Use of kinematical Lorentz invariants

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Crossing angle meeting
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with input from C. Weiss, A. Bacchetta

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Intro

- Based on several sources:
  1. Bacchetta, D'Alesio, Diehl, Miller, "SSAs: Trento conventions" https://arxiv.org/abs/hep-ph/0410050v2
  2. Bacchetta et al., "SIDIS at small p_T https://iopscience.iop.org/article/10.1088/1126-6708/2007/02/093
  3. Cosyn, Weiss, "Neutron spin structure from polarized deuteron DIS with proton tagging", https://journals.aps.org/prc/abstract/10.1103/PhysRevC.102.065204
- Aim is pedagogical, so sometimes the obvious could be stated

Lorentz Invariants

- Quantities built from (ratios of) fourvector products.
- Are frame independent
  - can be computed in any frame
  - can be used as a check between two frames, result should not change
- We are all familiar with

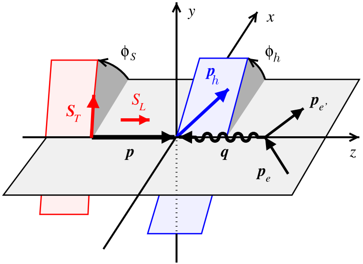
$$x = -\frac{q^2}{2(pq)} = \frac{Q^2}{2(pq)}\,, \qquad y = \frac{(pq)}{(pp_e)}\,,$$

$$s = (p_1 + p_2)^2\,, \qquad t = (p_1 - p_3)^2$$

- Also transverse momentum (pT, ϕ) and spin (S_L, S_T, ϕ_S) can be introduced through invariants

Collinear Frame

- Used in physics analysis of (SI)DIS, decomposition of cross sections with variables defined in this frame
- Target and virtual photon 3-momentum are aligned and define z-axis
  $\rightarrow$ target rest frame, Breit frame are special cases



- Electron momenta are in xz-plane ($p_{e,x}, p_{e',x} > 0$)
- Spin vector (later) and final-state particle momenta have longitudinal [L] and transverse components [T]
- Azimuthal angles ϕ_i defined in xy-plane relative to pos. x-axis (sign(ϕ) $\leftrightarrow$ right hand rule).

$\rightarrow$ orientation ϕ is opposite to Trento convention where z-axis is along $\boldsymbol{q}$

Collinear Frame: basis vectors

- Construct a set of 4 basis vectors (1 timelike, 3 spacelike) from $\{p, q, p_e\}$

1. Use $\{p, q\}$ to span time + longitudinal component. 2 choices

Start from q

$$L^\mu \equiv p^\mu - \frac{(pq)q^\mu}{q^2}, \quad (qL) = 0, \quad L^2 > 0,$$

$$L^2 = \frac{(pq)^2}{Q^2}(1+\gamma^2) = \frac{Q^2}{4x^2}(1+\gamma^2),$$

$$\gamma^2 \equiv \frac{M^2 Q^2}{(pq)^2} = \frac{4x^2 M^2}{Q^2},$$

$$e_L^\mu \equiv \frac{L^\mu}{\sqrt{L^2}}, \qquad e_q^\mu \equiv \frac{q^\mu}{\sqrt{-q^2}},$$

$$e_L^2 = 1, \qquad e_q^2 = -1, \qquad (e_L e_q) = 0.$$

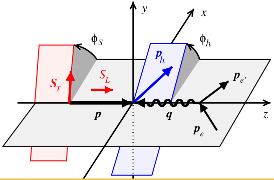
Start from p

$$L_*^\mu = q^\mu - \frac{(pq)}{p^2}p^\mu, \qquad (pL_*) = 0,$$

$$L_*^2 = -\frac{(pq)^2}{p^2}(1+\gamma^2) = -\frac{Q^2}{M^2}L^2$$

$$e_p^\mu \equiv \frac{p^\mu}{\sqrt{p^2}}, \qquad e_{L*}^\mu \equiv \frac{L_*^\mu}{\sqrt{-L_*^2}},$$

$$e_p^2 = 1, \qquad e_{L*}^2 = -1, \qquad (e_p e_{L*}) = 0$$

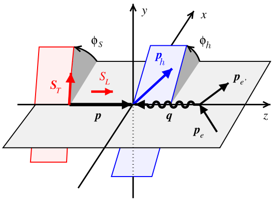


- 2 sets can be related: Eq. (3.29) Ref [3].

Collinear Frame: basis vectors

- Construct a set of 4 basis vectors (1 timelike, 3 spacelike) from $\{p, q, p_e\}$

1. Use $\{p, q\}$ to span time + longitudinal component.
   2 choices $\{e_q, e_L\}$,$\{e_p, e_{L*}\}$
2. Use p_e to construct two transverse unit vectors
   $\rightarrow$ two transverse 2D tensors



$$g_\perp^{\mu\nu} = g^{\mu\nu} + e_q^\mu e_q^\nu - e_L^\mu e_L^\nu = g^{\mu\nu} + e_{L*}^\mu e_{L*}^\nu - e_p^\mu e_p^\nu$$

$$\epsilon_\perp^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} e_{L,\rho} e_{q,\sigma} = \epsilon^{\mu\nu\rho\sigma} e_{p,\rho} e_{L*,\sigma} \qquad [\epsilon^{\mathbf{0123}} = \mathbf{1}]$$

$\rightarrow$ construct e_{T1}, e_{T2}:

$$p_{eT}^\mu = p_e^\mu - (e_p p_e)\, e_p^\mu + (e_{L*} p_e)\, e_{L*}^\mu = g_\perp^{\mu\nu} p_{e,\nu}, \qquad e_{T\mathbf{1}}^\mu \equiv \frac{p_{eT}^\mu}{\sqrt{-p_{eT}^{\mathbf{2}}}},$$

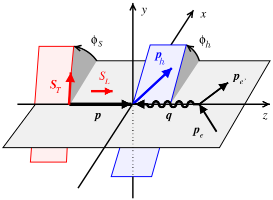
$$e_{T\mathbf{2}}^\mu \equiv \epsilon^{\mu\alpha\beta\gamma} e_{p,\alpha}\, e_{L*,\beta}\, e_{T\mathbf{1},\gamma} = \epsilon^{\mu\alpha\beta\gamma} e_{L,\alpha}\, e_{q,\beta}\, e_{T\mathbf{1},\gamma} = \epsilon_\perp^{\mu\nu} e_{T\mathbf{1},\nu}$$

$$e_{T\mathbf{1}}^{\mathbf{2}} = e_{T\mathbf{2}}^{\mathbf{2}} = -\mathbf{1}.$$

- In collinear frame $e_{T1} = e_x$, $e_{T2} = e_y$
- Definitions are completely covariant
  $\rightarrow$ these basis vectors can be constructed **in any frame**!

Kinematical variables: Lorentz invariants

- Using the set $\{e_p, e_{L*}, e_{T1}, e_{T2}\}$ we can define Lorentz invariants
  - can be calculated in **any** frame
  - kinematical interpretation specific to collinear frames
    →[cf. $p^2 = m^2$, interpretation in rest frame]



- $|\boldsymbol{p}_{hT}|, \phi_h$ correspond to length,azimuthal angle of transverse part of $\boldsymbol{p}_h$ **in collinear frame**

$$z = \frac{p \cdot p_h}{p \cdot q}$$

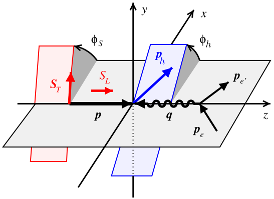
$$p^\mu_{hT} = p^\mu_h - (e_p p_h)\, e^\mu_p + (e_{L*} p_h)\, e^\mu_{L*} = g^{\mu\nu}_\perp p_{h,\nu}$$

$$|\boldsymbol{p}_{hT}| = \sqrt{-\boldsymbol{p}^{\mathbf{2}}_{hT}} = \sqrt{-p^\mu_h p^\nu_h g_{\perp,\mu\nu}}$$

$$|\boldsymbol{p}_{hT}| \cos\phi_h = (-e_{T\mathbf{1}} \cdot p_h)\,, \qquad |\boldsymbol{p}_{hT}| \sin\phi_h = (-e_{T\mathbf{2}} \cdot p_h)\,,$$

Spin vector: Lorentz invariants

- Polarization state of particle determined by density matrix
  - For spin 1/2 in rest frame characterized by 3D vector $\boldsymbol{S}$
  - For moving particle: covariant spin 4vector s^μ ($p \cdot s = 0$).
  - s^μ reached by boosting rest frame $s_R^\mu = (0, \boldsymbol{S})$ with the same canonical boost used to transform $p_R^\mu = (M, 0) \to p^\mu$



- s^μ can be decomposed as

$$s^\mu = -(s \cdot e_{L*}) e^\mu_{L*} + s^\mu_T \,, \qquad S_L \equiv (s \cdot e_{L*}) = \frac{(s \cdot q)}{(s \cdot p)} \frac{M}{\sqrt{1+\gamma^2}}$$

$$S_T \cos\phi_S \equiv -(e_{T\mathbf{1}} s) = -(e_{T\mathbf{1}} s_T) \,, \qquad S_T \sin\phi_S \equiv -(e_{T\mathbf{2}} s) = -(e_{T\mathbf{2}} s_T)$$

- $S_L = 1$ means polarization **along** $\boldsymbol{p}$ in coll. frames.
- Physical interpretation? Related to components of $\boldsymbol{S}$ in rest frame
  (z-axis opposite $\boldsymbol{q}$, x-axis in electron plane)

$$S_L \equiv S^z \,, \qquad (S_T \cos\phi_S, S_T \sin\phi_S) \equiv (S^x, S^y)$$

- S_L, S_T, ϕ_S differ event from event

Step by Step

- Can be carried out in any frame (lab frame, head-on, collinear, etc.)
- 4vectors of particles known in particular frame
- Construct basis $\{e_p, e_{L*}, e_{T1}, e_{T2}\}$ or $\{e_L, e_q, e_{T1}, e_{T2}\}$ in that frame
- Use both the particle four-vectors and basis vectors constructed **in that frame** to calculate invariants: $z, |\boldsymbol{p}_h T|, \phi_h, S_L, S_T, \phi_S \ldots$
  [Physical interpretation is in collinear frame (p_h) or rest frame (S)]
- Any procedure that incorporates explicit rotations and/or boosts can be validated by comparing the invariants computed in the lab frame with the invariants explicitly computed in the boosted frame.
- Do we need head-on [ep] frame?

Example

- Input
  - ep 18 $\times$ 275 GeV, $Q^2 = 20$ GeV^2, $x = 0.056$, random $\phi_{e'} = \pi/4$ $\rightarrow q = (0.085, -3.13, -3.13, -0.64)$
  - p_h: pion 30 GeV, $\theta_{\text{Lab}} = 10^o$, random ϕ_{Lab}
  - proton spin: longitudinal along proton beam, transverse along lab $+x$
- $\rightarrow$ Invariants evaluated in lab frame directly
  $\rightarrow$ Lorentz transformation (boost + rot) to collinear frame implemented in python script.
  $\rightarrow$ Invariants calculated directly in collinear frame **identical**
  $\Rightarrow$ can be done through 4vector contractions or from components

25mrad crossing angle, collider lab frame

$z = 0.91$, $p_{hT} = 9.634$ GeV

$\cos\phi_h = 0.954$, $\sin\phi_h = -0.300$

Longitudinal beam pol.

$S_L = 0.99$, $S_T = 0.023$,

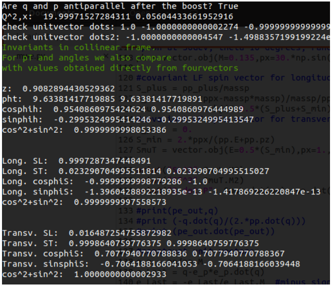
$\cos\phi_S = -1.00$, $\sin\phi_S = 0.00$

Transverse beam pol.

$S_L = 0.016$, $S_T = 0.99$,

$\cos\phi_S = 0.708$, $\sin\phi_S = -0.706$

25mrad crossing angle, collinear frame



Influence of crossing angle?

- Input
  - ep 18 × 275 GeV, $Q^2 = 20$ GeV^2, $x \approx 0.05$, random $\phi_{e'} = \pi/4$
    $\rightarrow q = (0.085, -3.13, -3.13, -0.64)$
  - p_h: pion 30 GeV, $\theta_{\text{Lab}} = 10^o$, random ϕ_{Lab}
  - proton spin: longitudinal along proton beam, transverse along lab $+x$
- $\longrightarrow$ Invariants evaluated in lab frame directly

two different physical situations in collider LAB frame!

reality: 25mrad crossing angle, $x = 0.056$

$z = 0.91$, $p_{hT} = 9.634$ GeV
$\cos\phi_h = 0.954$, $\sin\phi_h = -0.300$

Longitudinal beam pol.

$S_L = 0.99$, $S_T = 0.023$,
$\cos\phi_S = -1.00$, $\sin\phi_S = 0.00$

Transverse beam pol.

$S_L = 0.016$, $S_T = 0.99$,
$\cos\phi_S = 0.708$, $\sin\phi_S = -0.706$

hypothetical: 0^o angle, $x = 0.05$

$z = 0.63$, $p_{hT} = 7.78$ GeV,
$\cos\phi_h = 0.953$, $\sin\phi_h = -0.304$

Longitudinal beam pol.

$S_L = 0.99$, $S_T = 0.021$,
$\cos\phi_S = -1.00$, $\sin\phi_S = 0.00$

Transverse beam pol.

$S_L = 0.015$, $S_T = 0.99$,
$\cos\phi_S = 0.707$, $\sin\phi_S = -0.707$

Comparison of only one kinematic!