Probing conformal towers of states with Density Matrix Renormalization Group algorithm

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Scope

- Basics of DMRG
  - Area law
  - Graphical notations
  - Variational optimization (finite- and infinite-size DMRG)
- Energy spectra with DMRG
- Conformal towers of states
  - Ising
  - 3-state Potts
  - 4-state Potts
- Outlook

Area law

Why tensor networks work?

Area law

Exponential growth of the Hilbert space $\dim H = d^N$ Exact diagonalization is limited to small clusters.

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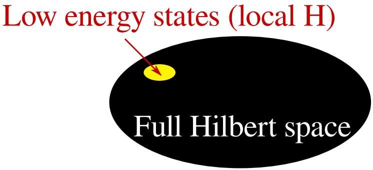
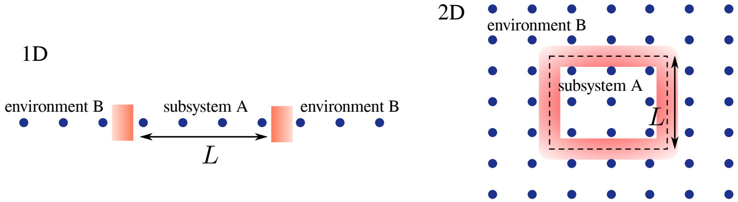
Area law for the entanglement entropy



Ground states of local Hamiltonians are less entangled than a random state in the Hilbert space

Area law



Entanglement entropy: $S_A = -\mathrm{tr}(\rho_A \log \rho_A)$

GS of local Hamiltonians

Area law: $S_A(L) \propto L^{d-1}$

1D: $S_A(L) = \mathrm{const}$
2D: $S_A(L) \propto L$

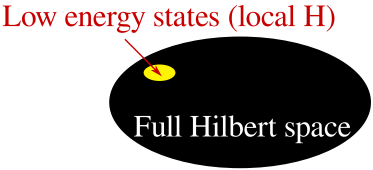
Random state

Volume law: $S_A(L) \propto L^d$

Critical state in 1D

$S_A(L) \propto \log(L)$

Area law



Our goal:
to diagonalize the Hamiltonian directly in the truncated basis

Number of relevant states $D \propto \exp(S)$

GS of local Hamiltonians

Area law: $S_A(L) \propto L^{d-1}$

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Random state

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Critical state in 1D

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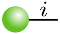
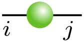
Graphical notations

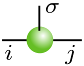
C Number



A_i Vector

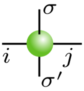


$A_{i,j}$ Matrix



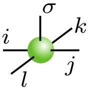
$A^{\sigma}_{i,j}$

rank-3 tensor (MPS)



$A^{\sigma,\sigma'}_{i,j}$

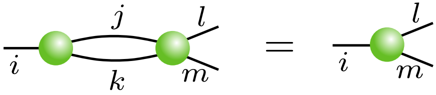
rank-4 tensor (MPO)



$A^{\sigma}_{i,j,k,l}$

rank-5 tensor (PEPS)

Contraction



$$\sum_{j,k} A_{i,j,k} \; B_{j,k,l,m} = T_{i,l,m}$$

- Summation over connected bonds
- In practice: reshape tensors into matrices and use optimized matrix multipliers
- Rank of the resulting tensor = number of open legs

SVD

singular values decomposition

Singular Values Decomposition (SVD)

For any rectangular matrix $M_{i,j}$ exists a decomposition

$$M = U_{i,k} S_{k,k} V^{\dagger}_{k,j}$$

such that:

- $U^{\dagger} U = \mathbb{I}$
- S is a diagonal matrix with non-negative entries
- $V^{\dagger} V = \mathbb{I}$

Schmidt decomposition

- Quantum state:

$$|\psi\rangle = \sum_{i,j} \Psi_{i,j} |i\rangle_A |j\rangle_B,$$

  where $|i\rangle_A$ and $|j\rangle_B$ are orthonormal basis of subsystems A and B.
- Treat $\Psi_{i,j}$ as a matrix and perform SVD
- Schmidt decomposition

$$|\psi\rangle = \sum_{i,j} \sum_{k} U_{i,k} S_{k,k} V^{\dagger}_{k,j} |i\rangle_A |j\rangle_B$$

Schmidt decomposition

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- Treat $\Psi_{i,j}$ as a matrix and perform SVD
- Area law - D relevant states only

$$|\psi\rangle = \sum_{i,j} \sum_{k}^{D} U_{i,k} S_{k,k} V^{\dagger}_{k,j} |i\rangle_A |j\rangle_B$$

Bring quantum states into MPS

$|\psi\rangle =$ 

Bring quantum states into MPS

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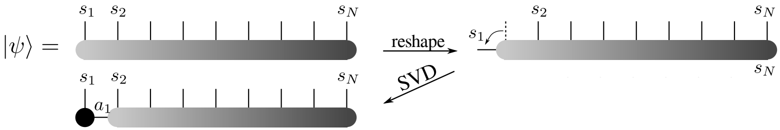

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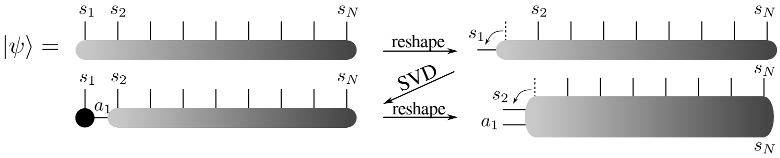

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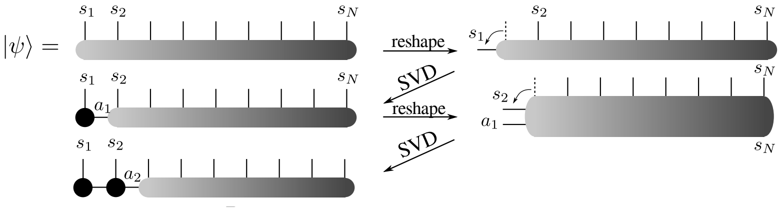

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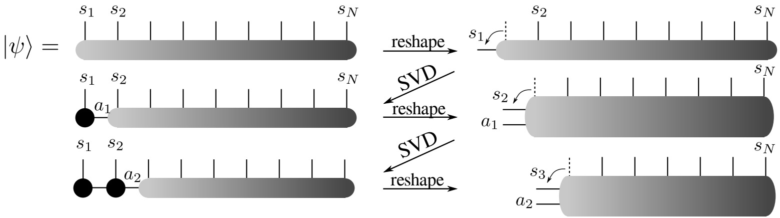

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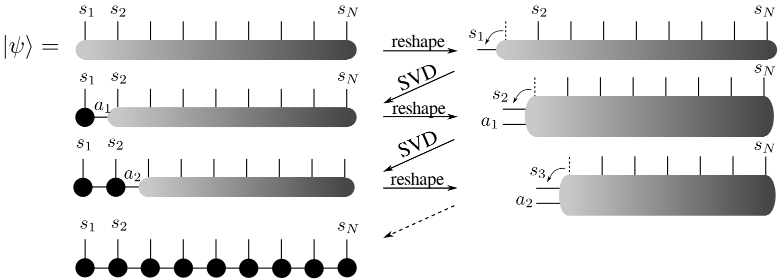

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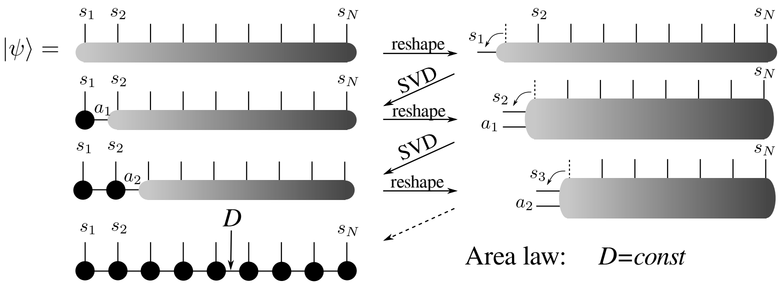

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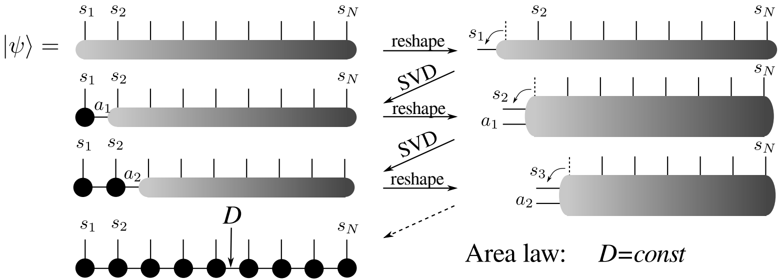
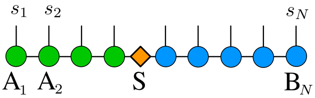


Mixed-canonical form



Normalization:

Normalization

The goal is to find $|\Psi\rangle$ that minimizes the energy:

$$E = \frac{\langle\Psi|\hat{H}|\Psi\rangle}{\langle\Psi|\Psi\rangle}$$

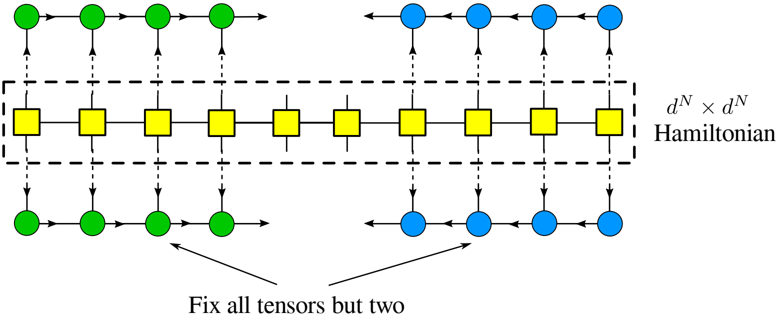
If norm is fixed $\langle\Psi|\Psi\rangle = 1$, it becomes

$$E = \langle\Psi|\hat{H}|\Psi\rangle$$

In variational optimization a generalized eigenvalue problem is reduced to a generalized eigvenvalue problem:

$$\hat{H}_{eff}|\psi\rangle = E|\psi\rangle \qquad \text{instead of} \qquad \hat{H}_{eff}|\psi\rangle = E\hat{N}_{eff}|\psi\rangle$$

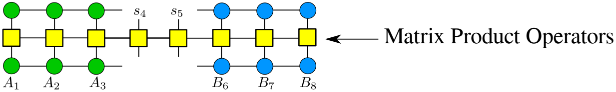
Variational optimization of the MPS



DMRG / variational MPS

White, PRL (1992); Östlund, Rommer, PRL (1995); Schollwöck Ann. of Phys. (2011)

DMRG sweep



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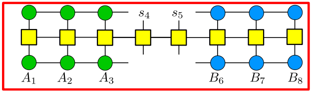
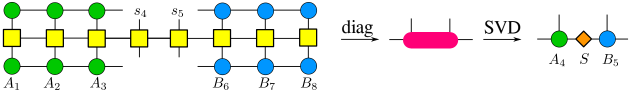


Group the legs and treat this rank-8 tensor as a matrix

DMRG sweep

Left-to-right:



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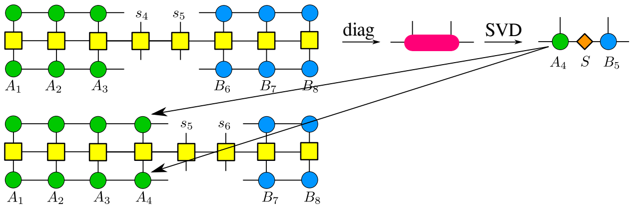

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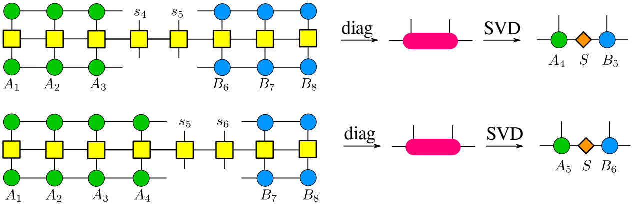

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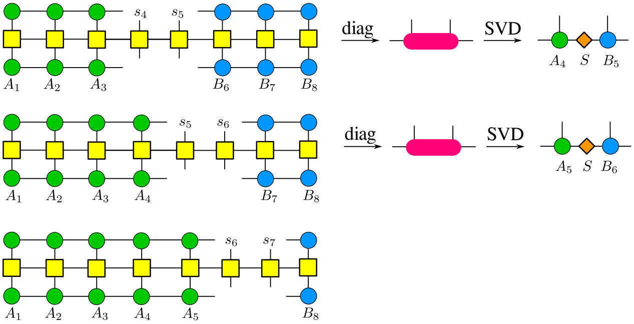

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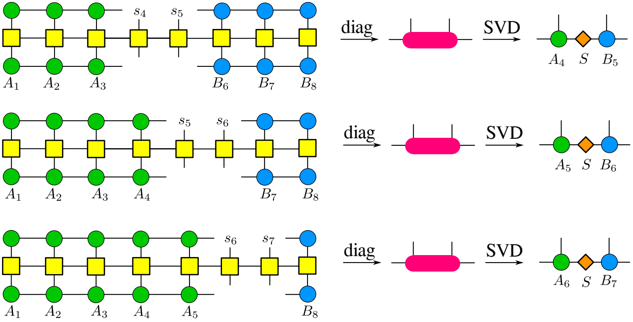

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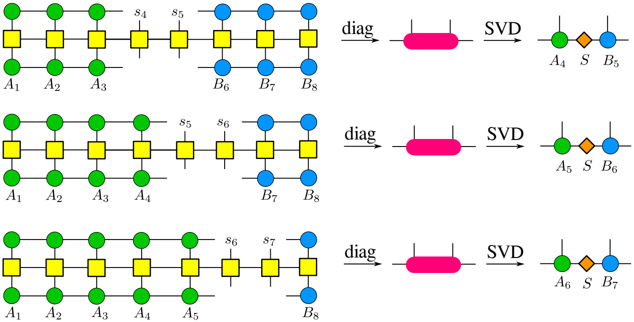
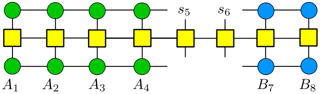


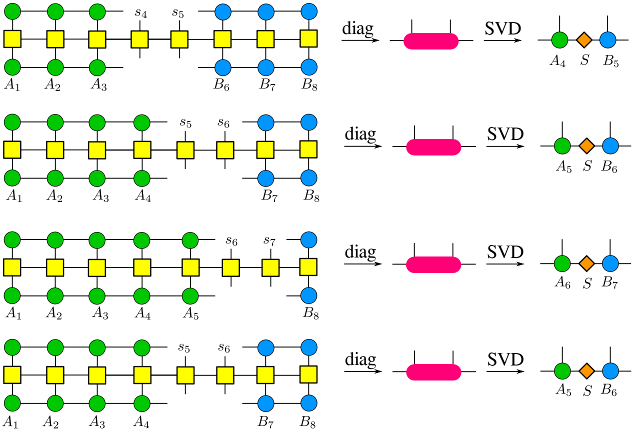
Right-to-left:



DMRG sweep

Left-to-right:

Right-to-left:



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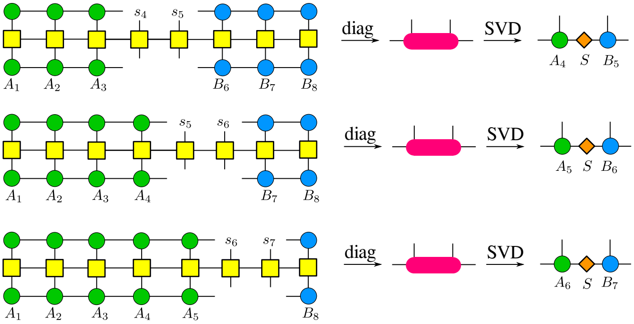

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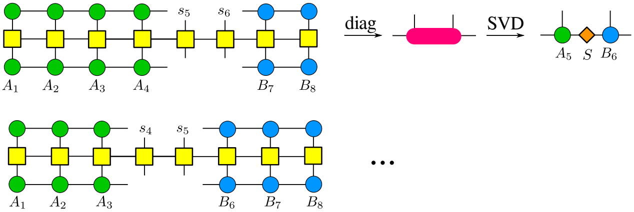
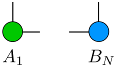
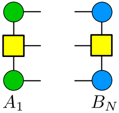


Initial guess

- Product state
- Random state
- Infinite-size DMRG

Infinite-size DMRG





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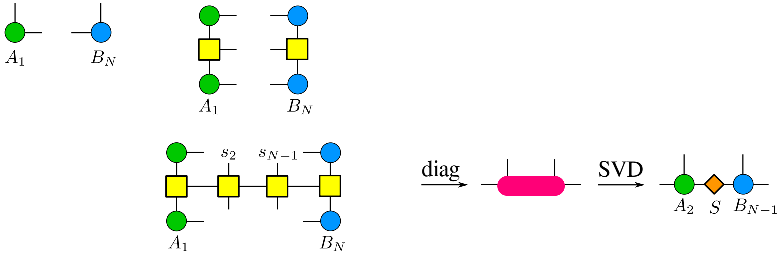

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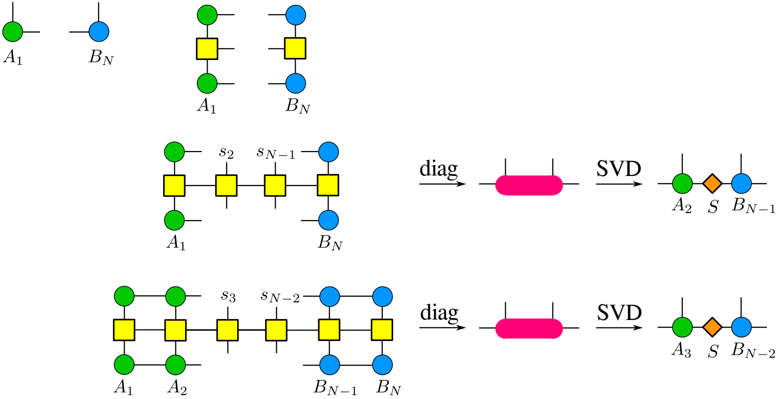

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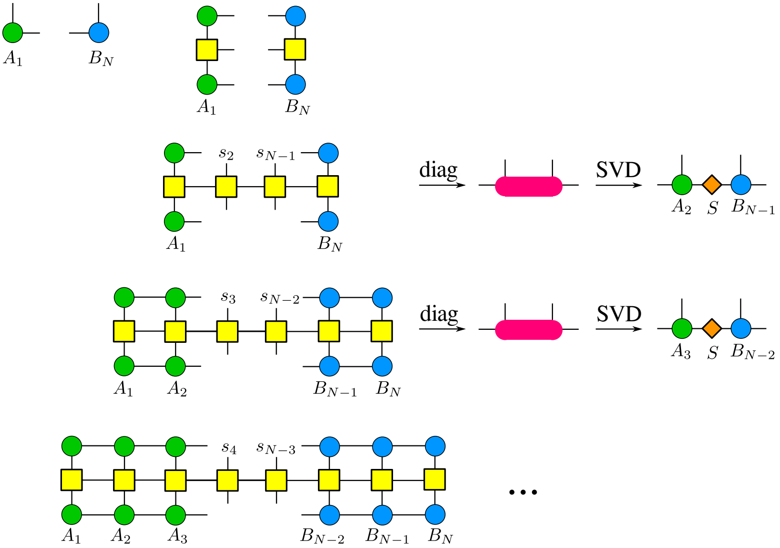
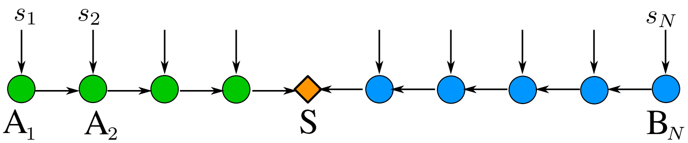
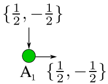


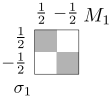
Abelian symmetry



- Assign quantum numbers - labels to physical bonds of MPS
- Using fusion rules of the symmetry, find quantum numbers on auxiliary legs
- When local basis is sorted according to the quantum number of states, the MPS takes a block-diagonal form

Abelian symmetry. Examples





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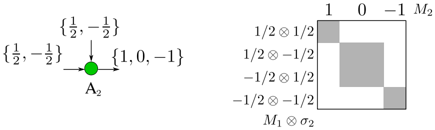

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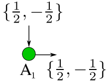
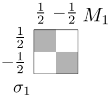
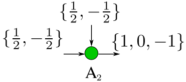
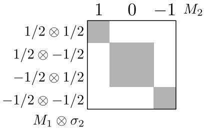
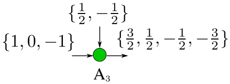
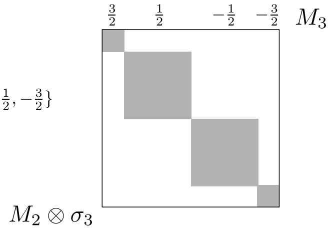








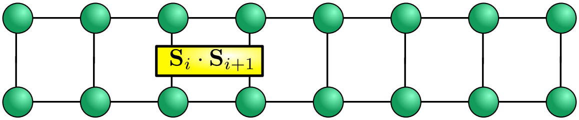




Observables

Observables

Nearest-neighbor correlations $\langle \Psi | \mathbf{S}_i \cdot \mathbf{S}_{i+1} | \Psi \rangle$



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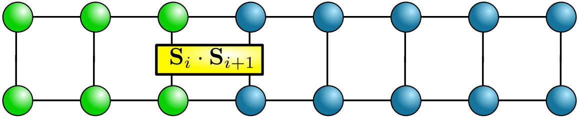

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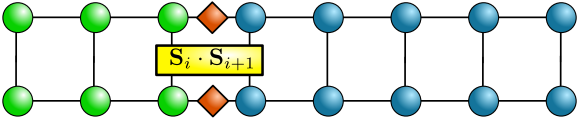

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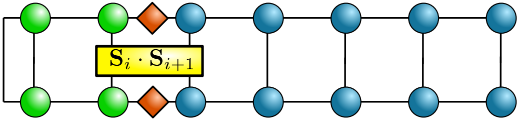

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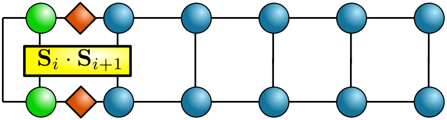

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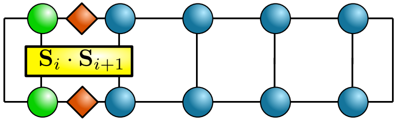

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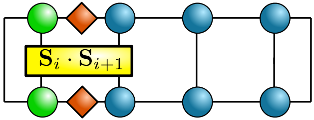

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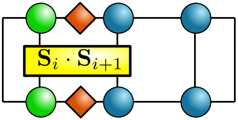

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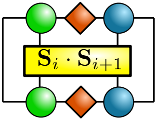

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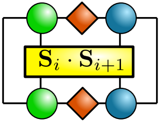
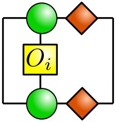
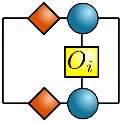


On-site measures $\langle\Psi|O_i|\Psi\rangle$



or



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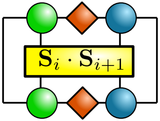
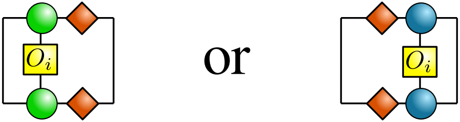
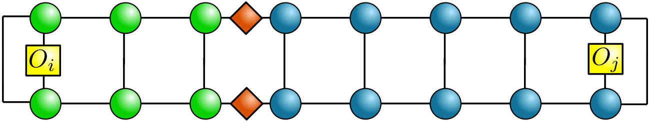
Nearest-neighbor correlations $\langle\Psi|\mathbf{S}_i \cdot \mathbf{S}_{i+1}|\Psi\rangle$



On-site measures $\langle\Psi|O_i|\Psi\rangle$



Long range correlations $\langle\Psi|O_i \cdot O_j|\Psi\rangle$



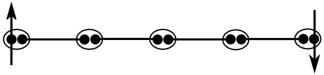
Energy excitation spectrum with DMRG

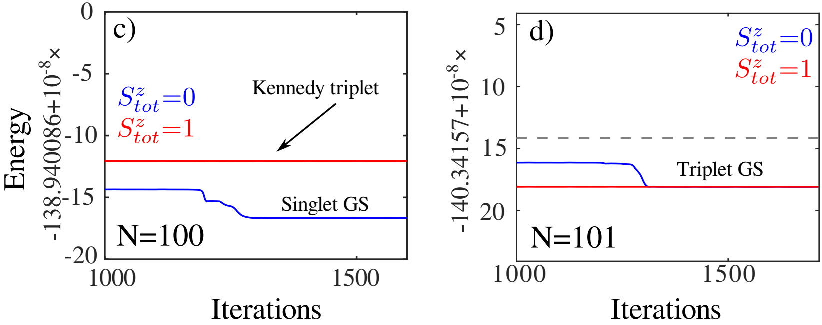
Excitation spectrum with DMRG/MPS

1. The excited state is the 'ground-state' of the different symmetry sector

Example: Kennedy triplet in Haldane chain

$$H = J \sum_j \mathbf{S}_j \cdot \mathbf{S}_{j+1}$$





Excitation spectrum with DMRG/MPS

1. The excited state is the 'ground-state' of the different symmetry sector
2. Conventional DMRG: Mixed states
   - The ground-state is spoilt
   - Heavy memory usage

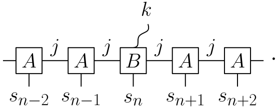
Excitation spectrum with DMRG/MPS

1. The excited state is the 'ground-state' of the different symmetry sector
2. Conventional DMRG: Mixed states
   - No longer variational
   - Heavy memory usage
3. MPS: Construct the lowest-energy state orthogonal to the previously constructed ones
   - Time consuming
   - Accumulation of the error

Excitation spectrum with DMRG/MPS

1. The excited state is the 'ground-state' of the different symmetry sector
2. Conventional DMRG: Mixed states
   - No longer variational
   - Heavy memory usage
3. MPS: Construct the lowest-energy state orthogonal to the previously constructed ones
   - Time consuming
   - Accumulation of the error
4. Elementary excitations + plane-wave superposition
   - Translation invariant MPS
   - Not suitable for OBC

$$|\Phi_p^k(B)\rangle = \sum_n \mathrm{e}^{ipn}$$



Vanderstraeten, Wybo, NC, Verstraete, Mila, PRB (2020)

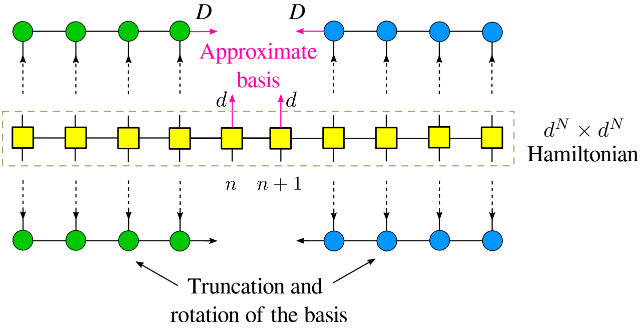
Excitation spectrum with DMRG/MPS

1. The excited state is the 'GS' of the different symmetry sector
2. Conventional DMRG: Mixed states
   - No longer variational & Heavy memory usage
3. MPS: Construct the lowest-energy state orthogonal to the previously constructed ones
   - Time consuming & Accumulation of the error
4. MPS: Domain wall or special tensor
   - Translation invariant MPS

There is a cheaper option:

Sometimes it is sufficient to target multiple eigenstates of the effective Hamiltonian and keep track of the energies as a function of iterations [NC, Mila, Phys.Rev.B **96**, 054425 (2017)]

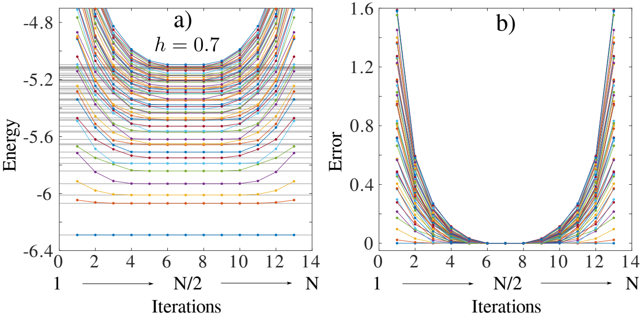
Effective Hamiltonian



- The Hamiltonian is written in a truncated and rotated basis.
- This basis is selected for the ground state.
- Could this basis be suitable for other low-energy states?

Trivial case - non-truncated MPS

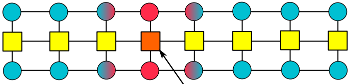
When no truncation is imposed and all basis states are kept in MPS, the DMRG is equivalent to exact diagonalization and one can access the entire spectrum!



When does it work?

Local impurities

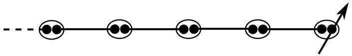
- Localized excitations
- MPS is the same except for a few sites



When does it work?

Edge states

- Edge spins are entangled through the entire network
- All edge states are in the basis



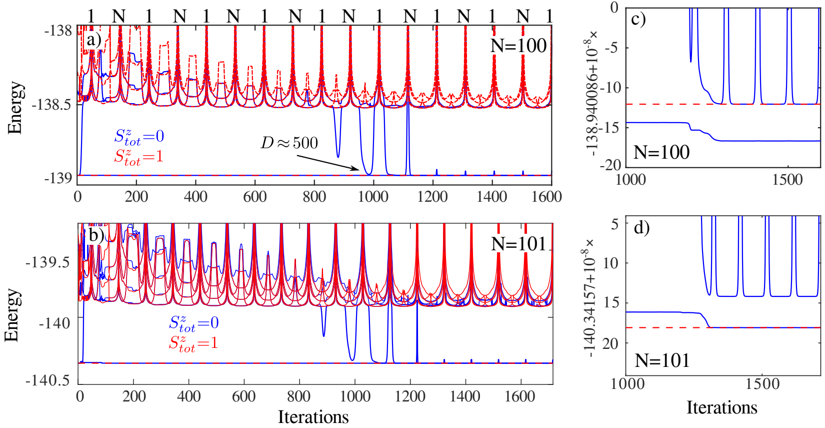
$$\left(\frac{B}{2} \otimes \uparrow\right) \oplus \left(\frac{B}{2} \otimes \downarrow\right)$$

Local impurities

- Localized excitations
- MPS is the same except for a few sites

Edge states in the Haldane chain

$$H = J \sum_j \mathbf{S}_j \cdot \mathbf{S}_{j+1}$$



When does it work?

Critical systems

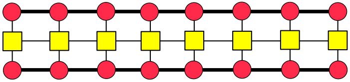
- Divergent correlation length
- Slow decay of Schmidt values
- Special structure of spectrum

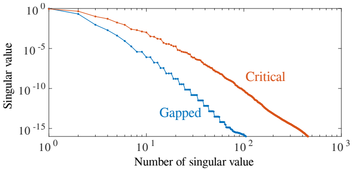
Edge states

- Edge spins are entangled through the entire network
- All edge states are in the basis

Local impurities

- Localized excitations
- MPS is the same except for a few sites



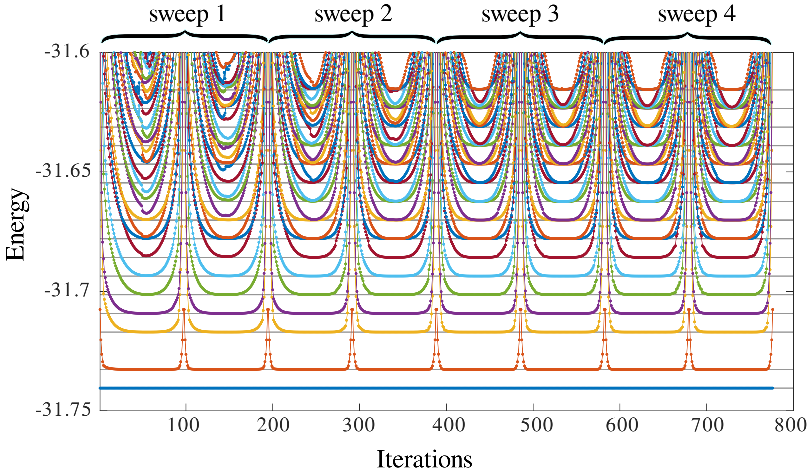


Transverse field Ising model

$$H = \sum_i J S_i^x S_{i+1}^x + h S_i^z$$

- Critical at $h = J/2$
- Solved by Jordan-Wigner transformation
- Corresponds to the minimal model (4,3) in CFT

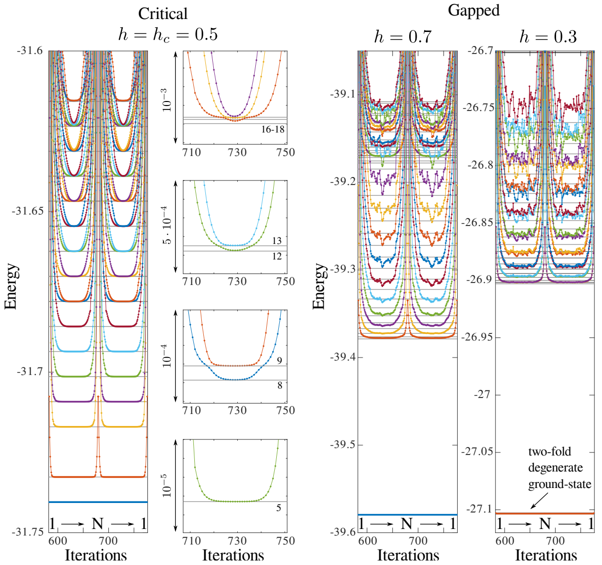
Transverse field Ising model. Excitation spectrum



- 30 states within a single run!
- Flat modes signal convergence

NC, F. Mila, Phys. Rev. B 96, 054425'17

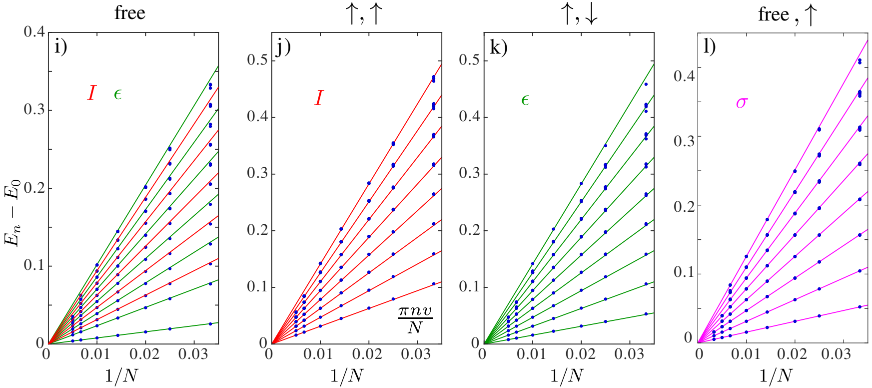
Transverse field Ising model. Excitation spectrum



- Remarkable accuracy for critical system
- Wrong spectrum for gapped system

[NC, F. Mila, Phys. Rev. B 96, 054425'17]

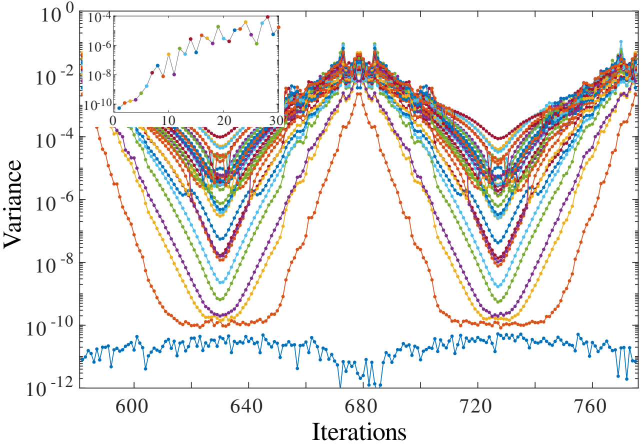
Finite-size scaling of the excitation energy



$$\chi_I(q) = q^{-1/48}\left(1 + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 3q^7 + 5q^8\right)$$
$$\chi_\epsilon(q) = q^{1/2-1/48}\left(1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 4q^7 + 5q^8\right)$$
$$\chi_\sigma = q^{1/16-1/48}\left(1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + 4q^6 + 5q^7 + 6q^8 + ...\right)$$

- BCFT prediction: Cardy, Nuc. Phys. B, **324** 581-596'89
- DMRG results: NC, Mila, Phys. Rev. B **96**, 054425'17

States are also good!



NC, Mila, Phys. Rev. B **96**, 054425'17

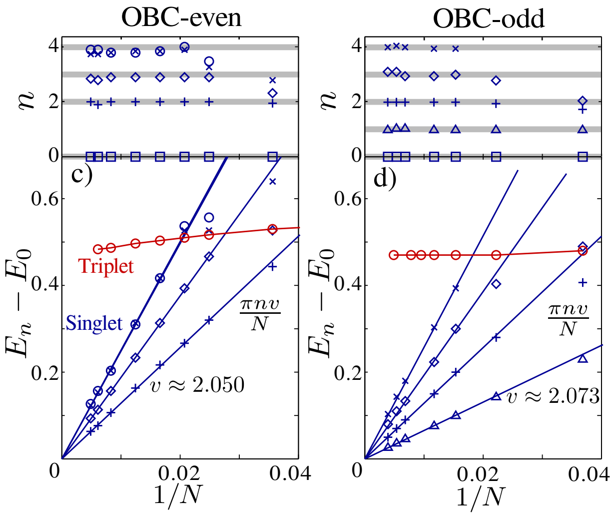
Not only for the simplest models:

Ising transition in spin-1 chain

$$H_{J_1J_2J_3} = J_1 \sum_j \mathbf{S}_j \cdot \mathbf{S}_{j+1} + J_2 \sum_j \mathbf{S}_j \cdot \mathbf{S}_{j+2} \\ + J_3 \sum_j \left[(\mathbf{S}_j \cdot \mathbf{S}_{j+1})(\mathbf{S}_{j+1} \cdot \mathbf{S}_{j+2}) + \text{h.c.}\right]$$

NC, Affleck, Mila, Phys. Rev. B **93** 241108, 2016

Ising conformal towers in spin-1 chain



- Singlet-triplet gap is **open**
- Critical scaling of the gap in the singlet sector
- N **even**
  I conformal tower
- N **odd**
  ϵ conformal tower

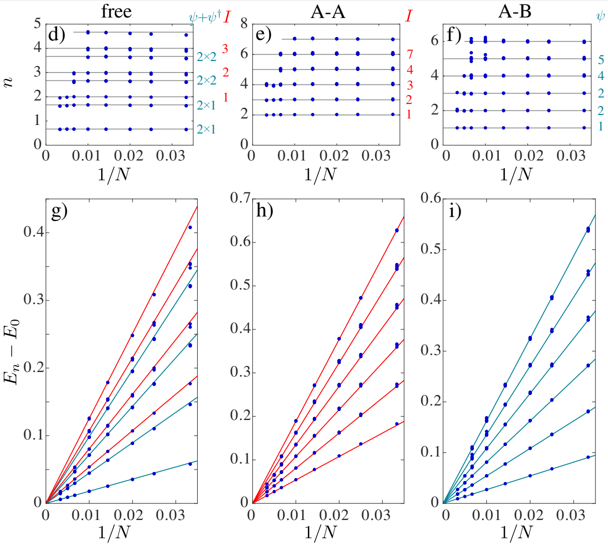
NC, Affleck, Mila, PRB **93**, 241108'16

3-state Potts critical model

$$H_{3-\text{state Potts}} = -J\sum_{i=1}^{N-1}\sum_{\mu=1}^{3} P_i^{\mu} P_{i+1}^{\mu} - h\sum_{i=1}^{N} P_i,$$

- $P_i^{\mu} = |\mu\rangle_{ii}\langle\mu| - 1/3$ - ferromagnetic interaction
- $P_i = |\lambda_0\rangle_{ii}\langle\lambda_0| - 1/3$ - generalized transverse field
  - align spins along $|\lambda_0\rangle_i = \sum_{\mu}|\mu\rangle\sqrt{3}$
- Critical at $h = J$
- Corresponds to the minimal model (6,5) in CFT

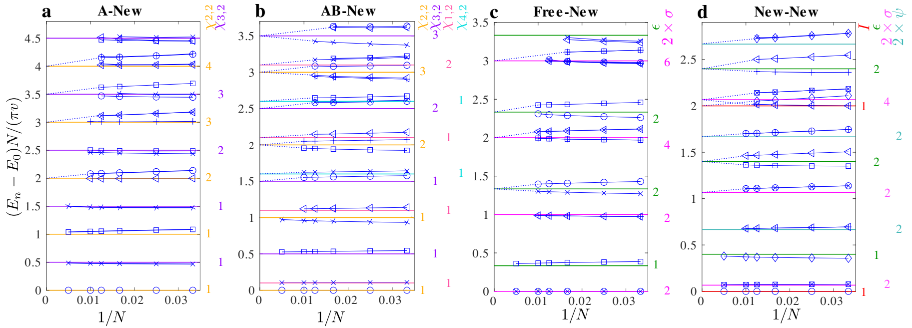
Some examples: Free and fixed BC



NC, Mila, Phys. Rev. B **96**, 054425'17

Less trivial: the "new" boundary conditions

- Dual to the mixed ones: AB, BC and AC
- In 2+0D: negative entries in the Boltzman weight matrix
- In 1+1D: transverse field at the edges



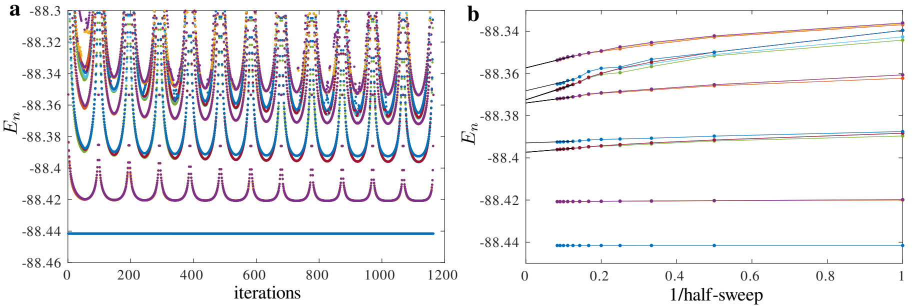
CFT: Affleck, Oshikawa, Saleur (1998); Numerics: NC, arXiv:2107.08899

4-state Potts critical model

$$H_{4-\text{state Potts}} = -J \sum_{i=1}^{N-1} \sum_{\mu=1}^{4} P_i^\mu P_{i+1}^\mu - h \sum_{i=1}^{N} P_i,$$

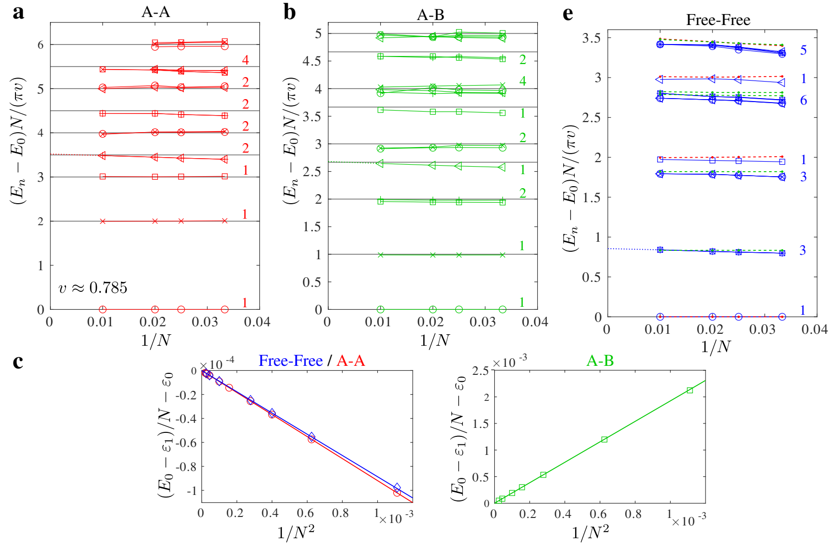
- $P_i^\mu = |\mu\rangle_{ii}\langle\mu| - 1/4$ - ferromagnetic interaction
- $P_i = |\lambda_0\rangle_{ii}\langle\lambda_0| - 1/4$ - generalized transverse field
- Critical at $h = J$
- Special case of the Ashkin-Teller critical point

The convergence is sometimes tricky



NC, arXiv:2107.08899

4-state Potts: Free are dual to fixed

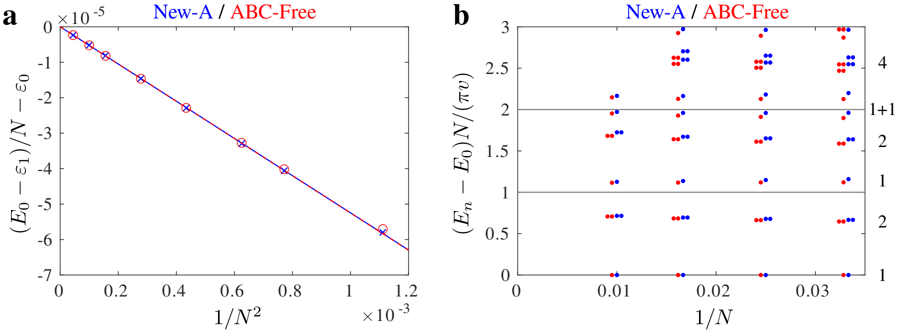


NC, arXiv:2107.08899

4-state Potts: direct comparision

Two sets of boundary conditions

Two identical spectra!



No adjustment parameters!

NC, arXiv:2107.08899

Outlook

- There are many ways to compute excitation spectra with DMRG
  - Optimal choice always depends on the problem
- For critical systems there is a cheap option:
  - Diagonalize the effective Hamiltonian targetting for several states
  - Track the energy as a function of iterations
  - Faithful energy appears as flat modes
  - Access to excited states
- The method of choice for boundary CFT

- "New" BC in 3-state Potts $\Leftarrow$ transverse field at the edges
- Also true for the 4-state Potts!
- "New" in 4-state Potts is dual to ABC boundary conditions

MPO

Full Hamiltonian as a product of local tensors

MPO construction

- For a given site j write all possible terms in the Hamiltonian:

  Transverse field Ising model: $H = \sum J S_i^x S_{i+1}^x + \sum h S_i^z$

$$
\begin{array}{l|c|r}
I...I & hS_j^z & I...I \\
I...IJS_{j-1}^x & S_j^x & I...I \\
I...I & JS_j^x & S_{j+1}^x I...I \\
I...JS_i^x S_{i+1}^x...I & I & I...I \\
I...hS_i^z...I & I & I...I \\
I...I & I & I...JS_i^x S_{i+1}^x...I \\
I...I & I & I...hS_i^z...I
\end{array}
$$

MPO construction

- For a given site j write all possible terms in the Hamiltonian:

  Transverse field Ising model: $H = \sum J S_i^x S_{i+1}^x + \sum h S_i^z$

$I...I$	hS_j^z	$I...I$
$I...IJS_{j-1}^x$	S_j^x	$I...I$
$I...I$	JS_j^x	$S_{j+1}^x I...I$
$I...JS_i^x S_{i+1}^x...I$	I	$I...I$
$I...hS_i^z...I$	I	$I...I$
$I...I$	I	$I...JS_i^x S_{i+1}^x...I$
$I...I$	I	$I...hS_i^z...I$

MPO construction

- For a given site j write all possible terms in the Hamiltonian:

  Transverse field Ising model: $H = \sum JS_i^x S_{i+1}^x + \sum hS_i^z$

$$
\begin{array}{l|c|r}
I...I & hS_j^z & I...I \\
I...IJS_{j-1}^x & S_j^x & I...I \\
I...I & JS_j^x & S_{j+1}^x I...I \\
I...\text{Full}...I & I & I...I \\
I...I & I & I...\text{Full}...I
\end{array}
$$

- Five non-trivial entries in the MPO

MPO construction

- For a given site j write all possible terms in the Hamiltonian:

  Transverse field Ising model: $H = \sum J S_i^x S_{i+1}^x + \sum h S_i^z$

$$\begin{array}{l|c|r} I ... I & h S_j^z & I ... I \\ I ... I J S_{j-1}^x & S_j^x & I ... I \\ I ... I & J S_j^x & S_{j+1}^x I ... I \\ I ... \text{Full} ... I & I & I ... I \\ I ... I & I & I ... \text{Full} ... I \end{array}$$

- Five non-trivial entries in the MPO
- Look at the left and right basis in which the MPO is going to be written

MPO construction

- For a given site j write all possible terms in the Hamiltonian:
- Five non-trivial entries in the MPO
- Look at the left and right basis in which the MPO is going to be written

$$\begin{array}{l|lll}
I\text{...Full...}I & & & \\
I\ldots IJS^x_{j-1} & & & \\
I\ldots I & & & \\
\hline
 & I\ldots I & S^x_{j+1}I\ldots I & I\text{...Full...}I
\end{array}$$

MPO construction

- For a given site j write all possible terms in the Hamiltonian:
- Five non-trivial entries in the MPO
- Look at the left and right basis in which the MPO is going to be written
- Fill-in the matrix:

$$
\begin{array}{l|ccc}
I\ldots\text{Full}\ldots I & I & 0 & 0 \\
I\ldots I J S^x_{j-1} & S^x_j & 0 & 0 \\
I\ldots I & hS^z_j & JS^x_j & I \\
\hline
 & I\ldots I & S^x_{j+1} I\ldots I & I\ldots\text{Full}\ldots I
\end{array}
$$

MPO exercise:

- Heisenberg nearest-neighbor:

$$H = J \sum_j \mathbf{S}_j \cdot \mathbf{S}_{j+1}$$
$$= J \sum_j \frac{1}{2} \left(S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+ \right) + S_j^z S_{j+1}^z \quad (1)$$

- $J_1 - J_2$ model:

$$H_{J_1 - J_2} = J_1 \sum_j \mathbf{S}_j \cdot \mathbf{S}_{j+1} + J_2 \sum_j \mathbf{S}_j \cdot \mathbf{S}_{j+2} \quad (2)$$

- Three-site interaction:

$$H = H_{J_1 - J_2} + J_3 \sum_j [(\mathbf{S}_j \cdot \mathbf{S}_{j+1})(\mathbf{S}_{j+1} \cdot \mathbf{S}_{j+2}) + \text{h.c.}] \quad (3)$$

MPO answers:

- Heisenberg nearest-neighbor:

$$H = J \sum_j \frac{1}{2}\left(S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+\right) + S_j^z S_{j+1}^z \qquad (4)$$

$$H_j = \begin{pmatrix} I & . & . & . & \\ S_j^- & . & . & . & \\ S_j^+ & . & . & . & \\ S_j^z & . & . & . & \\ . & \frac{J}{2} S_j^+ & \frac{J}{2} S_j^- & J S_j^z & I \end{pmatrix}$$

MPO answers:

- $J_1 - J_2$ model:

$$H_{J_1-J_2} = J_1 \sum_j \mathbf{S}_j \cdot \mathbf{S}_{j+1} + J_2 \sum_j \mathbf{S}_j \cdot \mathbf{S}_{j+2} \tag{5}$$

$$H_j = \begin{pmatrix} I & . & . & . & . & . & . & \\ S_j^- & . & . & . & . & . & . & \\ S_j^+ & . & . & . & . & . & . & \\ S_j^z & . & . & . & . & . & . & \\ . & I & . & . & . & . & . & \\ . & . & I & . & . & . & . & \\ . & . & . & I & . & . & . & \\ . & \frac{J_1}{2} S_j^+ & \frac{J_1}{2} S_j^- & J_1 S_j^z & \frac{J_2}{2} S_j^+ & \frac{J_2}{2} S_j^- & J_2 S_j^z & I \end{pmatrix}$$

MPO answers:

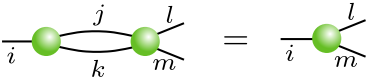
- Three-site interaction:

$$H = H_{J_1-J_2} + J_3 \sum_j \left[(\mathbf{S}_j \cdot \mathbf{S}_{j+1})(\mathbf{S}_{j+1} \cdot \mathbf{S}_{j+2}) + \text{h.c.} \right] \qquad (6)$$

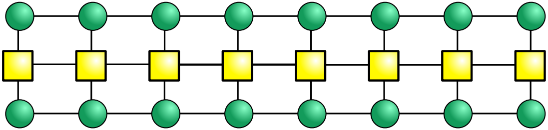
$$H_i = \begin{pmatrix} I & . & . & . & . & . & . & \\ S_i^- & . & . & . & . & . & . & \\ S_i^+ & . & . & . & . & . & . & \\ S_i^z & . & . & . & . & . & . & \\ . & \frac{J_2}{2} I + J_3 Q^{+-} & J_3 Q^{--} & J_3 Q^{-z} & . & . & . & \\ . & J_3 Q^{++} & \frac{J_2}{2} I + J_3 Q^{+-} & J_3 Q^{+z} & . & . & . & \\ . & J_3 Q^{+z} & J_3 Q^{-z} & J_2 I + J_3 Q^{zz} & . & . & . & \\ . & \frac{J_1}{2} S_i^+ & \frac{J_1}{2} S_i^- & J_1 S_i^z & S_i^+ & S_i^- & S_i^z & I \end{pmatrix},$$

where $Q_i^{\alpha\beta} = S_i^\alpha S_i^\beta + S_i^\beta S_i^\alpha$ with $S^\alpha = \{\frac{S^+}{\sqrt{2}}, \frac{S^-}{\sqrt{2}}, S^z\}$

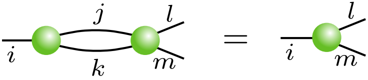
Contraction



- Complexity $\displaystyle\prod_{i\in \text{ connected legs}} D_i \cdot \prod_{j\in \text{ open legs}} D_j$
- The order of contraction matters!



Contraction



- Complexity $\prod_{i \in \text{ connected legs}} D_i \cdot \prod_{j \in \text{ open legs}} D_j$
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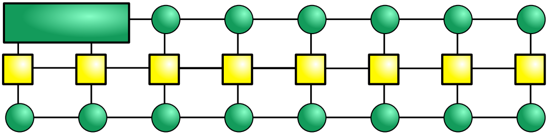
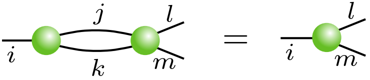


Contraction



- Complexity $\prod_{i \in \text{ connected legs}} D_i \cdot \prod_{j \in \text{ open legs}} D_j$
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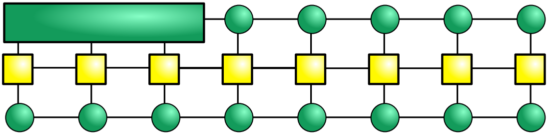
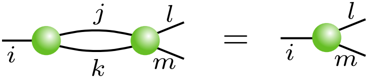
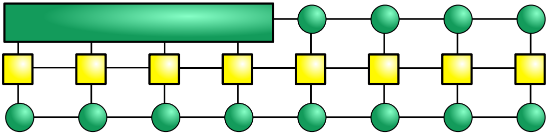


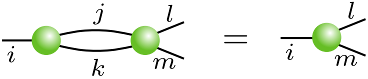
Contraction



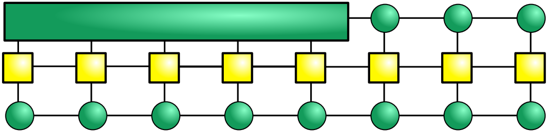
- Complexity $\prod_{i\in \text{ connected legs}} D_i \cdot \prod_{j\in \text{ open legs}} D_j$
- The order of contraction matters!



Contraction

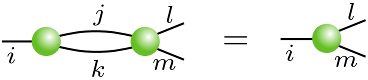


- Complexity $\prod_{i \in \text{ connected legs}} D_i \cdot \prod_{j \in \text{ open legs}} D_j$
- The order of contraction matters!



Exponential growth of complexity!

Contraction



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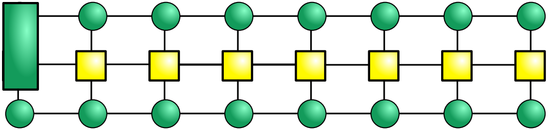
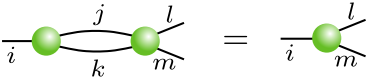


Contraction



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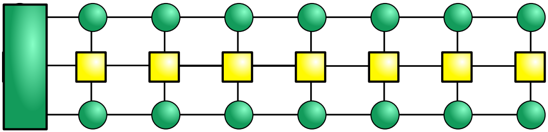
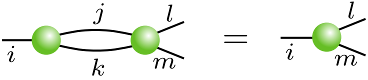


Contraction



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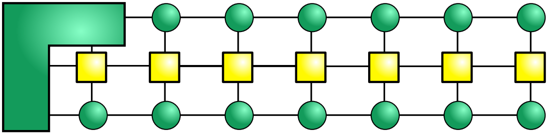
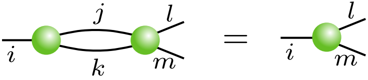


Contraction



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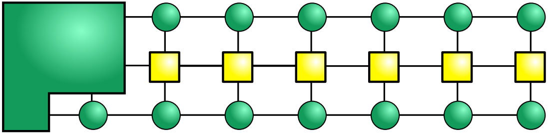
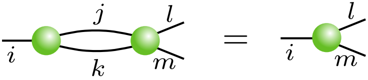


Contraction



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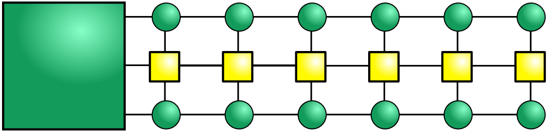
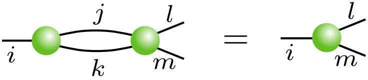


Contraction



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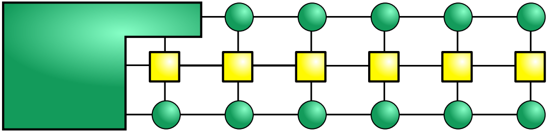
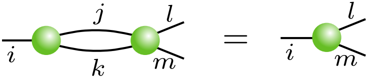


Contraction



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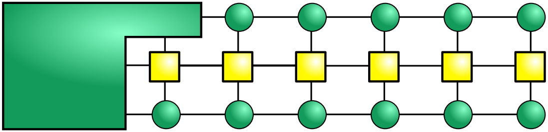
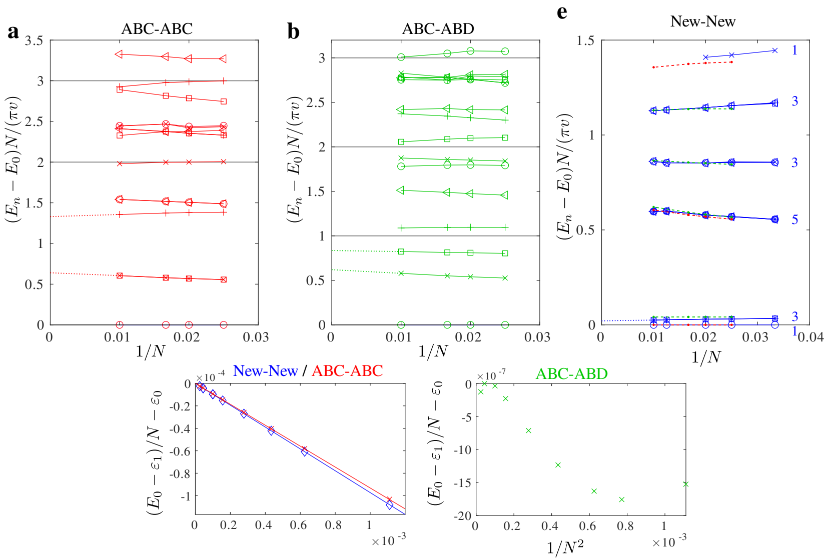


Complexity stays finite!

4-state Potts: "New" are dual to three-state-mixed



NC, arXiv:2107.08899