Anomalies and symmetric mass generation with Kähler–Dirac fermions

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Plan

- Staggered fermions: a puzzle
- Gapping fermions without breaking symmetries (SMG)
- Kähler–Dirac fermions and Dirac fermions. Discretization $\rightarrow$ staggered fermions.
- Gravitational anomaly for Kähler–Dirac fermions.
- Non-perturbative Z_4 anomaly $\rightarrow$ constraints on numbers of fermions.
- Massive Kähler–Dirac fermions in odd dimensions, induced gravity and topological insulators with Kähler–Dirac fermions.

Work with Nouman Butt, Arnab Pradhan and Goksu Can Toga
2101.01026, 1810.06117, 1806.07845

(Reduced) interacting staggered fermions

$$S = \sum_{x,\mu} \chi^a(x)\eta_\mu(x)D^S_\mu \chi^a(x) - \frac{G^2}{8}\sum_x \left[\chi^a(x)\chi^b(x)\right]^2_+$$

$\chi^a(x)$: 4 single component Grassmanns in fund of $SO(4)$

$\eta_\mu(x) = (-1)^{\sum_{i=1}^{\mu-1} x_i}$ and $[]_+$ projects to $(1,0)$ rep $SO(4)$

Describes 16 Majorana fermions in $D = 3, 4$ at $G = 0$

Symmetries

- $SO(4)$
- shift: $\chi(x) \to \xi_\mu(x)\chi(x+\mu)$ with $\xi_\mu(x) = (-1)^{\sum_{i=\mu+1}^{d} x_i}$
- Z_4: $\chi^a(x) \to i\epsilon(x)\chi^a(x)$ with $\epsilon(x) = (-1)^{\sum_i x_i}$

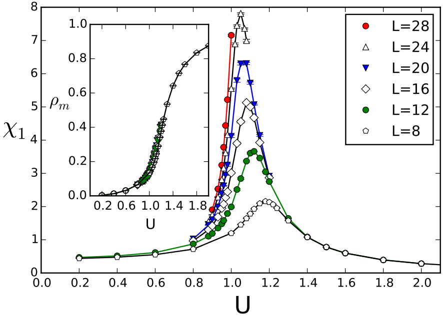
Symmetries prohibit all fermion bilinear terms.

An exotic phase diagram in three dimensions

$(U \sim G^2)$

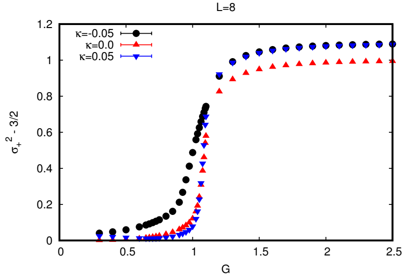
- $G \to \infty < \chi^1 \chi^2 \chi^3 \chi^4 > \neq 0$. Fermions massive. But condensate breaks no symmetries
- $G \to 0$. Massless fermions.

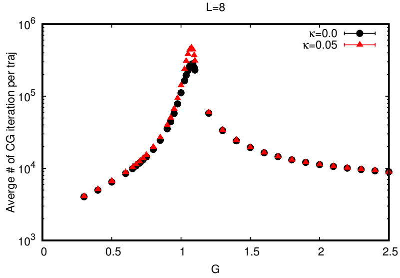
> Must be at least 1 phase transition. But no order parameter !



Chandrasekharan et al. Phys.Rev.D 93 (2016) 8, 081701.

Massive symmetric phase in four dimensions





Evidence for direct, continuous phase transition between massless and massive phases with no symmetry breaking (S.C et al. PRD98 (2018) 114514)

A puzzle

- Previous work with lattice fermions had seen symmetry breaking bilinear condensate for $G \to \infty$. What feature of current model is different ?
- What is the nature of the phase transition here ? – not of Landau-Ginzburg form ..

Resolution

- Staggered fermions arise from discretization of Kähler-Dirac (KD) fermions
- Symmetric mass generation tied to (novel) anomaly cancellation for KD fermions
- Anomaly structure survives intact on lattice

Fermion masses

Typically fermions acquire mass by breaking symmetries:

- Explicitly eg Dirac mass breaks axial symmetry.
- Spontaneously eg. chiral condensate $< \overline{q}q > \neq 0$ in QCD.
- Via anomalies eg η'

Does this exhaust the possibilities ?

No!

Fermion masses can arise without breaking global symmetries provided **all** 't Hooft anomalies vanish

Symmetric Mass Generation (SMG)

’t Hooft anomalies

Imagine gauging global symmetry $G \rightarrow$ non-zero anomaly coeff **A** (triangle diagram)

Anomaly is RG invariant $\rightarrow$
requires massless particles in I.R with same **A**
Options:

- Massless composite fermions
- G breaks spontaneously - massless Goldstone bosons

If we are to gap fermions in IR without breaking symmetries a necessary condition is that all ’t Hooft anomalies must vanish in U.V

Thus SMG for staggered fermions requires cancellation of (new) anomalies for Kähler–Dirac fermions

Kähler–Dirac fermions

An alternative solution to the problem of square rooting the Laplacian:

Kähler-Dirac equation

$$(K-m)\Phi = \left(d - d^\dagger - m\right)\Phi = 0$$

$K^2 = -\Box$. Note: Φ collection of p-forms ($p = 0 \ldots D$).

From Kähler-Dirac field $\Phi = (\phi, \phi_\mu, \phi_{\mu\nu}, \ldots)$ form matrix

$$\Psi = \sum_{p=0}^{D} \phi_{n_1 \ldots n_p}(x) \gamma_1^{n_1} \gamma_2^{n_2} \cdots \gamma_p^{n_p}$$

Can show that the Kähler-Dirac equation in flat space equivalent to:

$$(\gamma^\mu \partial_\mu - m)\Psi = 0$$

In $D = 4$:

Four copies of Dirac equation where Dirac spinors correspond to columns of Ψ.

Kähler–Dirac fermions continued ...

Representation in curved space

Form: $(K - m)\Omega = 0$ unchanged

Matrix form: $e^\mu_a \gamma^a \left(\partial_\mu \Psi + [\omega_\mu, \Psi]\right) - m\Psi = 0$

ω_μ – spin connection and e_μ – frame with $e^a_\mu e^b_\nu \delta_{ab} = g_{\mu\nu}$

Key feature:

Linear operator $\Gamma : \phi_{\mu_1 \ldots \mu_p} \to (-1)^p \, \phi_{\mu_1 \ldots \mu_p}$ with $\{\Gamma, K\}_+ = 0$

Generates exact $U(1)$ symmetry of massless action $\int \overline{\Phi} K \Phi$:

$$\Phi \to e^{i\alpha\Gamma}\Phi$$

$$\overline{\Phi} \to \overline{\Phi} e^{i\alpha\Gamma}$$

Reduced Kähler–Dirac fermion

$\Phi_\pm = \frac{1}{2}(1 \pm \Gamma)\,\Phi$ with $S_{\rm RDK} = \int \overline{\Phi}_+ K \Phi_- \equiv \int \Phi^T K \Phi$

Lattice Kähler–Dirac fermions

- Approximate continuum by (oriented) triangulation T
- Place p-forms on p-simplices $\phi_p \rightarrow \phi_{\mathrm{p-simplex}}$
- Replace $(d, d^\dagger)$ by $(\delta, \overline{\delta})$ where boundary op.
  $\delta(a_0 \ldots a_p) = \sum_{i=0}^{p} (-1)^i (a_0 \ldots \overline{a_k} \ldots a_p) \leftarrow (p-1)$ simplices
- Discrete Kähler–Dirac equation:

$$(\delta - \overline{\delta} - m)\Phi_{\mathrm{lat}} = 0$$

No fermion doubling! Lattice sols go smoothly into cont. (homology theory). Zero mode structure reproduced on lattice.

Valid for any (oriented) random triangulation of any topology.

Relation to staggered fermions

Write continuum KD fermion matrix Ψ as

$$\Psi(x) = \sum_{\text{unit hypercube b}} \chi(x+b)\gamma^{x+b}$$

where $\gamma^x = \gamma_1^{x_1}\gamma_2^{x_2}\ldots\gamma_D^{x_D}$ and $b_i = 0, 1$ span unit hypercube on lattice.
Replace derivatives by (symmetric) difference ops.

$$\int Tr(\overline{\Psi}\gamma_\mu\Delta_\mu\Psi) \to \sum_x \eta_\mu(x)\overline{\chi}(x)\Delta_\mu\chi(x+\mu)$$

Staggered action!
Staggered fermions correspond to discrete KD fermions on flat regular torus

Quick recap so far

- Staggered fermions (special) discretization of KD fermions.
- KD fermions can be discretized on any random triangulation with any topology.

In flat space

Kähler–Dirac fermions do not suffer chiral anomalies (vector-like)

BUT

They do suffer from a new gravitational anomaly
Remarkably this anomaly survives discretization

Perturbative gravitational anomaly for $U_\Gamma(1)$

Work on lattice

Under $(\Phi, \overline{\Phi}) \to e^{i\alpha\Gamma}(\Phi, \overline{\Phi})$

$$\delta S_{\rm KD}(\overline{\Phi}, \Phi) = 0$$

But measure not invariant

$$D\Phi D\overline{\Phi} = \prod_p d\phi_p d\overline{\phi}_p \to e^{2iN_0\alpha} e^{-2iN_1\alpha} .. e^{2i(-1)^d N_d\alpha} \prod_p d\phi_p d\overline{\phi}_p$$
$$= \; e^{2i\chi\alpha} D\overline{\Phi} D\Phi \quad \chi \equiv \text{Euler}$$

Anomaly in even dimensions

Compactify $R^{2n} \to S^{2n}$. Breaks $U(1) \to Z_4$. Latter prohibits mass terms but allows for eg. four fermion ops.

Note

Lattice calc. agrees with continuum

Example of QM anomaly for finite number dof ...

Interactions for Kähler–Dirac fermions

Decompose a KD field into pair of reduced fields Φ^a, $a = 1, 2$.
Z_4 invariant four fermion interactions via coupling to scalar:

$$\Phi^a \to i\Gamma\Phi^a$$
$$\sigma \to -\sigma$$

$$S = \int \sum_{a=1}^{2} \Phi^a K \Phi^a + G\sigma\Phi^a\Phi^b\epsilon_{ab} + \ldots$$

Integrate fermions:

$$\to \mathrm{Pf}\left(K\delta^{ab} + G\sigma\epsilon^{ab}\right)$$

Real antisymmetric matrix. Pfaffian defined as product of eigenvalues in upper halfplane in some reference $\sigma = \sigma_0$.

Require it be a smooth function of σ
Possibility of sign change if eigenvalues flow thru origin

A non-perturbative anomaly for Kähler–Dirac fermions

Let $\sigma = s\sigma_0$ where interpolates $s \in (1 \to -1)$.

Eigenvalues of near zero modes with $K\Phi = 0$ are $\pm is\sigma_0$
Change sign as $s = 0^+ \to 0^-$
$\to$ Pfaffian changes sign for 2 flavors of reduced Kähler–Dirac field

Therefore: $Z = \int D\sigma \mathrm{Pf}\,(K + \sigma\epsilon) = 0$

To avoid this anomaly require eigenvalues to flow in pairs through origin. i.e Z_4 is anomalous unless number of (reduced) flavors is multiple of 4

In flat 4d reduced Kähler–Dirac field $\to$ 4 Majorana spinors
So only theories possessing 16 Majorana spinors are consistent !

By cancelling Z_4 anomaly in U.V can achieve SMG for continuum KD fermions and hence for lattice staggered fermions

Discrete anomalies

Recent work on global anomalies for discrete symmetries:

Cancelling 't Hooft anomalies for these symmetries gives new constraints on fermion content of consistent QFTs
Equivalent to cancelling Z_4 anomaly of Kähler–Dirac !

D=1	Time reversal	8 Majorana	4 RKD
D=2	Chiral fermion parity	8 Majorana/Weyl	4 RKD
D=3	Time reversal	16 Majorana	4 RKD
D=4	Spin-Z_4 symmetry	16 Majorana/Weyl	4 RKD

eg. Spin-Z_4 symmetry

$$\psi_L \to -i\psi_L \quad \psi_R \to +i\psi_R$$

n_L, n_R number of L/R Weyl fermions

anomaly cancellation: $n_L - n_R = 0 \mod 16$

Kähler–Dirac fermions in three dimensions

- Can integrate out massive 4 component Kähler–Dirac matrix fermions. Find CS theory:

$$S = \frac{1}{32\pi}\frac{M}{|M|}\int_M d^3x \epsilon^{\mu\nu\lambda}\epsilon_{ABCD}\left(\Omega_\mu^{AB}\partial_\nu\Omega_\lambda^{CD} + \frac{2}{3}\Omega_\mu^{AB}\Omega_\nu^{CM}\Omega_\lambda^{MD}\right)$$

where $\Omega = \omega^{ab}T_{ab} + \frac{1}{\ell}e^a T_{a4} \quad a < b = 1\ldots 3 \quad T_{AB} = \frac{1}{4}[\gamma_A, \gamma_B]$

T are generators of $\mathrm{spin}(4)$

Forced by Kähler–Dirac nature. Naturally allows embedding of 3d (ω, e)

Reexpressing

$$\to S = \frac{1}{32\pi}\frac{M}{|M|}\frac{1}{\ell}\int d^3x\, \epsilon^{\mu\nu\lambda}\left(e_\mu^a R_{\nu\lambda}^{cd} - \frac{1}{3\ell^2}e_\mu^a e_\nu^b e_\lambda^c\right) \quad \text{Witten}$$

Topological insulators with KD fermions

In presence of domain wall $M(x_3) \to M_0 \mathrm{sgn}(x_3)$ massless 2d reduced KD fields appear on the wall which are coupled to $U(1)$ gauge field.

A would be gravitational anomaly is cancelled by anomaly inflow from the bulk gravitational theory

$$\Downarrow$$

Gravitational anomaly renders theories of RKD inconsistent unless they form the boundary of a space of one higher dimension

Still suffer from discrete Z_4 anomaly which constrains $n_F = 4k$

Summary

- Kähler–Dirac eqn. alternative to Dirac eqn. In flat space describes multiples of Dirac fermions.
- In curved space this equivalence not true. Kähler–Dirac fermions suffer from a (new) gravitational anomaly $U_\Gamma(1) \to Z_4$.
- This Z_4 suffers from a further global anomaly unless $n_F = 4k$. This implies 8 and 16 Majorana fermions in $d = 2, 4$.
- Anomaly depends on topology. KD fermions admit discretization that captures it exactly.
- Cancelling this global anomaly is necessary for SMG - explains staggered fermion results.
- In odd d massive Kähler–Dirac fermions yield Chern-Simons gravity theories. If $\partial M \neq 0$ anomaly inflow cancels off $U(1)$ gauge anomaly. Topological insulator !
- $6 million question: can these features allow progress on lattice chiral gauge theories ?

Thanks!

(Lattice) chiral gauge theories ...

Lack a non-perturbative definition of a chiral gauge theory
Weyl fields in complex representation of gauge group

- Naive lattice approach fails because of fermion doubling: Nielsen-Ninomiya theorem always leads to equal numbers of left ψ_L and right ψ_R fields.
- Mirror models: try to give mass to say ψ_R without touching ψ_L. Hard (impossible ?) to do.
- Perhaps SMG can be used. Invariant four fermion terms possible eg. $(\psi_R^T \Gamma_\mu \psi_R)^2$ with ψ_R in **8** of $\mathrm{spin}(7)$. Generate cut-off scale mass for ψ_R leaving ψ_L massless.
- Must cancel off all 't Hooft anomalies – embed in reduced Kähler–Dirac field ? How to get γ_5 from Γ (twisted chiral symmetry) ?