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General relativistic celestial mechanics of binary systems II. The post-Newtonian timing formula

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ABSTRACT. — Starting from a previously obtained « quasi-Newtonian » solution of the equations of motion of a binary system at the first post-Newtonian approximation of General Relativity, we derive a new « timing formula » giving the arrival times at the barycenter of the solar system of electromagnetic signals emitted by one member of a binary system. Our timing formula is simpler and more complete than presently existing timing formulas. We propose to use it as a timing model to be fitted to the arrival times of pulses from binary pulsars. Specifically we show that the use of this timing model in the analysis of the timing measurements of the Hulse-Taylor pulsar could determine more parameters than is presently done. This should lead to additional tests of the simplest model of this binary system and to the first test of relativistic theories of gravity independent of any hypothesis of « cleanness » of the system.

RÉSUMÉ. — A partir d'une solution « quasi-newtonienne » des équations du mouvement d'un système binaire à la première approximation post-newtonienne de la Relativité Générale précédemment obtenue, nous dérivons une nouvelle formule de chronométrage donnant les temps d'arrivée au barycentre du système solaire des signaux électromagnétiques émis par un des membres d'un système binaire. Notre formule de chronométrage est plus simple et plus complète que celles existant actuellement. Nous proposons de l'utiliser comme un modèle auquel ajuster les temps d'arrivée des impulsions en provenance de pulsars binaires. Plus précisément, nous montrons que l'utilisation de ce modèle de chronométrage dans l'analyse des données du pulsar de Hulse et Taylor pourrait permettre de déterminer plus de paramètres que jusqu'à présent. Cela devrait conduire à des tests supplémentaires du modèle le plus simple de ce système binaire et au premier test des théories relativistes de la gravitation indépendant de toute hypothèse concernant la « propreté » du système.

1. INTRODUCTION

Since its discovery by Hulse and Taylor [*13*], the binary pulsar PSR 1913+16 has been timed with steadily improving precision. The fit of the observed arrival times on Earth of the radio pulses to a « timing formula » derived from a simple model of the system (essentially a general relativistic two-points-masses model) has allowed Taylor and collaborators ([*17*] and references therein) and Boriakoff et al. [*4*] to measure all the parameters needed to determine the dynamics of the system. Moreover two checks of the model were performed. The measured rate of decay of the orbital period, $\dot{P}_b$ [*16*] [*17*], and the measured sine of the angle of inclination of the orbital plane onto the celestial sphere, sin i [*19*], have indeed been found to agree with the theoretical predictions as deduced from the model and evaluated using the other directly measured parameters. The observed $\dot{P}_b^{\text{obs}}$ agrees within 4 % with the calculated $\dot{P}_b^{\text{theory}}$ deduced from the so-called « quadrupole formula » which has been recently shown to be indeed a consequence of the general relativistic mechanics of two strongly self-gravitating bodies ([*5*] [*6*] [*7*] and references therein). As for the observed and calculated sin i they agree within 20 % [*19*] but the significance of that test is somewhat obscure because the parameter sin i which is fitted for does not represent a clear-cut relativistic effect but a « coherent combination » of many relativistic effects (the combination being defined by fixing, somewhat arbitrarily, some theoretical relations between the parameters appearing in these relativistic effects).

From those results Taylor and collaborators concluded that: 1) the simple model used is consistent with observation and therefore, as assumed in the model, the system is indeed « clean »; 2) the system provides a new and profound confirmation of the theory of General Relativity and plausibly rules out a number of alternative theories of gravitation (for a detailed discussion of these points see also Will [*21*] and references therein).

Because of the importance of such conclusions the chain of deductions must be critically examined. It is essentially the following: first *assume* that the system is « clean » and that General Relativity is valid (that is that the system is reducible to a general relativistic two-point-masses model); then interpret the measurements of $\dot{P}_b$ and $\sin i$ as consistency checks confirming the latter assumptions. A more cautious and more rigourous procedure would however start by pointing out that *secular* effects such as the advance of the periastron, $\dot{\omega}$, and the rate of decay of the orbital period, $\dot{P}_b$, are sensitive to many of the « noise sources » that could « dirty » the system, such as the quadrupole deformation of an extended companion, the presence of an external ring of matter or a third body, mass-loss from the companion, tidal dissipation, accretion of matter... (see e. g. [*15*] [*11*] [*2*] [*20*]). As there are no *a priori* reasons to believe that all possible noise sources are absent, it must rather be assumed that $\dot{\omega}^{obs}$ and $\dot{P}_b^{obs}$ do contain some unknown noise contributions. Then, if we still assume the validity of General Relativity, the *two* « consistency checks » (« $\dot{P}_b$ » and « $\sin i$ ») only show that the noise contributions to the *two* secular effects $\dot{\omega}$ and $\dot{P}_b$ happen to be small (smaller say than 20 %, the error on $\sin i$). From that point of view then, the present analysis of the observational data does *not* provide any confirmation of General Relativity. Furthermore, if we used a theory of gravity other than General Relativity (along the lines of Will [*20*] [*21*]) we would deduce from the observations the noise contributions to $\dot{\omega}$ and $\dot{P}_b$, and the values of the masses of the pulsar and its companion (differing from the general relativistic values by theory-dependent gravitational binding energy contributions). Therefore, strictly speaking, the present analysis of the timing measurements of the Hulse-Taylor pulsar furnishes, neither a clear-cut test of the relativistic theories of gravity, nor an unambiguous (theory-independent) determination of the masses of the pulsar and its companion. One can only argue that General Relativity provides a more *plausible* description of the system because it predicts $\dot{P}_b^{theory} \simeq \dot{P}_b^{obs}$, while most alternative theories seem to predict $|\dot{P}_b^{theory}| \gg |\dot{P}_b^{obs}|$, so that the observed decay rate should be interpreted as a small residual between much bigger « theoretical » and « noise » contributions (for instance in Rosen's theory one would have $\dot{P}_b^{theory} \simeq -\dot{P}_b^{noise} \sim -10^3\dot{P}_b^{obs}$).

The preceding critical remarks raise the question whether it is possible to extract additional informations from the observational data so as to reach reliable conclusions concerning the determination of the masses, and to get clear-cut tests of the theories of gravity. Because of the above mentioned « noise sensitivity » of secular effects, it seems appropriate to concentrate on the « quasi periodic » effects which are probably harder to mimic by « noise » sources. Specifically we have in mind a nearly « clean » system: whereas various « noises » contribute small, but unknown, amounts to $\dot{\omega}$ and $\dot{P}_b$, their contributions to the « quasi periodic » effects can be expected to be uncorrelated to the relativistic effects (in the sense that they will not interfere with the fitting procedure proposed in this article), or to be altogether negligible (in view of the low precision of determination of the post-Newtonian periodic effects). This led us to reexamine in detail the relativistic timing formulas presently used to extract informations from the raw observational data. The basic timing model is due to Blandford and Teukolsky [3] and Epstein [11]. Epstein's description of the post-Newtonian timing effects due to relativistic corrections to a Keplerian motion is formidably complicated. This complication entailed two regrettable consequences: 1) it hided for several years an oversight (replacement of the periastron argument by a linear function of time while it is a linear function of the true anomaly), and 2) it led Epstein to parametrize « globally » all the post-Newtonian effects by a single parameter sin i whose significance is obscure.

In this paper we show how the use of a new, remarkably simple, explicit solution of the post-Newtonian motion of a binary system [9] leads to a drastic simplification of the description of post-Newtonian timing effects. This simplified description will allow us to discuss the significance of the present measurement of sin i, as well as the possibility to measure other « post-Newtonian periodic » parameters. We shall moreover include in our timing model the effect of the relative motion of the binary system and the solar system as well as the effect of the aberration of the radio pulses emitted by the pulsar. The effect of the relative motion was discarded by Blandford and Teukolsky [3] on the grounds that neglecting it only introduces small $[O(v_{\text{relative}}/c)]$ and constant uncertainties in the measurements of the orbital elements. However, in view of the increasing accuracy of the « $\dot{P}_b$ test » it becomes necessary to have a better control of the exact influence of these intrinsic uncertainties. As for the aberration effects (which are as important as post-Newtonian effects, that is $\sim 12\ \mu$sec), they were considered by Smarr and Blandford [15] and Epstein [11], and discarded by Epstein on the grounds that they could be mimicked by small secularly variable corrections to the Newtonian and half-post-Newtonian parameters. However no proof of this statement was given. Moreover the latter authors do not use the full aberration effect but neglect a numerically important secularly changing aberration term. Since the detection of any effect due to the aberration would give the first direct evidence that pulsars are rotating beacons we decided to include explicitly these aberration effects in our timing formula.

This paper is organized as follows: in section 2 we derive a new timing formula in the framework of General Relativity (we show in § 3.4 that in other gravity theories the timing formulas have the same form). In section 3 we first discuss the effect of the relative motion between the solar and the binary systems (§ 3.1); then the effect of aberration (§ 3.2) and finally we examine which new « post-Newtonian periodic » parameters could be measured (§ 3.6-§ 3.9). In § 3.5 we give the explicit differential timing formula (useful for the linearized least-squares fit) corresponding to our timing formula. Section 4 summarizes our results. Appendix A gives some details about the mathematical derivation of the timing formula.

2. DERIVATION OF THE TIMING FORMULA

By « timing formula » it is meant the mathematical relation linking the time of arrival $\tau_a^{(\mathrm{Earth})}$, as measured by an observer on Earth, of the Nth pulse emitted by a pulsar in a binary system, to the integer N. This timing formula depends on many parameters which describe the various physical effects taking part in the history of the electromagnetic signal connecting the pulsar and the Earth. It is convenient to treat separately the latter physical effects by introducing intermediate time variables (τ_a, t_a, t_e, T_e, T). This will allow a step by step derivation of the timing formula [of the type: $\tau_a^{(\mathrm{Earth})} = f_1(\tau_a)$, $\tau_a = f_2(t_a)$, ..., $T = f_6(N)$].

2.1. The « infinite-frequency barycenter arrival time » : τ_a.

We shall first assume, following Blandford and Teukolsky [*3*], that, correcting for the motion of the Earth with respect to the barycenter of the solar system as well as for the gravitational redshift in the solar system and the effect of interstellar dispersion, one has computed from $\tau_a^{(\mathrm{Earth})}$ the (hypothetical) time of arrival τ_a of the Nth pulse at the barycenter of the solar system in absence of any solar gravitational redshift and interstellar dispersion [see equations (2.23)-(2.24) of Blandford and Teukolsky [*3*] defining $\tau_a \equiv t$, the « infinite-frequency barycenter arrival time »].

2.2. The coordinate time of arrival : t_a.

The relation linking the proper time τ_a to the integer N is clearly general-relativistically invariant and can be computed using any convenient coordinate system. Let us use a system of harmonic coordinates such that « the barycenter of the binary system » is at rest at the origin (see [*6*] for the precise definition of this « center-of-mass frame » to order c^{-5}). In this coordinate system the barycenter of the solar system is moving with the velocity $\vec{v}_b$ (assumed here constant); τ_a is therefore linked to the coordinate time of arrival t_a by:

$$\tau_a = (1 - \vec{v}_b^2/c^2)^{1/2} t_a + \text{const.} \tag{1}$$

Additive constants like the last term in equation (1) are unimportant and will be often omitted in the following.

2.3. The coordinate time of emission : t_e.

The coordinate time of arrival t_a is linked to the coordinate time of emission of the pulse t_e by:

$$t_a = t_e + \frac{1}{c} | \vec{r}_b(t_a) - \vec{r}(t_e) | + \Delta_S, \tag{2}$$

where $\vec{r}_b$ is the coordinate position vector of the barycenter of the solar system, $\vec{r}$ the coordinate position vector of the pulsar ($|\vec{V}|$ denoting the usual Euclidean length of $\vec{V}$: $[(V^1)^2 + (V^2)^2 + (V^3)^2]^{1/2}$), and Δ_S denotes the « Shapiro time delay » due to the propagation of the radio signal in a curved space-time. Δ_S has been computed by Blandford and Teukolsky [3] under the assumption of everywhere weak gravitational fields. This assumption is violated in the case at hand ($2Gm/(c^2 r) \sim 0.4$ at the surface of a neutron star, compared to $2GM_\odot/c^2R_\odot \sim 4.10^{-6}$). However it can be shown [8] that, in General Relativity, strong field effects modify their result only by an inessential additive constant:

$$\Delta_S(t_e) = -\frac{2Gm'}{c^3} \log \left[\vec{n}.(\vec{r}(t_e) - \vec{r}\,'(t_e)) + | \vec{r}(t_e) - \vec{r}\,'(t_e) | \right] + \text{const.} + O\left(\left(\frac{v}{c}\right)^{2/3} \frac{Gm'}{c^3} \right), \tag{3}$$

where $\vec{n}$ denotes $\vec{r}_b / | \vec{r}_b |$ (unit vector pointing from the binary system to the solar system) m' and $\vec{r}\,'$ being respectively the mass and the position vector of the companion (the corresponding quantities for the pulsar being denoted m and $\vec{r}$). In equation (2) the quantity $c^{-1} | \vec{r}_b(t_a) - \vec{r}(t_e) |$ depends on t_a through the motion of the solar system: $\vec{r}_b(t_a) = \vec{r}_b(0) + \vec{v}_b t_a$. Expanding it to first order in powers of $(\vec{v}_b t_a - \vec{r})/| \vec{r}_b(0) |$ and using equation (1) we find

$$D . \tau_a = t_e + \Delta_R(t_e) + \Delta_S(t_e) + \text{const.}, \tag{4}$$

where D is the following « Doppler factor »:

$$D := \frac{1 - \vec{n} . \vec{v}_b / c}{\sqrt{1 - \vec{v}_b^2/c^2}}, \tag{5}$$

and where Δ_R denotes what can be called the (coordinate) « Roemer time delay » i. e. the time of flight across the orbit (counted from the barycenter and projected on the line of sight $-\vec{n}$)

$$\Delta_R(t_e) := -\frac{1}{c}\vec{n}\,.\,\vec{r}(t_e)\,. \tag{6}$$

2.4. **The « proper time » of emission** (T_e, T).

It remains to relate the coordinate time of emission t_e of the N*th* pulse to N. This can be done in two steps.

First introduce a suitable « proper time » T for the pulsar associated to a suitable « comoving coordinate system » X^i such that in the coordinate system (T, X^i) the emission mechanism of the pulsar can be described (with a sufficient accuracy) as if the pulsar was isolated. Assume that a pulsar is a rotating beacon geared to the fast spinning motion of a neutron star. In the coordinate system (T, X^i) the « proper » angle Φ measuring the position of the emission spot which rotates around the spin axis, say $\vec{e}_3 = \partial/\partial X^3$, is then related to the « proper time » T by the following equation (valid for slowly spinning down noiseless pulsars):

$$\Phi/2\pi = \overline{N}_0 + \nu T + \frac{1}{2}\dot{\nu}T^2 + \frac{1}{6}\ddot{\nu}T^3\,, \tag{7}$$

where $\overline{N}_0$ is a constant (not necessarily an integer), ν is the proper rotation frequency of the pulsar (at T = 0) and $\dot{\nu}$ and $\ddot{\nu}$ the first and the second derivatives of the rotation frequency (at T = 0).

Now the proper time T_e of emission of the N*th* pulse is linked to N by the fact that the proper angle $\Phi_e = \Phi(T_e)$ of emission of the N*th* pulse is equal to

$$\Phi(T_e) = \Phi_0 + 2\pi N + \delta_A\Phi(T_e)\,, \tag{8}$$

where Φ_0 is a constant and where the non constant angular shift $\delta_A\Phi$ is caused by the aberration effects involved in the transformation between the comoving frame (T, X^i) and the center of mass frame (t, x^i). One finds ([*15*], see [*8*] for a proof that taking into account the strong gravitational field of the pulsar does not alter, in General Relativity, the result):

$$\delta_A\Phi = +\frac{\vec{v}\,.(\vec{n}\times\vec{e}_3)}{c(\vec{n}\times\vec{e}_3)^2} + O\left(\frac{v^2}{c^2}\right), \tag{9a}$$

where $\vec{v} = d\vec{r}/dt$ is the coordinate velocity of the pulsar and $\vec{e}_3$ the direction of the spin axis in the center of mass frame. We can associate to the angular shift $\delta_A\Phi$ an « aberration time delay » Δ_A defined as

$$\Delta_A := \delta_A\Phi/(2\pi\nu)\,. \tag{9b}$$

Introducing as additional intermediate variable the proper time

$$\mathrm{T} := \mathrm{T}_e - \Delta_\mathrm{A}\,, \tag{10}$$

at which the N*th* pulse would have been emitted if the pulsar mechanism had been a radial pulsation instead of a rotating beacon, one finds that T is implicity defined as a function of N by the relation:

$$\mathrm{N} = \mathrm{N}_0 + \nu\mathrm{T} + \frac{1}{2}\dot{\nu}\mathrm{T}^2 + \frac{1}{6}\ddot{\nu}\mathrm{T}^3\,, \tag{11}$$

where N_0 denotes $\overline{\mathrm{N}}_0 - \Phi_0/2\pi$.

Now the coordinate time of emission, t_e, is linked to the proper time of emission T_e by

$$t_e = \mathrm{T}_e + \Delta_\mathrm{E}\,, \tag{12}$$

where Δ_E (the « Einstein time delay ») contains contributions coming from the gravitational redshift caused by the companion and from the second order Doppler effect [see equation (19) below for the explicit expression of Δ_E].

Then, using equations (4), (10) and (12), we find that τ_a is linked to T through

$$\mathrm{D}.\tau_a = \mathrm{T} + \Delta_\mathrm{R} + \Delta_\mathrm{E} + \Delta_\mathrm{S} + \Delta_\mathrm{A}\,. \tag{13}$$

As Δ_A is a very small delay [$\Delta_\mathrm{A} \sim (v/c)$ (pulsar period) <<< (binary period)], the time delays $\Delta_\mathrm{R}, \ldots$ in equation (13) can be computed at the time T (instead of $\mathrm{T}_e = \mathrm{T} + \Delta_\mathrm{A}$).

2.5. The post-Newtonian motion in terms of the proper time.

Now in order to write down an explicit formula for τ_a one must have an explicit solution for the relativistic motion of a binary system. A recent reexamination of the relativistic mechanics of binary systems (Damour and Deruelle [9] hereafter quoted as paper I) has led to discover a remarkably simple « quasi-Newtonian » form of the motion at the post-Newtonian order. In the « center of mass frame » used here the motion of the pulsar lies in a plane, say $(\vec{e}_x, \vec{e}_y)$, whose position and orientation with respect to the plane of the sky, say $(\vec{e}_{\mathrm{X}_0}, \vec{e}_{\mathrm{Y}_0})$, are defined by two angles: the longitude of the ascending node $\Omega\,(0 \leqslant \Omega < 2\pi)$ and the inclination $i\,(0 \leqslant i < \pi)$. Our orientation conventions are the following: the (orthonormal right handed) triad $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$ [where $\vec{e}_x$ is directed towards the ascending node and where the plane $(\vec{e}_x, \vec{e}_y)$ is oriented in the sense of the motion] is deduced from the reference triad $(\vec{e}_{\mathrm{X}_0}, \vec{e}_{\mathrm{Y}_0}, \vec{e}_{\mathrm{Z}_0})$ (where $\vec{e}_{\mathrm{Z}_0} = -\vec{n}$

is the direction from the Earth to the pulsar) by two successive rotations, namely

$$\begin{cases} \vec{e}_x = \vec{e}_X, & (14\,a) \\ \vec{e}_y = \cos i\vec{e}_Y + \sin i\vec{e}_Z, & (14\,b) \\ \vec{e}_z = -\sin i\vec{e}_Y + \cos i\vec{e}_Z, & (14\,c) \end{cases}$$

with

$$\begin{cases} \vec{e}_X = \cos\Omega\vec{e}_{X_0} + \sin\Omega\vec{e}_{Y_0}, & (15\,a) \\ \vec{e}_Y = -\sin\Omega\vec{e}_{X_0} + \cos\Omega\vec{e}_{Y_0}, & (15\,b) \\ \vec{e}_Z = \vec{e}_{Z_0}. & (15\,c) \end{cases}$$

The motion of the pulsar in the plane $(\vec{e}_x, \vec{e}_y)$ is defined by the polar coordinates $r = |\vec{r}|$ and θ (counted from the ascending node in the sense of the motion) so that $\vec{r} = r\cos\theta\vec{e}_x + r\sin\theta\vec{e}_y$. Then one has the following « quasi-Newtonian » parametric representation of the motion of the pulsar [equations I(7.1) of paper I]:

$$\begin{cases} n(t - T_0) = U - e_t \sin U, & (16\,a) \\ r = a_r(1 - e_r \cos U), & (16\,b) \\ \theta = \omega_0 + (1 + k)A_{e_\theta}(U), & (16\,c) \end{cases}$$

where n, T_0, e_t, a_r, e_r, ω_0, k, e_θ are constants, $n = 2\pi/P_b$ with P_b being the time of return to the periastron (« binary period »), $k = \Delta\theta/2\pi$ with $\Delta\theta$ being the angle of periastron precession per orbit ($k = \dot{\omega}/n$), and where the function $A_e(U)$ is defined as [equation I(4.11 *b*) of paper I] (for $e < 1$):

$$A_e(U) := 2\arctan\left[\left(\frac{1+e}{1-e}\right)^{1/2} \tan\frac{U}{2}\right]. \qquad (17\,a)$$

The function $A_e(U)$ satisfies the identities ([*18*]), $A_e(p\pi) = p\pi(\forall p \in \mathbb{Z})$, and

$$\cos A_e(U) = \frac{\cos U - e}{1 - e\cos U}, \qquad (17\,b)$$

$$\sin A_e(U) = \frac{(1 - e^2)^{1/2}\sin U}{1 - e\cos U}, \qquad (17\,c)$$

$$A_e(U) = U + 2\sum_{q=1}^{\infty} q^{-1}[e/(1 + (1 - e^2)^{1/2})]^q \sin qU. \qquad (17\,d)$$

Let us also introduce for convenience the total mass of the system $M := m + m'$ and the (relativistic) « semi-major axis » of the relative orbit $(\vec{R} = \vec{r} - \vec{r}')$ [see equation I(6.3 *a*)]

$$a_R = \frac{M}{m'} a_r. \qquad (18)$$

Now the explicit link between « the proper time of the pulsar », T, and the coordinate time t, has been investigated (assuming everywhere weak fields) by Blandford and Teukolsky [*3*]. A recent work valid in the actual case involving strong gravitational fields and orbital motion has shown that the final result is the same in General Relativity [*8*]. Particular cases (involving no orbital motion) of this remarkable « field strength indifference» have been recently worked out by Will [*22*]. Renormalizing T so that it measures the same « orbital period » as t leads to the following explicit expression for the « Einstein delay » of equation (12):

$$\Delta_{\mathrm{E}} = \gamma \sin \mathrm{U} + \mathrm{O}\left(\left(\frac{v}{c}\right)^4 \mathrm{P}_b\right), \tag{19}$$

with $\gamma = e_t \delta / n$ where

$$\delta = \frac{\mathrm{G}}{c^2 a_{\mathrm{R}}} \cdot \frac{m'(m + 2m')}{\mathrm{M}}. \tag{20}$$

Replacing equation (19) in equations (12) and (16 *a*) allows one to relate the eccentric-anomaly-type parameter U to the proper time T:

$$n(\mathrm{T} - \mathrm{T}_0) = \mathrm{U} - \mathring{e}_{\mathrm{T}} \sin \mathrm{U}, \tag{21}$$

where $\mathring{e}_{\mathrm{T}} = e_t(1 + \delta)$. Equations (21), (16 *b*) and (16 *c*) give a parametric « quasi-Newtonian » representation of the motion of the pulsar expressed in proper time T. This is just what is needed to turn equation (13) into a well-defined timing formula linking τ_a to T. Remarkably enough it is shown in Appendix A that this timing formula can be further simplified by introducing a new eccentric-anomaly-type parameter u, defined as a function of $n(\mathrm{T} - \mathrm{T}_0)$ and of a new eccentricity-type parameter $e_{\mathrm{T}} := \mathring{e}_{\mathrm{T}} + e_\theta - e_r$ [say $u = \mathrm{K}_{e_{\mathrm{T}}}(n(\mathrm{T} - \mathrm{T}_0))$] through a usual Kepler equation:

$$n(\mathrm{T} - \mathrm{T}_0) = u - e_{\mathrm{T}} \sin u. \tag{22}$$

2.6. The explicit timing formula.

The use of the function $u(\mathrm{T})$ defined by equation (22), together with the equations (3), (6), (9), (11), (13), (14), (15), (16) and (19), lead to the following timing formula (see appendix A for some details):

$$\mathrm{D}.\tau_a = \mathrm{T} + \Delta_{\mathrm{R}}(\mathrm{T}) + \Delta_{\mathrm{E}}(\mathrm{T}) + \Delta_{\mathrm{S}}(\mathrm{T}) + \Delta_{\mathrm{A}}(\mathrm{T}) + \mathrm{O}\left(\left(\frac{v}{c}\right)^4 \mathrm{P}_b\right), \tag{23}$$

$$\Delta_{\mathrm{R}}(\mathrm{T}) = \frac{a_r \sin i}{c} \{ \sin \omega \, [\cos u - e_r] + (1 - e_\theta^2)^{1/2} \cos \omega \sin u \}, \tag{24}$$

$$\Delta_{\mathrm{E}}(\mathrm{T}) = \gamma \sin u, \tag{25}$$

$$\Delta_S(T) = -\frac{2Gm'}{c^3}\log\{1 - e\cos u - \sin i\,[\sin\omega(\cos u - e) + (1-e^2)^{1/2}\cos\omega\sin u]\}, \quad (26)$$

$$\Delta_A(T) = A\{\sin(\omega + A_e(u)) + e\sin\omega\} + B\{\cos(\omega + A_e(u)) + e\cos\omega\}, \quad (27)$$

where T is linked to the number of the pulse N by

$$N = N_0 + \nu T + \frac{1}{2}\dot{\nu}T^2 + \frac{1}{6}\ddot{\nu}T^3, \quad (28)$$

and where the slowly-precessing relativistic « argument of the periastron » $\omega(T)$ is given by

$$\omega = \omega_0 + kA_e(u), \quad (29)$$

where the function $A_e(x)$ has been defined by equations (17). (In the previous and following formulas when there appears an eccentricity e without index this means that it belongs already to a small relativistic correction and that e can indifferently be replaced by any eccentricity linked with the system, e. g. e_T of equation (22), because e_T, e_r, e_θ, ... differ only by a few times $e\ GM/a_Rc^2$). Posing $e_r = e_T(1 + \delta_r)$, $e_\theta = e_T(1 + \delta_\theta)$, we see that the RHS of equation (23), say $T + \Delta$, is a function not only of T but of the following set of 13 parameters:

$$\{\xi^i; 1 \leqslant i \leqslant 13\} = \{n, T_0, e_T, a_r\sin i, \omega_0, k, \delta_r, \delta_\theta, \gamma, m', \sin i, A, B\}, \quad (30)$$

so that by inverting equation (23) (see below) and replacing the result in equation (28) one would get N as a function of τ_a and of 18 parameters: D and $\{\xi^\mu; 1 \leqslant \mu \leqslant 17\}$,

$$N = \mathcal{N}(D.\tau_a, \xi^\mu), \quad (31)$$

$$\{\xi^\mu; 1 \leqslant \mu \leqslant 17\} = \{\xi^i, N_0, \nu, \dot{\nu}, \ddot{\nu}\}. \quad (32)$$

The theory of General Relativity gives the following equations between the parameters ξ^i and the two (Schwarzschild) masses of the binary system, $m = m_{\text{pulsar}}$, $m' = m_{\text{companion}}$ [remember $M := m + m'$, $a_R := (M/m')a_r$, and see equations (3.7), (3.8 *b*), (4.13), (4.15) and (6.3*b*) of paper I, and equations (9) and (20) of this paper]

$$n = \left(\frac{GM}{a_R^3}\right)^{1/2}\left[1 + \left(\frac{mm'}{M^2} - 9\right)\frac{GM}{2a_Rc^2}\right], \quad (33)$$

$$k = \frac{3GM}{c^2a_R(1-e^2)}, \quad (34)$$

$$\gamma = \frac{e}{n}\cdot\frac{Gm'(m+2m')}{c^2a_RM}, \quad (35)$$

$$\frac{e_r - e_T}{e_T} \equiv \delta_r = \frac{G}{c^2a_RM}\{3m^2 + 6mm' + 2m'^2\}, \quad (36)$$

$$\frac{e_\theta - e_T}{e_T} \equiv \delta_\theta = \frac{G}{c^2 a_R M}\left\{\frac{7}{2}m^2 + 6mm' + 2m'^2\right\}, \tag{37}$$

$$A = -\frac{na_r \sin\eta}{2\pi c\nu \sin\lambda(1-e^2)^{1/2}}, \tag{38}$$

$$B = -\frac{na_r \cos i \cos\eta}{2\pi c\nu \sin\lambda(1-e^2)^{1/2}}, \tag{39}$$

where λ and η are the polar angles of the spin vector of the pulsar, $\vec{e}_3$, with respect to the triad $(\vec{e}_X, \vec{e}_Y, \vec{e}_Z)$ defined by equations (15):

$$\vec{e}_3 = \sin\lambda \cos\eta \vec{e}_X + \sin\lambda \sin\eta \vec{e}_Y + \cos\lambda \vec{e}_Z.$$

It should be noted that our equation (27) for Δ_A differs in two respects from the corresponding result of Smarr and Blandford [*15*] [their equation (2.7)]: 1) there is an overall sign change due to their non standard convention $\vec{e}_Z = + \vec{n}$ and, 2) the slowly-varying terms $Ae \sin\omega$ and $Be \cos\omega$ are absent in their result (probably because they were interested only in the rapidly varying part of Δ_A). It can be noticed also that the dimensionless relativistic parameters k, δ_r, δ_θ and $\delta = n\gamma/e$ are all of order ε where $\varepsilon := GM/c^2 a_R = Gm'/c^2 a_r$ ($\varepsilon = 2.14 \times 10^{-6}$ for PSR 1913+16). More precisely, in order to estimate the relative magnitudes of the various effects contained in the timing formula let us introduce an adapted system of units such that $1 = a_R = n (= 2\pi/P_b)$. In this system the velocity of light becomes a pure number $c (= cP_b/2\pi a_R \simeq 683.2$ for PSR 1913+16) and $1/c$ can be taken as the small parameter of the formal post-Newtonian expansions (note that $\varepsilon \simeq 1/c^2$). Then the Roemer time delay Δ_R is (numerically) of order $1/c$, the Einstein time delay Δ_E is of order $1/c^2$, the Shapiro time delay Δ_S is of order $1/c^3$ and the aberration time delay Δ_A is of order $(P_p/P_b).(1/c)$ (where $P_p := 1/\nu$). For PSR 1913+16 $P_p/P_b \simeq 2.115\,10^{-6} \sim 1/c^2$, so that Δ_A is of order $1/c^3$ like Δ_S. We shall assume here that Δ_A is always small enough to be able to neglect any product of Δ_A by another small term.

2.7. The inverse timing formula.

Finally let us give explicit formulas for inverting equation (23), i.e., posing for simplicity's sake

$$t := D.\tau_a, \tag{40}$$

$$\Delta(T) := \Delta_R(T) + \Delta_E(T) + \Delta_S(T) + \Delta_A(T), \tag{41}$$

for solving the equation (23), $t = T + \Delta(T)$, for T:

$$T = t - \overline{\overline{\Delta}}(t). \tag{42}$$

A first, sufficiently accurate, explicit expression for the function $\bar{\bar{\Delta}}(t)$ is obtained by iteration and should be convenient to use in a computer program:

$$\bar{\bar{\Delta}}(t) = \Delta(t - \Delta(t - \Delta(t))) + O\left(\frac{1}{c^4}\right). \tag{43}$$

Posing

$$\Delta_{RE}(T) := \Delta_R(T) + \Delta_E(T), \tag{44}$$

equation (43) can also be written as:

$$\bar{\bar{\Delta}}(t) = \Delta_{RE}(t - \Delta_{RE}(t - \Delta_{RE}(t))) + \Delta_S(t) + \Delta_A(t) + O\left(\frac{1}{c^4}\right). \tag{45}$$

Let us emphasize that $\Delta(t)$, $\Delta_{RE}(t)$, ... denote the values of the mathematical *functions* $\Delta(T)$, $\Delta_{RE}(T)$, ... taken for the value $T = t$ of the independent variable, as computed by replacing T by t in equation (22) [for instance $\Delta_E(t) = \gamma \sin K_{e_T}(n(t - T_0))$]. It is easily checked that in computing the iterations of equation (45) ω can be considered as a constant [$= \omega_0 + kA_e(K_{e_T}(n(t - T_0)))$]. Let us also quote another explicit analytical expression for $\bar{\bar{\Delta}}(t)$. Posing

$$\alpha := \frac{a_r \sin i}{c} \sin \omega, \tag{46}$$

$$\beta := \frac{a_r \sin i}{c}(1 - e_\theta^2)^{1/2} \cos \omega, \tag{47}$$

$$\Delta_{RE} = \alpha(\cos u - e_r) + (\beta + \gamma) \sin u, \tag{48}$$

$$\Delta'_{RE} = -\alpha \sin u + (\beta + \gamma) \cos u, \tag{49}$$

$$\Delta''_{RE} = -\alpha \cos u - (\beta + \gamma) \sin u, \tag{50}$$

$$\hat{n} := n/(1 - e_T \cos u), \tag{51}$$

one has

$$\bar{\bar{\Delta}}(t) = \Delta_{RE} \times \left\{ 1 - \hat{n}\Delta'_{RE} + \hat{n}^2(\Delta'_{RE})^2 + \frac{1}{2}\hat{n}^2\Delta_{RE}\Delta''_{RE} - \frac{1}{2}\frac{e_T \sin u}{(1 - e_T \cos u)}\hat{n}^2\Delta_{RE}\Delta'_{RE} \right\} + \Delta_S(t) + \Delta_A(t), \tag{52}$$

where $u = K_{e_T}(n(t - T_0))$ is the solution of $u - e_T \sin u = n(t - T_0)$.

3. CONSEQUENCES OF THE TIMING FORMULA

The 18 parameters appearing in the full timing formula (31) can be classified in 9 classical parameters, $\{n, T_0, e_T, a_r \sin i, \omega_0, N_0, \nu, \dot{\nu}, \ddot{\nu}\}$, 6 relativistic parameters $\{k, \gamma, \delta_r, \delta_\theta, m', \sin i\}$, 2 aberration parameters

$\{A, B\}$ and the Doppler parameter D. The 9 classical parameters are easily measurable because they correspond to time delays of order $1/c$. In the timing model presently used by Taylor and coworkers for PSR 1913+16 only 3 relativistic parameters are explicitly introduced and A, B and D are not included. In this section we shall investigate in detail the effect of including the parameters D (§ 3.1), A, B (§ 3.2) and the full 6 relativistic parameters (§ 3.3-§ 3.8).

3.1. Effect of the relative motion between the solar system and the binary system: D.

In the timing model presently used for analyzing the arrival times of the signals from binary pulsars the Doppler factor D is replaced by one, on the grounds that probably $v_b/c \sim 10^{-3}$ so that D will introduce only small constant relative errors in the measured elements of the binary system. However in view of our total lack of *a priori* knowledge of the exact value of D it is important to investigate precisely its influence on the measured elements. Moreover it is especially important to control the effect of D on the tests of relativistic theories of gravity performed when enough parameters are measured. Indeed if all the « measured » elements contain relative errors $\sim D - 1 \sim$ a few per thousands then a test involving several of these elements (4 in the case of $\dot{P}_b$) might be intrinsically vitiated at the level of may be a percent. Another reason for investigating the precise role of D is that a recent work of Portilla and Lapiedra [*14*], analyzing the motion of the binary system in the center-of mass-frame of the *solar* system, has suggested that the relative motion between the two systems could induce additional apparent orbital period changes.

What is needed is to compare the « wrong » parameters ξ^μ_{wrong} obtained by fitting the measured arrival times τ_a by means of the formula

$$N = \mathcal{N}(\tau_a, \xi^\mu_{\text{wrong}}), \tag{53}$$

to the « true » parameters ξ^μ_{true} satisfying:

$$N = \mathcal{N}(D.\tau_a, \xi^\mu_{\text{true}}). \tag{54}$$

The answer is easily obtained if we notice that the number measuring the arrival times τ_a in the unit system $(\text{CGS})_D = (D \times \text{cm}, D \times \text{gr}, D \times \text{sec})$ is D^{-1} times the number measuring τ_a in the usual (CGS) system. Moreover the numbers measuring Newton's constant G and the velocity of light c are the *same* in $(\text{CGS})_D$ and in (CGS). Then it is easily seen that the effect of « forgetting » D and still using equations of the type (33)-(39) to connect the « measured » parameters ξ^μ_{wrong} to the dynamical parameters of the system (a_R, m, P_b, e_T, ...) is equivalent to measuring inadvertently the latter dynamical parameters in the $(GGS)_D$ system instead of the (CGS) one (hence $a_R^{wrong} = a_R^{true}/D$, $m^{wrong} = m^{true}/D$, $P_b^{wrong} = P_b^{true}/D$, $e_T^{wrong} = e_T^{true}$, etc...). An important consequence of this type of error is that, because of dimensional analysis, it will leave « invariant » any test of relativistic theories of gravity (involving only G and c): for instance the « $\dot{P}_b$ » and « sin i » tests now performed in PSR 1913+16 [*19*] must be equally verified or falsified using the « true » or the « wrong » parameters. This conclusion does not apply to tests involving other physical constants (e. g. h, $m_{protron}$, ...) whose numerical values are changed in the $(CGS)_D$ system: an example would be the test that m must be smaller than the maximal mass of a neutron star. We reach also the conclusion that the work of Portilla and Lapiedra [*14*] is misleading and concerns only coordinate effects (assuming that both $\vec{n}$ and $\vec{v}_b$ are constant in time to a high enough accuracy for allowing one to consider the orbital period shift; $D = P_b^{true}/P_b^{wrong}$ as constant).

3.2. **Effect of the aberration:** A, B.

As was noticed at the end of § 2 the aberration time delay Δ_A is of the same order of magnitude ($1/c^3 \sim 20\ \mu$ sec) as the Shapiro time delay in the binary pulsar PSR 1913 + 16. Moreover as our formula for Δ_A, equation (27), contains secularly changing terms omitted by Smarr and Blandford [*15*] it is important to reconsider the possibility (apparently excluded by Epstein [*11*] although without detailed proof) that the aberration effects might be detected. Such a detection would give the first direct evidence that pulsars are indeed rotating beacons.

A straightforward calculation shows that the function Y (T, σ, x, e_T, e_c, e_s) introduced in appendix A satisfies the following relations:

$$x\frac{\partial Y}{\partial x} + e\frac{\partial Y}{\partial e_T} - 2e\frac{\partial Y}{\partial e_c} = x\sin(\omega + A_e(u)) + xe\sin\omega + O\left(\frac{1}{c^2}\right), \qquad (55)$$

$$(1-e^2)^{1/2}\frac{\partial Y}{\partial \sigma} = x\cos(\omega + A_e(u)) + xe\cos\omega + O\left(\frac{1}{c^2}\right). \qquad (56)$$

From these relations, and the fact that $\Delta_R(T) = Y(T, -nT_0, (a_r \sin i)/c, e_T, e_r, e_\theta)$ we deduce that we can absorb *exactly* Δ_A into Δ_R by making the following replacements in Δ_R: $-nT_0 \to -nT_0 + (1-e^2)^{1/2}\varepsilon_B$, $(a_r \sin i)/c \to (a_r \sin i)/c.(1 + \varepsilon_A)$, $e_T \to e_T(1+\varepsilon_A)$, $e_r \to e_r(1-2\varepsilon_A)$ and $e_\theta \to e_\theta$ where we have introduced $\varepsilon_A := cA/a_r \sin i$ and $\varepsilon_B := cB/a_r \sin i$. Therefore if we omit Δ_A from the timing formula, as is done by Taylor and coworkers, we shall « measure » for instance $(a_r \sin i)^{wrong} = (1+\varepsilon_A) \times (a_r \sin i)^{true}$, $e_T^{wrong} = (1 + \varepsilon_A)e_T^{true}$, etc... Now if the spin axis of the pulsar $\vec{e}_3$ is not orthogonal to the plane of the orbit it will slowly precess by a few degrees per year for PSR 1913+16 [*10*] [*1*]. This will induce secular variations of

$(a_r \sin i)^{\text{wrong}}$ and e_T^{wrong} of relative order $\sim 10^{-6}$ in 10 years. It would be interesting to reanalyze the observational data and to look for such secular variations of $(a_r \sin i)^{\text{wrong}}$ and e_T^{wrong}. If no such variations are found this will confirm the hypothesis (already suggested by the absence of variation of the average pulse shape) that $\vec{e}_3$ is orthogonal to the orbital plane. If we further assume that $\vec{e}_3$ is parallel, rather than antiparallel, to the orbital angular momentum (as seems plausible from evolutionary considerations) we can determine λ and η: $\lambda = i$ and $\eta = -\pi/2$. Hence we can compute Δ_A:

$$\Delta_A = A_0 \left[\sin(\omega + A_e(u)) + e \sin \omega\right], \tag{57 a}$$

with

$$A_0 = na_r/(2\pi c v \sin i(1 - e^2)^{1/2}). \tag{57 b}$$

Numerically one finds for PSR 1913 + 16, $A_0 = 12.1\ \mu$ sec and $\varepsilon_A = 5.18 \times 10^{-6}$. Note that ε_A is comparable to the present relative error for e as measured by Weisberg and Taylor [*19*]. In order to be able to meaningfully take advantage of the ever increasing precision on the classical parameters of the system, we therefore propose, granted that it has been checked that no secular variations of $(a_r \sin i)^{\text{wrong}}$ and e_T^{wrong} are present, to include $\Delta_A = A_0(\sin(\omega + A_e(u)) + e \sin \omega)$ in the timing formula as a parameter-free contribution (i. e. A_0 being computed, $A_0 = 12.1\ \mu$ sec, instead of fitted for).

3.3. The post-Newtonian periodic (PNP) parameters.

Among the six relativistic parameters, three $(k, \delta_r, \delta_\theta)$ appear at lowest order in the Roemer time delay,

$$\Delta_R = \frac{a_r \sin i}{c} \{ \sin(\omega_0 + kA_e(u)) [\cos u - e_T(1+\delta_r)] + (1 - e_T^2(1+\delta_\theta)^2)^{1/2} \cos(\omega_0 + kA_e(u)) \sin u \}, \tag{58}$$

one (γ) appears at lowest order in the Einstein time delay,

$$\Delta_E = \gamma \sin u, \tag{59}$$

and the remaining two $(m', \sin i)$ in the Shapiro time delay

$$\Delta_S = -\frac{2Gm'}{c^3} \log \{ 1 - e \cos u - \sin i [\sin \omega (\cos u - e) + (1 - e^2)^{1/2} \cos \omega \sin u] \}, \tag{60}$$

where $\omega = \omega_0 + kA_e(u)$.

γ contributes at the highest order $(1/c^2)$ among the relativistic parameters but can be separated only with difficulty from the measurement of the classical parameter $(1 - e_T^2)^{1/2} a_r \sin i \cos \omega_0$ (it is now measured with a relative precision of about 3 % [*19*]).

k contributes only at the formal order $1/c^3$ but in a secular way which makes its measurement both easy and precise (relative precision better than 10^{-4} for $kn = \dot{\omega}$ [*19*]).

The four remaining parameters, δ_r, δ_θ, m' and $\sin i$, all contribute at the order $1/c^3$ in a quasi-periodic way. In the post-Newtonian timing model of Epstein [*11*] only one such Post-Newtonian-Periodic (PNP) parameter, say $(\sin i)_E$, was introduced. This reduction in the number of PNP parameters was achieved by expressing all PNP parameters (in our model δ_r, δ_θ, m' and $\sin i$) in terms of $\sin i$, of the classical parameters n, e_T, $a_r \sin i$, and of γ by means of equations (18), (33) and (35), and then by holding n, e_T, $a_r \sin i$ *and* γ fixed at their already measured value. This procedure has been applied by Weisberg and Taylor [*19*] and has led, after replacement of the wrong equation (A 28) of Epstein [*11*] by the correct periastron precession (see e. g. equation (29) above), to the measurement of $(\sin i)_E$ with a relative precision of 20 %.

We wish instead to investigate here the possibility of measuring simultaneously several PNP parameters, thereby hoping to get new tests of the model, and in fact, strictly speaking, (as discussed in the introduction) to get the first real tests of the relativistic theory of gravity used in the description of the binary system.

3.4. The timing formula in other relativistic theories of gravity.

Let us first emphasize that the structure of the timing formula (22)-(29) will probably be the same in most viable relativistic theories of gravity: only the precise equations (18) and (33)-(39) linking the relativistic parameters between themselves and to the « inertial masses » of the two objects will depend upon the theory of gravity used to study the motion of the binary.

Indeed the simple quasi-Newtonian structure of $\Delta_R(T)$, equations (24) with (22), comprising 3 relativistic parameters k, $(e_r - e_T)/e_T$ and $(e_\theta - e_T)/e_T$, was a straightforward consequence of the structure of the quasi-Newtonian solution (16) and of the structure (19) for Δ_E. Now, as can be seen from the proof used in paper I, the quasi-Newtonian structure for the motion is a consequence of the existence of a general post-Newtonian Lagrangian $L = L_0 + c^{-2}L_2$ (where L_2 is a polynomial in the velocities and in the interaction potential) and of a relativistic center of mass theorem. Now such a Lagrangian and such a theorem exist for most viable relativistic theories of gravity (even for the motion of condensed objects, see [*20*]). Moreover it seems that Δ_E has the same structure in all theories of gravity [*20*]. Besides, the contribution to Δ_S due to the companion, being a weak field effect, will have always the same structure (but it can be shown that the coefficient in front, $-2Gm'/c^3$, will be replaced by $-G(\alpha_2^* + \gamma_2^*)m_2/c^3$

in the notations of Will [20], chap. 11). Therefore if we assume that the strong field effects of the various relativistic theories of gravity can be treated, with respect to their influence on the timing formula, in a way at least qualitatively similar to the strong field effects of General Relativity a final timing formula of the type (22-29) should result. Consequently any attempt, as suggested here, to measure independently (i. e. without assuming any relations, à la Epstein, between them) several PNP parameters will yield direct tests of the theories of gravity.

3.5. The differential timing formula.

Let us first recall that the way the various parameters (except D, see § 3.1) appearing in the full timing formula (31) are « measured », starting from initial estimates of the parameters, say $\xi^{\mu}_{(1)}$, is the following: 1) determine the ordinal number N_a of the pulse which arrived at the « time » $t_a \equiv D\tau_a$ as being the integer nearest to $\mathcal{N}(t_a, \xi^{\mu}_{(1)})$, 2) compute the actual « residual » $R^{\text{actual}}(t_a) := (\mathcal{N}(t_a, \xi^{\mu}_{(1)}) - N_a)/(\partial\mathcal{N}/\partial t)_{(1)}$, 3) estimate the corrections, $\delta\xi^{\mu}_{(1)} = \xi^{\mu} - \xi^{\mu}_{(1)}$, to be added to $\xi^{\mu}_{(1)}$ by a least-squares fit of

$$R_{(1)}(t_a) = -(\partial\mathcal{N}/\partial t)^{-1}_{(1)} \sum_{\mu} (\partial\mathcal{N}/\partial\xi^{\mu})_{(1)}\delta\xi^{\mu}_{(1)}$$

to the actual residuals $R^{\text{actual}}(t_a)$, 4) iterate the process (see [3] [11]).

With sufficient accuracy one finds for the contributions of the parameter corrections to the residuals (« differential timing formula »):

$$R_{(1)}(t_a) = \left[-\frac{\delta N_0}{v} - \frac{t_a}{v}\delta v - \frac{t_a^2}{2v}\delta\dot{v} - \frac{t_a^3}{6v}\delta\ddot{v} + \sum_{i=1}^{11} C_i(t_a, \xi)\delta\xi^i \right]_{(1)}, \quad (61)$$

with $C_i(t, \xi^j) = \partial\Delta(t, \xi^j)/\partial\xi^i$, $\Delta(t, \xi^i)$ being the function (41), defined by equations (22)-(27), taken at the value $T = t$, with $\{ \xi^i ; 1 \leqslant i \leqslant 11 \}$ being the set of parameters (30) minus A and B. One finds explicitly (posing $x := a_r \sin i/c$, and using $\sigma := -nT_0$ instead of T_0)

$$C_{\sigma} = x \frac{[-\sin\omega\sin u + (1-e^2)^{1/2}\cos\omega\cos u]}{1 - e\cos u}, \quad (62a)$$

$$C_n = t . C_{\sigma}, \quad (62b)$$

$$C_{e_T} = \sin u C_{\sigma} - x\sin\omega - (1-e^2)^{-1/2} e x \cos\omega \sin u, \quad (62c)$$

$$C_x = \sin\omega(\cos u - e) + (1-e^2)^{1/2}\cos\omega\sin u, \quad (62d)$$

$$C_{\omega_0} = x[\cos\omega(\cos u - e) - (1-e^2)^{1/2}\sin\omega\sin u], \quad (62e)$$

$$C_k = A_e(u) . C_{\omega_0}, \quad (62f)$$

$$C_\gamma = \sin u\,, \tag{62 g}$$

$$C_{\delta_r} = -\,ex\sin\omega\,, \tag{62 h}$$

$$C_{\delta_\theta} = -\,(1-e^2)^{-1/2}e^2 x\cos\omega\sin u\,, \tag{62 i}$$

$$C_{m'} = -\frac{2G}{c^3}\log\{\,1-e\cos u-\sin i\,[\sin\omega(\cos u-e)+(1-e^2)^{1/2}\cos\omega\sin u]\,\}, \tag{62 j}$$

$$C_{\sin i} = \frac{2Gm'}{c^3}\frac{\sin\omega(\cos u-e)+(1-e^2)^{1/2}\cos\omega\sin u}{1-e\cos u-\sin i\,[\sin\omega(\cos u-e)+(1-e^2)^{1/2}\cos\omega\sin u]}, \tag{62 k}$$

with

$$nt+\sigma = u - e_T\sin u\,, \tag{63 a}$$

$$\omega = \omega_0 + kA_e(u)\,. \tag{63 b}$$

As the fit is an iterative procedure, we have kept only the largest term in each partial derivative (62) and we have replaced (after differentiation) all the eccentricities by $e(=e_T$ say).

Secular variations (beyond post-Newtonian order) of the elements x, e_T and $P_b = 2\pi/n$ can be incorporated by making the replacements $\delta x \to \delta x + t\delta\dot{x}$, $\delta e_T \to \delta e_T + t\delta\dot{e}_T$, $\delta P_b \to \delta P_b + \frac{1}{2}t\delta\dot{P}_b$ (corresponding to $x \to x + t\dot{x}$, etc. in (31)) [3].

Then one can estimate the accuracy with which the ξ^μ $(1 \leqslant \mu \leqslant 17)$ will be determined when fitting the $R_{(1)}(t_a) = \Sigma_{\mu=1}^{17} C_\mu(t, \xi_{(1)})\delta\xi^\mu_{(1)}$ to the actual $R^{\text{actual}}(t_a)(= C_\mu\delta\xi^\mu_{\text{actual}} + \text{noise})$ by constructing the covariance matrix of the $\delta\xi^\mu_{(1)}$: $V^{\mu\nu} = \langle(\delta\xi^\mu_{(1)} - \langle\delta\xi^\mu_{(1)}\rangle).(\delta\xi^\nu_{(1)} - \langle\delta\xi^\nu_{(1)}\rangle)\rangle$. Assuming that the timing noises of the arrival times t_a are uncorrelated, and have the same variance, say ε^2 (if this last condition is not satisfied it is sufficient to introduce appropriate weights in all the quadratic forms) the covariance matrix $V^{\mu\nu}$ is given by inverting a 17×17 matrix:

$$V^{\mu\nu} = \varepsilon^2\left(\sum_a C_\mu(t_a, \xi_{(1)})C_\nu(t_a, \xi_{(1)})\right)^{-1} \tag{64}$$

(where we use the subscript a as an index numbering the arrival times).

3.6. **Primary fit and « periodic approximation ».**

The contributions of the 4 post-Newtonian periodic parameters δ_r, δ_θ, m' and $\sin i$ to $\Delta(t)$ are only of about 20 μ sec, i. e. at the level of the present timing errors ε. Therefore there will be some hope of measuring them only if they can be cleanly separated from the stronger signals associated with the classical parameters even when neglecting all secular effects (indeed the parameter γ contributes much stronger signals but has been measured only with a reduced accuracy because it is separable from the classical parameter $(1 - e_T^2)^{1/2} a_r \sin i \cos \omega$ *only* in presence of a secular periastron precession). Hence it seems appropriate to estimate the measurability of the four PNP parameters by following the procedure used by Blandford and Teukolsky [*3*], hereafter referred to as BT, in their primary fit concerning an ensemble of measurements done during a time interval much smaller than the post-Newtonian time scale ($\sim$ 90 years for PSR 1913+16). This procedure consists in introducing instead of the 17 ξ^μ a convenient set of independent parameters which absorb all the purely secular time dependence in the sense that these parameters can be expressed as short polynomials in the time t and that the coefficients of the differential timing formula corresponding to these parameters have only a periodic explicit dependence on t. These restrictions (useful if one wanted to do a secondary fit as in BT) oblige us to split the slowly precessing periastron ω (equation (29)) in a purely secular part ω_E (as used by Epstein) and a purely periodic part $k\mathrm{B}_e(u)$ (wrongly omitted by Epstein)

$$\omega = \omega_E + k\mathrm{B}_e(u), \tag{65}$$

where

$$\omega_E := \omega_0 + \dot{\omega}(t - \mathrm{T}_0), \tag{66}$$

($\dot{\omega}$ denoting $k.n$) and

$$\mathrm{B}_e(u) := \mathrm{A}_e(u) - u + e \sin u. \tag{67}$$

Introducing $\beta := e/(1 + (1 - e^2)^{1/2})$, the purely periodic function $\mathrm{B}_e(u)$ can be expanded as (using formula C of chapter XIII of Tisserand [*18*])

$$\mathrm{B}_e(u) = e \sin u + 2\left[\beta \sin u + \frac{1}{2}\beta^2 \sin 2u + \frac{1}{3}\beta^3 \sin 3u + \dots\right]. \tag{68}$$

Now a convenient set of parameters adapted to a primary fit covering a time interval $\ll \mathrm{P}_b/k$ is $\{ \eta^i ; 1 \leqslant i \leqslant 10 \}$ with

$$\delta\eta^1 = -\nu^{-1}\left[\delta\mathrm{N}_0 + t\delta\nu + \frac{1}{2}t^2\delta\dot{\nu} + \frac{1}{6}t^3\delta\ddot{\nu}\right],$$
$$\eta^2 = x \sin \omega_E, \quad \eta^3 = (1 - e_T^2)^{1/2} x \cos \omega_E + \gamma,$$
$$\delta\eta^4 = -\delta\sigma - t\delta n, \quad \eta^5 = e_T, \quad \{ \eta^i ; 6 \leqslant i \leqslant 10 \} = \{ \delta_r, \delta_\theta, m', \sin i, k \}.$$

The corresponding differential timing formula reads (to leading order)

$$\mathrm{R}(t) = \sum_{i=1}^{10} \mathrm{B}_i \delta\eta^i, \tag{69}$$

with

$$B_1 = 1\,, \tag{70 a}$$

$$B_2 = \cos u - e\,, \tag{70 b}$$

$$B_3 = \sin u\,, \tag{70 c}$$

$$B_4 = (\alpha_E \sin u - \beta_E \cos u)/(1 - e\cos u)\,, \tag{70 d}$$

$$B_5 = -\,[\alpha_E + (\alpha_E \sin u - \beta_E \cos u)\sin u/(1 - e\cos u)]\,, \tag{70 e}$$

$$B_{\delta_r} = -\,e\alpha_E\,, \tag{70 f}$$

$$B_{\delta_\theta} = -\,(1 - e^2)^{-1}e^2\beta_E \sin u\,, \tag{70 g}$$

$$B_{m'} = C_{m'}\,, \tag{70 h}$$

$$B_{\sin i} = C_{\sin i}\,, \tag{70 i}$$

$$B_k = [(1 - e^2)^{-1/2}\beta_E(\cos u - e) - (1 - e^2)^{1/2}\alpha_E \sin u].B_e(u)\,, \tag{70 j}$$

where $\alpha_E = x \sin \omega_E$, $\beta_E = (1 - e_T^2)^{1/2} x \cos \omega_E$.

The first 5 parameters η^i ($1 \leqslant i \leqslant 5$) correspond to the five parameters K, α, η, Σ and e of BT. The last term $B_k = (\partial\Delta/\partial\omega).\,B_e(u)$ comes from keeping in the full coefficient of k in the residual, $C_k = (\partial\Delta/\partial\omega).\,A_e(u)$, only its purely periodic part (its purely secular part $(\partial\Delta/\partial\omega).\,n(t-T_0)$ being contained in $(B_2\partial\eta^2/\partial\omega_E + B_3\partial\eta^3/\partial\omega_E)\delta\omega_E$). However $k = \dot\omega/n$ is best measured with good accuracy because of its secular influence in ω_E and not because of its tiny periodic contribution $B_k\delta k$ to the residuals. Therefore when estimating the measurability of the 4 PNP parameters $\{\delta_r, \delta_\theta, m', \sin i\}$ we can consider that k is already accurately known ($\delta k = 0$) and that it is sufficient to fit a reduced timing formula, $R(t) = \sum_{i=1}^{9} B_i\delta\eta^i$; involving only the 5 BT parameters and the 4 extra PNP parameters. The covariance matrix corresponding to such a primary fit will be ε^2 times the inverse of the 9×9 matrix $B_{ij} = \sum_a B_i(t_a)B_j(t_a)$.

Following BT we shall further assume that the measurements are regularly distributed over the orbital phase so that the latter matrix can be approximated by

$$B_{ij} = N \times \frac{1}{2\pi}\int_0^{2\pi} du(1 - e\cos u)B_i(u)B_j(u)\,, \tag{71}$$

where N is the total number of measurements, and where, as we consider an ensemble of measurements done in a time $\ll P_b/k$, we can approximate α_E and β_E by constants during the integration (this defines what one can call the « periodic approximation »).

3.7. « Unmeasurability » of the periodic relativistic corrections to the motion : δ_r, δ_θ.

The PNP parameters δ_r and δ_θ describe purely periodic relativistic corrections to a Keplerian motion. Measuring them would provide stringent tests of the theories of gravity. However it is apparent from the explicit expressions (70) that, in the « periodic approximation », the corresponding « signals » B_{δ_r} and B_{δ_θ} are constant multiples of, respectively, B_1 and B_3. Therefore we get the remarkable result that, in the latter approximation, the post-Newtonian periodic parameters δ_r and δ_θ are not separable from the measurement of the respective classical parameters $-\nu^{-1}\delta\varphi(t)$ (φ being the phase of the pulsar) and $\beta_E(+\gamma)$. Moreover, from what has been said in § 3.4 this result holds not only in General Relativity but in all relativistic theories of gravity. Note however that in the long term δ_θ can in principle be measured when ω_E has changed by a sufficient amount, but previous experience with the measurement of γ suggests that it will be very difficult to measure δ_θ with any decent accuracy. On the other hand it seems hopeless to measure δ_r because the secular variation of $B_{\delta_r} = -e\alpha_E$ can be absorbed in $\delta\varphi(t) = \delta N_0 + t\delta\nu + t^2\delta\dot{\nu}/2 + \dots$

Besides, the « unmeasurability » of δ_r and δ_θ has the interesting consequence of clarifying the significance of the « global » parameter $(\sin i)_E$ which is not as global as it seemed *a priori*. Indeed the only physical effects playing a role in the measurement of $(\sin i)_E$ will come from the Shapiro effect.

The results of this section lead us to propose (like for the aberration effect) to replace, in the timing formula, δ_r and δ_θ by their numerical values (as computed from the already known parameters) without trying to fit for them.

3.8. Measurability of the Shapiro effect : m', sin i.

In view of the results of the preceding section the only hope to get genuine tests of the theories of gravity is to try to measure simultaneously the remaining two PNP parameters m' and $\sin i$ (which describe the Shapiro effect). To estimate the measurability of $\sin i$ and m', we must now consider the 7×7 covariance matrix corresponding to the 5 BT parameters and m' and $\sin i$. We have estimated this matrix numerically as ε^2 times the inverse of the corresponding 7×7 matrix B_{ij}, equation (71). In order to compare the relative accuracies obtainable on $\sin i$ and m' it is convenient to introduce the dimensionless variables s and μ such that $\sin i = (\sin i)_{(1)}(1 + s)$, $m' = (m')_{(1)}(1 + \mu)$, ($(\sin i)_{(1)}$ and $(m')_{(1)}$ being current estimated values). We find that the matrix B_{ij} is indeed invertible, and that taking $e_{(1)} = 0.617$, $x_{(1)} = 2.34$ sec, $(\sin i)_{(1)} = 0.72$, $Gm'_{(1)}/c^3 = 6.86$ μsec and

$\omega_E = 5\pi/4$ (values appropriate to PSR 1913+16 in March 1985) the correlation and standard deviation of s and μ are (keeping only three digits):

$$\rho_{\mu s} = -0.836, \tag{72 a}$$

$$\sigma_s = 1.01\,\varepsilon/\sqrt{N}, \tag{72 b}$$

$$\sigma_\mu = 4.74\,\varepsilon/\sqrt{N}. \tag{72 c}$$

If instead of measuring simultaneously s and μ one measures only one of them, fixing the other one to its calculated value, the corresponding σ's are smaller than their respective values above by the factor $(1-\rho_{\mu s}^2)^{1/2}=0.549$. On the other hand if instead of trying to measure both s and μ one decides to fix some relation between them (say $\mu = \lambda s$ with a fixed slope λ, which is equivalent to posing $m' \propto (\sin i)^\lambda$ and only then to measure say s, then one can show that the corresponding standard deviation will be

$$\sigma_s^{(\lambda)} = \sigma_s \frac{(1-\rho_{\mu s}^2)^{1/2}}{[1 - 2\rho_{\mu s}(\lambda\sigma_s/\sigma_\mu) + (\lambda\sigma_s/\sigma_\mu)^2]^{1/2}}. \tag{73}$$

Such a procedure, measuring « globally » sin i by fixing a relation between m' and sin i, corresponds to the sin i test proposed by Epstein (taking into account the irrelevance of the other PNP parameters as proven above). The relation between m' and sin i chosen by Epstein (γ = fixed quantity) amounts to choosing a slope $\lambda = (m + 5m')/(m + m') = 3$ for PSR 1913+16 (as easily obtained from Epstein's equation (A 29)). For such a positive value of λ, the negative correlation $\rho_{\mu s}$ implies that the measurement of sin i à la Epstein is easier than the measurement of sin i alone and therefore *a fortiori* easier than the measurement of sin i simultaneously with m' (according to equation (73) one finds $\sigma_s^{(3)} = \sigma_s/2.87$). Now Weisberg and Taylor [*19*] who have recently measured « sin i » in « following essentially the procedure outlined by Blandford and Teukolsky and Epstein » (see however § 3.9) attribute to their measurement a relative uncertainty of 20 % which however is, according to what they say, at least twice the formal standard deviation (which should correspond to our $\sigma_s^{(3)}$) (*). We can bring together this information with our estimates (72), (73) which should give an indication of the *relative* measurability of s, μ and $s^{(\lambda)}$ through the values of the ratios $\sigma_\mu/\sigma_s = 4.69$, $\sigma_s/\sigma_s^{(3)} = 2.87$, etc... (the absolute values of our estimates for σ_s, ... are somewhat less reliable because the actual fitting procedure combines data spanning 10 years while we have used a « quasi-periodic approximation » which neglected secular effects). This indicates that if one tries to measure *simultaneously* sin i and m', it should be possible

(*) According to a recent report [*23*] the precision in the measurement of « sin i » by Taylor and Weisberg has recently been improved by about a factor 3. This should make much easier the simultaneous measurement of sin i and m' proposed here.

to get $\sin i$ with an acceptable accuracy ($\sigma_s \sim 30\,\%$). Such a measurement would have a clear physical significance contrarily to the present « global » $\sin i$ measurement. On the other hand the same procedure indicates that in such a measurement the relative uncertainty on m' could be quite high ($\sigma_\mu > 100\,\%$). However we must remember that our estimates above (72), (73) on the one hand are probably pessimistic (because taking into account secular effects should improve the measurability of m' and $\sin i$) and on the other hand are based on a *linearized* least-squares fit and are therefore valid only for small corrections $\delta \sin i$, $\delta m'$. As m' plays the role of an overall « amplitude » factor in the Shapiro time delay Δ_S (60), it seems plausible that if $\sin i$ (which influences the « shape » of Δ_S) is indeed measurable, when trying to measure it simultaneously with m' (as our estimates suggest), then it should be also possible to constrain m' to belong to a useful range of values. Therefore a (may be non-linearized) fit to the fully parametrized Shapiro time delay Δ_S ($\sin i$, m') should give valuable constraints on the simultaneous values of $\sin i$ and m' as some type of elliptic-like region in the ($\sin i$, m') plane (elongated in the m' direction). Using the accurately known mass function of the system this information will lead to a valuable constraint on the simultaneous values of m and m' as some allowed bounded region of the (m, m') plane. This would represent a definite improvement on the presently available $\sin i$-constraint which leads to an allowed infinite strip in the (m, m') plane. In alternative theories of gravity the analysis of this constraint must take into account the facts that the « amplitude factor » m' of Δ_S becomes $(\alpha_2^* + \gamma_2^*)m_2/2$ (in the notations of Will [*20*], chap. 11) and that the equations (33)-(39) are also modified (see [*20*]).

3.9. **Comments on a work of Haugan.**

While finishing to prepare this work for publication we received a preprint by Haugan [*12*] which gives in detail the timing formula actually used by Weisberg and Taylor in their latest work which led to the measure of one post-Newtonian quasi-periodic parameter say $(\sin i)_H$, H standing for Haugan to distinguish it from the $(\sin i)_E = (\sin i)^{(3)}\,(\lambda = 3)$ à la Epstein and from our ($\sin i$). This timing formula is essentially the original one of Epstein [*11*] with correction of the oversight of Epstein replacing ω, equation (29), by ω_E, equation (66). Haugan separates explicitly the effects of the secular (ω_E) and of the periodic ($kB_e(u)$, equation (67), in our notations) parts of ω both in the full timing formula and in the differential timing formula. In our notations this consists in expanding the Roemer time delay $\Delta_R = x\,[\sin\omega\,(\cos u - e_r) + (1 - e_\theta^2)^{1/2}\cos\omega\sin u]$ according to:

$$\Delta_R = x\,[\sin\omega_E\,(\cos u - e_r) + (1 - e_\theta^2)^{1/2}\cos\omega_E\sin u] \\ + kxB_e(u)\,.\,[\cos\omega_E\,(\cos u - e) - (1 - e^2)^{1/2}\sin\omega_E\sin u]\,, \qquad (74)$$

and in replacing everywhere else ω by ω_E (these operations lead to errors that are uniformly-in-time negligible). The same separation is also performed in the corresponding differential timing formula (similarly to what we have done in § 3.6 in our discussion of the primary fit for the PNP parameters). Then the measure of the post-Newtonian parameter $(\sin i)_H$ is defined in even a more « global » manner than $(\sin i)_E$ by constraining not only m' to be function of $\sin i$ via fixing γ but also constraining k (as it appears explicitly in the second term of equation (74)) to be function of $\sin i$ via the equation (34) with fixed $x = a_r \sin i/c$. In our opinion the « global » nature of such parameters (which can imply, as discussed in § 3.8, a somewhat artificial improvement in their measurability) makes their measurement only of weakened significance as tests of the model. Stronger, cleaner and more numerous tests are obtained when measuring several independent parameters.

Pursuing this philosophy we can even propose to try to get further tests of the model by fitting simultaneously for the post-Newtonian parameters m', $\sin i$, $\dot{\omega}$ and k. The parameter k denoting now the factor appearing explicitly in the second term of the expanded Roemer time delay (74) (replacing our previous equation (24)), and $\dot{\omega}$ being the parameter appearing in equation (66) defining ω_E ($\dot{\omega}$ and k being considered as independent parameters parametrizing respectively the secular and the quasi-periodic effects linked with the relativistic advance of the periastron). Then the *a posteriori* verification or falsification of the relation $\dot{\omega} = kn$ will provide a test of the « cleanness » of the system (no non relativistic contributions to $\dot{\omega}$). Note that the test $\dot{\omega} = kn$ will not provide a test of the relativistic theories of gravity because in all such theories the periastron advances at the rate $\omega = \omega_0 + k\mathrm{A}_e(u) = \omega_0 + kn(t - \mathrm{T}_0) + k\mathrm{B}_e(u)$.

If this extra-test could not be performed with an adequate accuracy it should anyway be replaced by a study of the average post-fit residuals plotted as a function of the orbital phase. This plot should exhibit no significant dependence on the orbital phase and should show a $1/\sqrt{\mathrm{N}}$ decrease in function of the number N of measurements used for any given orbital phase. Indeed the fact that the corresponding plot in Taylor and Weisberg [*17*] (their figure 4), using the uncorrected Epstein timing model, does not show such a $1/\sqrt{\mathrm{N}}$ reduction prompted us to reexamine ab initio the problem treated in this article.

4. RECAPITULATION

In this article we have derived a new timing formula for the arrival time analysis of a binary pulsar (§ 2). The full inverse timing formula, $\mathrm{N} = \mathrm{N}(\tau_a)$, can be written compactly by iterating the relativistic time delays [equations (42)-(45)] [the corresponding differential timing formula (residuals) is given in § 3.5]. This timing formula is much simpler (in its description of the effects due to the relativistic motion of the pulsar) and more complete (in its description of relative motion and aberration effects) than the presently existing timing formulas which are being used to fit the observational data of the Hulse-Taylor pulsar PSR 1913+16.

Using this formula we have first proven (§ 3.1) that the relative motion between the solar system and the binary system cannot be detected in the observational data and leads to measuring « wrong » parameters for the binary system corresponding to using unawares a modified system of units: (D × cm, D × gr, D × sec), D being given by equation (5).

As for aberration effects (§ 3.2) we have shown that, if the spin of the pulsar is not orthogonal to the orbital plane, part of the aberration effects can in principle be detected by looking for secular variations of $(a_r \sin i)^{\text{wrong}}$ and e_T^{wrong} as deduced by fitting the data to the presently existing timing formulas (which do not include aberration effects). Such a detection would provide the first direct evidence that a pulsar is a rotating beacon. If no such variations are found we propose to assume that the spin vector of the pulsar is parallel to the orbital angular momentum and to include the correspondingly simplified and calculable (from the existing data) aberration effect as a parameter-free contribution to the timing formula (i. e. without trying to fit for it).

Our simplified treatment of the relativistic effects due to the orbital motion of the pulsar (based on our previous work I) has moreover allowed us to reach the following conclusions not easily obtainable from the intricate timing formulas presently in use. The first conclusion (§ 3.7) is that the quasi-periodic relativistic effects coming (in a harmonic coordinate system) from the non secular relativistic corrections to the Keplerian orbital motion of the pulsar are practically not detectable in the data (only one of them, parametrized by $\delta_\theta = (e_\theta - e_T)/e_T$, may be measurable, although probably with reduced accuracy, in the long term thanks to the secular periastron precession). This result holds in all relativistic theories of gravity, showing that the only post-Newtonian timing effects that are detectable are the effects linked with the relativistic advance of the periastron and the effects linked (in harmonic coordinates) with the variable Shapiro time delay due to the propagation of the radio pulses in the weak relativistic gravitational field of the companion.

We therefore propose, and this is our second conclusion, to try to extract more information from these effects than is done now. In the presently used fitting procedure only one parameter linked globally with all the quasi-periodic post-Newtonian timing effects is measured. We propose instead to measure simultaneously two independent parameters (m' and $\sin i$) giving the amplitude and the shape of the Shapiro time delay.

This measurement could provide not only a new test of the model but in fact the first test of the model which is independent of any hypothesis of « cleanness » of the system and thereby the first real test of the relativistic theory of gravity used in describing the motion. We have given a rough estimate (§ 3.8) of the relative accuracy which could be achieved in such a test by means of a « periodic approximation » (§ 3.6). This estimate shows that such a new test is possible in principle ($\rho_{m' \sin i} \neq \pm 1$) but that probably some kind of non-linearized fit will be necessary to get useful constraints on the allowed values of m' and $\sin i$.

Finally in § 3.9 we suggest, without estimating its feasibility, an additional test (based on the effects linked with the relativistic advance of the periastron) which however will not be a test of the relativistic theories of gravity, but only of the « cleanness » of the system.

APPENDIX A

Let us define a function Y of the 11 variables T, σ, n, e_T, e_c, e_s, e, ω_0, h, k and x by the following parametric representation:

$$Y = x\,[\sin\omega\,[\cos u - e_c] + (1 - e_s^2)^{1/2}\cos\omega\sin u]\,, \tag{A1}$$

$$\omega = \omega_0 + h\sin f + kf, \tag{A2}$$

$$f = A_e(u) \equiv 2\arctan\left[\left(\frac{1+e}{1-e}\right)^{1/2}\tan\frac{u}{2}\right], \tag{A3}$$

$$nT + \sigma = u - e_T\sin u\,. \tag{A4}$$

In order to obtain the explicit expression of Y(T, ...) one must solve the Kepler equation (A4) in u (we assume $e_T < 1$), and replace the result, say $u = K_{e_T}(nT + \sigma)$, in (A1-3). In our problem Y will represent the Roemer time delay Δ_R, i. e. $(\sin i)/c$ times the ordinate of the position of the pulsar in the orbital plane expressed as a function of the « proper time » T of the pulsar and of various other Newtonian and post-Newtonian parameters. We can consider as basic Newtonian parameters $\sigma\,(= -nT_0)$, n, e_T, ω_0 and x. The post-Newtonian parameters (numerically of order Gm'/c^2a_r) will be $(e_c - e_T)/e_T$, $(e_s - e_T)/e_T$, h and k. Because of these orders of magnitude Y is in fact independent, to order $O(1/c^4)$, of the eccentricity-like parameter e as long as $e = e_T + O(1/c^2)$ which will be the case in our problem.

It now happens that the function Y (T, e_T, e_c, e_s, h, ...) satisfies some remarkable mathematical identities which have important physical implications for the measurability of various physical effects: for instance equations (55)-(56) of the text and equation (A5) below. Using well-known techniques for differentiating implicit functions and the equation (17 c), a straightforward calculation indeed shows that:

$$\frac{\partial Y}{\partial e_T} + \frac{\partial Y}{\partial e_s} = \frac{1}{1-e^2}\frac{\partial Y}{\partial h} + O\left(\frac{1}{c^2}\right). \tag{A5}$$

This means that if ε is an *arbitrary* number of order $O(1/c^2)$ then

$$Y\left(T, e_T + \varepsilon, e_c, e_s + \varepsilon, h - \frac{\varepsilon}{1-e^2}, \ldots\right) = Y(T, e_T, e_c, e_s, h, \ldots) + O\left(\frac{1}{c^4}\right), \tag{A6}$$

in other words Y is invariant under a proper simultaneous shift of e_T, e_s and h of post-Newtonian order.

Now the parametric representation of the Roemer time delay,

$$\Delta_R = -\frac{1}{c}\,\vec{n}\,.\,\vec{r} = +\frac{1}{c}\sin i\,.\,r\,.\sin\theta\,, \tag{A7}$$

using the eccentric-anomaly-type parameter of the « quasi-Newtonian » solution of paper I (denoted U here, and u in I) i. e. using equations (16 b), (16 c) and (21) of this paper, is

$$\Delta_R = x(1 - e_r\cos U)\sin[\omega_0 + (1+k)A_{e_\theta}(U)]\,, \tag{A8}$$

where $x := a_r\sin i/c$. If we were to develop directly (A8) using $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ with $\alpha = \omega_0 + kA_{e_\theta}(U)$ and $\beta = A_{e_\theta}(U)$ this would lead (with equations (17)) to a troublesome ratio $(1 - e_r\cos U)/(1 - e_\theta\cos U)$. One way for eliminating such a ratio is to use the following trick. It is easy to check that

$$\frac{\partial A_e(U)}{\partial e} = \frac{1}{1-e^2}\sin A_e(U)\,, \tag{A9}$$

so that we can write

$$A_{e_\theta}(U) = A_{e_r}(U) + \overset{\circ}{h} \sin A_e(U) + O\left(\frac{1}{c4}\right), \tag{A10}$$

with

$$\overset{\circ}{h} = \frac{e_\theta - e_r}{1 - e^2}. \tag{A11}$$

This allows to decompose θ in $\alpha' + \beta'$ with

$$\alpha' = \omega_0 + \overset{\circ}{h} \sin f + kf \tag{A12}$$

and

$$\beta' = A_{e_r}(U). \tag{A13}$$

Now the standard addition formula for $\sin(\alpha' + \beta')$ introduces the ratio $(1 - e_r \cos U)/(1 - e_r \cos U) \equiv 1$. Remembering equation (21) this leads to the following expression for Δ_R expressed as a function of T:

$$\Delta_R = Y(T, \sigma = -nT_0, e_T = \overset{\circ}{e}_T, e_c = e_r, e_s = e_r, h = \bar{h}, x = a_r \sin i/c). \tag{A14}$$

This expression is already simple and « quasi-Newtonian » (compared to the much more intricate corresponding expression of Epstein [*11*]) but it can be even further simplified. Indeed by using the transformation (A6) with $\varepsilon = e_\theta - e_r$ we can transform h to zero, i. e. get a simple advance of the « periastron argument » $\omega = \omega_0 + kf$, instead of equation (A2) $\equiv$ (A12). This leads to

$$\Delta_R = Y(T, e_T = \overset{\circ}{e}_T + e_\theta - e_r, e_c = e_r, e_s = e_\theta, h = 0), \tag{A15}$$

with still $\sigma = -nT_0$ and $x = a_r \sin i/c$, which is precisely the result given in equations (22), (24) and (29) of the text.

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