Lattice Calculation of Δm_K

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$K^0 - \overline{K^0}$ mixing and Δm_K

$K^0(S = -1)$ and $\overline{K^0}(S = +1)$, each having definite strangeness, which is conserved in the strong processes, mix through second order weak interactions.

$$i\frac{d}{dt}\begin{pmatrix} K^0(t) \\ \overline{K^0}(t) \end{pmatrix} = (M - \frac{i}{2}\Gamma)\begin{pmatrix} K^0(t) \\ \overline{K^0}(t) \end{pmatrix} \quad (1)$$

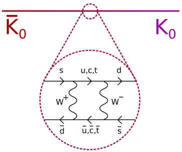


Figure: from wikipedia

where the matrix M is given by:

$$M_{ij} = m_K^{(0)}\chi_{ij} + \mathcal{P}\sum_n \frac{\langle K_i^0|H_W|n\rangle\langle n|H_W|K_j^0\rangle}{m_K - E_n} \quad (2)$$

If the small effects of CP violation are neglected, long-lived (K_L) and short-lived (K_S) are the two eigenstates:

$$K_S \approx \frac{K^0 - \overline{K^0}}{\sqrt{2}}, \quad K_L \approx \frac{K^0 + \overline{K^0}}{\sqrt{2}}. \quad (3)$$

$$\Delta m_K \equiv m_{K_L} - m_{K_S} = 2\mathrm{Re}M_{12}. \quad (4)$$

Different life times:

$K_S \xrightarrow[\text{even}]{\text{CP}} \pi\pi$,

$2m_\pi \approx 280\text{MeV} \lesssim m_K$

$K_L \xrightarrow[\text{odd}]{\text{CP}} \pi\pi\pi$,

$3m_\pi \approx 420\text{MeV} \lesssim m_K$

Physics motivation

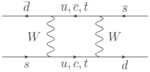
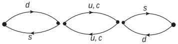
Δm_K is given by:

$$\Delta m_K \equiv m_{K_L} - m_{K_S} = 2\mathrm{Re}M_{12} = 2\mathcal{P}\sum_n \frac{\langle \overline{K^0}|H_W|n\rangle\langle n|H_W|K^0\rangle}{m_K - E_n}. \quad (5)$$

- This quantity is:
  1. Tiny compared to the K^0 mass $\sim$ 498 MeV, and precisely measured
     $$\Delta m_{K\text{-}exp} = 3{\cdot}483(6) \times 10^{-12}\ \mathrm{MeV}$$
  2. Sensitive to new physics: FCNC via 2nd order weak interaction
  3. Significant contribution from scale of m_c(GIM mechanism)
  4. Di□cult to compute by treating charm quark perturbatively: strong coupling at m_c scale

J. Brod and M. Gorbahn, Phys. Rev. Lett. 108, 121801 (2012)

Non-perturbative calculation of Δm_K using a renormalization scale above the charm quark mass

H_W^{SM}	$H_{e□}^{\Delta S=1}$
	

- Re $\textcircled{X}_{GIM} \simeq -^2_u \textcircled{X}^{(u-c)(u-c)}$ [arXiv:1402.2577]
- The $\Delta S = 1$ effective weak Hamiltonian:

$$H_{e□}^{\Delta S\ =1} = \frac{G_F}{\sqrt{2}} \underset{q\text{-}q'=u\text{-}c}{\tilde{O}} V_{qd} V^*_{q's} (C_1 Q_1^{qq'} + C_2 Q_2^{qq'}) \tag{6}$$

where the $Q_i^{qq'}{}_{i=1\text{-}2}$ are current-current opeartors, defined as:

$$Q_1^{qq'} = (s_i W^` (1 - W^5) d_i)(q_j W^` (1 - W^5) q'_j)$$

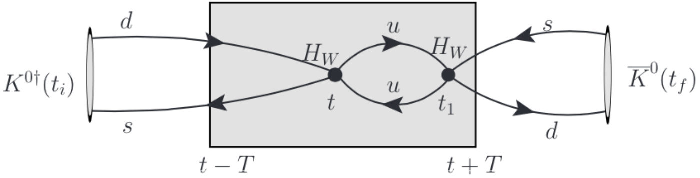
$$Q_2^{qq'} = (s_i W^` (1 - W^5) d_j)(q_j W^` (1 - W^5) q'_i)$$

- The single-integrated correlator is defined as:

$$\mathcal{A}^s(t\text{-}T) \equiv \frac{1}{2!}\sum_{t_1=t-T}^{t+T} \langle 0|T\{\overline{K^0}(t_f)H_W(t_1)H_W(t)K^0(t_i)\}|0\rangle \qquad (7)$$

- If we insert a complete set of intermediate states, we find:

$$\mathcal{A}^s = N_K^2 e^{-m_K(t_f-t_i)} \sum_n \frac{\langle\overline{K^0}|H_W|n\rangle\langle n|H_W|K^0\rangle}{m_K - E_n}(-1+e^{(m_K-E_n)(T+1)}) \qquad (8)$$



Status of the calculation

- **"$K_L - K_S$ mass difference from Lattice QCD"**
  Z. Bai, N. H. Christ, T. Izubuchi, C. T. Sachrajda, A. Soni and J. Yu
  Phys. Rev. Lett. 113(2014), 112003
  All diagrams included on a $24^3 \times 64$ lattice with unphysical masses
- **"The $K_L - K_S$ mass difference"**
  Z. Bai, N. H. Christ, C. T. Sachrajda
  EPJ Web of Conferences 175(2018), 13017
  All diagrams included on a $64^3 \times 128$ lattice with physical masses on 59 configurations: $\Delta m_k = 5.5(1.7)_{stat} \times 10^{-12}$ MeV.
- **"Calculation of the $K_L - K_S$ mass difference for physical quark masses"**
  B. Wang
  PoS LATTICE2019 (2019) 093
  All diagrams included on a $64^3 \times 128$ lattice with physical masses on 152 configurations: $\Delta m_k = 6.7(0.6)_{stat}(1.7)_{sys} \times 10^{-12}$ MeV.
- In this talk I will present the most recent Δm_K results and address studies performed on smaller lattices to estimate the systematic errors in our result.

m_K calculation with physical quark masses

$64^3 \times 128 \times 12$ lattice with Möbius DWF and the Iwasaki gauge action with physical pion mass (136 MeV).

Lattice ensemble	Action (F+G)	a^{-1} (GeV)	Lattice Volume	V	b+c	L_s	m_l	m_h	m_{res}
64I	MDWF+I	2.359(7)	$64^3 \times 128 \times 12$	2.25	2.0	12	0.000678	0.02661	0.000314

Data analysis:

Sample AMA correction: [Phys. Rev. D88(9), 094503 (2013)]

data type	CG stop residual
Sloppy	$1e{-}4$
Exact	$1e{-}8$

Diagram types	sample AMA correction	# of Sloppy	# of Exact
Type-3&4	Y	116	36
Type-1&2	N	0	36

The super-jackknife method is used to estimate the statistical errors for the AMA corrected data.

Disconnected Type4 diagrams:

save left- and right-pieces separately and use multiple source-sink separation for fitting.

Calculation of m_K using single-integrated correlators

Subtract light states from the averaged unintegrated correlator:

$$\tilde{G}^{sub}_{ij}(X) = \tilde{G}_{ij}(X) - \sum_{n \in \{n_l\}} \langle \overline{K^0}|Q_i'|n\rangle\langle n|Q_j'|K^0\rangle e^{(m_K - E_n)X} \tag{9}$$

Perform a single-integration over X for the subtracted correlator between $X = 0$ and $X = T$ to obtain:

$$\tilde{A}^S_{ij}(T) = \sum_{X=1}^{T} \tilde{G}^{sub}_{ij}(X) + \frac{1}{2}\tilde{G}^{sub}_{ij}(0) \tag{10}$$

Results for m_K preliminary

We choose to use the results from the single-integration method:

Analysis method	m_K /10 ^{12}MeV	m_K (type1&2)	m_K (type3&4)
Double-integration	6.31(0.98)	6.71(0.48)	-0.20(0.65)
Single-integration	6.34(0.57)	6.24(0.24)	0.33(0.50)

Systematic errors:

Finite-volume corrections: small

"E ects of nite volume on the K_L K_S mass di erence"

N.H. Christ, X. Feng, G. Martinelli and C.T. Sachrajda, arXiv:1504.01170

$X^1\ m_K{}^{oFV} = 0{\bullet}54^1 18^o\ 10^{12}$MeV.

Discretization e ects are the largest source of systematic error:

e ects from low-energy scale $_{QCD}$

heavy charm quark, $^1\ m_c a^{o2}$ gives 25%

Another estimate based on HVP calculation is 15%

Finite lattice spacing e ects: scaling tests preliminary

- Scaling tests: perform calculations of three- and four-point quantities on tw lattices with di erent lattice spacings. We need a coarser lattice to be compared with a ner lattice.
  - 64I(2.4 GeV) $ 96I(2.8 GeV) Hard to do
  - 24I(1.8 GeV) $ 32I(2.4 GeV) 3
- We would like to see how large the discrepancy is
  - at a relatively small $m_c a$, which shows the nite lattice spacing e ects from low-energy scale $_{QCD}$.
  - at a relatively large $m_c a$ 0•32, which corresponds to the physical mass in our calculation on the 64I ensemble.

Lattice name	Action (F+G)	a [1] (GeV)	Lattice Volume	V	b+c	L_s	m_l	m_h	m_{res}
24I	DWF+I	1.785(5)	24^3 64 16	2.13	1.0	16	0.0050	0.0400	0.00308
32I	DWF+I	2.383(9)	32^3 64 16	2.25	1.0	16	0.0040	0.0300	0.000664

Finite lattice spacing effects: scaling tests preliminary

Set up input masses to keep physics consistent on the two ensembles:

Lattice	m_x	m_y	m_c	m_K	m_c's
24I	0.00667	0.0321	0.2079	0.3125	0.15:0.05:0.35
32I	0.00649	0.0249	0.1557	0.2332	(0.15:0.05:0.35) $\frac{1.785}{2.383}$

Quantities to be compared are:

2-point:

m_c , m_K ! confirm the valence masses yield consistent physical values.

Lattice	N_{conf}	m_c /MeV	m_K /MeV
24I	186	371.3(7)	556.2(7)
32I	222	371.4(6)	557.5(6)

m_D ! to calculate m_c ¹m_D º to match the physics.

3-point figure-8 diagrams: light charm which has degenerate mass with m_u
hcjZ$_{184-1}$ºQ¸ jK^{0}i , hcjZ$_{120-1}$ºQ jK^{0}i ,
4-point single-integrated correlators(connected diagrams only) with m_c dependency:
Integrated correlators: with operator combinations Q¸ Q¸ and Q Q

` /GeV	Op.	Z factors 32I (a^{-1} = 2.38GeV)	Z factors 24I (a^{-1} = 1.78GeV)	Matrix elements in physical Unit 32I (a^{-1} = 2.38GeV)	Matrix elements in physical Unit 24I (a^{-1} = 1.78GeV)	Scaling violation
2.15	Q¸	0.52997(11)	0.47143(8)	0.003957(18)	0.004045(18)	-2.19 %
	Q	0.58755(14)	0.57493(26)	0.011949(65)	0.009936(59)	18.39 %
2.64	Q¸	0.52489(6)	0.46996(6)	0.003919(18)	0.004032(18)	-2.84 %
	Q	0.60358(11)	0.58239(11)	0.012275(67)	0.010065(60)	19.78 %

Table: The Z factors of NPR in ¹W – W⁰ scheme and hcjQ jK ^{0}i (gure-8 only) in physical units on the two lattice ensembles and di erent scale`.

Comment: K ! cc matrix elements,O 2 (8,8) irrep also show similarly large nite lattice spacing errors. [PhysRevD.91.07450]

Scaling test - four-point Q¸ Q¸ single-integrated correlators

preliminary

- at low m_D values, the ratio is consistent with 1.
- at $m_c a = 0{\cdot}32$, which corresponds to the physical charm mass on 64I, the scaling violation is about 5%.

Scaling test - four-point Q Q single-integrated correlators

preliminary

at low m_D values, the ratio is about 1.45.
The scaling violation: 40%.

at $m_c a = 0.32$, the scaling violation: 14%.

Quantities compared between the two lattice spacings are:

- 2-point:
  - m_c, m_K ✓
  - m_D ✓
- 3-point: light charm which has degenerate mass with m_u:
  $\langle \bar{c}|Z_{184-1}Q_{\theta}|K^0\rangle$ ✓ 3% difference
  $\langle \bar{c}|Z_{120-1}Q|K^0\rangle$ ✗ 20% difference
  Comment: $K \to \bar{c}c$ matrix elements 2 (8,8) irrep also show similarly large finite lattice spacing errors. [PhysRevD.91.07450].
- 4-point(connected only) with m_c dependency:
  single-integrated correlators: with operators
  $Z^2_{184-1}Q_{\theta}Q_{\theta}$ ✓ 5% difference
  $Z^2_{120-1}Q\,Q$ ✗ 40% difference

We estimate the finite lattice spacing error in our m_K calculation to be of order of 40%.

Conclusion and outlook

Our preliminary result for m_K based on 152 configurations is:

$$m_K = 5.8(0.6)_{stat}(2.3)_{sys} \times 10^{-12}\ \text{MeV}, \qquad (11)$$

to be compared to the experimental value:

$$m_K^{exp} = 3.483(6) \times 10^{-12}\ \text{MeV}. \qquad (12)$$

We find reasonable agreement given the large finite lattice spacing errors.

Outlook:
Future calculations:

- m_K: on $96^3 \times 192$ lattice with $a^{-1} = 2.8$ GeV
  - Better estimate of finite lattice spacing effect:
  - 64I(2.4 GeV) $\rightarrow$ 96I(2.8 GeV) continuum limit to be explored
  - Further improvement of the precision to 5% level.
- long-distance ϵ_K: Joe Karpie, improve the accuracy of ϵ_K to sub-percent level.

Thanks for your attention!

Backup slides

K^0 $\overline{K^0}$ mixing and m_K

Two particles: K^0: $|d\bar{s}\rangle$, $S = +1$ and $\overline{K^0}$ $|\bar{d}s\rangle$, $S = -1$.

- Strong interactions: conserve strangeness
- Weak interactions: H_W changes strangeness, $S = 1$.

Second-order weak process: $K^0 \leftrightarrow \overline{K^0}$

$$i\frac{d}{dt}\begin{pmatrix} K^0(t) \\ \overline{K^0}(t) \end{pmatrix} = \left(M - \frac{i}{2}\right)\begin{pmatrix} K^0(t) \\ \overline{K^0}(t) \end{pmatrix} \qquad (13)$$

If the small effects of CP violation are neglected, long-lived (K_L) and short-lived (K_S) are the two eigenstates:

$$K_S = \frac{K^0 - \overline{K^0}}{\sqrt{2}}, \quad K_L = \frac{K^0 + \overline{K^0}}{\sqrt{2}} \qquad (14)$$

Figure: gure from wikipedia

Different life times:

$K_S \xrightarrow[\text{even}]{CP} \pi\pi$, $2m_\pi \approx 280\,\text{MeV} \lesssim m_K$

$K_L \xrightarrow[\text{odd}]{CP} \pi\pi\pi$, $3m_\pi \approx 420\,\text{MeV} \lesssim m_K$

The operator product expansion(OPE) and Δm_K

OPE: full theory $H_W \xrightarrow[\text{heavy particles}]{\text{integrate out}} H_{eff} = \sum_j C_j(\mu) O_j(\mu)$, renormalized at scale μ

$C_j(\mu)$: short-distance, perturbative; $O_j(\mu)$: long-distance, non-perturbative

H_W^{SM}	$H_{eff}^{\Delta S=1}$	$H_{eff}^{\Delta S=2}$

The speci c division` Ÿ m_c in OPE where charm quark is integrated out.

short-distance box only:

H_W^{SM} ! $H_e^{S=2}$	

$$H_e^{S=2} = C^{1`o}O_{LL}^{1`o}– \quad (15)$$

$$O_{LL} = {}^1\bar{s}d^o{}_{V\ A}{}^1\bar{s}d^o{}_{V\ A}– \quad (16)$$

leaving out:

QCD penguin	disconnected

long-distance box	

Only 36% accuracy in the next-to-next-to-leading-order(NNLO) calculation of the QCD correction factors using perturbation theory: slow convergence of the perturbative series

J. Brod and M. Gorbahn, Phys. Rev. Lett. 108, 121801 (2012)

! better to treat charm quark non-perturbatively on the lattice

GIM mechanism and the short- and long-distance characteristics of m_K

GIM mechanism: avor-changing neutral currents(FCNC) are suppressed in loo
diagrams! charm quark! the CKM matrix

Quark mixing: at each weak vertex
! a product of CKM matrix elements $V_{qd} V_{q^0 s}$, where $q, q^0 = u, c, t$.

De ne $_q = V_{qd} V_{qs}$, $q = u, c, t$,
unitarity of the CKM matrix ! $_u + _c + _t = 0$! $_c = _u \; _t$

Speci c diagram with GIM mechanism:

$$(X)_{GIM} = {}_u^2 (X)^{(u\ c)(u\ c)} + {}_t^2 (X)^{(t\ c)(t\ c)} + 2 {}_u {}_t (X)^{(u\ c)(t\ c)}$$

For $m_K = 2\mathrm{Re}M_{12}$, the rst term dominates.

Overview of the calculation of Δm_K

Quantities to be calculated are:

- two-point correlation functions:
  - meson masses: m_π, m_K, $m_{\pi\pi}$, m_η
  - normlization factors of meson interpolating operators: N_π, N_K, $N_{\pi\pi}$, N_η
- three-point correlation functions:
  - light state matrix elements to be subtracted: $\langle\pi|Q_i'|K^0\rangle = \langle\pi|Q_i|K^0\rangle - c_{si}\langle\pi|\bar{s}d|K^0\rangle$, and $\langle\pi\pi_{I=0}|Q_i - c_{pi}\bar{s}\gamma_5 d|K^0\rangle$.
  - coefficients of the $\bar{s}d$ and $\bar{s}\gamma_5 d$ operators: $c_{si} = \frac{\langle\eta|Q_i|K^0\rangle}{\langle\eta|\bar{s}d|K^0\rangle}$, $c_{pi} = \frac{\langle 0|Q_i|K^0\rangle}{\langle 0|\bar{s}\gamma_5 d|K^0\rangle}$.
- four-point correlation functions:
  - unintegrated correlation functions calculated from diagrams having light state contribution subtracted:
    $\mathcal{G}^{sub}(\tau) = \mathcal{G}(\tau) - \sum_{n\in\{n_l\}} \langle K^0|H_W|n\rangle\langle n|H_W|K^0\rangle e^{(m_K - E_n)\tau}$
  - single-integrated correlation functions:
    $\mathcal{A}^S(T) = \sum_{\tau=1}^{T} \mathcal{G}^{sub}(\tau) + \frac{1}{2}\mathcal{G}^{sub}(0) \to \Delta m_K$

Calculation of m_K using double-integrated correlators

Analysis method	m_K
Double-integration	6.31(0.98)
Single-integration	6.34(0.57)

Scaling of diagrams with a speci c topology

- A collection of diagrams to be studied in isolation in a lattice calculation
  - Fermion propagators contracted with a xed topology.
  - The path integral provides a sum over all possible gluon emissions, gluon self-interactions and closed fermion loop insertions.
- Conditions for a continuum limit:
  - The quark propagator topology DOES NOT introduce new divergent sub-diagrams not present in QCD. The renormalizability and chiral symmetry of DWF QCD will lead to a continuum limit with a ca^2 scaling behavior.
  - The quark propagator topology DOES introduce new divergent sub-diagrams not present in QCD: include these same diagrams when performing the NPR subtraction.

Non-perturbative renormalizations

- Renormalization of lattice operator $Q_{1\text{-}2}$ in 3 steps:

$$C_i^{lat} = C_a^{\overline{MS}}(1+\Delta r)_{ab}^{RI\to\overline{MS}} Z_{bi}^{lat\to RI}$$

Non-perturbative Renormalization: from the lattice to the RI-SMOM

$$Z^{lat\to RI} = \begin{pmatrix} 0{\bullet}5642 & -0{\bullet}03934 \\ -0{\bullet}03934 & 0{\bullet}5642 \end{pmatrix} \quad (17)$$

Perturbation theory: from the RI-SMOM to the $\overline{MS}$

C. Lehner, C. Sturm, Phys. Rev. D 84(2011), 014001

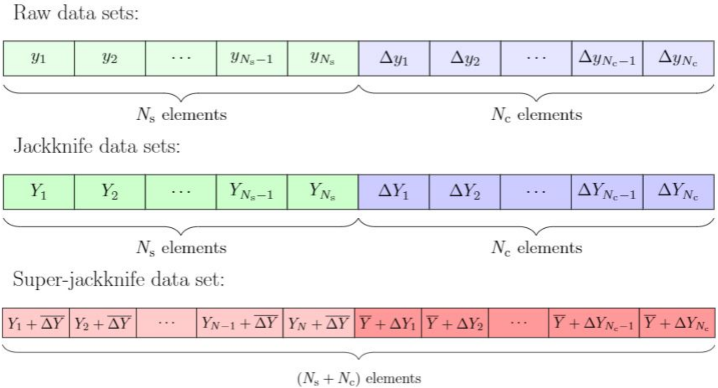
$$\Delta r^{RI\to\overline{MS}} = 10^{-3} \times \begin{pmatrix} -2{\bullet}28 & 6{\bullet}85 \\ 6{\bullet}85 & -2{\bullet}28 \end{pmatrix} \quad (18)$$

Use Wilson coe□cients in the $\overline{MS}$ scheme

G. Buchalla, A.J. Buras and M.E. Lautenbacher, arXiv:hep-ph/9512380

$$C^{\overline{MS}} = 10^{-3} \times \begin{pmatrix} -0{\bullet}260 & 1{\bullet}118 \end{pmatrix} \quad (19)$$

The jackknife and super-jackknife method



The mean of the fitting parameter Θ is given by:

$$\overline{\Theta} = \frac{1}{N_s+N_c}\sum_{i=1}^{N_s+N_c}\Theta_i, \quad \sigma^2_{\Theta} = \frac{N_s+N_c-1}{N_s+N_c}\sum_{i=1}^{N_s+N_c}(\Theta_i-\overline{\Theta})^2. \tag{20}$$

Scaling test - three-point $K \to c$ diagrams