TPOT SUPPORT STRUCTURE FINITE ELEMENT ANALYSIS

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ABSTRACT

This paper is a discussion of the results of finite element analysis (FEA) of the TPOT detector support frame. The analysis of the complete support frame is discussed in section 5. Earlier sections are used to verify assumptions in the model. The maximum deflection of the full support frame is 20.04mm if pinned boundary conditions are assumed at each end of the 80/20 beams. The maximum deflection of the support frame is 7.09mm assuming fixed connections at the ends of the 80/20 beams.

Introduction

The TPOT detector support frame is constructed of 80/20 aluminum extrusion. Aluminum U-channel is bolted to the 80/20 beams in five locations along the beam. Eight TPOT detectors will be supported by this frame, weighing 3kg each. The 80/20 beams that make up the frame are profile 1004 and they are each 3.6576m (144in) long. The 80/20 beams are supported by a patch panel at the end of the beams. Drawings of the support frame and mounting panels of the TPOT detectors are shown in figures 1 and 2. The complete TPOT support frame without the mounting frames is shown in figure 3.

There are not established techniques on how to analyze 80/20 beams using finite element method (FEM). This issue is due to the complex geometry of the 80/20 profile. Therefore, part of this report will be used to establish rules regarding FEA of 80/20 beams. First, an element type and mesh scheme will need to be established. Second, a method to apply the TPOT detector load to the support frame will need to be chosen. Third, rules will need to be established regarding the attachment of the aluminum U-channel to the 80/20 beam. The U-channel is bolted to the 80/20 in multiple locations meaning the model can become over constrained if connections are not carefully chosen. Fourth, similar rules will need to be established on how to model the 150° connections between adjacent 80/20 beams.This paper will be organized as follows:

1. Discuss feasibility of using solid elements to mesh the support frame
2. Apply an offset point load to a single 80/20 beam. Compare the deflection and angle of rotation solutions to a solid element model, a beam element model, and classical textbook formula.
3. Repeat item 2 for a single 80/20 beam bolted to a U-channel
4. Briefly discuss solutions to the two beam model
5. Describe boundary conditions, connections and loads applied to the full support frame assembly. Show results.

1. SOLID ELEMENTS VS. BEAM ELEMENTS

Solid elements are multi-node elements that can be quadrilateral, triangular, or tetrahedral in shape. They are often used to perform stress analysis on parts. The original part geometry can be imported into the analysis software and meshed. Many parts can be successfully meshed this way with sufficiently small element size to retain features of the part. However, the 80/20 beams used in the support frame have highly complex geometry and are not readily meshed using solid elements. The mesh of a solid element model of a single beam of the TPOT frame is shown in figure 4. The element size across the face is rigidly defined as 1mm. The element size along the length of the beam is 10mm. The total number of elements in this mesh is 116,754. It should be noted the 80/20 beam was defeatured to minimize the number of elements needed to mesh the part and to allow for rectangular elements to be used.

In contrast, the same 80/20 beam can be meshed using beam elements. Beam elements are 2-node elements based on Timoshenko beam theory. They can model translation and rotation about the x,y,z axes. The mesh of a beam element model of the single 80/20 beam is shown in figure 5. The beam elements are 10mm long, which is equivalent to the length of the solid elements used above. The number of elements in this model is 366, which is 0.3% of the number of elements in the solid element model. Defeaturing was not required in this model because cross section information is captured in the tabulated values for the moment of inertia in the x,y,z axes.

2. DEFLECTION AND ROTATION OF A SINGLE 80/20 BEAM

A test case is considered and the results are compared to textbook solutions for solid element and beam element models of a single 80/20 beam. The test load shown in figure 6 and 7 is chosen to approximate how the TPOT detectors will deflect and rotate beams on the full support structure.

Two separate boundary conditions are considered. The first boundary condition is a pinned connection at both ends of the beam. Pinned connections prohibit translation in any direction, and prohibit rotation about the z-axis. The problem is therefore that of a simply supported beam with a point load applied to the center. Textbook formulae shown in equations 1 and 3 can be used to compute the deflection and rotation of the 80/20 beam under these conditions.

The second boundary condition considered is a fixed end connection. Both ends of the 80/20 beam are prohibited from translation and rotation about any axis. Textbook solutions exist for this load configuration and equations 2 and 3 can be used to compute the deflection and rotation of the 80/20 beam under these conditions. The values in table 1 are used in the textbook formulae.

A separate analysis including gravity will be discussed at the end of the section to show how the deflection of the bar increases under its own weight. Gravity is not included in every analysis because it is simpler to compare ANSYS results to textbook formulae without gravity.

$$\delta_{pinned} = \frac{PL^3}{48EI} \tag{1}$$

$$\delta_{fixed} = \frac{PL^3}{192EI} \tag{2}$$

$$\theta = \frac{1}{2}\frac{TL_\theta}{JG} \tag{3}$$

P	30 N	Applied Load
L	3.6576 m	Total length of beam
E	71000 MPa	Young's Modulus
I	18540 mm^4	Area Moment of Inertia
T	P * 203.2 mm	Applied torque to beam
L_θ	L/2	Length to compute rotation
J	9283.5 mm^4	Polar moment of inertia
G	26692 MPa	Shear modulus
w	8.45 N/mm	beam weight per mm

Table 1: List of properties of a single 80/20 beam member used in equations 1 and 3.

The test load is applied via a remote force in ANSYS in both the solid element and beam element models. The remote force is scoped to the 80/20 beam and assigned 'beam' behavior. This uses a 2-node beam element to make the connection and allows the load to properly transmit moment. The beam material is aluminum alloy and the beam radius is set to 3mm. The beam radius was found through trial and error. 3mm seems to accurately capture the bending moment and torsional moment in the 80/20 beam. Larger beam radii tend to add rigidity to the model and lead to incorrect solutions such as zero bending moment at the center location of the beam. Other behaviors such as deformable behavior also add rigidity to the model and lead to incorrect solutions. The load is scoped to 4 nodes on each beam, meaning the load is applied over a 30mm region and should closely approximate a point load.

Pinned end connections in the solid element are modeled through a single 'Universal Joint' scoped to both of the beam end faces. The y-axis of the local joint coordinate system is directed along the central axis of the beam. The pinned end connections in the beam element model are applied by setting the nodal displacement and nodal z-axis rotation equal to zero. The nodal constraints are scoped to the end node at both ends of the beam. It should be noted the constraints need not be scoped to the nodes directly. They can be scoped to the vertices instead. The deflection, δ_{pinned}, and rotation, θ, computed by ANSYS for the solid element model and the beam element model are compared to textbook values in table 2. The deflection computed by ANSYS for the beam element model is shown in figure 8.

	Textbook	ANSYS Solid	ANSYS Beam
δ_{pinned} [mm]	23.23	21.17	23.24
θ [degrees]	1.289	1.324	1.28

Table 2: The computed deflection and rotation of a single 80/20 beam with **pinned** boundary conditions. The three methods used are equations 1 and 3, ANSYS using solid elements, and ANSYS using beam elements.

Fixed end connections in the solid element are modeled through a single 'Fixed Joint' scoped to both of the beam end faces. The fixed end connections in the beam element model are applied by setting the nodal displacement and nodal rotation equal to zero in all directions. The nodal constraints are scoped to the end node at both ends of the beam. The deflection, δ_{fixed}, and rotation, θ, computed by ANSYS for the solid element model and the beam element model are compared to textbook values in table 3. The deflection computed by ANSYS for the beam element model is shown in figure 9. The angle of rotation does not change because the two boundary conditions considered are equivalent in preventing rotation about the z-axis.

	Textbook	ANSYS Solid	ANSYS Beam
δ_{fixed} [mm]	5.808	5.681	5.8115
θ [degrees]	1.289	1.3262	1.2842

Table 3: The computed deflection and rotation of a single 80/20 beam with **fixed** boundary conditions. The three methods used are equations 1 and 3, ANSYS using solid elements, and ANSYS using beam elements.

The textbook solution for the maximum deflection of the 80/20 beam under a point load and a distributed load (gravity) is given in equation 4 for pinned boundary conditions, and equation 5 for fixed boundary conditions. w is the product of the specific weight of the beam with the cross sectional area, $w = \rho g A$. Table 4 compares textbook results to ANSYS results.

$$\delta_{pinned} = \frac{PL^3}{48EI} + \frac{5wL^4}{384EI} \tag{4}$$

$$\delta_{fixed} = \frac{PL^3}{192EI} + \frac{wL^4}{384EI} \tag{5}$$

	Textbook	ANSYS Solid	ANSYS Beam
δ_{pinned} [mm]	38.19	35.87	38.19
δ_{fixed} [mm]	8.8	8.50	8.80

Table 4: The computed deflection of a single 80/20 beam due to a point load and a distributed load using equations 4 and 5 and ANSYS. Both pinned and fixed boundary conditions are considered.

The solution produced by each method closely agree indicating that beam elements can be used to accurately model the 80/20 beams. Remote forces with 'beam' behavior can be used to model forces acting at a distance from the central axis of the 80/20 beam.

3. DEFLECTION AND ROTATION OF A SINGLE 80/20 BEAM CONNECTED TO A U-CHANNEL

This section will build on the model discussed in section 2. The U-channel is attached to the 80/20 in five places as shown in figure 10. Effectively, this will increase the resistance to bending and twisting by increasing the moment of inertia and polar moment of inertia of the system. Both pinned and fixed boundary conditions will be considered in this section. Analysis including gravity will not be considered in this section.

Table 6 shows the joint information for the U-channel attachments to the 80/20. The center joint is made completely fixed. The non-center joints are allowed to translate in the y direction to keep from over-constraining the model. The joints are set to beam behavior, the beam material is aluminum alloy, and the beam radius is set to 3mm. The pinball radius of each joint is set to 25.4mm.

The first load case to consider is the deflection and rotation for a system with pinned boundary conditions and a single offset point load applied to the center of the 80/20 beam. Equations 6 and 8 can be used to compute the deflection and rotation of the system with pinned end connections. Equation 6 is derived in Appendix A. The values that will be used in equations 6 and 8 are shown in table 5. The results for the textbook and ANSYS computations are shown in table 7. Figure 11 shows the deflection of the system with pinned boundary conditions as computed by ANSYS.

Note, it is possible to compute the deflection of the system using equation 1 rather than equation 6. The correction equation 6 provides is actually quite small. If equation 1 is used to compute the deflection, and the moment of inertia is assumed to be I_2 throughout the entire length, then the computed deflection is $\delta_{pinned} = 5.56$mm. This is only a difference of 0.04mm between the two results. It is recommended that equation 1 is used to compute the deflection with $I = I_2$.

The second case to consider is the 80/20 beam, U-channel system with fixed boundary conditions. It was shown in section 2 that the rotation is the same for fixed and pinned boundary conditions, therefore the rotation will not be computed again. The corrections Appendix A provide are minimal, therefore, the recommendations in the previous paragraph are followed. Equation 7 is used to compute the deflection of the system with fixed boundary conditions. Figure 12 shows the deflection of the system with fixed boundary conditions as computed by ANSYS. The results are also listed in table 7.

$$\delta_{pinned} = f(P, L_1, L_2, E, I_1, I_2) \tag{6}$$

$$\delta_{fixed} = \frac{PL^3}{192EI_2} \tag{7}$$

$$\theta = \frac{1}{2}(\frac{TL_1}{J_1 G} + \frac{TL_2}{J_2 G}) \quad (8)$$

P	30 N	Applied Load
L	3.6576 m	Total length of beam
E	71000 MPa	Young's Modulus
I_1	18540 mm^4	Region 1 Area Moment of Inertia
I_2	77476 mm^4	Region 2 Area Moment of Inertia
T	P * 203.2 mm	Applied torque to beam
L_1	241.3 mm	Region 1 Length
L_2	1587.5 mm	Region 2 Length
J_1	9283.5 mm^4	Region 1 Polar moment of inertia
J_2	14866 mm^4	Region 2 Polar moment of inertia
G	26692 MPa	Shear modulus
w	16.67 N/mm	Region 2 beam weight per mm

Table 5: List of properties of a single 80/20 beam member attached to a U-channel used in equations 6 and 8. See appendix for definitions of regions 1 and 2.

Location	Translation DOF	Rotation DOF
Center (1)	Fixed X,Y,Z	Fixed Y,Z Free X
Outer (4)	Fixed X,Z Free Y	Fixed Y,Z Free X

Table 6: Description of joints used to connect the U-channel to the 80/20.

	Textbook	ANSYS Beam
δ_{pinned} [mm]	5.6	7.24
δ_{fixed} [mm]	1.39	3.25
θ [degrees]	0.87	1.07

Table 7: The deflection and rotation of an 80/20 beam, U-channel system with pinned and fixed boundary conditions. These results are computed using two methods - equations 6, 7 and 8, and ANSYS using beam elements. Equation 1 can be used to compute the deflection by assuming the moment of inertia is I_2 throughout the entire length, $\delta = 5.56$mm.

The ANSYS computation estimates a larger deflection and rotation than the textbook computation. This is likely due to the assumptions made in each model. The textbook model assumes the U-channel and the 80/20 cross section are continuous cross section, and bending moment is computed assuming a neutral axis through the continuous cross section. ANSYS connects a U-channel beam and an 80/20 beam together at discrete locations. ANSYS distributes the bending moment between the two beams.

4. DEFLECTION AND ROTATION OF A PAIR OF 80/20 BEAMS CONNECTED TO U-CHANNELS

This section will discuss the computational results of a model with two 80/20 beams that support a 120N point load at the center of the assembly. The 120N load is distributed across the two 80/20 members in this configuration. Equations 6 and 8 can be used to compute the deflection and rotation of each beam. Note that gravity is not included in the analysis in this section. See figure 13 for the load configuration.

The remote force of 120N is scoped to both 80/20 beams. The pinball radius is set to 204mm which encircles 4 nodes on each beam. This indicates the load will be applied over a 30mm region on each beam. The remote force is assigned beam behavior, the beam material is an aluminum alloy, and the beam radius is set as low as possible at 0.2mm. ANSYS implements these settings by connecting the remote force to the 80/20 beam using a 2 node fixed-fixed beam element. In this way the load to the 80/20 beams is the reaction force and moment for a fixed-fixed beam problem with a single point load applied at the center of the beam. The applied force to the 80/20 will be $(120N/2)$, and the applied torque will be $(120N\,406.4mm/8)$, where 406.4mm is the distance between the two 80/20 beams. This load behavior is representative of the actual mounting frames that will be used to support the TPOT detectors. The mounting frames are plates that are supported by the 80/20 as shown in figures 1 and 2.

Table 8 shows the parameters that will be used in equations 6, 7 and 8 to compute the deflection and rotation of each beam for pinned and fixed boundary conditions. Table 9 compares the results from ANSYS and equations 6, 7 and 8. The ANSYS result for pinned boundary conditions is shown in figure 14, and the result for fixed boundary conditions is shown in figure 15.

P	60 N	Applied Load
L	3.6576 m	Total length of beam
E	71000 MPa	Young's Modulus
I_1	18540 mm^4	Region 1 Area Moment of Inertia
I_2	77476 mm^4	Region 2 Area Moment of Inertia
T	P * 203.2 mm	Applied torque to beam
L_1	241.3 mm	Region 1 Length
L_2	1587.5 mm	Region 2 Length
J_1	9283.5 mm^4	Region 1 Polar moment of inertia
J_2	14866 mm^4	Region 2 Polar moment of inertia
G	26692 MPa	Shear modulus
w	16.67 N/mm	Region 2 beam weight per mm

Table 8: List of properties used in the analysis of two 80/20 beam members with attached U-channel. The properties are used in equations 6, 7, and 8 to compute the deflection and rotation of the beams. See appendix for definitions of regions 1 and 2.

	Textbook	ANSYS Beam
δ_{pinned} [mm]	11.12	14.32
δ_{fixed} [mm]	2.78	6.24
θ [degrees]	0.87	0.97

Table 9: Deflection and rotation of two 80/20 beams connected to a U-channel. Results are computed using two methods - equations 1 and 3, and ANSYS using beam elements. The angle of rotation computed indicates that each beam will rotate about 1° towards the center of the assembly where the load is applied.

The discrepancy in the results of table 9 is likely due to the differences in the model assumptions. See the end of section 3.

5. FULL TPOT SUPPORT FRAME ANALYSIS

The full TPOT frame analysis will utilize the information learned in the previous sections.

1. Each 80/20 beam is separately constrained at the ends. Pinned and fixed boundary conditions are each considered.
2. The U-channel is fixed to the 80/20 in 5 locations using joints as specified in table 6.
3. The loads are applied to the 80/20 beams as described by section 4. The load scheme is shown in figure 16.

The loads are placed at the center locations of the mounting frames on the TPOT structure. The mounting frames bear the 3kg TPOT detector load. The frames attach to the 80/20 in six locations, although they will be modeled as attaching in one location similar to a point load. This will lead to a more conservative analysis as the loading is no longer distributed but concentrated at a point. Figure 16 shows the location of each load.

Connectors are used to fix adjacent 80/20 beams at 150° relative to each other. The connectors tie the adjacent 80/20 beams together and prevent adjacent beams from rotating relative to each other. General joints with 'beam' behavior are used to implement these connections. The beam material is aluminum alloy, the beam radius is 3mm, and the joint pinball radius is 25.4mm. Joint locations are shown in figure 17. All joints are the same and joint DOF information is listed in table 10. y-translation is allowed to prevent over-constraining the model. If y-translation is fixed fictitious axial loads arise in the model.

Translation DOF	Rotation DOF
Fixed X,Z Free Y	Fixed X,Y,Z

Table 10: Description of joints used to model 150 ° connectors.

Figure 18 shows the ANSYS result for the deflection of the TPOT support frame including gravity and assuming pinned boundary conditions. Figure 19 shows the ANSYS result for the deflection of the TPOT support frame including gravity and assuming fixed boundary conditions. The maximum deflection and rotation with and without gravity, are listed in table 11.

	without gravity	with gravity
δ_{pinned} [mm]	10.19	20.04
δ_{fixed} [mm]	3.89	7.09
θ [degrees]	0.42	0.44

Table 11: Deflection and rotation of the TPOT support frame for pinned and fixed boundary conditions. Columns for models including gravity and excluding gravity are shown.

The deflection of the TPOT support frame highly depends on the boundary conditions. In reality the boundary conditions of the support frame will not be pinned and will provide some rigidity. However, the deflection reported for δ_{pinned} in table 11 can be used as a worst case value.

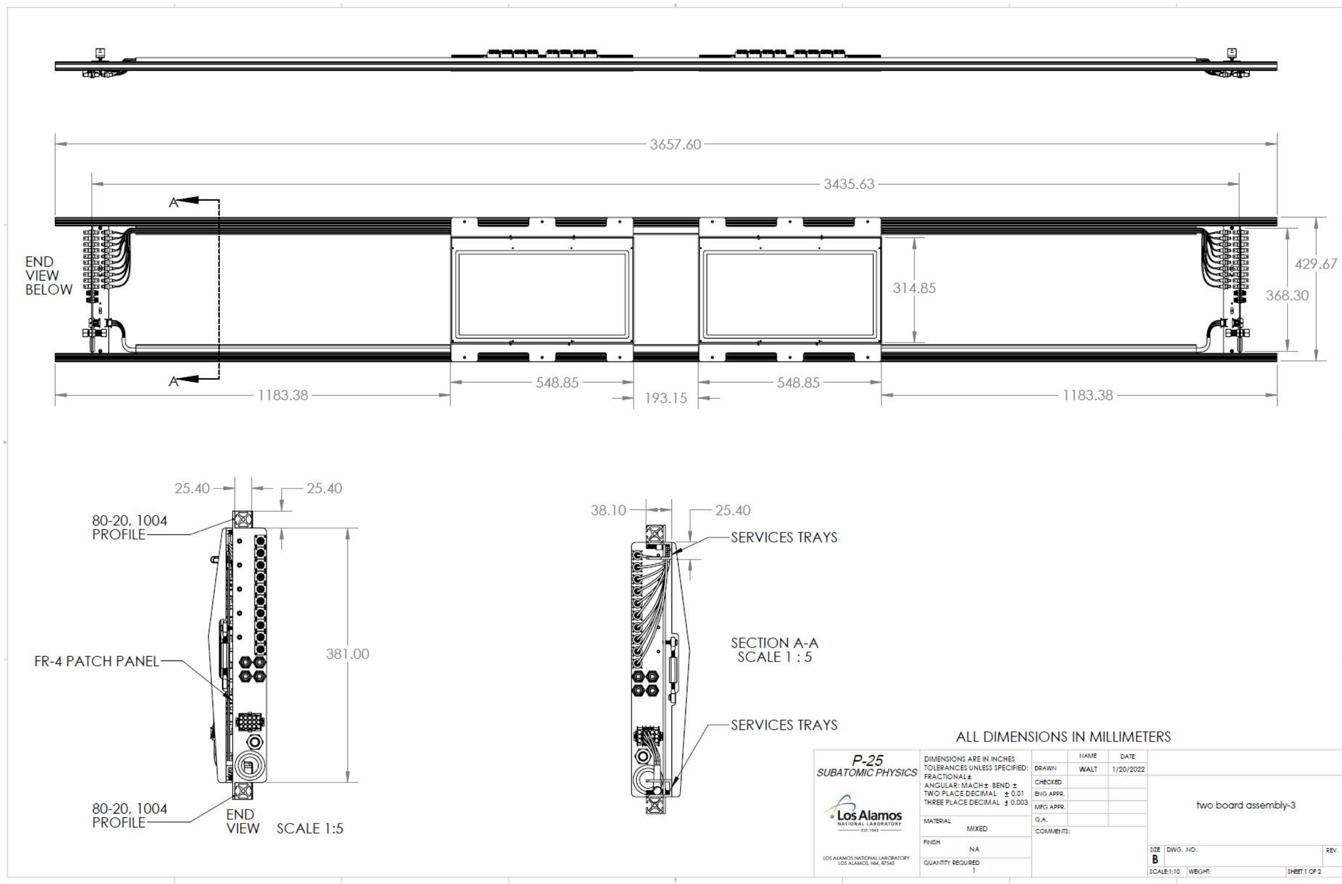

Fig. 1: Drawing of TPOT frame with two detector frames attached the top face of the 80/20 beams. U-channel service trays are used for electronic cables, cooling, etc.

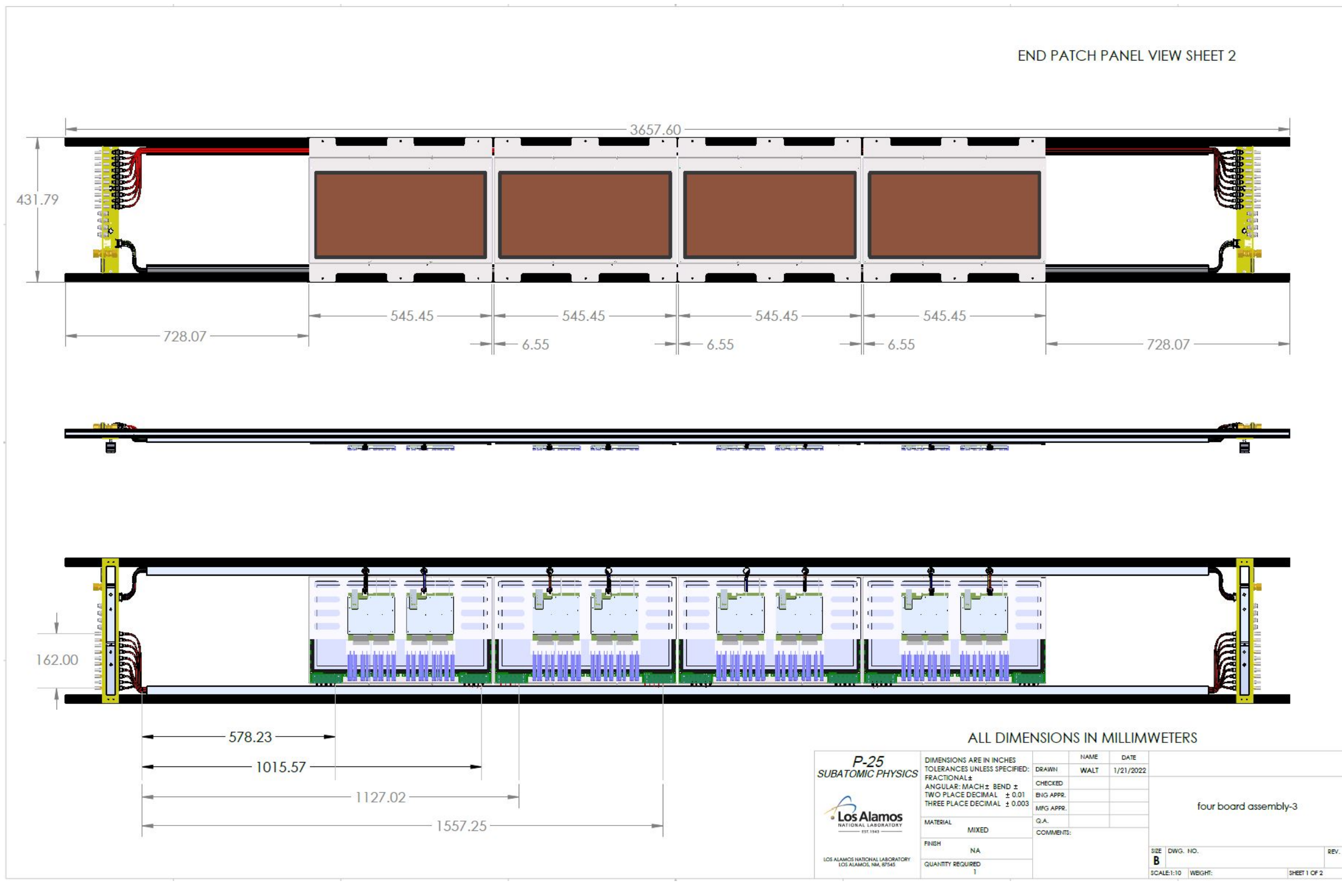

Fig. 2: Drawing of TPOT frame with four detector frames attached the top face of the 80/20 beams. U-channel service trays are used for electronic cables, cooling, etc.

Fig. 3: TPOT support frame assembly. Mounting frames are not shown.

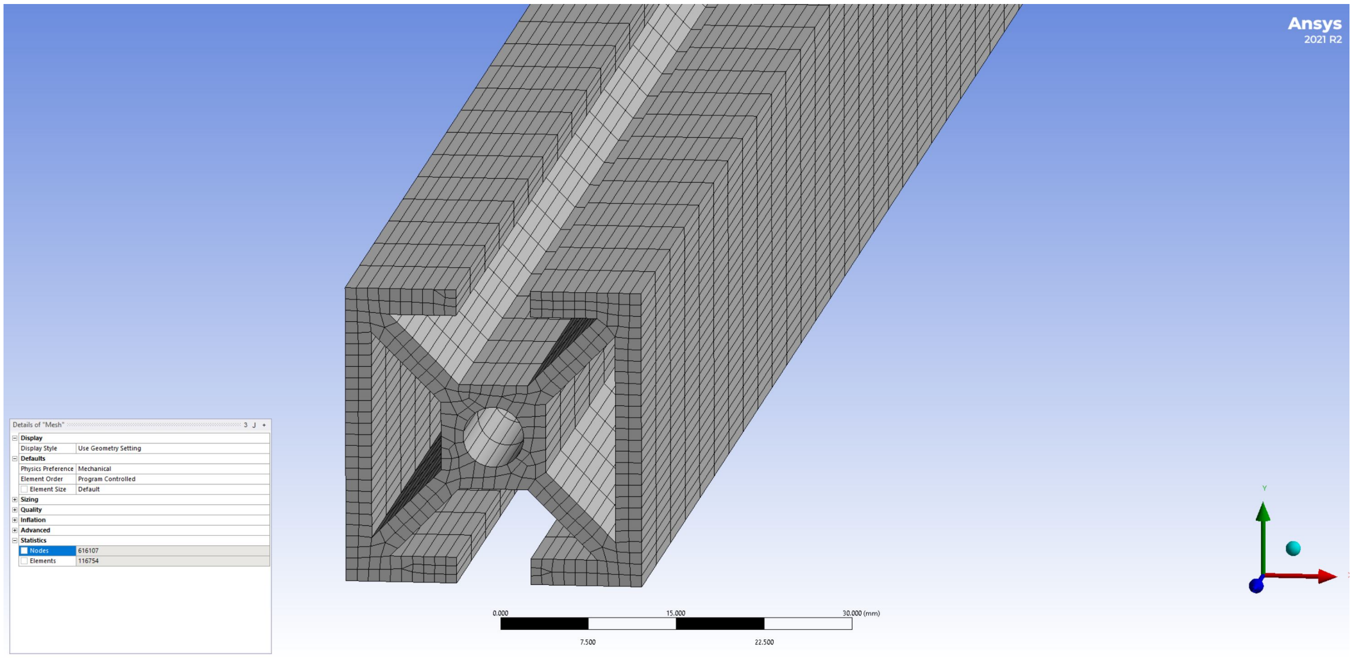

Fig. 4: Solid element mesh. The element size on the face of the beam is 1mm and 10mm along the length of the beam. Total number of solid elements required is 116,754. There are about 32 elements across the face of the beam.

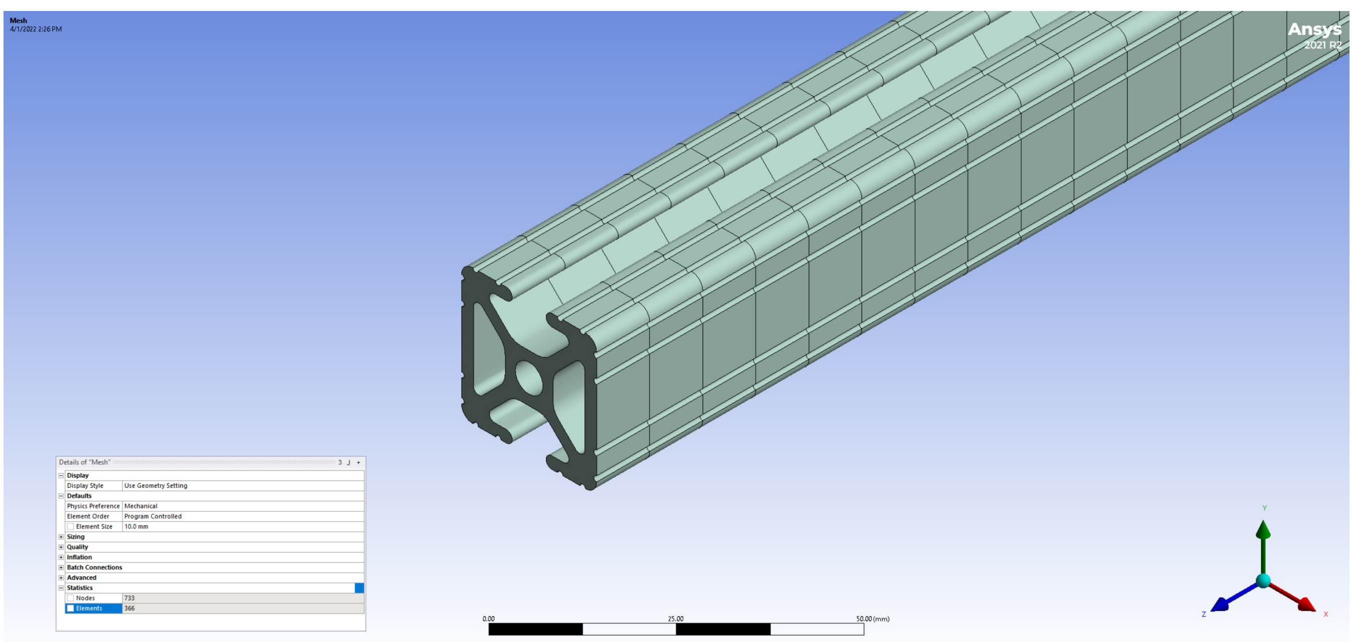

Fig. 5: Beam element mesh. Beam elements are 10mm long. Total beam elements required to mesh a single 80/20 beam is 366.

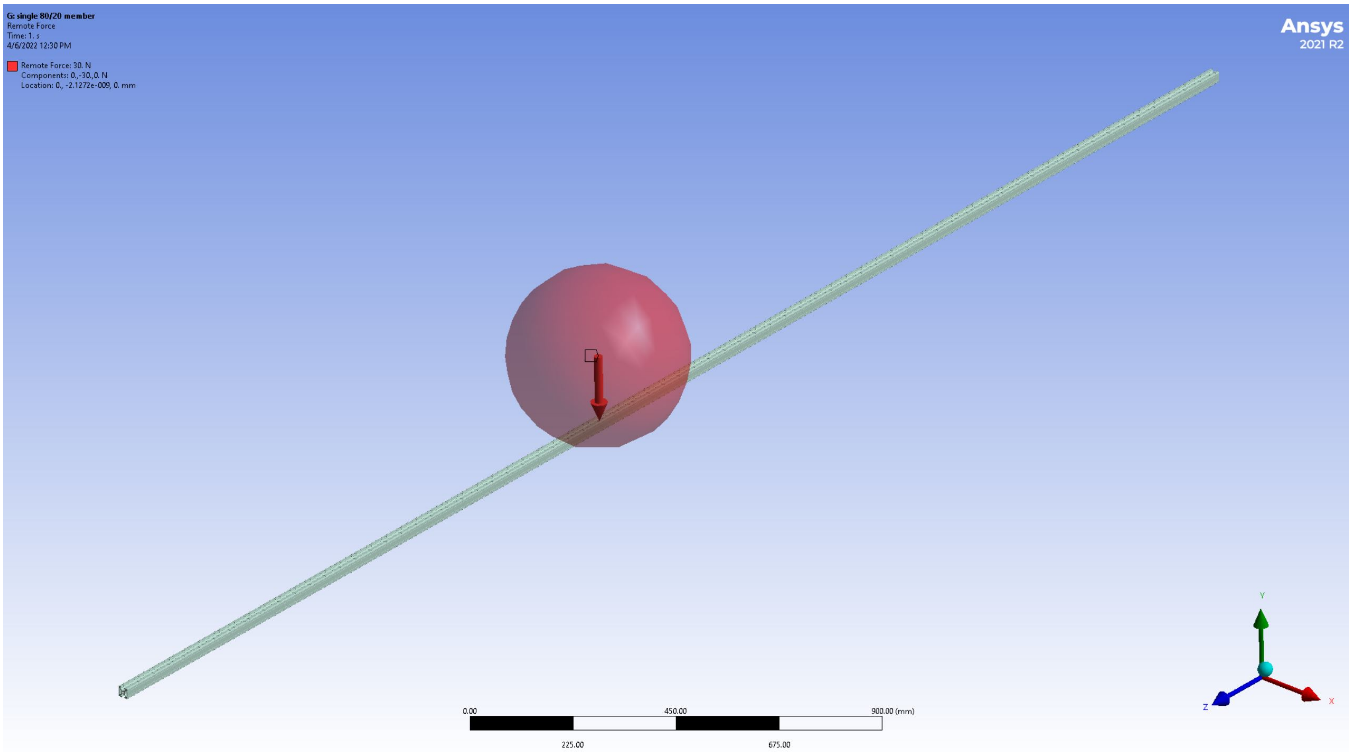

Fig. 6: Offset load applied to single 80/20 beam. Horizontal offset load of 30N is placed 203.2mm from the beam axis.

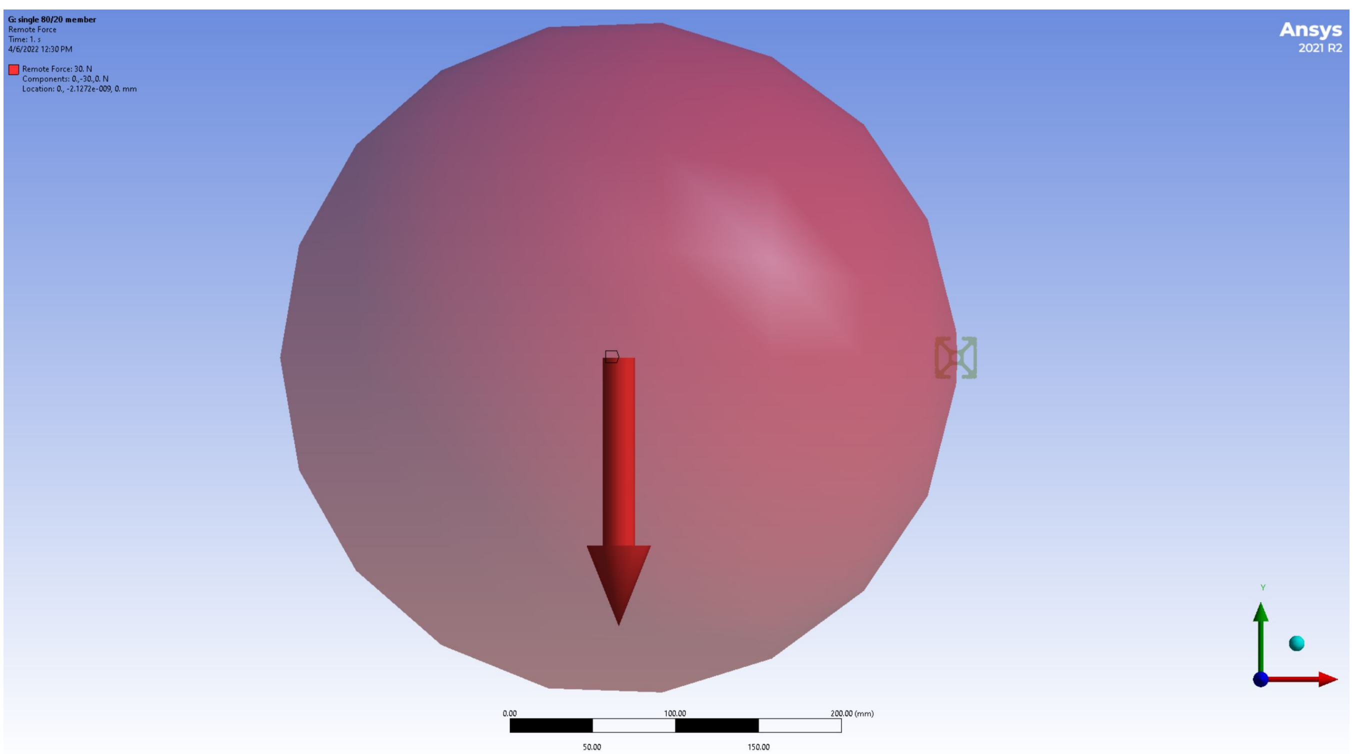

Fig. 7: Front view of offset load applied to single 80/20 beam. Horizontal offset load of 30N is placed 203.2mm from the beam axis.

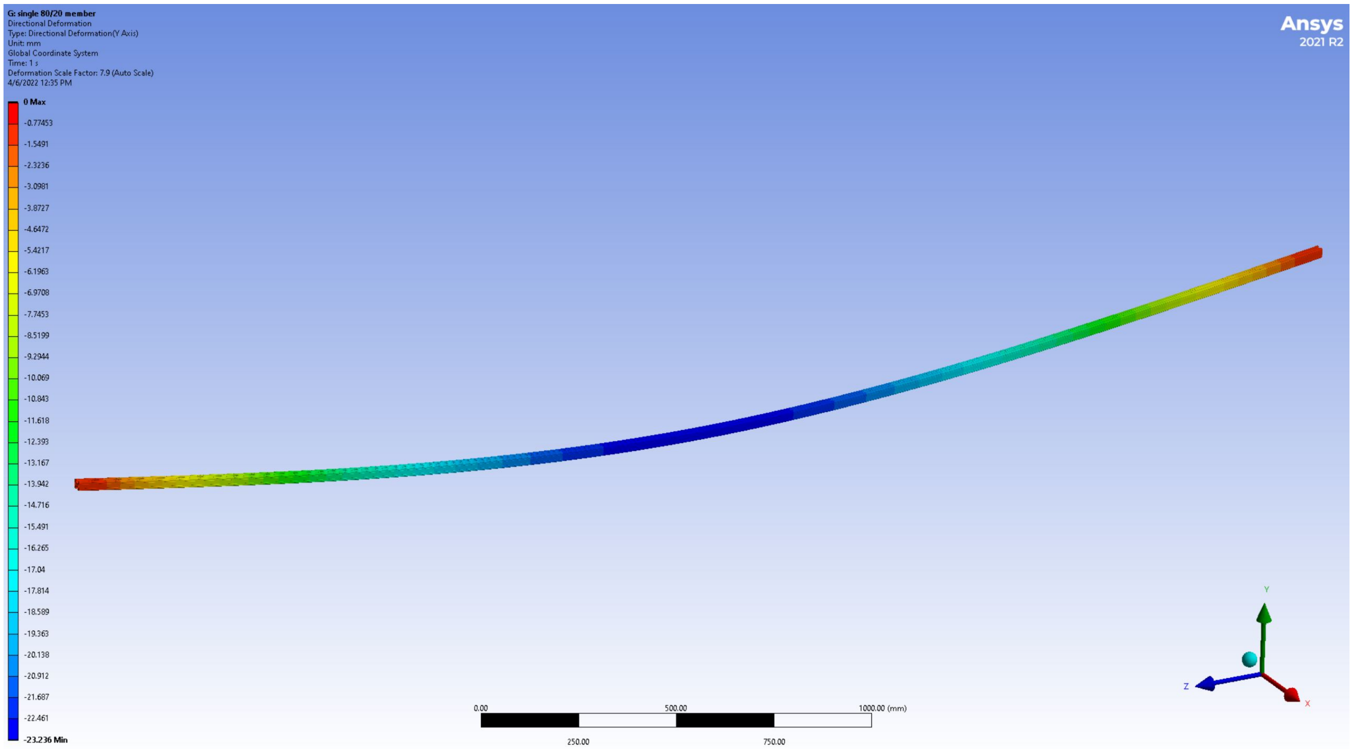

Fig. 8: Deflection of a single 80/20 member under an offset 30N load assuming pinned end connections. The maximum deflection is 23.236mm.

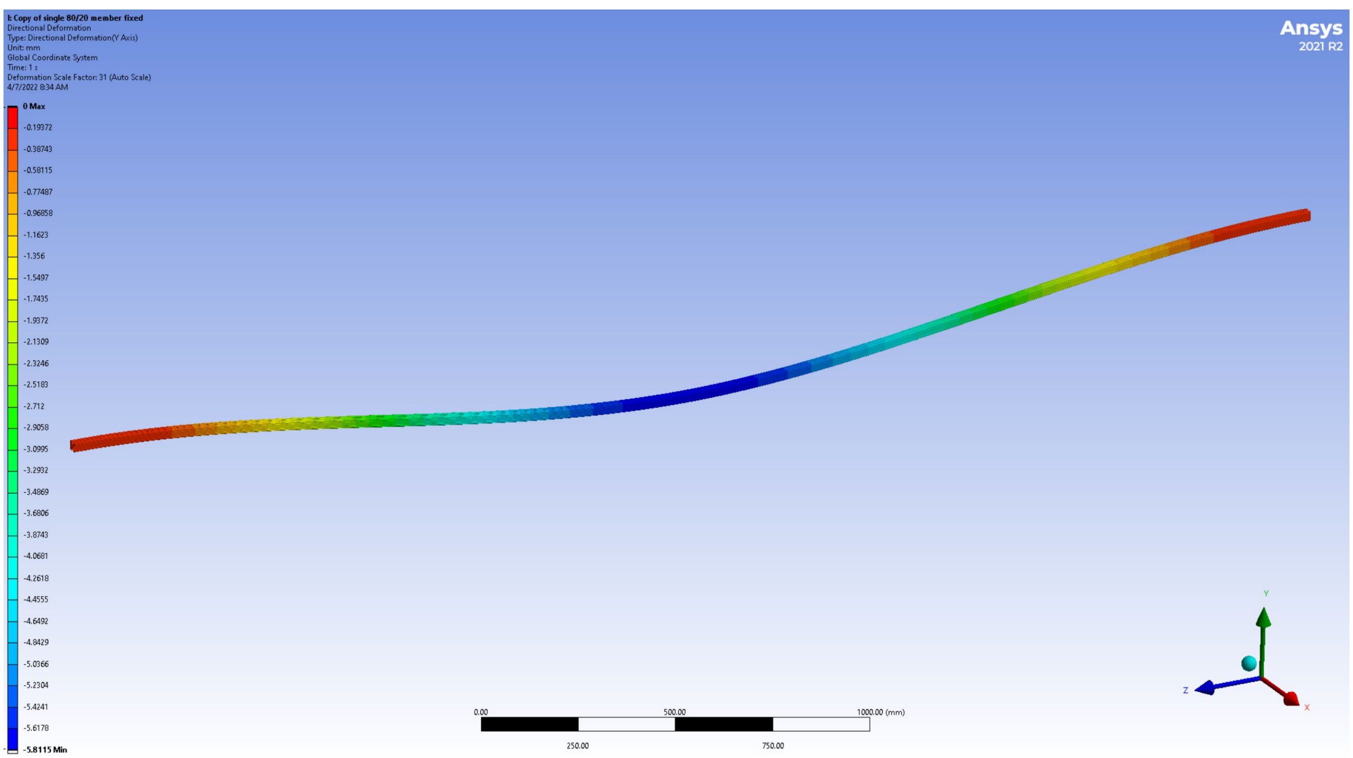

Fig. 9: Deflection of a single 80/20 member under an offset 30N load assuming fixed end connections. The maximum deflection is 5.8115mm.

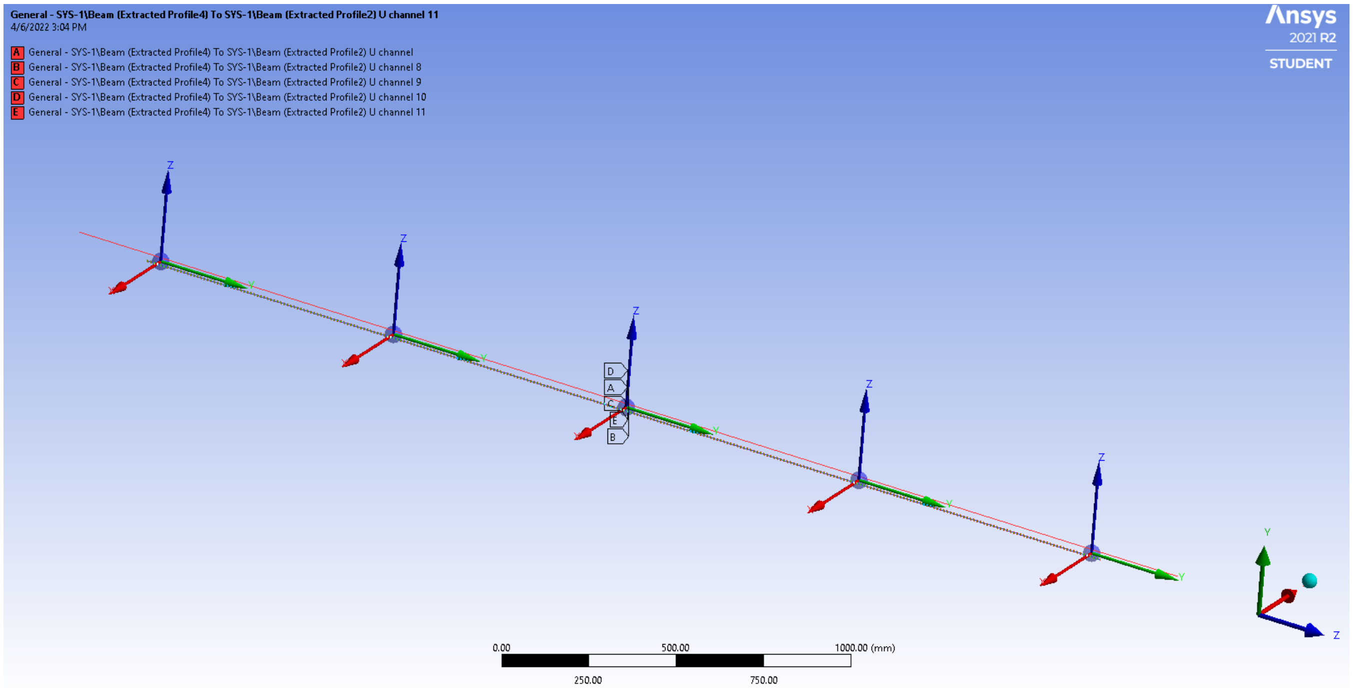

Fig. 10: Joints to attach the U-channel to the 80/20 beam. Five joints are spaced 775mm apart. Joint information can be found in table 6.

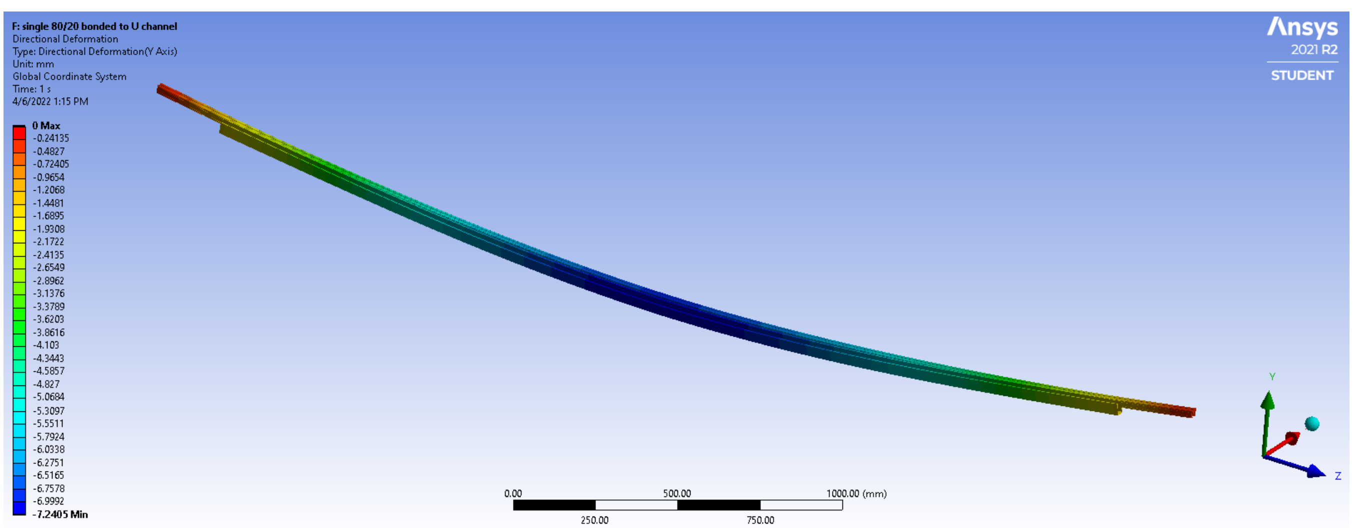

Fig. 11: Deflection of a single 80/20 member with a U-channel attached under an offset 30N load and pinned boundary conditions. The maximum deflection is 7.2405mm.

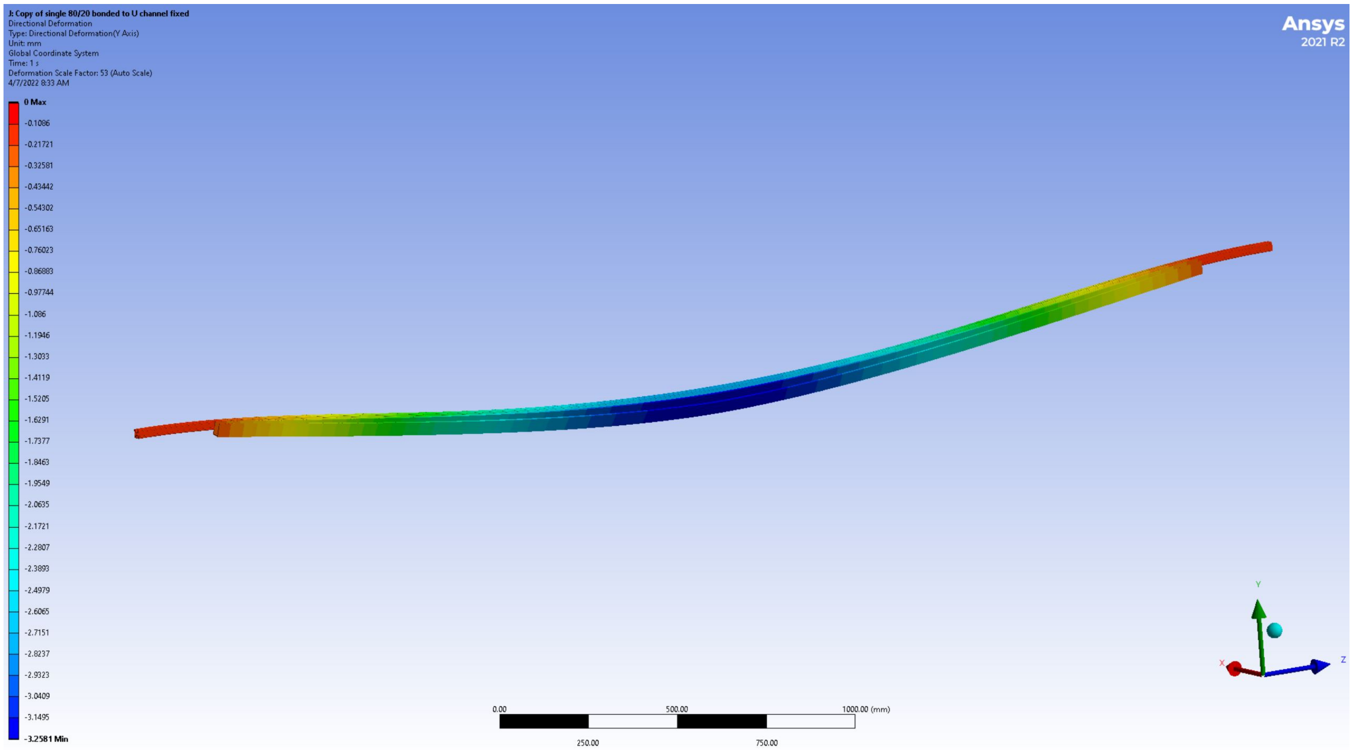

Fig. 12: Deflection of a single 80/20 member with a U-channel attached under an offset 30N load and fixed boundary conditions. The maximum deflection is 3.2581mm.

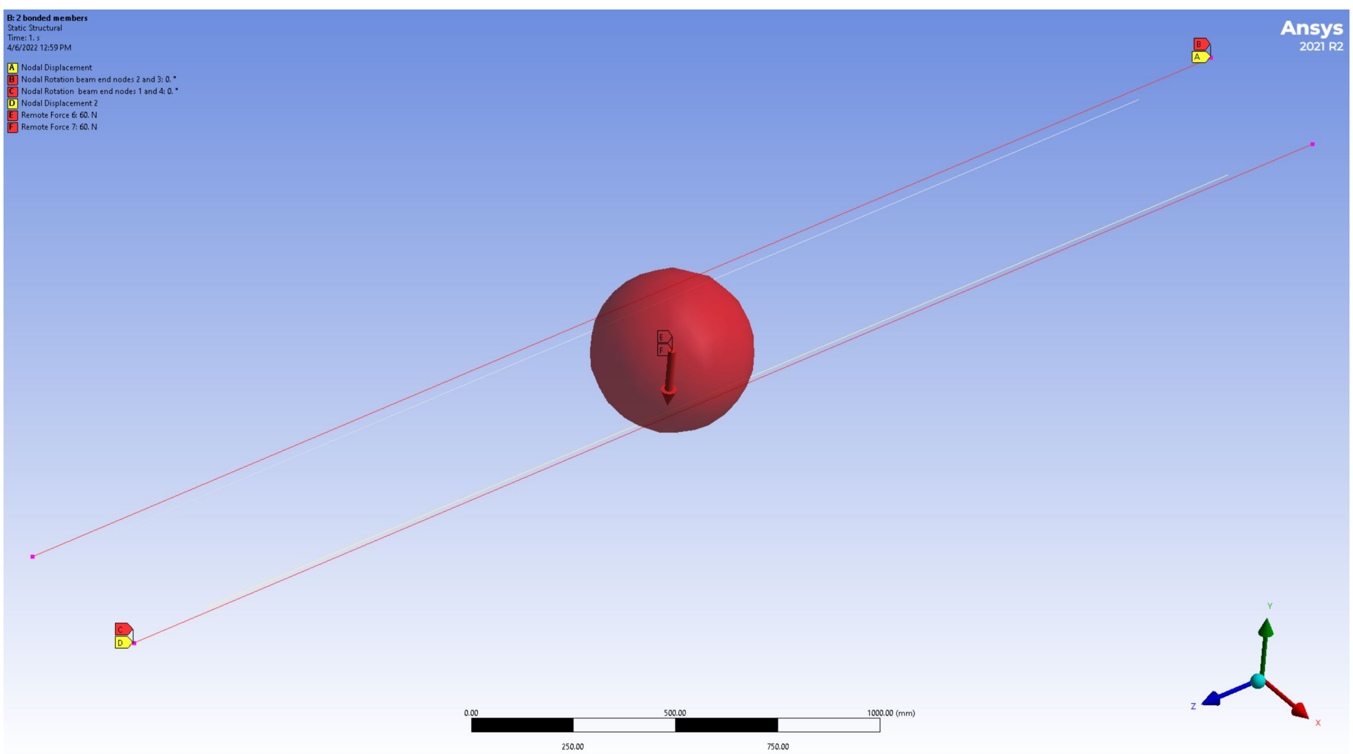

Fig. 13: Load scheme for two beam model. Load is applied in the center of the assembly causing rotation and deflection of the two beams.

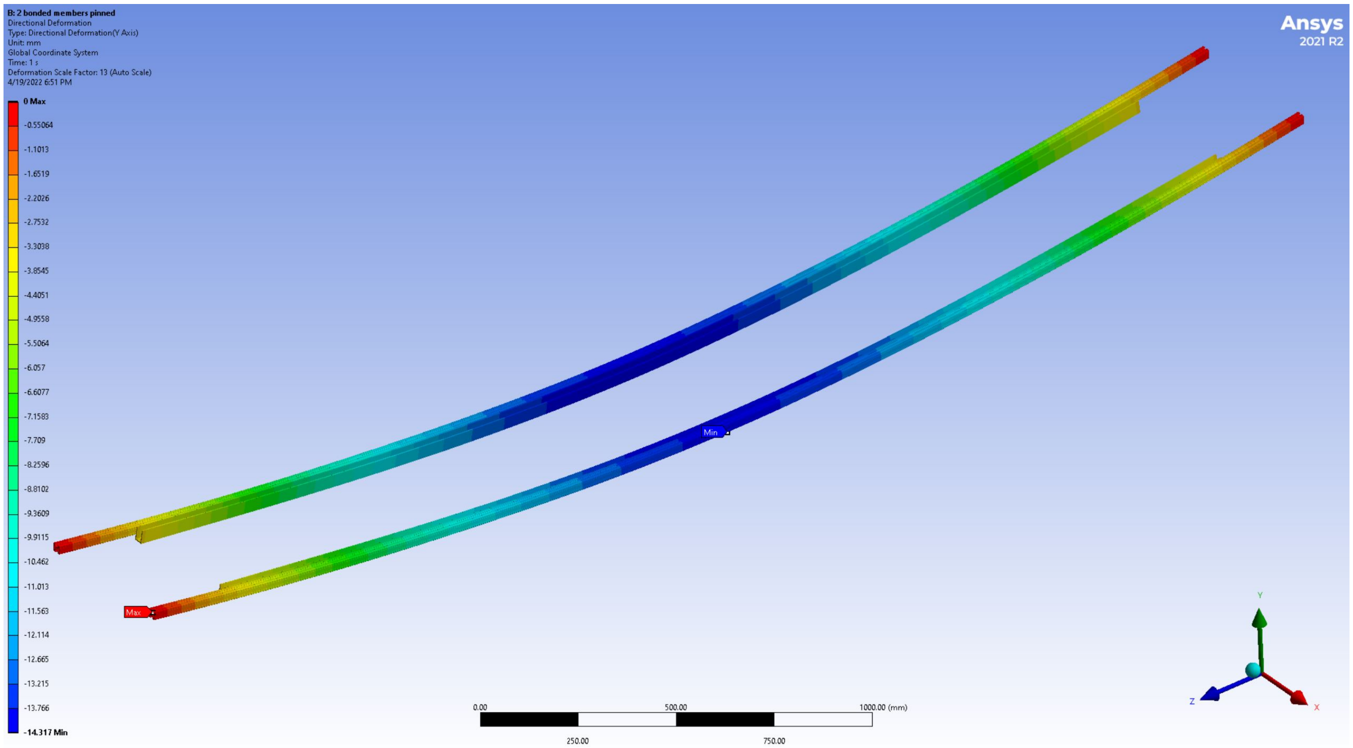

Fig. 14: Deflection of two 80/20 members with a U-channel attached under an offset 120N load with pinned boundary conditions. The maximum deflection is 14.317mm.

8020 pair def fix

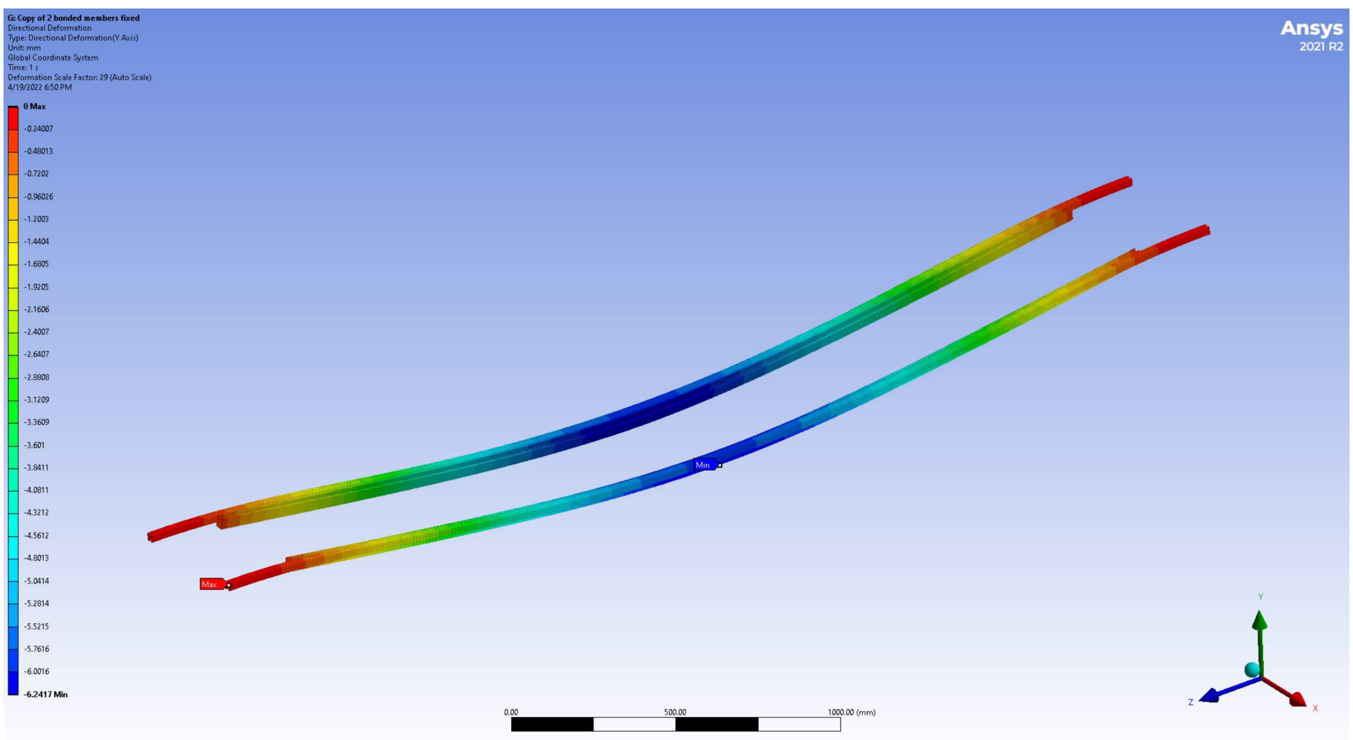

Fig. 15: Deflection of two 80/20 members with a U-channel attached under an offset 120N load with fixed boundary conditions. The maximum deflection is 6.2417mm.

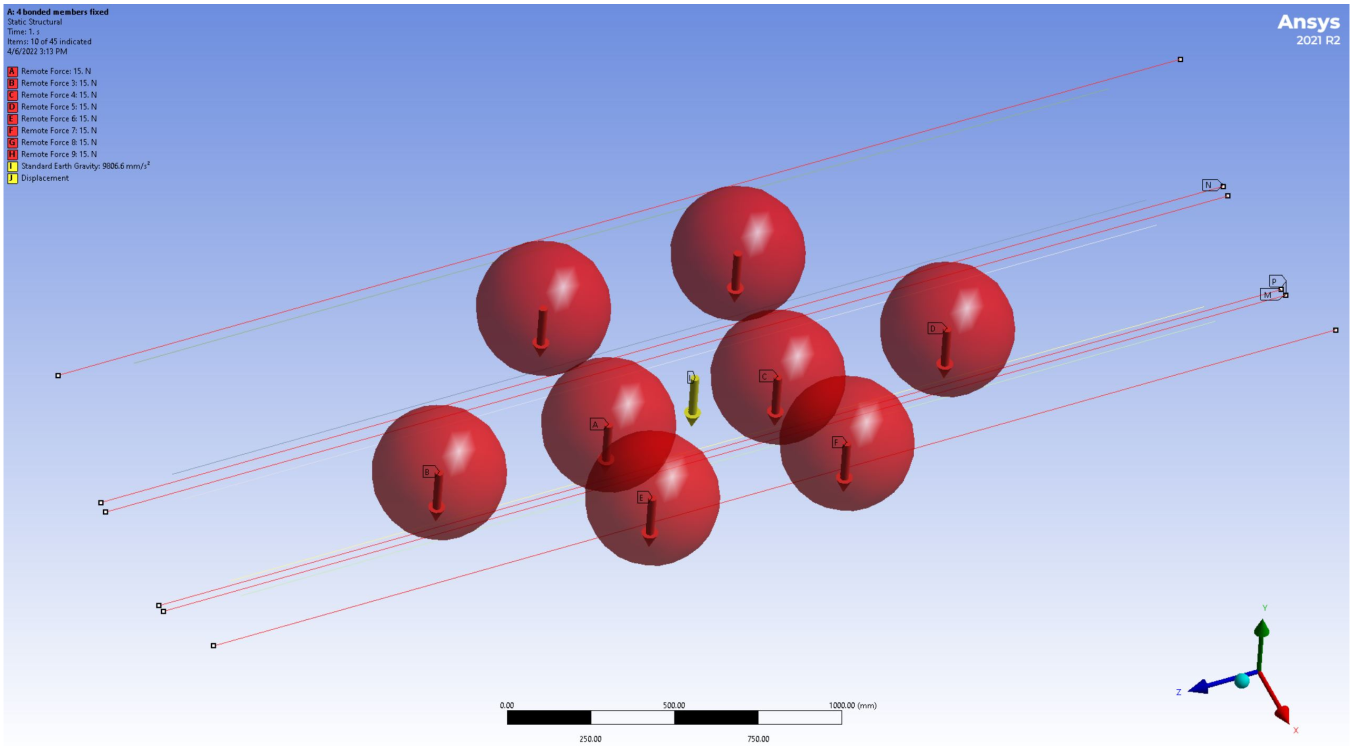

Fig. 16: The load scheme used to simulate the full TPOT support structure. Each load represents a TPOT detector. The TPOT detector weighs about 3kg, therefore a 30N load is suitable. Gravity is included in the full TPOT support structure analysis.

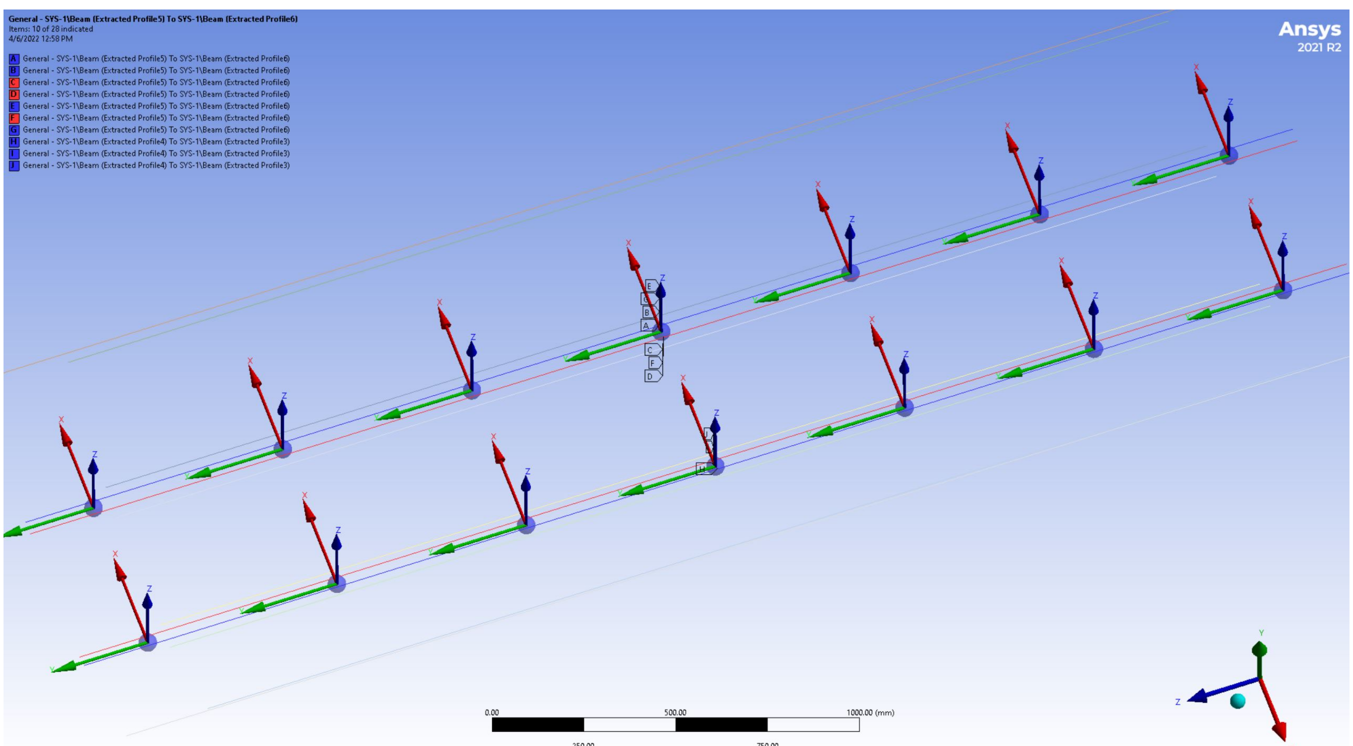

Fig. 17: Joints used to model 150° connectors. See table 10 for joint information.

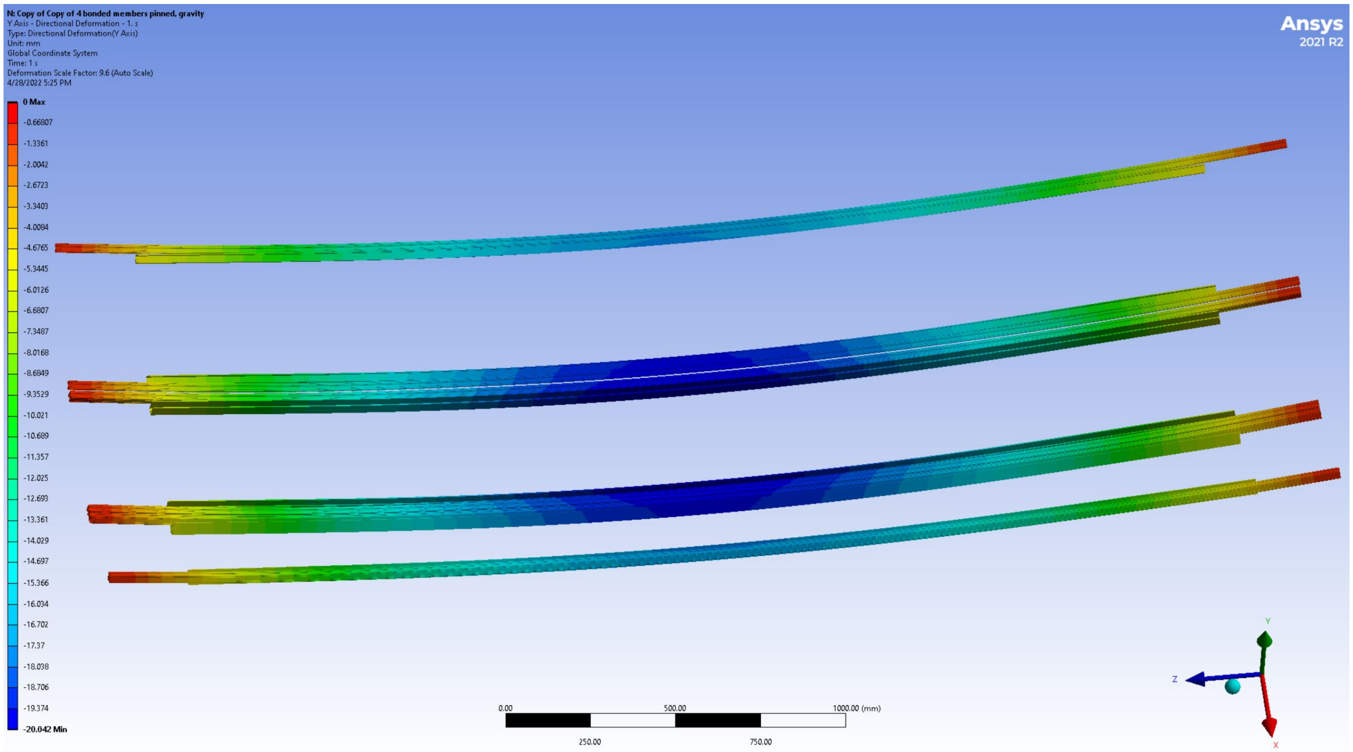

Fig. 18: TPOT support frame deflection assuming pinned end connections. Gravity is included. The maximum deflection is 20.042mm.

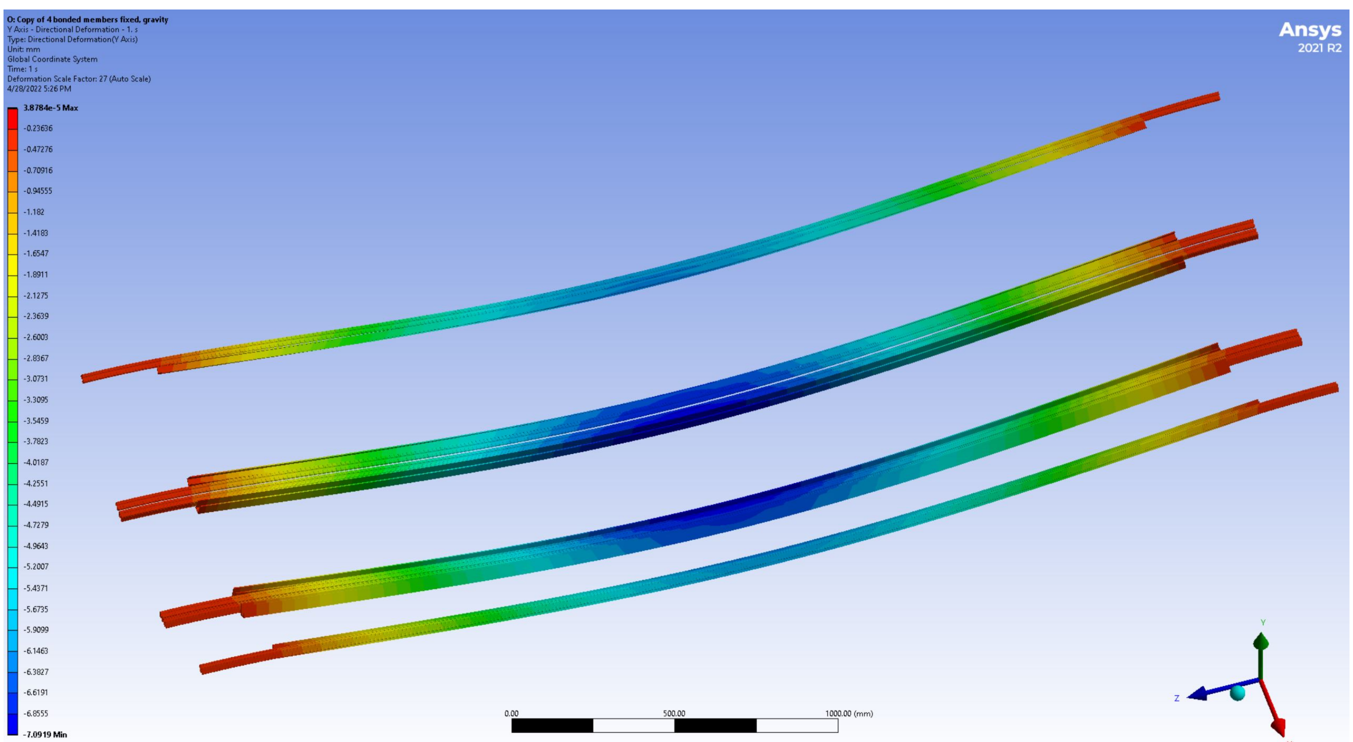

Fig. 19: TPOT support frame deflection assuming fixed end connections. Gravity is included. The maximum deflection is 7.0919mm.

Appendix A: Deflection of beam with two different moment of inertia

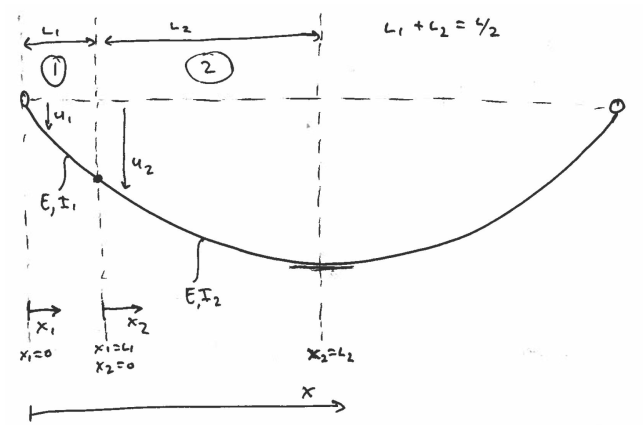

Fig. 1

Figure 1 shows the deflection of a beam with two different moment of inertia through the length of the beam. The maximum deflection of the beam occurs at the center of the beam, $x_2 = L_2$. This section details the solution to the deflection of the beam. The boundary conditions used to solve the problem are given in table 1.

BC	Location	u	$\frac{du}{dx}$
1	$x_1 = 0$	$u_1 = 0$	–
2	$x_1 = L_1, x_2 = 0$	$u_1 = u_2$	$\frac{du_1}{dx_1} = \frac{du_2}{dx_2}$
3	$x_2 = L_2$	–	$\frac{du_2}{dx_2} = 0$

Table 1: Boundary conditions to solve the elastic beam problem with two beam sections.

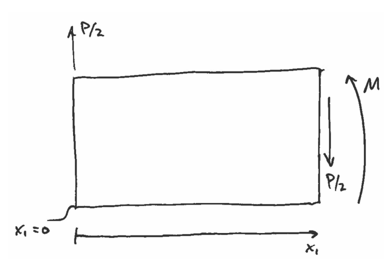

Fig. 2

The free body diagram for the first section is given in figure 2. The internal moment in the beam, M, can be computed by summing the moments about the neutral axis of the beam and setting the sum equal to 0 as shown in equation 1. M is related to the deflection of the beam, u_1, by equation 2. The differential equation in equation 2 can be integrated twice to solve for the deflection of the beam with respect to x_1. This is shown in equations 3 and 4

$$\sum M_{NA} = M - \frac{P}{2}x_1 = 0 \tag{1}$$

$$\frac{d^2u_1}{{dx_1}^2} = \frac{M}{EI_1} = \frac{P}{2EI_1}x_1 \tag{2}$$

$$\frac{du_1}{dx_1} = \frac{P}{4EI_1}x_1^2 + C_1 \tag{3}$$

$$u_1 = \frac{P}{12EI_1}x_1^3 + C_1x_1 + C_2 \tag{4}$$

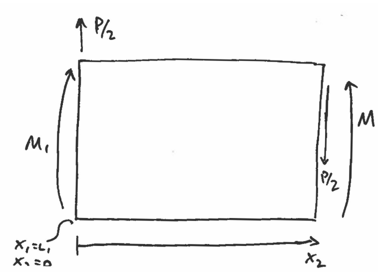

Fig. 3

Similarly, the free body diagram for the second section is given in figure 3. The internal moment in the beam, M, can be computed using equation 5. M is related to the deflection of the beam, u_2, by equation 6. M_1 is the internal moment in the beam at $x_1 = L_1$, which is $\frac{P}{2}x_1$. The differential equation in equation 6 can be integrated twice to solve for the deflection of the beam with respect to x_2. This is shown in equations 7 and 8.

$$\sum M_{NA} = M - \frac{P}{2}x_2 - M_1 = 0 \tag{5}$$

$$\frac{d^2u_2}{{dx_2}^2} = \frac{M}{EI_2} = \frac{P}{2EI_2}(x_2 + L_1) \tag{6}$$

$$\frac{du_2}{dx_2} = \frac{P}{2EI_2}(\frac{1}{2}x_2^2 + L_1x_2) + C_3 \tag{7}$$

$$u_2 = \frac{P}{2EI_2}(\frac{1}{6}x_2^3 + \frac{1}{2}L_1x_2^2) + C_3x_2 + C_4 \tag{8}$$

The boundary conditions are now applied to equations 3, 4, 7, and 8. The results are shown in equations 9-12.

$$C_2 = 0 \tag{9}$$

$$\frac{P}{12EI_1}L_1^3 + C_1L_1 = C_4 \tag{10}$$

$$\frac{P}{4EI_1}L_1^2 + C_1 = C_3 \tag{11}$$

$$\frac{P}{2EI_2}(\frac{1}{2}L_2^2 + L_1L_2) + C_3 = 0 \tag{12}$$

The constants C_1, C_2, C_3, and C_4, can be found and the deflection, u_1 and u_2, can be computed. This solution should reduce to the solution found in most textbooks, equation 13, when the moment of inertia are the same, $I_2 = I_1$. Figure 4 shows the deflection for the special case where the moment of inertia in each beam section is the same, $I_2 = I_1$. u_1 and u_2 in the figure are computed using equations 4 and 8 and the parameters in table 2. The maximum deflection is 23.23mm, which is exactly the solution to equation 13. Equation 13 computes the deflection of a simply supported beam due to a point load at the center of the beam. This equation assumes constant moment of inertia throughout the beam and can be found in most engineering mechanics textbooks.

$$\delta = \frac{PL^3}{48EI} \tag{13}$$

P	30 N	Applied Load
L	3.6576 m	Total length of beam
E	71000 MPa	Young's Modulus
I	18540 mm^4	Area Moment of Inertia
T	P * 203.2 mm	Applied torque to beam
L_θ	L/2	Length to compute rotation
J	9283.5 mm^4	Polar moment of inertia
G	26692 MPa	Shear modulus

Table 2: List of properties of a single 80/20 beam member used in equation 13.

Figure 5 was generated using equations 4 and 8 and the parameters in table 3. The maximum deflection is 5.6mm.

P	30 N	Applied Load
L	3.6576 m	Total length of beam
E	71000 MPa	Young's Modulus
I_1	18540 mm^4	Region 1 Area Moment of Inertia
I_2	77476 mm^4	Region 2 Area Moment of Inertia
T	P * 203.2 mm	Applied torque to beam
L_1	241.3 mm	Region 1 Length
L_2	1587.5 mm	Region 2 Length
J_1	9283.5 mm^4	Region 1 Polar moment of inertia
J_2	14866 mm^4	Region 2 Polar moment of inertia
G	26692 MPa	Shear modulus

Table 3: List of properties of a single 80/20 beam member attached to a U-channel used in equations 4 and 8.

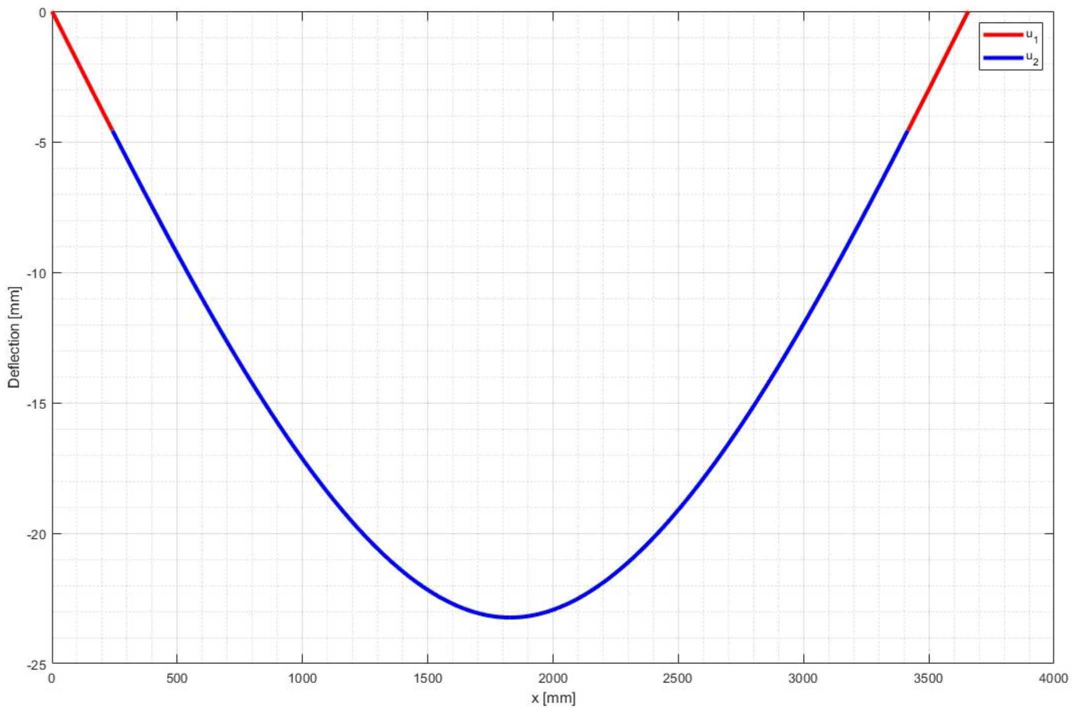

Fig. 4: Deflection computed using equations 4 and 8 for the case, $I_2 = I_1$. Properties in table 2 were used to produce the elastic curve. The maximum deflection is 23.23mm. Equation 13 produces the same answer using the properties in table 2.

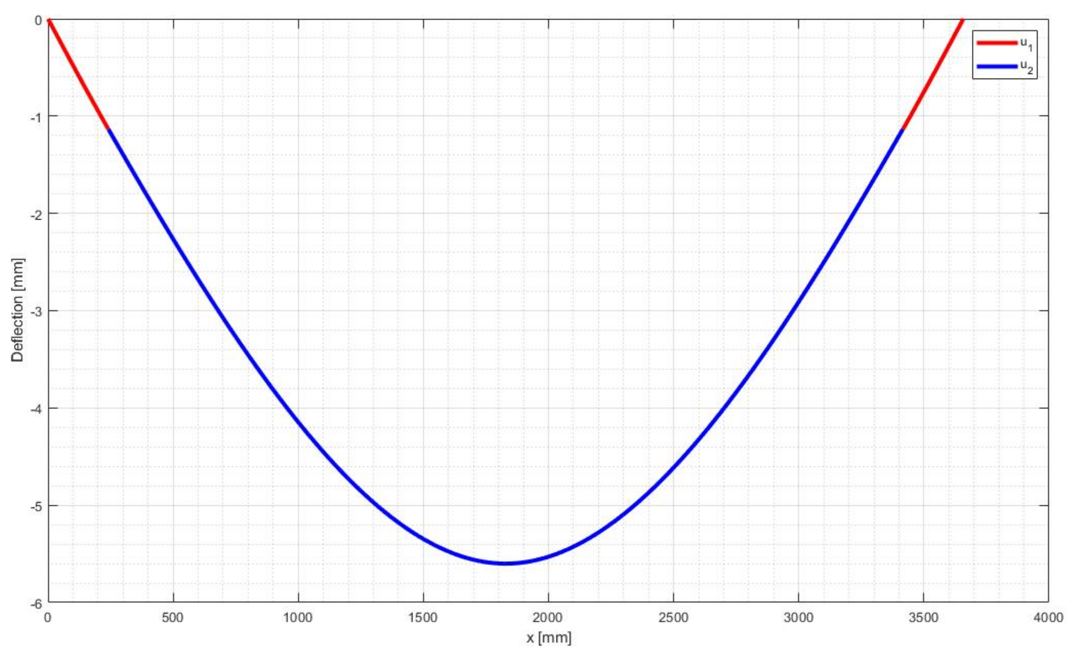

Fig. 5: Deflection computed using equations 4 and 8 for the case, $I_2 \neq I_1$. Properties in table 3 were used to produce the elastic curve. The maximum deflection is 5.6mm.