s+N_c-1}{N_s+N_c}\sum_{i=1}^{N_s+N_c}(\Theta_i-\overline{\Theta})^2. \tag{20}$$

Scaling test - three-point $K \longrightarrow \pi$ diagrams