Shear, Ellipticity and All That (Part 1)

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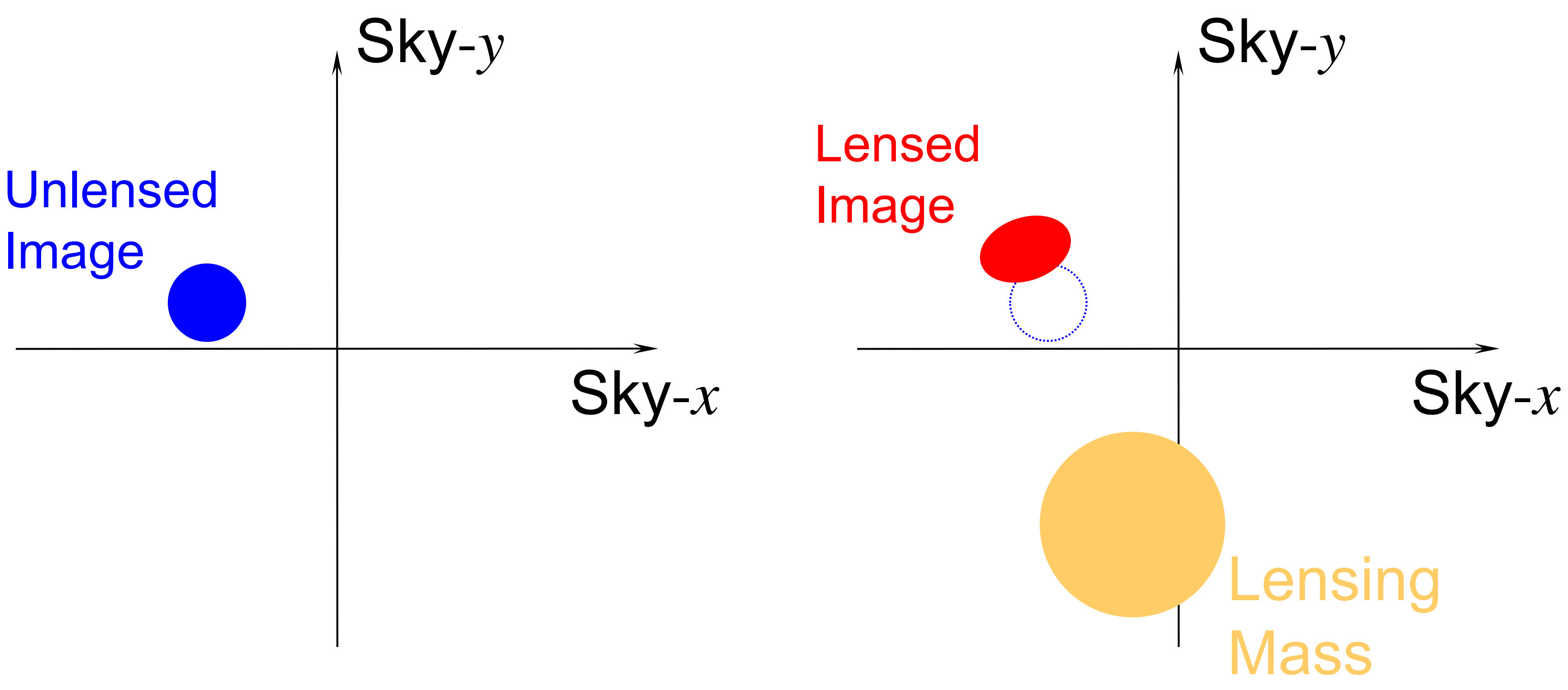
"Look, up in the sky…"



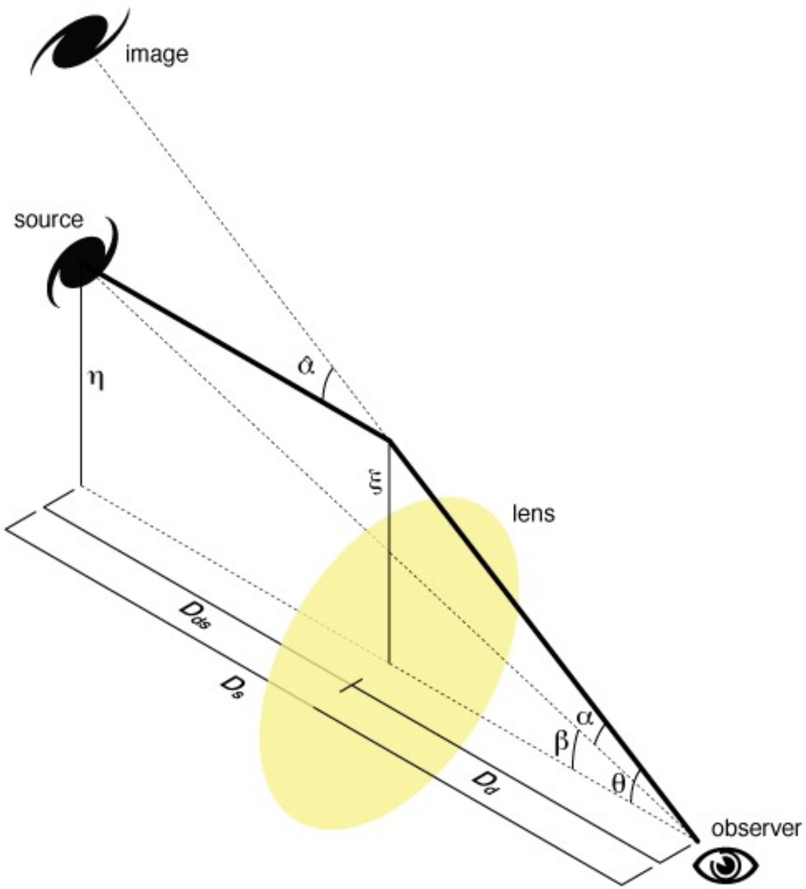
Part 1: Describe re-mapping, maps from weak lensing

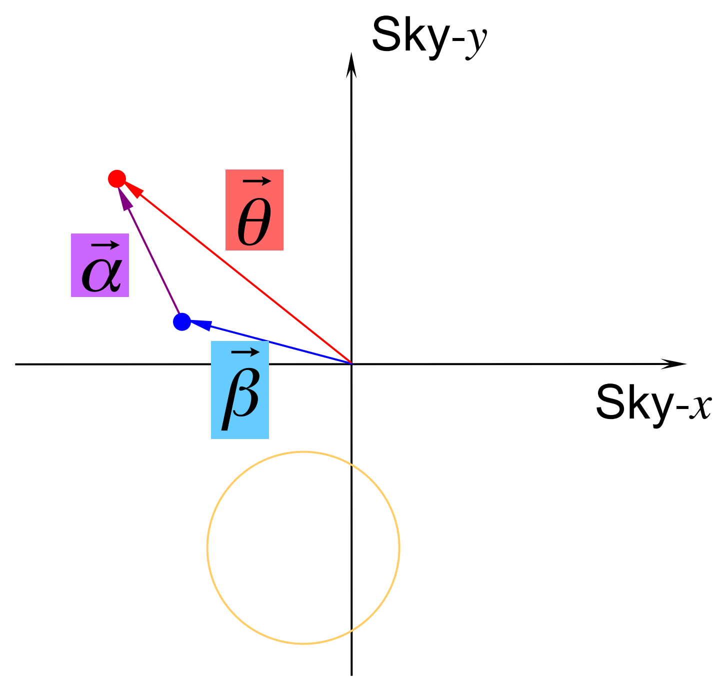
Part 2: Measuring map features, connect to physics

Displacement on the sky

$\vec{\beta}$ Unlensed/Source

$\vec{\theta}$ Lensed/Image





$$\vec{\beta}(\vec{\theta}) = \vec{\theta} - \vec{\alpha}(\vec{\theta})$$ Simple

$$\vec{\theta}(\vec{\beta}) = \vec{\beta} + \vec{\alpha}(\vec{\beta})$$ Awkward

Single-bend/"thin-screen" approximation

(figure from Wikipedia)

General 2D→2D mappings

$$\vec{\beta}(\vec{\theta})$$

$$\vec{\beta}_0 = \vec{\beta}(\vec{\theta}_0)$$ Taylor expand around $\vec{\theta}_0$

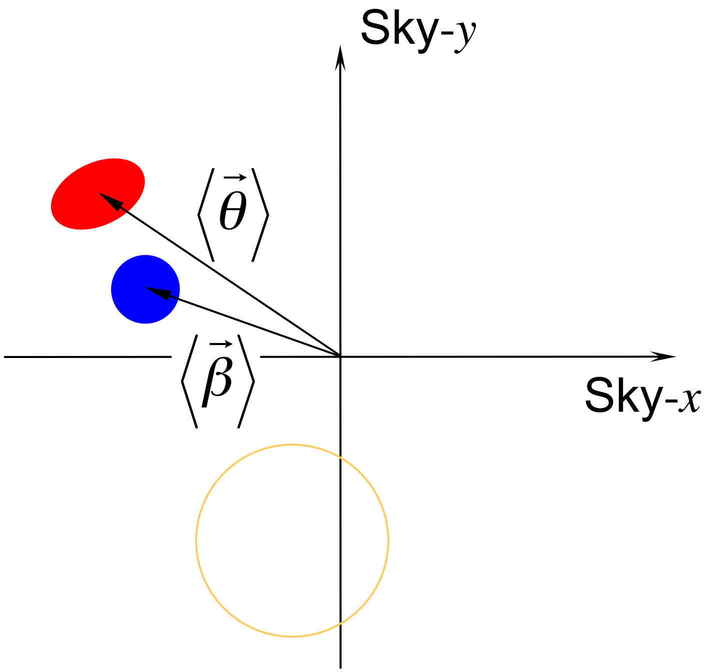
$$\vec{\beta} = \vec{\beta}_0 + \left.\frac{\partial \vec{\beta}}{\partial \vec{\theta}}\right)_{\vec{\theta}_0} (\vec{\theta} - \vec{\theta}_0) \quad + O((\vec{\theta} - \vec{\theta}_0)^2)$$

$$\begin{bmatrix} \beta_x - \beta_{0x} \\ \beta_y - \beta_{0y} \end{bmatrix} = \begin{bmatrix} \frac{\partial \beta_x}{\partial \theta_x} & \frac{\partial \beta_x}{\partial \theta_y} \\ \frac{\partial \beta_y}{\partial \theta_x} & \frac{\partial \beta_y}{\partial \theta_y} \end{bmatrix} \begin{bmatrix} \theta_x - \theta_{0x} \\ \theta_y - \theta_{0y} \end{bmatrix}$$ to 1st order

$$\underbrace{\vec{\beta} - \vec{\beta}_0}_{\text{Source}} = \underbrace{\mathbf{A}}_{\substack{\text{Jacobi}\\ \text{Matrix}}} \underbrace{(\vec{\theta} - \vec{\theta}_0)}_{\text{Image}} \qquad \underbrace{(\vec{\theta} - \vec{\theta}_0)}_{\text{Image}} = \underbrace{\mathbf{A}^{-1}}_{\substack{\text{Magnification}\\ \text{Matrix}}} \underbrace{(\vec{\beta} - \vec{\beta}_0)}_{\text{Source}}$$

Effect of 1st-order map on mean

$$\langle X \rangle \equiv \frac{\sum_{\text{Stars } i} X(i)}{N_{\text{Stars}}}$$



$$\vec{\beta} = \vec{\beta}_0 + \mathbf{A}(\vec{\theta} - \vec{\theta}_0)$$

$$\begin{aligned} \langle \vec{\beta} \rangle &= \left\langle \vec{\beta}_0 + \mathbf{A}(\vec{\theta} - \vec{\theta}_0) \right\rangle \\ &= \vec{\beta}_0 + \mathbf{A}\left(\langle \vec{\theta} \rangle - \vec{\theta}_0 \right) \\ &= \vec{\beta}\left(\langle \vec{\theta} \rangle \right) \end{aligned}$$

The mean of the image is the image of the mean

Effect of 1st-order map on covariance

Taylor expand around means, take $\vec{\beta}_0 = \langle \vec{\beta} \rangle$ and $\vec{\theta}_0 = \langle \vec{\theta} \rangle$

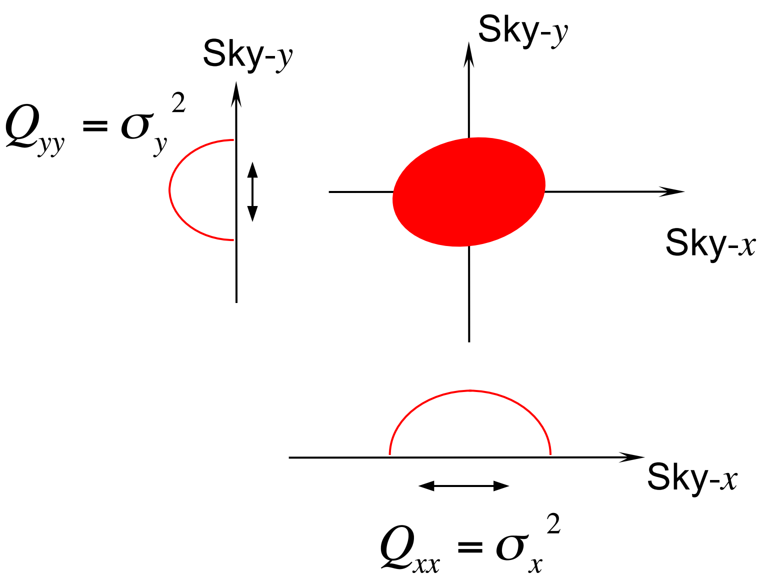
Covariance matrix: $\Sigma_{\mathrm{ij}}^{\mathrm{Source}} \equiv \left\langle (\beta_i - \langle \beta_i \rangle)(\beta_j - \langle \beta_j \rangle) \right\rangle \quad i,j \in \{x,y\}$

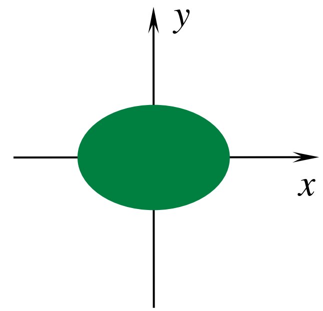
$$\begin{aligned}
\Sigma^{\mathrm{Source}} &= \left\langle (\vec{\beta} - \vec{\beta}_0)(\vec{\beta} - \vec{\beta}_0)^{\mathrm{T}} \right\rangle \\
&= \left\langle \mathbf{A}(\vec{\theta} - \vec{\theta}_0)\left[\mathbf{A}(\vec{\theta} - \vec{\theta}_0)\right]^{\mathrm{T}} \right\rangle \\
&= \mathbf{A}\left\langle (\vec{\theta} - \vec{\theta}_0)(\vec{\theta} - \vec{\theta}_0)^{\mathrm{T}} \right\rangle \mathbf{A}^{\mathrm{T}} \\
&= \mathbf{A}\,\Sigma^{\mathrm{Image}}\mathbf{A}^{\mathrm{T}}
\end{aligned}$$

$$\Sigma^{\mathrm{Image}} = \mathbf{A}^{-1}\,\Sigma^{\mathrm{Source}}\left[\mathbf{A}^{\mathrm{T}}\right]^{-1}$$

This is how lensing **A** affects the image covariance Σ

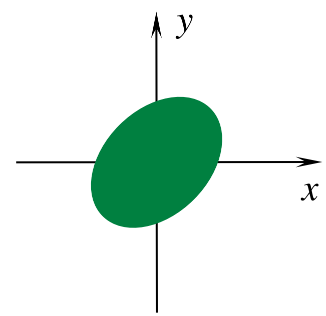
Astronomers refer to Σ as Q





$Q_{xx} > Q_{yy}$

$Q_{xy} = 0$



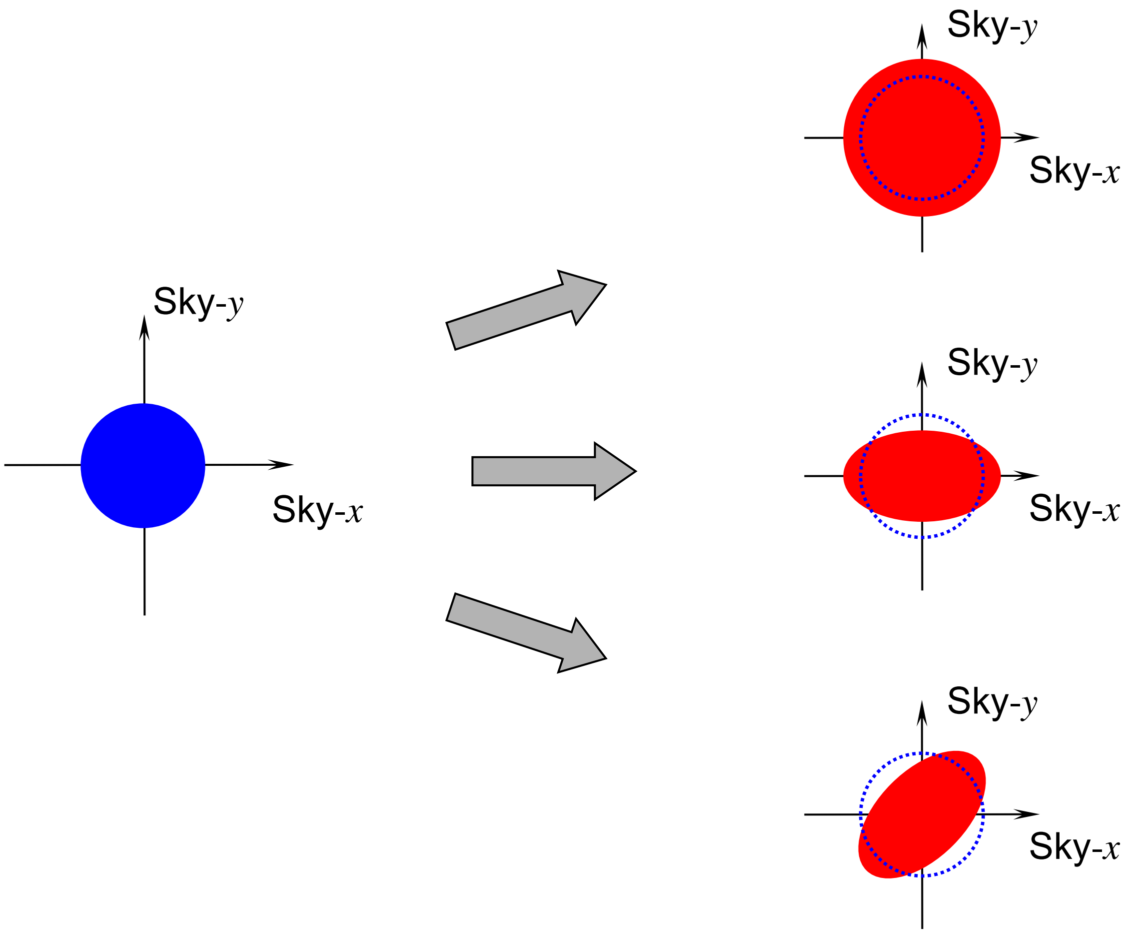
$Q_{xx} = Q_{yy}$

$Q_{xy} > 0$

How do we relate properties of $\mathbf{A}$ to features of $\boldsymbol{Q}$?

Symmetric Traceless Decomposition

$$\mathbf{A} = \mathbf{I} - (\text{Convergence})\mathbf{I} - \begin{bmatrix} \text{Shear}_+ & \text{Shear}_\times \\ \text{Shear}_\times & -\text{Shear}_+ \end{bmatrix}$$



$$\text{Conv} > 0 \quad \text{Shear}_+ = 0 \quad \text{Shear}_\times = 0$$

$$\text{Conv} = 0 \quad \text{Shear}_+ > 0 \quad \text{Shear}_\times = 0$$

$$\text{Conv} = 0 \quad \text{Shear}_+ = 0 \quad \text{Shear}_\times > 0$$

Deflection potential ψ

M. Bartelmann, P. Schneider / Physics Reports 340 (2001) 291–472

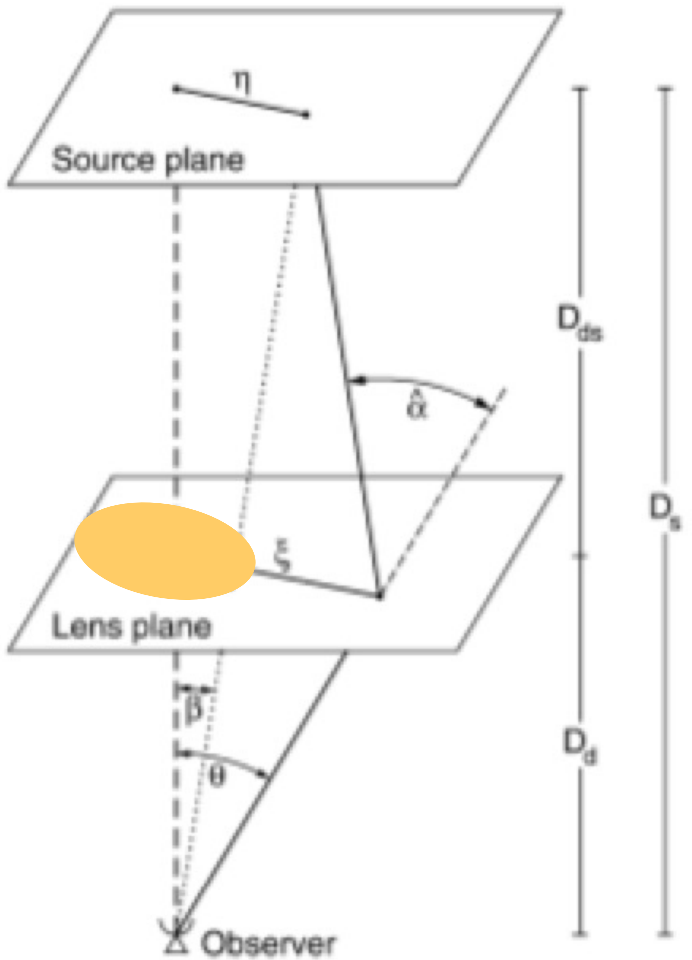


Fig. 11. Sketch of a typical gravitational lens system.

Deflection angle $\vec{\hat{\alpha}}(\vec{\theta})$ depends on local projected mass density $\Sigma(\vec{\xi}) = \int \rho(\vec{\xi}, z)\,dz$ at location $\vec{\xi} = \vec{\theta}\,D_{\mathrm{d}}$ in the lens plane.

Define **deflection potential** $\psi(\theta)$ such that displacement angle

$$\vec{\alpha}(\vec{\theta}) = \vec{\nabla}\psi(\vec{\theta})$$

The deflection potential is related to the projected mass density:

$$\frac{1}{2}\nabla^2\psi(\vec{\theta}) = \frac{\Sigma(\vec{\theta}\,D_{\mathrm{d}})}{\Sigma_C} \equiv \kappa(\vec{\theta})$$

The A matrix for a thin screen

$$\vec{\beta}(\vec{\theta}) = \vec{\theta} - \vec{\alpha}(\vec{\theta}) = \vec{\theta} - \vec{\nabla}\psi(\vec{\theta}) \quad \text{or} \quad \beta_i = \theta_i - \frac{\partial \psi(\vec{\theta})}{\partial \theta_i} \quad i \in \{x,y\}$$

$$A_{ij} = \frac{\partial \beta_i}{\partial \theta_j} = \frac{\partial \theta_i}{\partial \theta_j} - \frac{\partial^2 \psi(\vec{\theta})}{\partial \theta_j \partial \theta_i} \quad i,j \in \{x,y\}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \dfrac{\partial^2 \psi}{\partial {\theta_x}^2} & \dfrac{\partial^2 \psi}{\partial \theta_x \partial \theta_y} \\ \dfrac{\partial^2 \psi}{\partial \theta_y \partial \theta_x} & \dfrac{\partial^2 \psi}{\partial {\theta_y}^2} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2}\nabla^2 \psi & 0 \\ 0 & \frac{1}{2}\nabla^2 \psi \end{bmatrix} - \underbrace{\begin{bmatrix} \dfrac{1}{2}\left(\dfrac{\partial^2 \psi}{\partial {\theta_x}^2} - \dfrac{\partial^2 \psi}{\partial {\theta_y}^2}\right) & \dfrac{\partial^2 \psi}{\partial \theta_x \partial \theta_y} \\ \dfrac{\partial^2 \psi}{\partial \theta_y \partial \theta_x} & -\dfrac{1}{2}\left(\dfrac{\partial^2 \psi}{\partial {\theta_x}^2} - \dfrac{\partial^2 \psi}{\partial {\theta_y}^2}\right) \end{bmatrix}}_{\text{Symmetric and Traceless}}$$

Convergence, shear$_+$ and shear$_\times$

$$\mathbf{A} = \mathbf{I} - \kappa\mathbf{I} - \begin{bmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{bmatrix}$$

$$\kappa = \frac{1}{2}\left(\frac{\partial^2\psi}{\partial\theta_x^{\ 2}} + \frac{\partial^2\psi}{\partial\theta_y^{\ 2}}\right) = \frac{1}{2}\nabla^2\psi = \frac{\Sigma(\vec{\theta}D_d)}{\Sigma_C}$$ Convergence

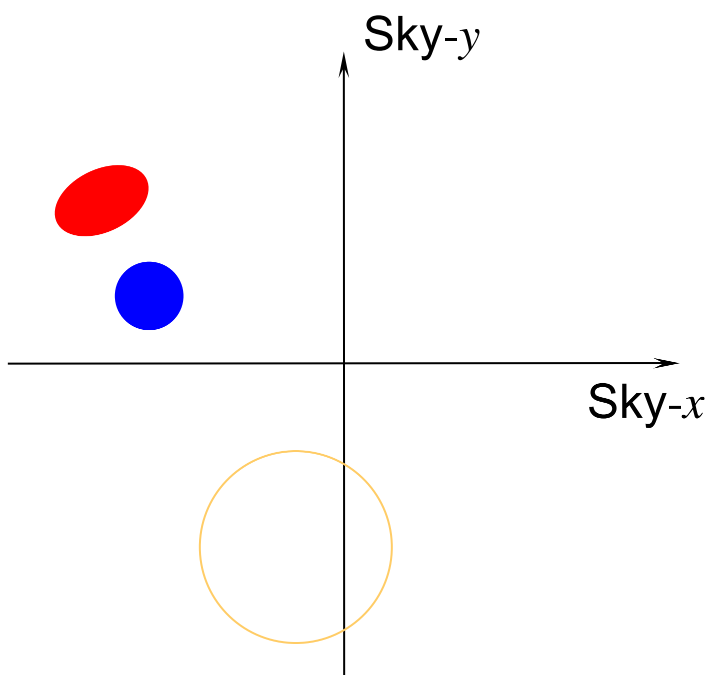
$$\gamma_1 = \frac{1}{2}\left(\frac{\partial^2\psi}{\partial\theta_x^{\ 2}} - \frac{\partial^2\psi}{\partial\theta_y^{\ 2}}\right)$$ Shear$_+$

$$\gamma_2 = \frac{\partial^2\psi}{\partial\theta_y\partial\theta_x}$$ Shear$_\times$

All 1st order in ψ and so 1st order in lensing mass.

Object brightness

Surface brightness = (sources per area) x (brightness of each source)



$$\underbrace{(\vec{\theta}-\vec{\theta}_0)}_{\text{Image}} = \underbrace{\mathbf{A}^{-1}}_{\text{Magnification Matrix}} \underbrace{(\vec{\beta}-\vec{\beta}_0)}_{\text{Source}}$$

Area( ) = Det[$\mathbf{A}^{-1}$] Area( )

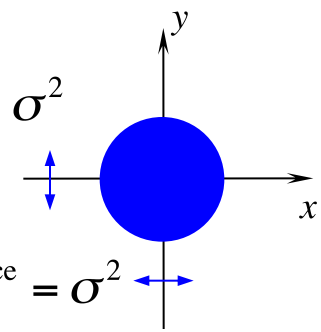
Since surface brightness is preserved by lensing, spreading the sources over a larger area, Det[$\mathbf{A}^{-1}$] >1, must increase the brightness by that same factor:

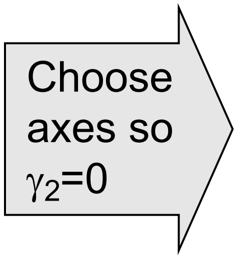
$$\frac{\text{Imaged brightness}}{\text{Original brightness}} = \left|\mathbf{A}^{-1}\right| = \frac{1}{(1-\kappa)^2 - ({\gamma_1}^2 + {\gamma_2}^2)}$$

"Mass-sheet Degeneracy"

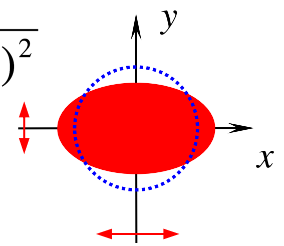
$$Q_{yy}^{\text{Source}} = \sigma^2$$

$$Q_{xx}^{\text{Source}} = \sigma^2$$





$$Q_{yy}^{\text{Image}} = \frac{\sigma^2}{(1-\kappa+\gamma_1)^2}$$



$$Q_{xx}^{\text{Image}} = \frac{\sigma^2}{(1-\kappa-\gamma_1)^2}$$

All shape analysis boils down to measuring:

$$\frac{\text{Long axis}}{\text{Short axis}} = \frac{a}{b} = \frac{\sigma_x}{\sigma_y} = \frac{\sqrt{Q_{xx}}}{\sqrt{Q_{yy}}} = \frac{1-\kappa+\gamma_1}{1-\kappa-\gamma_1}$$

Measuring a/b cannot determine κ and γ_1 independently!

Reduced shears g_1 and g_2

$$\frac{\text{Long axis}}{\text{Short axis}} = \frac{a}{b} = \frac{1-\kappa+\gamma_1}{1-\kappa-\gamma_1} = \frac{1+\gamma_1/(1-\kappa)}{1-\gamma_1/(1-\kappa)} \equiv \frac{1+g_1}{1-g_1}$$

So measuring a/b fixes g_1, in axes where $\gamma_2 = g_2 = 0$.

For general axes (e.g. pixels) define *reduced shears*: $g_1 \equiv \frac{\gamma_1}{1-\kappa}$ $g_2 \equiv \frac{\gamma_2}{1-\kappa}$

In general measuring a/b fixes $\sqrt{(g_1^2 + g_2^2)}$

$$\mathbf{A} = \mathbf{I} - \kappa\mathbf{I} - \begin{bmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{bmatrix} = (1-\kappa)\begin{bmatrix} 1-g_1 & -g_2 \\ -g_2 & 1+g_1 \end{bmatrix}$$

Sequential dilation & shear

Note: inverse ⟺ negative shear

If

$$\mathbf{A} = (\text{Constant})\begin{bmatrix} 1-g_1 & -g_2 \\ -g_2 & 1+g_1 \end{bmatrix}$$

Then

$$\mathbf{A}^{-1} = (\text{Constant}')\begin{bmatrix} 1+g_1 & +g_2 \\ +g_2 & 1-g_1 \end{bmatrix}$$

Measuring (g_1,g_2) from Q

$$Q^{\text{Image}} = \mathbf{A}^{-1} Q^{\text{Source}} \left[\mathbf{A}^{\text{T}}\right]^{-1}$$

$$Q^{\text{Image}} = \begin{bmatrix} 1-g_1 & -g_2 \\ -g_2 & 1+g_1 \end{bmatrix}^{-1} \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix} \begin{bmatrix} 1-g_1 & -g_2 \\ -g_2 & 1+g_1 \end{bmatrix}^{-1}$$

$$= (Const)\,\sigma^2 \begin{bmatrix} 1+2g_1+g_1{}^2+g_2{}^2 & 2g_2 \\ 2g_2 & 1-2g_1+g_1{}^2+g_2{}^2 \end{bmatrix}$$

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \propto \begin{pmatrix} Q_{xx}^{\text{Image}} - Q_{yy}^{\text{Image}} \\ 2Q_{xy}^{\text{Image}} \end{pmatrix}$$