ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 43, nº 1 (1985), p. 107-132
<http://www.numdam.org/item?id=AIHPA_1985__43_1_107_0>

Ann. Inst. Henri Poincaré,
Vol. 43, n° 1, 1985, p.107-132. *Physique théorique*

General relativistic celestial mechanics of binary systems I. The post-Newtonian motion

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ABSTRACT. — We present a new method for solving explicitly the equations of motion of a binary system at the first post-Newtonian approximation of General Relativity. We show how to express the solution in a simple, quasi-Newtonian form. The results are compared and constrasted with other results existing in the literature.

RÉSUMÉ. — Nous présentons une nouvelle méthode de résolution explicite des équations du mouvement d'un système binaire à la première approximation post-newtonienne de la Relativité Générale. Nous montrons comment exprimer la solution sous une forme particulièrement simple, quasi-newtonienne. Les résultats sont confrontés à ceux précédemment obtenus dans la littérature.

1. INTRODUCTION

The non-relativistic two-body problem consists in two sub-problems: 1) deriving the equations of orbital motion for two gravitationally inter-

acting extended bodies and 2) solving these equations of motion. In the case of widely separated objects one can simplify the sub-problem 1) by neglecting the contribution of the quadrupole and higher multipole moments of the bodies to their external gravitational field, thereby approximating the equations of orbital motion of two extended bodies by the equations of motion of two point masses (located at the Newtonian centres of mass of the extended objects). Then the sub-problem 2) can be exactly solved (cf. Appendix A).

The two-body problem in General Relativity is more complicated; because of the non-linear hyperbolic structure of Einstein's field equations one is not sure of the « good » boundary conditions at infinity so that the problem is not even well posed (see e. g. [*18*]). Moreover since in Einstein's theory the local equations of motion are contained in the gravitational field equations, it is *a priori* difficult to separate the problem in two sub-problems as in the non relativistic case where one can compute the gravitational field as a linear functional of the matter distribution independently of its motion. Furthermore, even when one can (approximately) achieve such a separation and derive some equations of orbital motion for the two bodies, these equations will *a priori* not be ordinary differential equations, but, because of the finite velocity of propagation of gravity, will consist in some kind of retarded-integro-differential system. However all these difficulties can be somehow dealt with if one resorts to approximation procedures and breaks the general covariance by selecting special classes of coordinate systems (for some exact results on the « laws of motion » rather than what we call here « equations of motion » see [*16*]).

Two physically different situations amenable to perturbation treatments have been considered in the literature. The first one is the problem of two *weakly self-gravitating, slowly moving, widely separated* fluid bodies which has been treated by the so-called post-Newtonian approximation schemes (for references to the abundant relevant literature, see e. g. [*30*] [*7*] [*5*] and [*8*]). The second case is the problem of two *strongly self-gravitating, widely separated* bodies which has been treated by matching a (strong field) « internal » approximation scheme (in and near the objects) to a (weak field) « external » approximation scheme (outside the objects). This has been done both for slowly moving objects, either black holes ([*15*]) or in general strongly self-gravitating objects ([*23*]), and for strongly self-gravitating objects moving with arbitrary velocities ([*3*] [*8*]). In the latter case one obtained equations of orbital motion in the form of a retarded-integro-differential system which could however be transformed into ordinary differential equations and which, when attention was restricted to slowly moving bodies, were expanded in power series of v/c ([*11*] [*8*]). When keeping only the first relativistic corrections to Newton's law (first post-Newtonian approximation), it turns out that the equations of orbital

motion of widely separated, slowly moving, *strongly* self-gravitating objects depend only on two parameters (the « Schwarzschild masses ») and are identical to the equations of motion of weakly self-gravitating objects (when using in both cases a coordinate system which is « harmonic » at lowest order). This result is in fact a non trivial consequence of the structure of Einstein's theory (« effacing principle »: see e. g. [*8*]) and does not seem to be valid in most other theories of gravity ([*17*] [*35*] [*34*]).

The sub-problem 1) having been thus attacked, and in fact solved at the post-Newtonian level (as early as 1917 in the case of weakly self-gravitating incompressible fluid balls (see [*26*])) it would seem that the sub-problem 2) (*solving* the post-Newtonian equations of motion) would be thoroughly understood and already fully treated in the literature. This does not seem to be the case. Indeed most treatments work out only the secular effects of the motion: the acceleration of the center of mass and the precession of the periastron ([*28*] [*22*] [*25*] [*6*] [*1*] [*31*] [*27*]). The reason for deriving only the *secular* effects caused by relativistic corrections was that the precision of the observations of binary systems (and even of our planetary system) was, for a long time, such that there was no hope to detect the (quasi-)*periodic* relativistic effects. However the recent discovery of binary pulsars and especially the extremely precise tracking of the orbital motion of the Hulse-Taylor pulsar PSR 1913 + 16 (see e. g. [*32*]) have made it necessary to work out explicitly all the post-Newtonian effects (both secular and periodic) in the motion. This has been done ([*33*] [*19*] [*20*]) but the results have been expressed only in a quite unwieldy form. The purpose of this article is then to present a method for solving explicitly the post-Newtonian equations of motion which is simple and systematic. Indeed we shall show that the post-Newtonian motion (including secular and periodic effects) can be written down in a *quasi-Newtonian* form (see § 7 below). In a sequel paper we shall apply our results to the astrophysical problem of the « timing of binary pulsars »; our simple « quasi-Newtonian » solution will allow us to derive a correspondingly simple formula giving the arrival times on Earth of radio-pulses emitted by a pulsar in a binary system – like PSR 1913 + 16 – (see [*13*]).

Let us stress that one considers here only the *first* post-Newtonian periodic corrections to the motion. This is justified because the next order relativistic corrections yield negligible periodic effects: indeed the present precision in the measurement of the arrival times of the radio pulses from the Hulse-Taylor binary pulsar is of the order of *20* μsec which is of the same order of magnitude as the periodic post-Newtonian corrections ($\sim Gm_{\text{pulsar}}/c^2 \sim 7$ μsec). As for the *second* post-Newtonian *periodic* corrections they are of order $(v/c)^2$. $Gm_{\text{pulsar}}/c^2 \sim 10^{-5}$ μsec and therefore completely unobservable. However when dealing with *secular* effects

one must consider also higher order approximations, as done recently for the secular acceleration of the mean orbital motion caused by terms coming from the second and a half post-Newtonian approximation (see [*9*] [*10*]).

2. THE POST-NEWTONIAN CENTER OF MASS AND THE LAGRANGIAN FOR THE RELATIVE MOTION

The (first) post-Newtonian equations of orbital motion of a binary system constrain the evolution in (coordinate) time t of the « positions » $\boldsymbol{r}$ and $\boldsymbol{r}'$ of the two objects—these « positions » represent the « centers of mass » in the case of weakly self-gravitating objects (see e. g. [*30*]) and the « centers of field » in the case of strongly self-gravitating objects (see [*8*]). They can be derived from a Lagrangian which is a function of the positions $\boldsymbol{r}(t)$, $\boldsymbol{r}'(t)$, and velocities $\boldsymbol{v}(t) := d\boldsymbol{r}/dt$, $\boldsymbol{v}'(t) := d\boldsymbol{r}'/dt$ *simultaneous* in a given harmonic coordinate system, and of two constant parameters the (« Schwarzschild ») masses of the objects m, m':

$$\mathrm{L_{PN}}(\boldsymbol{r}(t), \boldsymbol{r}'(t), \boldsymbol{v}(t), \boldsymbol{v}'(t)) = \mathrm{L_N} + \frac{1}{c^2}\mathrm{L_2} \tag{2.1 a}$$

with

$$\mathrm{L_N} = \frac{1}{2}mv^2 + \frac{1}{2}m'v'^2 + \frac{\mathrm{G}mm'}{\mathrm{R}} \tag{2.1 b}$$

$$\mathrm{L_2} = \frac{1}{8}mv^4 + \frac{1}{8}m'v'^4 + \frac{\mathrm{G}mm'}{2\mathrm{R}}\left[3v^2 + 3v'^2 - 7(vv') - (\mathrm{N}v)(\mathrm{N}v') - \mathrm{G}\frac{m+m'}{\mathrm{R}}\right] \tag{2.1 c}$$

where we have introduced the (instantaneous) relative position vector $\mathbf{R} := \boldsymbol{r} - \boldsymbol{r}'$ and $\mathrm{R} := |\mathbf{R}|$, $\mathbf{N} := \mathbf{R}/\mathrm{R}$; where we have used the abbreviated notations: $\boldsymbol{v}\,.\,\boldsymbol{v} = |\boldsymbol{v}|^2 = v^2$, $\boldsymbol{v}\,.\,\boldsymbol{v}' = (vv')$ for the ordinary euclidean scalar products, and where G is Newton's constant and c the velocity of light.

The invariance, at the post-Newtonian approximation, and modulo an exact time derivative, of $\mathrm{L_{PN}}$ under spatial translations and Lorentz boosts implies, via Noether's theorem, the conservation of the total linear momentum of the system:

$$\mathbf{P}_{\mathrm{PN}} = \frac{\partial \mathrm{L_{PN}}}{\partial \boldsymbol{v}} + \frac{\partial \mathrm{L_{PN}}}{\partial \boldsymbol{v}'} \tag{2.2}$$

and of the relativistic center of mass integral

$$\mathbf{K}_{\mathrm{PN}} = \mathbf{G}_{\mathrm{PN}} - t\mathbf{P}_{\mathrm{PN}} \tag{2.3 a}$$

$$\mathbf{G}_{\mathrm{PN}} = \Sigma\left(m + \frac{1}{2}\frac{mv^2}{c^2} - \frac{1}{2}\frac{\mathrm{G}mm'}{\mathrm{R}c^2}\right)\boldsymbol{r} \tag{2.3 b}$$

Σ denoting a sum over the two objects, (see e. g. [*11*] and [*8*] for a direct proof of the link between the conservation of $\mathbf{K}_{PN}$ and the Lorentz boosts).

By a Poincaré transformation it is possible to go to a post-Newtonian center of mass frame where $\mathbf{P}_{PN} = \mathbf{K}_{PN} = 0$. In this frame one has:

$$\boldsymbol{r} = \frac{\mu}{m}\mathbf{R} + \frac{\mu(m - m')}{2M^2c^2}\left(V^2 - \frac{GM}{R}\right)\mathbf{R} \tag{2.4 a}$$

$$\boldsymbol{r}' = -\frac{\mu}{m'}\mathbf{R} + \frac{\mu(m - m')}{2M^2c^2}\left(V^2 - \frac{GM}{R}\right)\mathbf{R} \tag{2.4 b}$$

where $\mathbf{V} := d\mathbf{R}/dt = \boldsymbol{v} - \boldsymbol{v}'$ is the instantaneous relative velocity, $M := m + m'$ the total mass and $\mu := mm'/M$ the reduced mass. The problem of solving the motion of the binary system is then reduced to the simpler problem of solving the *relative motion* in the post-Newtonian center of mass frame. For the sake of completeness, let us write down these equations of motion, derived from (2.1), and where, after variation, the positions and velocities are replaced by their centre of mass expressions (2.4):

$$\frac{d\mathbf{V}}{dt} = -\frac{GM}{R^2}\mathbf{N} + \frac{GM}{c^2R^2}\left\{\mathbf{N}\left[\frac{GM}{R}(4 + 2\nu) - V^2(1 + 3\nu) + \frac{3}{2}\nu(\mathbf{NV})^2\right] + (4 - 2\nu)\mathbf{V}(\mathbf{NV})\right\} \tag{2.5}$$

where we have introduced the notation $\nu := \mu/M = mm'/(m + m')^2$ $(0 \leqslant \nu \leqslant 1/4)$.

At this point it is worth noticing that in spite of the fact that it is in general incorrect to use, *before variation*, in a Lagrangian a consequence, like eq. (2.4), of the equations of motion, which are obtained only *after variation*, it turns out that the relative motion in the post-Newtonian center of mass frame, eq. (2.5), can be correctly derived from a « relative Lagrangian » obtained by replacing in the total Lagrangian (divided by μ) $\mu^{-1}L_{PN}(\boldsymbol{r}, \boldsymbol{r}', \boldsymbol{v}, \boldsymbol{v}')$ the positions and velocities by their post-Newtonian center of mass expressions obtained from (2.4) and that moreover it is surprisingly even sufficient to use the *non-relativistic* center of mass expressions:

$$\boldsymbol{r}_N = \frac{\mu}{m}\mathbf{R} \tag{2.6 a}$$

$$\boldsymbol{r}'_N = -\frac{\mu}{m'}\mathbf{R} \tag{2.6 b}$$

$$\boldsymbol{v}_N = \frac{\mu}{m}\mathbf{V} \tag{2.6 c}$$

$$\boldsymbol{v}'_N = -\frac{\mu}{m'}\mathbf{V} \tag{2.6 d}$$

The proof goes as follows. Let us introduce the following linear change of spatial variables in the post-Newtonian Lagrangian $L_{PN}(\boldsymbol{r}-\boldsymbol{r}', d\boldsymbol{r}/dt, d\boldsymbol{r}'/dt)$: $(\boldsymbol{r}, \boldsymbol{r}') \to (\mathbf{R}, \mathbf{X})$ with $\mathbf{R} := \boldsymbol{r} - \boldsymbol{r}'$ and $\mathbf{X} := (m\boldsymbol{r} + m'\boldsymbol{r}')/M$, that is:

$$\boldsymbol{r} = \boldsymbol{r}_N + \mathbf{X} \tag{2.7 a}$$

$$\boldsymbol{r}' = \boldsymbol{r}'_N + \mathbf{X} \tag{2.7 b}$$

which implies (denoting $d\mathbf{X}/dt =: \mathbf{W}$):

$$\boldsymbol{v} = \boldsymbol{v}_N + \mathbf{W} \tag{2.7 c}$$

$$\boldsymbol{v}' = \boldsymbol{v}'_N + \mathbf{W} \tag{2.7 d}$$

Expressing $L_{PN} = L_N(\boldsymbol{r} - \boldsymbol{r}', \boldsymbol{v}, \boldsymbol{v}') + (1/c^2)L_2(\boldsymbol{r} - \boldsymbol{r}', \boldsymbol{v}, \boldsymbol{v}')$, given by eq. (2.1) in terms of the new variables one finds:

$$L_{PN} = \frac{1}{2}MW^2 + \frac{1}{2}\mu V^2 + \frac{G\mu M}{R} + \frac{1}{c^2}L_2\left(\mathbf{R}, \frac{\mu \mathbf{V}}{m} + \mathbf{W}, -\frac{\mu}{m'}\mathbf{V} + \mathbf{W}\right) \tag{2.8}$$

Hence one obtains as a consequence of the equations of the post-Newtonian motion:

$$\begin{aligned} O &= \frac{1}{\mu}\frac{\delta L_{PN}}{\delta \mathbf{R}} \\ &= \left(\frac{\partial}{\partial \mathbf{R}} - \frac{d}{dt}\frac{\partial}{\partial \mathbf{V}}\right)\left[\frac{1}{2}V^2 + \frac{GM}{R} + \frac{1}{\mu c^2}L_2\left(\mathbf{R}, \frac{\mu \mathbf{V}}{m} + \mathbf{W}, -\frac{\mu}{m'}\mathbf{V} + \mathbf{W}\right)\right] \end{aligned} \tag{2.9}$$

where in the last bracket we have discarded $1/2\, MW^2$ which gives no contribution. The first two terms in the RHS of eq. (2.9) yield the Newtonian relative motion. We wish to evaluate the relativistic corrections to the relative motion: $(\delta/\delta\mathbf{R})(L_2/\mu c^2)$ in the post-Newtonian centre of mass system. Now L_2 is a polynomial in the velocities and therefore a polynomial in $\mathbf{W}$, and from eq. (2.4) one sees that in the post-Newtonian centre of mass frame $\mathbf{W} = O(1/c^2)$. Therefore as $\delta/\delta\mathbf{R}$ does not act on $\mathbf{W}$, we see that the contributions coming from $\mathbf{W}$ to the RHS of eq. (2.9) are of the second post-Newtonian order $O(1/c^4)$ that we shall consistently neglect throughout this work. In other words one obtains as a consequence of the equations of the post-Newtonian motion in the post-Newtonian centre of mass system:

$$\frac{\delta}{\delta \mathbf{R}}\left[\frac{1}{2}V^2 + \frac{GM}{R} + \frac{1}{\mu c^2}L_2\left(\mathbf{R}, \frac{\mu \mathbf{V}}{m}, -\frac{\mu \mathbf{V}}{m'}\right)\right] = O(1/c^4) \tag{2.10}$$

This shows that the equations of the relative motion in the post-Newtonian centre of mass frame derive from the following « relative Lagrangian »:

$$L^R_{PN}(\mathbf{R}, \mathbf{V}) = \frac{1}{2}V^2 + \frac{GM}{R} + \frac{1}{\mu c^2}L_2\left(\mathbf{R}, \frac{\mu \mathbf{V}}{m}, -\frac{\mu \mathbf{V}}{m'}\right) \tag{2.11}$$

which happens to be obtainable by replacing in the full post-Newtonian Lagrangian, see eq. (2.8) above, **X** and **W** by zero, i. e. the original variables by eq. (2.6) (and by dividing by μ).

The explicit expression of L^R_{PN} reads:

$$L^R_{PN}(\mathbf{R}, \mathbf{V}) = \frac{1}{2}V^2 + \frac{GM}{R} + \frac{1}{8}(1 - 3\nu)\frac{V^4}{c^2} + \frac{GM}{2Rc^2}\left[(3 + \nu)V^2 + \nu(NV)^2 - \frac{GM}{R}\right] \quad (2.12)$$

The Lagrangian (2.12) was obtained by Infeld and Plebanski [*22*] although without a clear justification (see appendix C).

The integration of the equations of motion (2.5) can be done in several different ways. A standard approach: Lagrange's method of variation of the osculating elements, is discussed in Appendix B. The Hamilton-Jacobi equation approach which takes advantage of the existence of the post-Newtonian relative Lagrangian is the route which has been taken by Landau and Lifshitz [*25*] who worked out only the secular precession of the periastron. Another approach, based on the Maupertuis principle, which reduces the post-Newtonian problem to a simple auxiliary Newtonian problem is presented in Appendix C. However, in order to fully describe the motion, it is most convenient to use a more straightforward approach, which follows closely one of the standard methods for solving the non-relativistic two-body problem (see Appendix A) and which consists in exploiting the symmetries of the relative Lagrangian L^R_{PN}. The invariance of L^R_{PN} under time translations and space rotations implies the existence of four first integrals: $E = \mathbf{V} . \partial L^R_{PN}/\partial \mathbf{V} - L^R_{PN}$ and $\mathbf{J} = \mathbf{R} \times \partial L^R_{PN}/\partial \mathbf{V}$:

$$E = \frac{1}{2}V^2 - \frac{GM}{R} + \frac{3}{8}(1-3\nu)\frac{V^4}{c^2} + \frac{GM}{2Rc^2}\left[(3+\nu)V^2 + \nu(NV)^2 + \frac{GM}{R}\right] \quad (2.13)$$

$$\mathbf{J} = \mathbf{R} \times \mathbf{V}\left[1 + \frac{1}{2}(1-3\nu)\frac{V^2}{c^2} + (3+\nu)\frac{GM}{Rc^2}\right] \quad (2.14)$$

It is straightforwardly checked that these quantities coincide respectively with μ^{-1} times the total Noetherian energy and the total Noetherian angular momentum of the binary system when computed in the post-Newtonian center of mass frame (see e. g. [*33*]).

Eq. (2.14) implies that the motion takes place in a (coordinate) plane, therefore one can introduce polar coordinates R, θ in that plane (i. e. there exists a spatial rotation after which one can write $R_x = R \cos\theta$, $R_y = R \sin\theta$, $R_z = 0$). Then starting from the first integrals (2.13)-(2.14) and using the identities: $V^2 = (dR/dt)^2 + R^2(d\theta/dt)^2$, $|\mathbf{R} \times \mathbf{V}| = R^2 d\theta/dt$, $(NV) = dR/dt$,

we obtain by iteration (in these and the following equations we consistently neglect terms of the second post-Newtonian order $O(1/c^4)$):

$$\left(\frac{dR}{dt}\right)^2 = A + 2B/R + C/R^2 + D/R^3 \tag{2.15}$$

$$\frac{d\theta}{dt} = H/R^2 + I/R^3 \tag{2.16}$$

where the coefficients A, B, C, D, H, I are the following polynomials in E and $J := |\mathbf{J}|$:

$$A = 2E\left(1 + \frac{3}{2}(3\nu - 1)\frac{E}{c^2}\right) \tag{2.17 a}$$

$$B = GM\left(1 + (7\nu - 6)\frac{E}{c^2}\right) \tag{2.17 b}$$

$$C = -J^2\left(1 + 2(3\nu - 1)\frac{E}{c^2}\right) + (5\nu - 10)\frac{G^2M^2}{c^2} \tag{2.17 c}$$

$$D = (-3\nu + 8)GMJ^2/c^2 \tag{2.17 d}$$

$$H = J\left(1 + (3\nu - 1)\frac{E}{c^2}\right) \tag{2.17 e}$$

$$I = (2\nu - 4)GMJ/c^2 \tag{2.17 f}$$

3. THE POST-NEWTONIAN RADIAL MOTION

The relativistic relative radial motion, i. e. the solution of eq. (2.15) can be very simply reduced to the integration of an auxiliary *non-relativistic* radial motion (i. e. to eq. A.10). Indeed let us consider the following change of radial variable:

$$R = \overline{R} + D/2C_0 \tag{3.1}$$

where C_0 is the limit of C when $c^{-1} \to 0$ ($C_0 = -J^2$). In ordinary geometry the transformation which is expressed in polar coordinates by the equations: $R' = R + \text{Const.}$, $\theta' = \theta$, is called a « conchoidal transformation ». We shall use systematically in the following such « conchoidal » transformations. Taking into account the fact that D is $O(1/c^2)$ and that we can consistently neglect all terms of order $O(1/c^4)$ we find that replacing eq. (3.1) in eq. (2.15) leads to:

$$\left(\frac{d\overline{R}}{dt}\right)^2 = A + \frac{2B}{\overline{R}} + \frac{\overline{C}}{\overline{R}^2} \tag{3.2 a}$$

with

$$\overline{C} = C - BD/C_0 \tag{3.2 b}$$

The solution of eq. (3.2) in parametric form is recalled in Appendix A. In the case of quasi-elliptic motion ($E < 0$; $A < 0$) (the quasi-hyperbolic and quasi-parabolic cases will be dealt with below), $\bar{R}$ is a linear function of $\cos u$, u being an « eccentric anomaly » and the same is true of $R = \bar{R} + D/2C_0$. We then obtain the post-Newtonian radial motion in quasi-newtonian parametric form (t_0 being a constant of integration):

$$n(t - t_0) = u - e_t \sin u \tag{3.3}$$

$$R = a_R(1 - e_R \cos u) \tag{3.4}$$

with

$$n = \frac{(-A)^{3/2}}{B} \tag{3.5 a}$$

$$e_t = \left[1 - \frac{A}{B^2}\left(C - \frac{BD}{C_0}\right)\right]^{1/2} \tag{3.5 b}$$

$$a_R = -\frac{B}{A} + \frac{D}{2C_0} \tag{3.5 c}$$

$$e_R = \left(1 + \frac{AD}{2BC_0}\right)e_t \tag{3.5 d}$$

The main difference between the relativistic radial motion and the non-relativistic one is the appearance of two different eccentricities: the « time eccentricity » e_t appearing in the Kepler equation (3.3) and the « relative radial eccentricity » e_R appearing in (3.4). Using eq. (2.17) we can express a_R, e_R, e_t and n in terms of E and J:

$$a_R = -\frac{GM}{2E}\left[1 - \frac{1}{2}(\nu - 7)\frac{E}{c^2}\right] \tag{3.6 a}$$

$$e_R = \left\{1 + \frac{2E}{G^2M^2}\left[1 + \left(\frac{5}{2}\nu - \frac{15}{2}\right)\frac{E}{c^2}\right]\left[J^2 + (\nu - 6)\frac{G^2M^2}{c^2}\right]\right\}^{1/2} \tag{3.6 b}$$

$$e_t = \left\{1 + \frac{2E}{G^2M^2}\left[1 + \left(-\frac{7}{2}\nu + \frac{17}{2}\right)\frac{E}{c^2}\right]\left[J^2 + (-2\nu + 2)\frac{G^2M^2}{c^2}\right]\right\}^{1/2} \tag{3.6 c}$$

$$n = \frac{(-2E)^{3/2}}{GM}\left[1 - \frac{1}{4}(\nu - 15)\frac{E}{c^2}\right] \tag{3.6 d}$$

It is remarkable that a well-known result of the Newtonian elliptic motion is still valid at the post-Newtonian level: both the relative semi-major axis a_R and the mean motion n depend only on the center of mass energy E. The same is true for the time of return to the periastron (« period »): $P := 2\pi/n$. (The corresponding results of Spyrou [*31*], his eqs. (26-28), are

incorrect, see below). As a consequence we can also express n in terms of a_R:

$$n = \left(\frac{GM}{a_R^3}\right)^{1/2}\left[1 + \frac{GM}{2a_R c^2}(-9 + \nu)\right] \tag{3.7}$$

Let us note also the relationship between e_t and e_R:

$$\frac{e_R}{e_t} = 1 + (3\nu - 8)\frac{E}{c^2} \tag{3.8 a}$$

$$\frac{e_R}{e_t} = 1 + \frac{GM}{a_R c^2}\left(4 - \frac{3}{2}\nu\right) \tag{3.8 b}$$

4. THE POST-NEWTONIAN ANGULAR MOTION

The relativistic angular motion, i. e. the solution of eq. (2.16) can also be simply reduced to the integration of an auxiliary non-relativistic angular motion, i. e. to eq. (A.11). Indeed let us first make the following $O(1/c^2)$ conchoidal transformation:

$$R = \tilde{R} + I/2H \tag{4.1}$$

which transforms eq. (2.16) into:

$$\frac{d\theta}{dt} = \frac{H}{\tilde{R}^2} \tag{4.2}$$

where $\tilde{R}$ can be expressed as:

$$\tilde{R} = \tilde{a}(1 - \tilde{e}\cos u) \tag{4.3}$$

with

$$\tilde{a} = a_R - I/2H \tag{4.4}$$

$$\tilde{e} = e_R\left(1 - \frac{AI}{2BH}\right) \tag{4.5}$$

The time differential is given from eq. (3.3) by:

$$dt = n^{-1}(1 - e_t \cos u)du \tag{4.6}$$

Hence we get:

$$d\theta = \frac{H}{n\tilde{a}^2}\frac{1 - e_t \cos u}{(1 - \tilde{e}\cos u)^2}du \tag{4.7}$$

As can be seen from eq. (3.8) and (4.5) e_t and $\tilde{e}$ differ only by small terms of order $1/c^2$. Now if we introduce any new eccentricity say e_θ also very near e_t, so that we can write: $e_t = (e_t + e_\theta)/2 + \varepsilon$, $e_\theta = (e_t + e_\theta)/2 - \varepsilon$, with $\varepsilon = O(c^{-2})$ then:

$$(1 - e_t \cos u)(1 - e_\theta \cos u) \equiv \left(1 - \frac{e_t + e_\theta}{2}\cos u\right)^2 - \varepsilon^2 \cos^2 u \tag{4.8}$$

Therefore if we choose e_θ such that the average of e_t and e_θ is equal to $\tilde{e}$, i. e. $e_\theta := 2\tilde{e} - e_t$ we have:

$$\frac{1 - e_t \cos u}{(1 - \tilde{e} \cos u)^2} = \frac{1}{1 - e_\theta \cos u} + O(1/c^4) \tag{4.9}$$

which transforms eq. (4.7) into a Newtonian like angular motion equation, similar to eq. (A.14 *a*):

$$d\theta = \frac{H}{n\tilde{a}^2} \frac{du}{1 - e_\theta \cos u} \tag{4.10}$$

which is easily integrated:

$$\theta - \theta_0 = KA_{e_\theta}(u) \tag{4.11 a}$$

θ_0 being a constant of integration and where for the sake of simplicity we have introduced the notations:

$$A_e(u) := 2 \arctan\left[\left(\frac{1 + e}{1 - e}\right)^{1/2} \tan \frac{u}{2}\right] \tag{4.11 b}$$

and

$$K := H/[n\tilde{a}^2(1 - e_\theta^2)^{1/2}] \tag{4.11 c}$$

From eq. (4.5) and (3.5 *d*) and the definition of $e_\theta := 2\tilde{e} - e_t$ we have:

$$e_\theta = e_t\left(1 + \frac{AD}{BC_0} - \frac{AI}{BH}\right) = e_R\left(1 + \frac{AD}{2BC_0} - \frac{AI}{BH}\right) \tag{4.12}$$

then, as shown by straightforward calculations:

$$\begin{aligned} e_\theta &= e_R\left(1 + \frac{G\mu}{2a_R c^2}\right) \\ &= \left\{1 + \frac{2E}{G^2M^2}\left[1 + \left(\frac{1}{2}\nu - \frac{15}{2}\right)\frac{E}{c^2}\right]\left[J^2 - 6\frac{G^2M^2}{c^2}\right]\right\}^{1/2} \end{aligned} \tag{4.13}$$

and

$$K = \frac{J}{(J^2 - 6G^2M^2/c^2)^{1/2}} \tag{4.14}$$

As is clear from eq. (4.1, 4.3) the radial variable R reaches its successive minima (« periastron passages ») for $u = 0, 2\pi, 4\pi, \ldots$ The periastron therefore precesses at each turn by the angle $\Delta\theta = 2\pi(K - 1)$, which if $J \gg GM/c$ reduces to the well-known result ([*28*]):

$$\Delta\theta = 6\pi \frac{G^2M^2}{J^2c^2} + O(1/c^4) = \frac{6\pi GM}{a_R(1 - e_R^2)c^2} + O(1/c^4). \tag{4.15}$$

5. THE POST-NEWTONIAN RELATIVE ORBIT

Contrarily to the usual approach which derives first the orbit by eliminating the time between eq. (2.15) and eq. (2.16) before working out the

motion on the orbit we find the orbit by eliminating u between eq. (3.4) and eq. (4.11). In order to simplify the formulae we introduce the notation f for the polar angle counted from a periastron and corrected for the periastron precession i. e.:

$$f := \frac{\theta - \theta_0}{\mathrm{K}} \tag{5.1}$$

We must eliminate u between:

$$\mathrm{R} = a_{\mathrm{R}}(1 - e_{\mathrm{R}} \cos u) \tag{5.2}$$

and

$$f = \mathrm{A}_{e_\theta}(u) \tag{5.3}$$

For doing this it is convenient to play a new « conchoidal » trick on R and write:

$$\mathrm{R} = \frac{e_{\mathrm{R}}}{e_\theta} a_{\mathrm{R}}(1 - e_\theta \cos u) + a_{\mathrm{R}}\left(1 - \frac{e_{\mathrm{R}}}{e_\theta}\right) \tag{5.4}$$

From the definition of $\mathrm{A}_{e_\theta}(u)$ we have:

$$1 - e_\theta \cos u = \frac{1 - e_\theta^2}{1 + e_\theta \cos \mathrm{A}_{e_\theta}(u)} = \frac{1 - e_\theta^2}{1 + e_\theta \cos f} \tag{5.5}$$

Moreover we find from eq. (4.13) that the radial displacement appearing in eq. (5.4) is simply:

$$a_{\mathrm{R}}\left(1 - \frac{e_{\mathrm{R}}}{e_\theta}\right) = \frac{\mathrm{G}\mu}{2c^2} \tag{5.6}$$

so that we find the polar equation of the relative orbit as:

$$\mathrm{R} = \left(a_{\mathrm{R}} - \frac{\mathrm{G}\mu}{2c^2}\right) \frac{1 - e_\theta^2}{1 + e_\theta \cos f} + \frac{\mathrm{G}\mu}{2c^2} \tag{5.7}$$

This equation means that the relative orbit is the *conchoïd* of a *precessing ellipse*, which means that it is obtained from an ellipse: $\mathrm{R}' = p(1 + e \cos \theta')^{-1}$ by a radial displacement $\mathrm{R} = \mathrm{R}' + \text{const.}$ together with an angular homothetic transformation: $\theta = (\text{const.}) . \theta'$. The result (5.7), already written down by Infeld and Plebanski [*22*], has often ([*33*] [*31*]) been written in the more complicated form:

$$\mathrm{R} = \frac{p_f}{1 + e_f \cos f + e_{2f} \cos 2f} + \mathrm{O}\left(\frac{1}{c^4}\right) \tag{5.8}$$

where

$$e_{2f} = -\frac{1}{4} e_{\mathrm{R}}^2 \frac{\mathrm{G}\mu}{a_{\mathrm{R}}(1 - e_{\mathrm{R}}^2)c^2} \tag{5.9 a}$$

$$e_f = e_{\mathrm{R}}(1 + e_{2f}) \tag{5.9 b}$$

$$p_f = (1 + e_{2f}) a_{\mathrm{R}}(1 - e_{\mathrm{R}}^2) \tag{5.9 c}$$

As noticed by Esnault and Holleaux [*20*], the corresponding results of Spyrou [*31*] (his eqs. (24) (26) (27) and (28)) are incorrect because $R_{min} = p_f(1 + e_f + e_{2f})^{-1}$ but $R_{max} = p_f(1 - e_f + e_{2f})^{-1}$; this vitiates his results for a_R (his a) and the « period », as well as his statement that $\lambda e = e_f + e_{2f}$ can be used as a usual Newtonian eccentricity.

Let us finally note that the relative orbit can also be written as:

$$R = \frac{a_R(1 - e_R^2)}{1 + e_R \cos f'} \tag{5.10 a}$$

with

$$f' = f + 2(e_{2f}/e_R) \sin f \tag{5.10 b}$$

6. THE POST-NEWTONIAN MOTIONS OF EACH BODY

The relativistic motions of each body are obtained by replacing the solution for the relative motion, $t(u)$, $R(u)$, $\theta(u)$, in the post-Newtonian center of mass formulae eq. (2.4). We see first that the polar angle of the first object (of mass m) is the same as the relative polar angle and that the polar angle of the second object (mass m') is simply $\theta + \pi$. Therefore it is sufficient to work out the radial motions. From eq. (2.4) we have by replacing V^2 in the relativistic corrections by $2GM/R + 2E \simeq 2GM/R - GM/a_R$:

$$r = \frac{m'}{M} R + \frac{G\mu(m - m')}{2Mc^2}\left(1 - \frac{R}{a_R}\right) \tag{6.1}$$

(and similar results for the second object by exchanging m and m') which shows remarkably enough, that r can also be written in a quasi-Newtonian form:

$$r = a_r(1 - e_r \cos u) \tag{6.2}$$

with

$$a_r = \frac{m'}{M} a_R \tag{6.3 a}$$

$$e_r = e_R\left[1 - \frac{Gm(m - m')}{2Ma_R c^2}\right] \tag{6.3 b}$$

and where as before:

$$n(t - t_0) = u - e_t \sin u \tag{6.4 a}$$

$$\theta - \theta_0 = KA_{e_\theta}(u) \tag{6.4 b}$$

The orbit in space of the first object can be written down by using the same method as in the preceding section for the relative orbit:

$$r = \frac{e_r}{e_\theta} a_r(1 - e_\theta \cos u) + a_r\left(1 - \frac{e_r}{e_\theta}\right) \tag{6.5}$$

One finds:

$$a_r\left(1-\frac{e_r}{e_\theta}\right)=\frac{Gm^2m'}{2M^2c^2} \tag{6.6}$$

hence we find also that the orbit is the conchoid of a precessing ellipse:

$$r=\left(a_r-\frac{Gm^2m'}{2M^2c^2}\right)\frac{1-e_\theta^2}{1+e_\theta\cos\left(\dfrac{\theta-\theta_0}{K}\right)}+\frac{Gm^2m'}{2M^2c^2} \tag{6.7}$$

7. RECAPITULATION

Gathering our results for the elliptic-like ($E<0$) post-Newtonian motion in the post-Newtonian center of mass frame, we have:

$$n(t-t_0)=u-e_t\sin u \tag{7.1 a}$$

$$\theta-\theta_0=K\times 2\arctan\left[\left(\frac{1+e_\theta}{1-e_\theta}\right)^{1/2}\tan\frac{u}{2}\right] \tag{7.1 b}$$

$$R=a_R(1-e_R\cos u) \tag{7.1 c}$$

$$r=a_r(1-e_r\cos u) \tag{7.1 d}$$

$$r'=a_{r'}(1-e_{r'}\cos u) \tag{7.1 e}$$

with

$$a_R=-\frac{GM}{2E}\left[1-\frac{1}{2}(\nu-7)\frac{E}{c^2}\right] \tag{7.2 a}$$

$$n=\frac{(-2E)^{3/2}}{GM}\left[1-\frac{1}{4}(\nu-15)\frac{E}{c^2}\right] \tag{7.2 b}$$

and K, e_t, e_θ, e_R, e_r, a_r, $e_{r'}$, $a_{r'}$, given in terms of the total energy and total angular momentum by unit reduced mass in the center of mass frame, E and J, by eq. (4.14) (3.6 *c*) (4.13) (3.6 *b*) (6.3 *b*) (6.3 *a*) and the interchange of m and m' for $e_{r'}$, $a_{r'}$. Eqs. (7.1) are very similar to the standard Newtonian solution of the non-relativistic two body problem (see Appendix A).

The simplest method for obtaining the post-Newtonian motion in the hyperbolic-like case ($E>0$) consists simply in making in eq. (7.1) (7.2), the analytic continuation in E from $E<0$ to $E>0$, passing below $E=0$ in the complex E plane and replacing the parameter u by iv ($i^2=-1$). The proof that this yields a correct parametric solution of the motion consists simply in noticing on one hand that K and the various eccentricities are analytic in E near $E=0$ and that if one replaces the parametric solution (7.1, 7.2) (and the corresponding expressions of e_t, e_θ, e_R, e_r, a_r, $e_{r'}$, $a_{r'}$ in terms of E and J) in $(dR/dt)^2$ and in $(d\theta/dt)^2$ one finds that eq. (2.15)

and the square of eq. (2.16) are satisfied identically, modulo $O(1/c^4)$, and that the resulting identities are *analytic* in E and u and are therefore still satisfied if E is continued to positive values and u to purely imaginary ones. One finds:

$$\bar{n}(t - t_0) = e_t \sinh v - v \tag{7.3 a}$$

$$\theta - \theta_0 = \mathrm{K}2 \arctan\left[\left(\frac{e_\theta + 1}{e_\theta - 1}\right)^{1/2} \tanh\frac{v}{2}\right] \tag{7.3 b}$$

$$\mathrm{R} = \bar{a}_\mathrm{R}(e_\mathrm{R} \cosh v - 1) \tag{7.3 c}$$

$$r = \bar{a}_r(e_r \cosh v - 1) \tag{7.3 d}$$

$$r' = \bar{a}_{r'}(e_{r'} \cosh v - 1) \tag{7.3 e}$$

where K, e_t, e_θ, e_R, e_r, $e_{r'}$, are the same functions of E and J as before but where it has been convenient to introduce the opposites of the analytic continuations of the semi-major axes:

$$\bar{a}_\mathrm{R} = \frac{\mathrm{GM}}{2\mathrm{E}}\left[1 - \frac{1}{2}(v - 7)\frac{\mathrm{E}}{c^2}\right] \tag{7.4}$$

(and $\bar{a}_r = m'\bar{a}_\mathrm{R}/\mathrm{M}$) and the modulus of the analytic continuation of the mean motion:

$$\bar{n} = \frac{(2\mathrm{E})^{3/2}}{\mathrm{GM}}\left[1 - \frac{1}{4}(v - 15)\frac{\mathrm{E}}{c^2}\right]. \tag{7.5}$$

Then the quasi-parabolic post-Newtonian motion (E = 0) can be obtained as a limit when $\mathrm{E} \to 0$. For instance let us start from the quasi-elliptic solution eq. (7.1) and pose:

$$u = \left(\frac{-2\mathrm{E}}{\mathrm{G}^2\mathrm{M}^2}\right)^{1/2} x \tag{7.6}$$

Taking now the limit $\mathrm{E} \to 0^-$, holding x fixed, yields the following parametric representation of the quasi-parabolic motion:

$$t - t_0 = \frac{1}{2\mathrm{G}^2\mathrm{M}^2}\left[\left(\mathrm{J}^2 + (2 - 2v)\frac{\mathrm{G}^2\mathrm{M}^2}{c^2}\right)x + \frac{1}{3}x^3\right] \tag{7.7 a}$$

$$\theta - \theta_0 = \frac{\mathrm{J}}{(\mathrm{J}^2 - 6\mathrm{G}^2\mathrm{M}^2/c^2)^{1/2}} \quad 2\arctan\frac{x}{(\mathrm{J}^2 - 6\mathrm{G}^2\mathrm{M}^2/c^2)^{1/2}} \tag{7.7 b}$$

$$\mathrm{R} = \frac{1}{2\mathrm{GM}}\left[\mathrm{J}^2 + (v - 6)\frac{\mathrm{G}^2\mathrm{M}^2}{c^2} + x^2\right] \tag{7.7 c}$$

Moreover let us point out that our solutions (for the three cases $\mathrm{E} < 0$, $\mathrm{E} > 0$, and $\mathrm{E} = 0$) have been written in a form suitable when $\mathrm{J}^2 > 6(\mathrm{GM}/c)^2$. However the validity of our solutions can be extended to the range

$J^2 \leqslant 6(GM/c)^2$ by first replacing in the solutions for the angular motion —eq. (7.1 *b*, 7.3 *b*, 7.7 *b*)—the function arctan by $-$ arccotan (at the price of a simple modification of θ_0) and then by simply making an analytic continuation in J. One ends up with an angular motion expressed with an argcoth which can also be approximately replaced by its asymptotic behaviour for large arguments: argcoth $X \sim 1/X$. The case of purely radial motion ($J = 0$) is also obtained by taking the limit $J \to 0$ (at u, v or respectively x fixed).

Finally a parametric representation of the general post-Newtonian motion in an arbitrary (post-Newtonian harmonic) coordinate system is obtained from our preceding centre-of-mass solution by applying a general transformation of the Poincaré group $(x'^a = L^a_b x^b + T^a)$.

APPENDIX A

THE NEWTONIAN TWO-BODY PROBLEM : A COMPENDIUM

The Newtonian equations of motion for two point-masses m and m' derive from the Lagrangian:

$$L_N(\boldsymbol{r}, \boldsymbol{r}', \boldsymbol{v}, \boldsymbol{v}') = \frac{1}{2} m v^2 + \frac{1}{2} m' v'^2 + \frac{Gmm'}{R} \tag{A.1}$$

where $(\boldsymbol{r}, \boldsymbol{r}')$ are the positions and $\boldsymbol{v} := d\boldsymbol{r}/dt$, $\boldsymbol{v}' := d\boldsymbol{r}'/dt$ the velocities of the masses in an inertial frame; $R := |\boldsymbol{r} - \boldsymbol{r}'|$ is their relative distance and G is Newton's constant. The invariance (modulo an exact time derivative) of this Lagrangian under spatial translations and Galileo transformations implies, via Noether's theorem, the conservation of the total momentum of the system:

$$\mathbf{P}_N = m\boldsymbol{v} + m'\boldsymbol{v}' \tag{A.2}$$

and of the centre-of-mass integral:

$$\mathbf{K}_N = m\boldsymbol{r} + m'\boldsymbol{r}' - t\mathbf{P}_N \tag{A.3}$$

In the centre of mass frame defined by $\mathbf{P}_N = \mathbf{K}_N = 0$ so that:

$$\boldsymbol{r} = \mu\mathbf{R}/m; \qquad \boldsymbol{r}' = -\mu\mathbf{R}/m' \tag{A.4 a}$$

$$\boldsymbol{v} = \mu\mathbf{V}/m; \qquad \boldsymbol{v}' = -\mu\mathbf{V}/m' \tag{A.4 b}$$

(with $\mathbf{R} := R\mathbf{N} := \boldsymbol{r} - \boldsymbol{r}'$; $\mathbf{V} := \boldsymbol{v} - \boldsymbol{v}'$; $M := m + m'$; $\mu := mm'/M$), the Newtonian equations of motion reduce to:

$$d\mathbf{V}/dt = -GMN/R^2 \tag{A.5}$$

As is easily checked the equation of motion (A.5) can also be derived directly from the reduced relative Lagrangian $L_N^R(\mathbf{R}, \mathbf{V})$ obtained by replacing in $\mu^{-1}L_N(\boldsymbol{r}, \boldsymbol{r}', \boldsymbol{v}, \boldsymbol{v}')$ the positions and velocities by their centre of mass expressions (A.4):

$$L_N^R(\mathbf{R}, \mathbf{V}) = \frac{1}{2} V^2 + \frac{GM}{R} \tag{A.6}$$

The invariance of the reduced relative Lagrangian (A.6) under time translations and space rotations implies, via Noether's theorem, the conservation of:

$$E = \frac{1}{2} V^2 - \frac{GM}{R} \tag{A.7}$$

$$\mathbf{J} = \mathbf{R} \times \mathbf{V} \tag{A.8}$$

which are the reduced (μ^{-1}) energy and angular momentum of the binary system in the centre of mass frame.

Eq. (A.8) implies that the motion takes place in a plane. Using polar coordinates (R, θ) in that plane and the relations:

$$\begin{cases} V^2 = (dR/dt)^2 + R^2(d\theta/dt)^2 \\ |\mathbf{R} \times \mathbf{V}| = R^2 d\theta/dt \end{cases} \tag{A.9}$$

we obtain from (A.7-A.8):

$$(dR/dt)^2 = A + 2B/R + C/R^2 \tag{A.10}$$

$$d\theta/dt = H/R^2 \tag{A.11}$$

where

$$A = 2E\,;\ B = GM\,;\ C = -J^2\,;\ H = |\mathbf{J}| =: J\,. \tag{A.12}$$

The solution of equations (A.10-A.11) is well known. In the case where $A < 0$ (elliptic motion), one introduces a parametrisation by means of the eccentric anomaly u:

$$n(t - t_0) = u - e \sin u \tag{A.13 a}$$

$$R = a(1 - e \cos u) \tag{A.13 b}$$

and as

$$d\theta = \frac{H}{\sqrt{-C}} \frac{\sqrt{1 - e^2}du}{1 - e \cos u} \tag{A.14 a}$$

$$\theta - \theta_0 = \frac{H}{\sqrt{-C}} A_e(u) \tag{A.14 b}$$

with

$$A_e(u) = 2 \arctan\left[\left(\frac{1 + e}{1 - e}\right)^{1/2} \tan \frac{u}{2}\right] \tag{A.14 c}$$

The orbit is obtained by eliminating u between the radial and angular motions (A.13-A.14). When $H = (-C)^{1/2}$ [as implied by eq. (A.12)] it is an ellipse:

$$R = \frac{p}{1 + e \cos(\theta - \theta_0)} \tag{A.15}$$

In (A.13-A.15) a, the semi major axis, and e the eccentricity are given in terms of A, B, C by:

$$a = -B/A\,; \qquad e = (1 - AC/B^2)^{1/2} \tag{A.16}$$

The semi-latus rectum p is given by

$$p = a(1 - e^2) = -C/B \tag{A.17}$$

and the mean motion $n = 2\pi/\text{period}$ is:

$$n = (-A)^{3/2}/B = (B/a^3)^{1/2} \tag{A.18}$$

Let us note that if one adds a term proportional to R^{-2} to the interaction potential (as in Appendix C) then eq. (A.10-A.11) still hold but the factor $K := H.(-C)^{-1/2}$ appearing in the RHS of eq. (A.14) is different from one. Then one must replace in eq. (A.15) $\theta - \theta_0$ by $(\theta - \theta_0)/K$, and one finds a « precessing ellipse » which precesses at each turn by an angle $2\pi(K - 1)$.

In the case $A > 0$ (hyperbolic motion) the solution is obtained from (A.13, A.14) by analytic continuation in A, together with the replacement $u = \sqrt{-1}v$, which yields:

$$\begin{cases} \bar{n}(t - t_0) = e \sinh v - v \\ \theta - \theta_0 = K2 \arctan\left[\left(\dfrac{e + 1}{e - 1}\right)^{1/2} \tanh \dfrac{v}{2}\right] \\ R = \bar{a}(e \cosh v - 1) \end{cases} \tag{A.20}$$

with

$$\bar{a} = B/A\,; \qquad \bar{n} = (B/\bar{a}^3)^{1/2} \tag{A.23}$$

The parabolic case ($A = 0$) is simply obtained as a limit as $A \to 0$. For instance one can start from the elliptic solution (A.13-A.15) and pose:

$$u = (1 - e^2)^{1/2} w \tag{A.22}$$

or:

$$u = (- A/B^2)^{1/2} x \tag{A.23}$$

and take the limit $A \to 0^-$ keeping w (or x) fixed.

As for the motions of each object they are deduced from (A.4 a):

$$\begin{cases} r = m'R/M & \text{(angle } \theta) \\ r' = mR/M & \text{(angle } \theta + \pi) \end{cases} \tag{A.24}$$

R and θ being given by (A.13, A.14).

APPENDIX B

THE POST-NEWTONIAN QUASI-ELLIPTIC MOTION BY LAGRANGE'S METHOD OF VARIATION OF CONSTANTS

The equations of motion for two bodies at the post-Newtonian approximation of General Relativity read, in the post-Newtonian centre of mass coordinate system:

$$\frac{d\mathbf{V}}{dt} = -\frac{\mathrm{GM}}{\mathrm{R}^2}\mathbf{N} + \frac{1}{c^2}a_{\mathrm{PN}} + \mathrm{O}(1/c^4) \tag{B.1}$$

where $\mathbf{R} := \mathrm{R}\mathbf{N}$ and $\mathbf{V}$ are the relative distance and velocity; $\mathrm{M} := m + m'$ is the total mass; G is Newton's constant, c the speed of light and:

$$\frac{1}{c^2}\boldsymbol{a}_{\mathrm{PN}} = \mathscr{R}\mathbf{N} + \mathscr{T}\mathbf{T} \tag{B.2}$$

where

$$\mathscr{R} = \frac{\mathrm{GM}}{c^2\mathrm{R}^2}\left[-\left(1 + \frac{3\mu}{\mathrm{M}}\right)\mathrm{V}^2 + \left(4 - \frac{1}{2}\frac{\mu}{\mathrm{M}}\right)(\mathrm{NV})^2 + \left(4 + \frac{2\mu}{\mathrm{M}}\right)\frac{\mathrm{GM}}{\mathrm{R}}\right] \tag{B.3}$$

is the component of the perturbing acceleration along the unit vector $\mathbf{N}$ in the direction of the instantaneous radius vector $\mathbf{R}$, and where

$$\mathscr{T} = \frac{\mathrm{GM}}{c^2\mathrm{R}}\left(4 - \frac{2\mu}{\mathrm{M}}\right)(\mathrm{NV})\frac{d\theta}{dt}. \tag{B.4}$$

is the component of the perturbing acceleration along the unit vector $\mathbf{T}$ in the instantaneous orbital plane, perpendicular to the radius vector in the sense of motion, θ being the polar angle of $\mathbf{N}$; μ is the reduced mass $\mu = mm'/\mathrm{M}$ (cf. [*33*]).

In the absence of the post-Newtonian perturbing acceleration (B.2-B.4), the motion of the two bodies is Newtonian (cf. Appendix A). It is characterized by 6 constants of integration, the 6 Keplerian orbital elements: Ω (the longitude of ascending node) and i (the inclination of the orbit) which determine the orientation of the plane of the orbit; ω (the longitude of periastron, denoted θ_0 in Appendix A) which determines the orientation of the orbit, an ellipse, in that plane; a (the semi-major axis) and e (the eccentricity) which determine the shape of the ellipse and τ (the epoch of periastron passage, denoted t_0 in Appendix A) which fixes the origin of time.

The post-Newtonian perturbing acceleration (B.2-B.4) makes the 6 osculating Keplerian orbital elements to vary with time. Since this acceleration lays in the orbital plane, Ω and i remain constant. The equations for the 4 other elements (the Gauss equations) read (cf. e. g. [*21*]):

$$\frac{da}{dt} = \frac{2}{n\sqrt{1-e^2}}\left[\mathscr{R}e\sin\phi + (1 + e\cos\phi)\mathscr{T}\right] \tag{B.5 a}$$

$$\frac{de}{dt} = \frac{\sqrt{1-e^2}}{na}\left[\mathscr{R}\sin\phi + \frac{(e + 2\cos\phi + e\cos^2\phi)}{1 + e\cos\phi}\mathscr{T}\right] \tag{B.5 b}$$

$$\frac{d\omega}{dt} = \frac{\sqrt{1-e^2}}{nae}\left[-\mathscr{R}\cos\phi + \frac{\sin\phi(2+e\cos\phi)}{1+e\cos\phi}\mathscr{T}\right] \tag{B.5 c}$$

$$\frac{dl}{dt} = n + \frac{1-e^2}{nae}\left[\mathscr{R}\frac{(-2e+\cos\phi+e\cos^2\phi)}{1+e\cos\phi} - \frac{\sin\phi(2+e\cos\phi)}{1+e\cos\phi}\mathscr{T}\right] \tag{B.5 d}$$

where the mean anomaly $l = n(t-\tau)$ is used instead of τ. The time dependence of the true anomaly $\phi = \theta - \omega$ can be expressed parametrically:

$$\phi = \theta - \omega = 2\arctan\left[\left(\frac{1+e}{1-e}\right)^{1/2}\tan\frac{u}{2}\right] \tag{B.6}$$

$$n(t-\tau) = u - e\sin u \tag{B.7}$$

u being the eccentric anomaly and n the mean motion:

$$n = (\mathrm{GM}/a^3)^{1/2}\,. \tag{B.8}$$

Since we have, by definition of the osculating elements:

$$\begin{cases} \mathrm{R} = \dfrac{p}{1+e\cos\phi}; \qquad (\mathrm{NV}) = \left(\dfrac{\mathrm{GM}}{p}\right)^{1/2} e\sin\phi \\ \mathrm{V}^2 = \dfrac{\mathrm{GM}}{p}(1+e^2+2e\cos\phi) \\ \dfrac{d\theta}{dt} = n(1+e\cos\phi)^2/(1-e^2)^{3/2} \end{cases} \tag{B.9}$$

$p = a(1-e^2)$ being the semi-latus rectum, we can write:

$$\mathscr{R} = \frac{\mathrm{G}^2\mathrm{M}^2}{c^2p^3}(1+e\cos\phi)^2\left[3 - \frac{\mu}{\mathrm{M}} + 3e^2 - \frac{7}{2}\frac{\mu}{\mathrm{M}}e^2 + \left(2 - 4\frac{\mu}{\mathrm{M}}\right)e\cos\phi + \left(-4 + \frac{1}{2}\frac{\mu}{\mathrm{M}}\right)e^2\cos^2\phi\right]$$

$$\mathscr{T} = \frac{\mathrm{G}^2\mathrm{M}^2}{c^2p^3}(1+e\cos\phi)^3\left(4 - \frac{2\mu}{\mathrm{M}}\right)e\sin\phi \tag{B.10}$$

Equations (B.5) can then easily be integrated if one approximates in their RHS the osculating elements by constants. One finds:

$$a(t) = \bar{a}\left\{1 + \frac{\mathrm{GM}}{c^2p(1-e^2)}\left[\left(-14 + 6\frac{\mu}{\mathrm{M}} - 6e^2 + \frac{31}{4}e^2\frac{\mu}{\mathrm{M}}\right)e\cos\phi + \left(-5 + 4\frac{\mu}{\mathrm{M}}\right)e^2\cos 2\phi + \frac{1}{4}\frac{\mu}{\mathrm{M}}e^3\cos 3\phi\right]\right\}$$

$$e(t) = \bar{e}\left\{1 + \frac{\mathrm{GM}}{c^2pe^2}\left[\left(-3 + \frac{\mu}{\mathrm{M}} - 7e^2 + \frac{47}{8}e^2\frac{\mu}{\mathrm{M}}\right)e\cos\phi + \left(-\frac{5}{2} + 2\frac{\mu}{\mathrm{M}}\right)e^2\cos 2\phi + \frac{1}{8}\frac{\mu}{\mathrm{M}}e^3\cos 3\phi\right]\right\} \tag{B.11}$$

$$\omega(t) = \bar{\omega} + \frac{\mathrm{GM}}{c^2pe}\left\{3e\phi + \left[-3 + \frac{\mu}{\mathrm{M}} + e^2\left(1 + \frac{21}{8}\frac{\mu}{\mathrm{M}}\right)\right]\sin\phi + e\left(-\frac{5}{2} + 2\frac{\mu}{\mathrm{M}}\right)\sin 2\phi + \frac{1}{8}\frac{\mu}{\mathrm{M}}e^2\sin 3\phi\right\}$$

$$l(t) = \bar{n}(t - \bar{\tau}) + \frac{\mathrm{GM}}{ec^2\sqrt{pa}}\left[\left(3 - \frac{\mu}{\mathrm{M}} + 7e^2 - \frac{41}{8}e^2\frac{\mu}{\mathrm{M}}\right)\sin\phi \right.$$
$$+ \left(\frac{5}{2} - 2\frac{\mu}{\mathrm{M}}\right)e\sin 2\phi - \frac{1}{8}\frac{\mu}{\mathrm{M}}e^2\sin 3\phi$$
$$\left. + e^2\frac{\sin\phi}{1 + e\cos\phi}(1 - e^2)\left(-3 + \frac{7}{2}\frac{\mu}{\mathrm{M}}\right)\right]$$

where $\bar{a}$, $\bar{e}$, $\bar{\omega}$ and $\bar{\tau}$ are constants of integration and where

$$\bar{n} = \left(\frac{\mathrm{GM}}{\bar{a}^3}\right)^{1/2}\left[1 + \frac{\mathrm{GM}a}{c^2p^2}\left(-9 + 2\frac{\mu}{\mathrm{M}} - \frac{21}{2}e^2 + \frac{19}{2}e^2\frac{\mu}{\mathrm{M}} - 3e^4 + \frac{7}{2}\frac{\mu}{\mathrm{M}}e^4\right)\right] \quad \text{(B.12)}$$

(In the limit when $\mu/\mathrm{M} = 0$ — « restricted two-body problem » —corresponding formulae were obtained by Brumberg [*6*]). The problem is then solved but the solution is very heavy. Note the appearance of « second harmonics » of the motion, that is of terms in $\cos 3\phi$ in eqs. (B.11). Several partial simplifications were proposed in the literature: see e. g. [*33*] [*4*] [*19*]. However, rather than manipulating the osculating elements, it is much simpler to integrate directly the equations of motion (B.1-B.4): see the text where the solution is expressed only in terms of « zeroth harmonics » (fundamental frequency). We have checked that the solutions coincide.

APPENDIX C

THE POST-NEWTONIAN MOTION OBTAINED VIA THE MAUPERTUIS PRINCIPLE

Infeld and Plebanski ([*22*] Chapter 5) have tried to take advantage of the existence of a relative Lagrangian to reduce the problem of the post-Newtonian motion to a simpler auxiliary problem of a *Newtonian* motion by means of several « transformations » both in space and time. However their method is conceptually incorrect because their « time transformations » do not constitute a licit operation that can be applied to a variational principle (for a discussion of the allowed time and space transformations and their effects on a variational principle see [*29*] [*2*] [*14*]). Moreover they ignore in the Lagrangian terms proportional to $(V^2/2 - GM/R)^2$ which is also incorrect as will be seen below. However we shall sketch here a method based on the Maupertuis principle which succeeds in reducing our post-Newtonian problem to a simpler Newtonian one. This method uses three tricks: the first trick consists in making a conchoidal transformation, i. e. precisely to replace the relative radial variable R in terms of a new radial variable R′ by means of:

$$R = R' + \frac{1}{2}\frac{G\mu}{c^2} \tag{C.1}$$

keeping the angular variable unchanged ($\theta = \theta'$).

The relative Lagrangian expressed in terms of $\mathbf{R}'$ and $\mathbf{V}' = d\mathbf{R}'/dt$ is:

$$L^{R}_{PN} = \frac{1}{2}V'^2 + \frac{GM}{R'} + \frac{1}{2c^2}(1 - 3\nu)(E'(R', V'))^2 + (4 - \nu)\frac{GM}{R'c^2}E'(R', V') + \frac{3G^2M^2}{R'^2c^2} \tag{C.2}$$

with

$$E'(R', V') := \frac{1}{2}V'^2 - \frac{GM}{R'} \tag{C.3}$$

Now one checks easily that if one first computes the variational derivative $\delta(E'(R', V'))^2/\delta\vec{\mathbf{R}}'$ and *then* (« after variation ») takes advantage of the fact that $E'(R', V') = E'_0 + O(1/c^2)$ where E'_0 is a constant number, one finds:

$$\frac{\delta}{\delta\mathbf{R}'}[(E'(R', V'))^2] = -4E'_0\frac{\delta}{\delta\mathbf{R}'}\frac{GM}{R'} + O(1/c^2) \tag{C.4}$$

Then the second trick consists in noticing that because of eq. (C.4) one can, modulo $O(1/c^4)$, replace in the Lagrangian (C.2) $(E'(R', V'))^2$ by $-4GME'_0/R'$ where E'_0 is a constant number which, *after variation*, will be put equal to the constant energy of the relative motion. In other words this second trick consists in replacing the total mass M by:

$$M' = M\left(1 - 2(1 - 3\nu)\frac{E'_0}{c^2}\right) \tag{C.5}$$

and the relative Lagrangian by:

$$L'^{R}_{PN} = \frac{1}{2}V'^2 + \frac{GM'}{R'} + \frac{3G^2M^2}{R'^2c^2} + (4 - \nu)\frac{GM}{R'c^2}E'(R', V') \tag{C.6}$$

where in the expression (C.3) for $E'(R', V')$ one can consistently replace M by M′.

Now let us more generally consider a Lagrangian of the type:

$$L\left(\boldsymbol{r}, \frac{d\boldsymbol{r}}{dt}\right) = \frac{1}{2}v^2 + \frac{m}{r} + \frac{1}{c^2}W(\boldsymbol{r}) + \frac{1}{c^2}\varepsilon(\boldsymbol{r})\left(\frac{1}{2}v^2 - \frac{m}{r}\right) \tag{C.7}$$

L does not explicitly depend on time, so that one can define a constant energy:

$$E_0 = \boldsymbol{v} \cdot \frac{\partial L}{\partial \boldsymbol{v}} - L \tag{C.8}$$

Then the spatial trajectory will be derivable from the Maupertuis principle (see e. g. [*24*]) at fixed energy:

$$\delta \int \left[\frac{\partial L}{\partial \boldsymbol{v}} \cdot d\boldsymbol{r}\right]_{E_0 = \text{const}} = 0 \tag{C.9}$$

One checks easily that the Maupertuis principle of (C.7) is, modulo $O(1/c^4)$:

$$\delta \int \left[2\left(E_0 + \frac{m}{r} + \frac{1}{c^2}W(\boldsymbol{r}) + \frac{\varepsilon(\boldsymbol{r})}{c^2}E_0\right)\right]^{1/2} |d\boldsymbol{r}| = 0 \tag{C.10}$$

Now let us consider the auxiliary variational problem expressed with a time t' and a Lagrangian:

$$L'\left(\boldsymbol{r}, \frac{d\boldsymbol{r}}{dt'}\right) = \frac{1}{2}\left(\frac{d\boldsymbol{r}}{dt'}\right)^2 + \frac{m}{r} + \frac{1}{c^2}W(\boldsymbol{r}) + \frac{\varepsilon(\boldsymbol{r})}{c^2}E_0 \tag{C.11}$$

where E_0 is a constant which, after variation, will be put equal to the energy. One sees that the Maupertuis principle of (C.11) for a constant energy:

$$E_0 = \frac{d\boldsymbol{r}}{dt'} \cdot \frac{\partial L'}{\partial(d\boldsymbol{r}/dt')} - L' \tag{C.12}$$

is precisely (C.10). Therefore the Lagrangian L of (C.7) has the same spatial trajectories as the Lagrangian L′ of (C.11). However the motion along the trajectories is not the same for L and L′. Computing the respective velocities $|\boldsymbol{v}| = |d\boldsymbol{r}/dt|$ and $|d\boldsymbol{r}/dt'|$ from eq. (C.8) and (C.12) one finds that along the trajectories the time variables associated with L and L′ are linked by:

$$dt = \frac{|d\boldsymbol{r}|}{|\boldsymbol{v}|} = \left(1 + \frac{\varepsilon(\boldsymbol{r})}{c^2}\right)\frac{|d\boldsymbol{r}|}{|d\boldsymbol{r}/dt'|} = \left(1 + \frac{\varepsilon(\boldsymbol{r})}{c^2}\right)dt' \tag{C.13}$$

Then our third trick consists in applying the general result just discussed to the Lagrangian $L'^{\mathbf{R}}_{PN}$ (C.6). In that case we have $\boldsymbol{r} = \mathbf{R}'$ and:

$$\frac{\varepsilon}{c^2} = (4 - \nu)\frac{GM}{R'c^2} \tag{C.14}$$

Therefore our post-Newtonian problem is equivalent to the following auxiliary simple Newtonian problem:

$$L'_N(\mathbf{R}', \quad \mathbf{V}'') = \frac{1}{2}(\mathbf{V}'')^2 + \frac{GM''}{R'} + \frac{3G^2M^2}{c^2R'^2} \tag{C.15}$$

where we use an auxiliary time t', and where we have denoted:

$$\mathbf{V}'' := \frac{d\mathbf{R}'}{dt'} \tag{C.16}$$

and:

$$\begin{cases} M'' = M' + (4 - \nu)ME'_0/c^2 \\ M'' = M[1 + (2 + 5\nu)E'_0/c^2] \end{cases} \tag{C.17}$$

(E'_0 being, modulo $O(1/c^4)$, the same constant as defined above). The real time t is linked to t' by:

$$dt = \left[1 + (4 - v)\frac{GM}{R'c^2}\right]dt' \tag{C.18}$$

Now the motion $R'(t')$ corresponding to L'_N eq. (C.15) is very simply obtained, for instance by the method of Appendix A. One finds for the coefficients A, B, C and H : $A = 2E'_0$, $B = GM''$, $C = -J'^2_0 + 6G^2M^2/c^2$ and $H = J'_0$. In particular the orbit is a « precessing ellipse » with a precession factor (see appendix A) $K = H.(-C)^{-1/2} = J'_0/(J'^2_0 - 6G^2M^2/c^2)^{1/2}$. Finally expressing E'_0 and J'_0 (the first integrals of L'_N) in terms of E and J (the first integrals of L^R_{PN}), we have checked that one recovers the solution obtained by a more direct procedure in the text.

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(Manuscrit reçu le 3 janvier 1985)