# *State of the art for QCD corrections in DIS*

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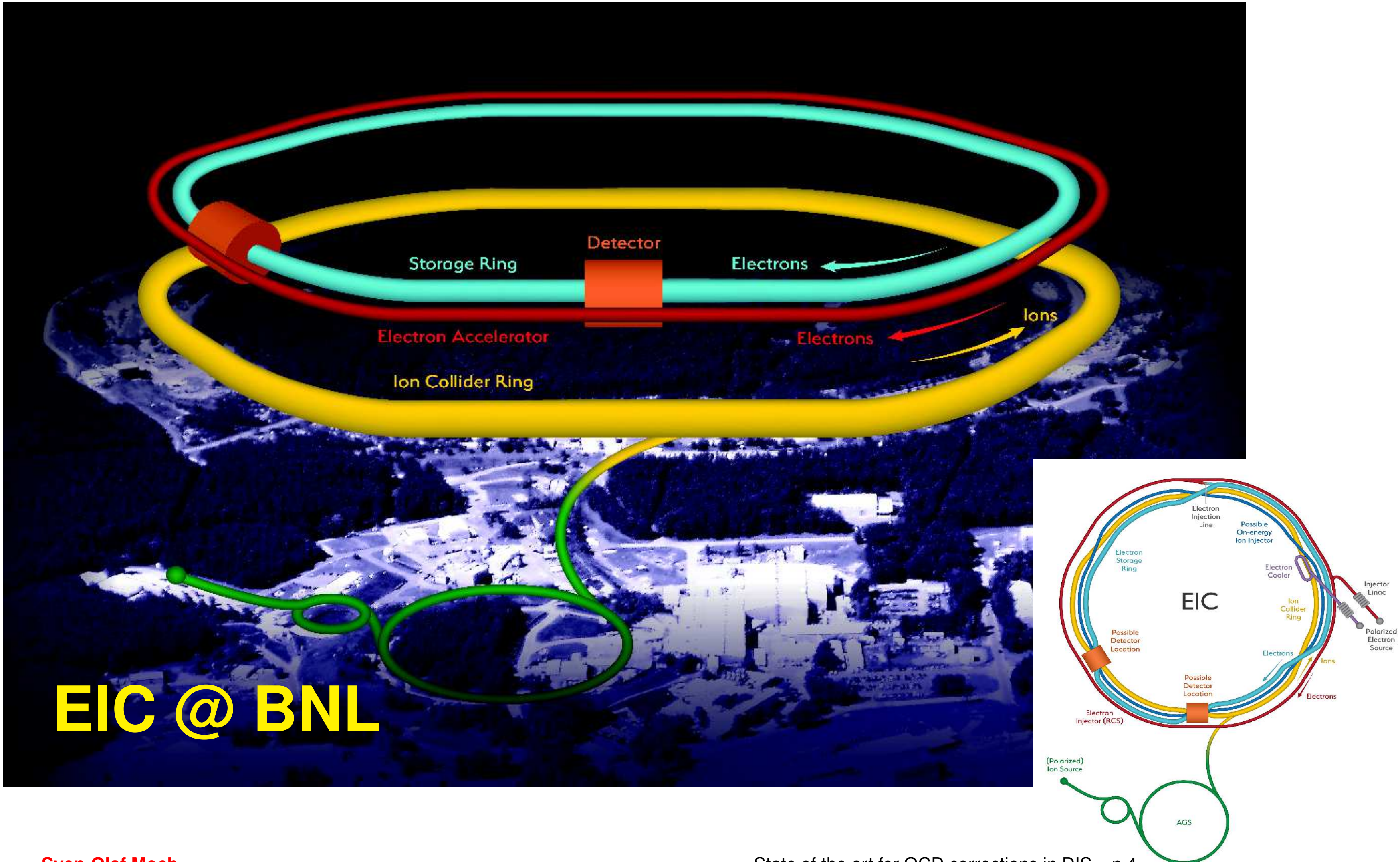
# *Deep-inelastic scattering*

# *Once upon a time ...*

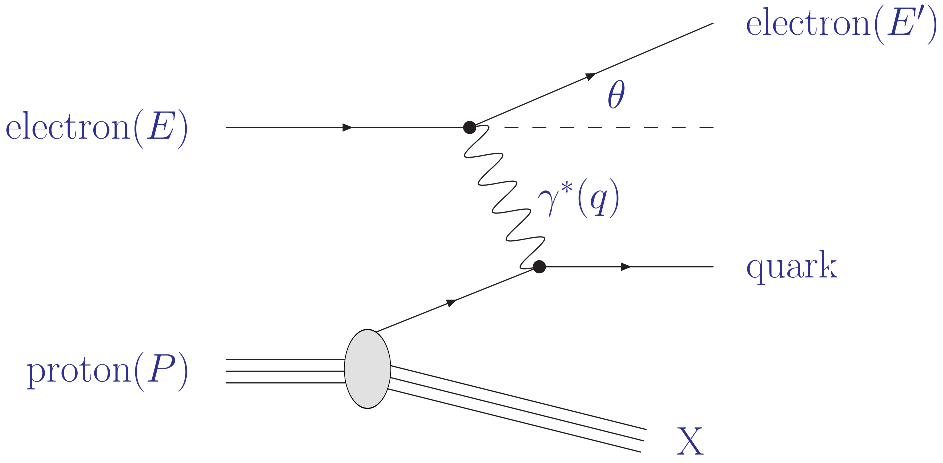
- HERA: deep structure of proton at highest $Q^2$ and smallest $x$

# *Bright future for precision hadron physics*

- Electron-Ion Collider
  *A machine that will unlock the secrets of the strongest force in Nature*



# Inelastic electron-proton scattering




- Virtuality of photon: resolution

  $$Q^2 \equiv -q^2 = 4EE' \sin^2(\theta/2)$$

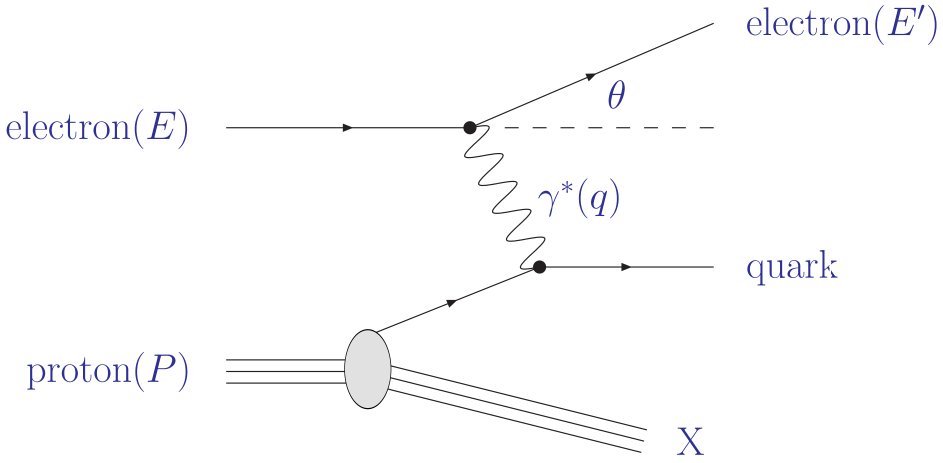
- Bjorken variable: inelasticity

  $$x = \frac{Q^2}{2P \cdot q} < 1$$

- Cross section ($X$ inclusive): proton structure function $F_i^p$

$$(E - E')\frac{d\sigma}{d\Omega\, dE'} \overset{\text{lab}}{=} \underbrace{\frac{\alpha^2 \cos^2 \frac{\theta}{2}}{4E^2 \sin^4 \frac{\theta}{2}}}_{\text{Mott-scattering (point-like)}} \left\{ F_2^p(x, Q^2) + \tan^2 \frac{\theta}{2}\, F_1^p(x, Q^2) \right\}$$

# *Inelastic electron-proton scattering*




- Virtuality of photon: resolution
  $$Q^2 \equiv -q^2 = 4EE' \sin^2(\theta/2)$$
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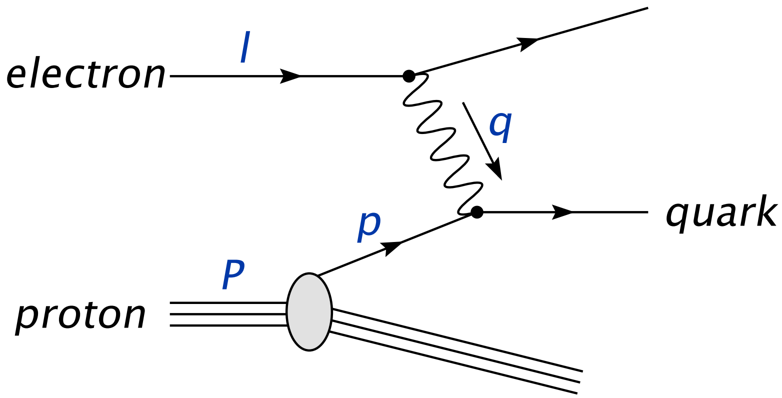
- Deep-inelastic scattering (Bjorken limit: $Q^2 \to \infty$ and $x$ fixed)
  Parton modell (quasi-free point-like constituents, incoherence)
  $$F_2(x,Q^2) \simeq F_2(x) = \sum_i e_i^2\, x f_i(x)$$
  - $x f_i(x)$ distribution for momentum fraction $x$ of parton $i$

# Deep-inelastic scattering




*Kinematic variables*

- momentum transfer $Q^2 = -q^2$
- Bjorken variable $x = Q^2/(2p \cdot q)$

- Structure functions (up to order $\mathcal{O}(1/Q^2)$)

$$F_a\left(x, Q^2\right) = \sum_i \left[C_{a,i}\left(\alpha_s(\mu^2), \mu^2/Q^2\right) \otimes f_i\left(\mu^2\right)\right](x)$$

- Coefficient functions up to N$^4$LO (work in progress)

$$C_{a,i} = \alpha_s^n \left(c_{a,i}^{(0)} + \alpha_s c_{a,i}^{(1)} + \alpha_s^2 c_{a,i}^{(2)} + \alpha_s^3 c_{a,i}^{(3)} + \alpha_s^3 c_{a,i}^{(4)} + \ldots\right)$$

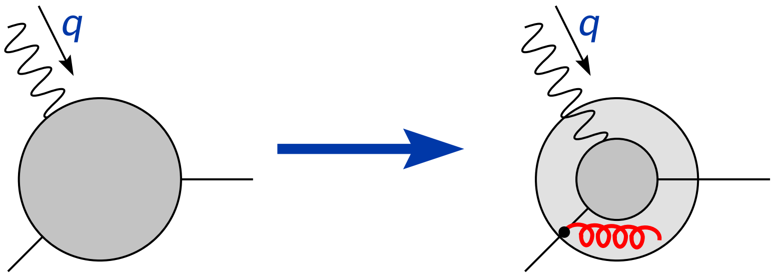
- Evolution equations up to N$^3$LO (work in progress)
  - non-singlet ($2n_f - 1$ scalar) and singlet ($2 \times 2$ matrix) equations

$$\frac{d}{d \ln \mu^2} f_i(x, \mu^2) = \left[P_{ij}(\alpha_s(\mu^2)) \otimes f_j(\mu^2)\right](x)$$

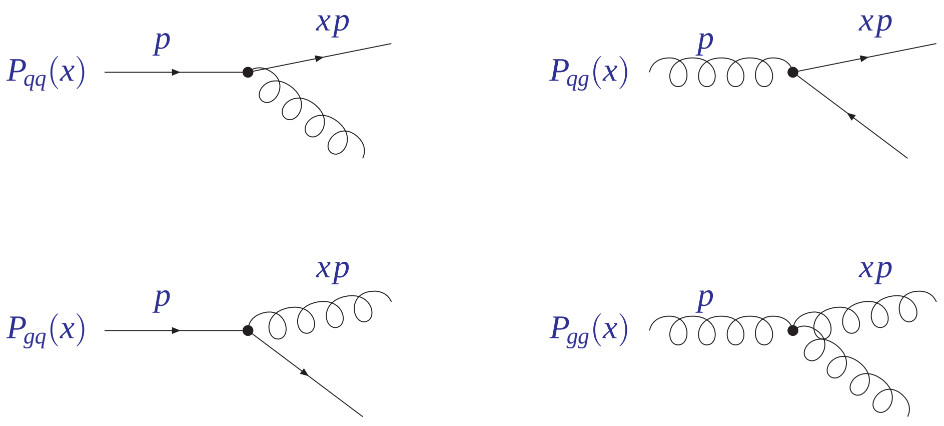
  - splitting functions $P_{ij} = \alpha_s P_{ij}^{(0)} + \alpha_s^2 P_{ij}^{(1)} + \alpha_s^3 P_{ij}^{(2)} + \alpha_s^4 P_{ij}^{(3)} + \ldots$

# *QCD evolution at 1% precision*

# *Parton evolution*




- Feynman diagrams in leading order




- Proton in resolution $1/Q \longrightarrow$ sensitive to lower momentum partons

- Evolution equations for parton distributions $f_i$
  - predictions from fits to reference processes (universality)

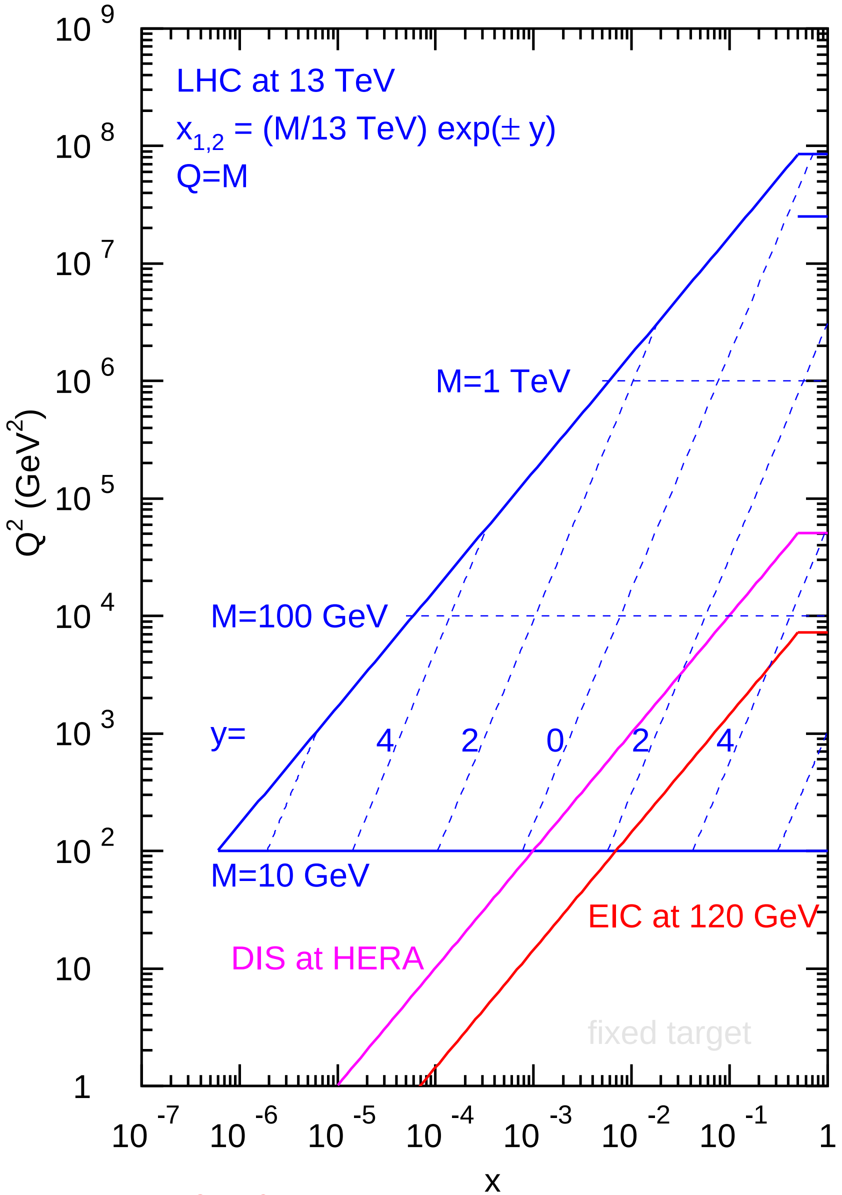
$$\frac{d}{d\ln\mu^2} f_i(x,\mu^2) = \sum_j \left[P_{ij}(\alpha_s(\mu^2)) \otimes f_j(\mu^2)\right](x)$$

- Splitting functions $P$ up to N$^3$LO (work in progress)

$$P_{ij} \;=\; \underbrace{\alpha_s\, P_{ij}^{(0)} + \alpha_s^2\, P_{ij}^{(1)} + \alpha_s^3\, P_{ij}^{(2)}}_{\text{NNLO: standard approximation}} + \alpha_s^4\, P_{ij}^{(3)} + \ldots$$

# *Parton kinematics at colliders*

- Information on proton structure depends on kinematic coverage



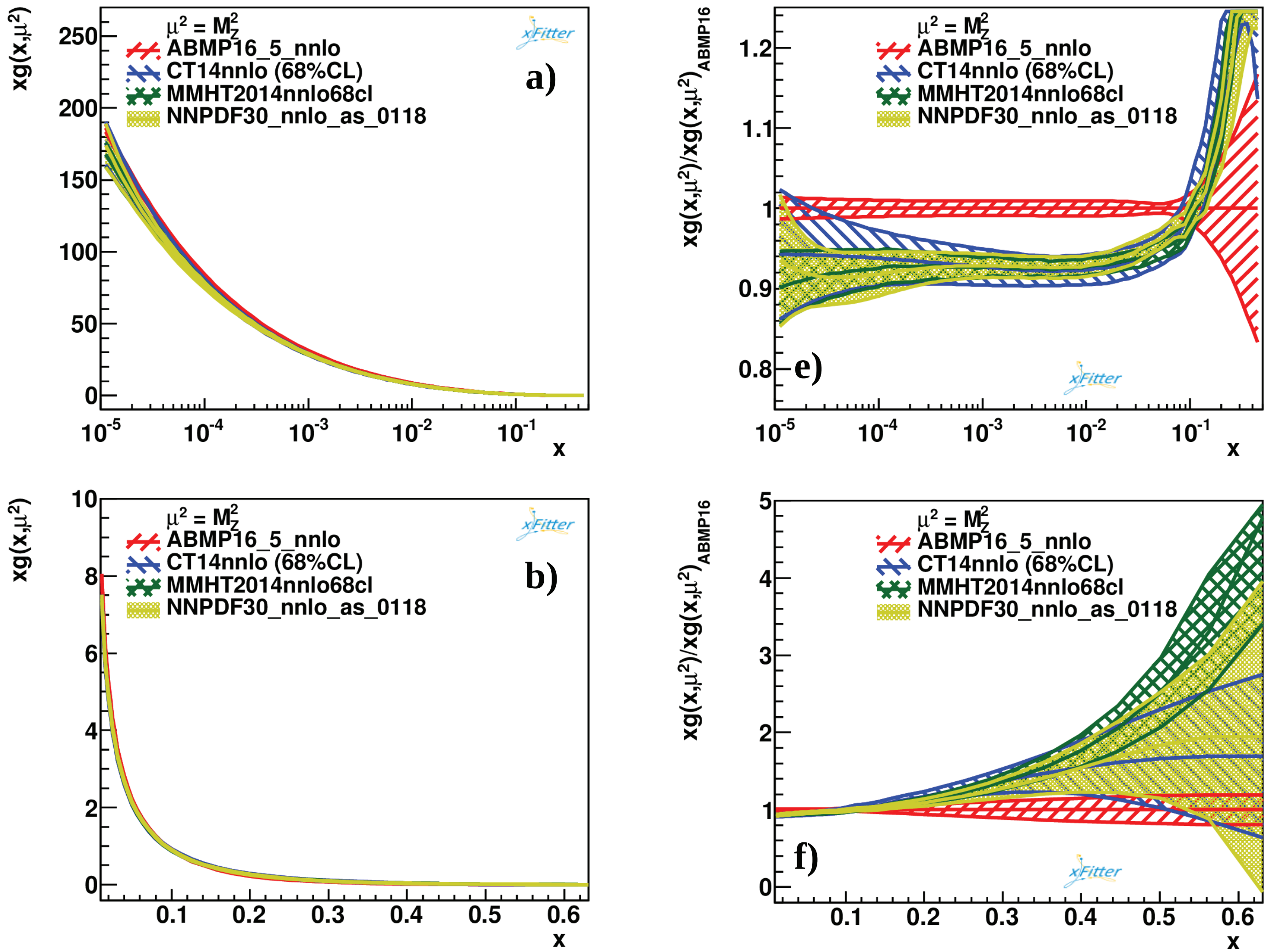

- LHC run at $\sqrt{s}$ = 13 TeV
  - parton kinematics well covered by HERA and fixed target experiments
- Parton kinematics with $x_{1,2} = M/\sqrt{S}e^{\pm y}$
  - forward rapidities sensitive to small-$x$
- EIC run at $\sqrt{s}$ = 120 GeV
  - parton kinematics provide complementary information
- Cross section depends on convolution of parton distributions
  - small-$x$ part of $f_i$ and large-$x$ PDFs $f_j$

$$\sigma_{ep\to X} = \sum_{ij} f_i(\mu^2) \otimes \left[ \ldots \right]$$

$$\sigma_{pp\to X} = \sum_{ij} f_i(\mu^2) \otimes f_j(\mu^2) \otimes \left[ \ldots \right]$$

# *Results for parton distributions*

- PDFs with $1\sigma$ uncertainty bands; compare ABMP16, CT14, MMHT14 NNPDF3.0
- Gluon $g(x)$



# *Quality of fit*

## *Statistical tests*

- Goodness-of-fit estimator
  - $\chi^2$ values compared to number of data points (typically a few thousand in global fit)

## *Covariance matrix*

- Positive-definite covariance matrix
  - correlations for fit parameters of ABMP16 PDFs

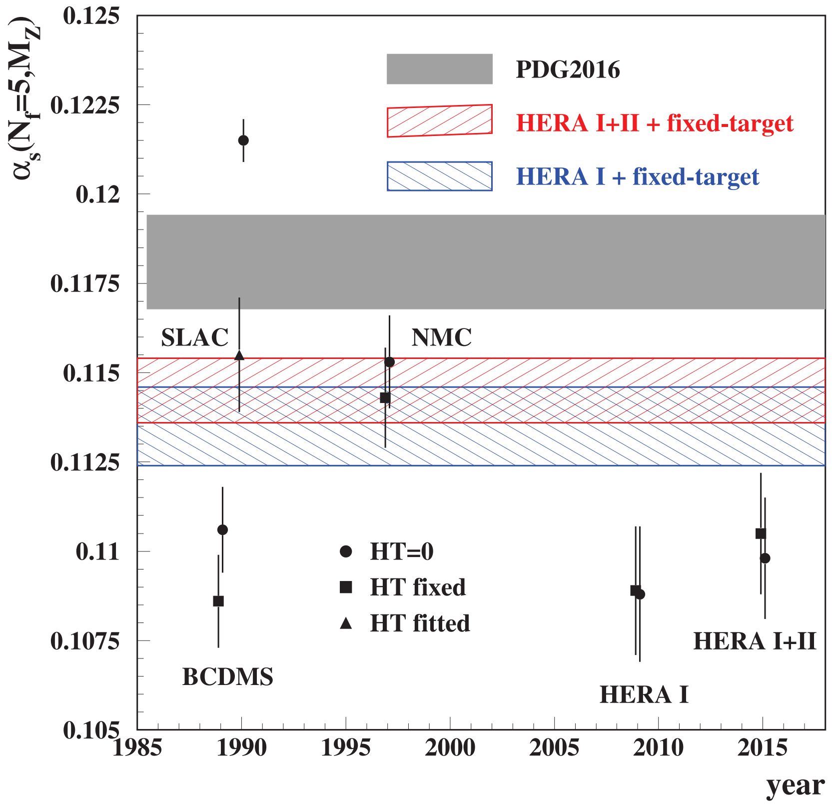
| | $a_u$ | $b_u$ | $\gamma_{1,u}$ | $\gamma_{2,u}$ | $\gamma_{3,u}$ | $a_d$ | $b_d$ | $\gamma_{1,d}$ | $\gamma_{2,d}$ | $\gamma_{3,d}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $a_u$ | 1.0 | 0.7617 | 0.9372 | -0.5078 | 0.4839 | 0.4069 | 0.3591 | 0.4344 | -0.3475 | 0.0001 |
| $b_u$ | 0.7617 | 1.0 | 0.6124 | -0.1533 | -0.0346 | 0.3596 | 0.2958 | 0.3748 | -0.2748 | 0.0001 |
| $\gamma_{1,u}$ | 0.9372 | 0.6124 | 1.0 | -0.7526 | 0.7154 | 0.2231 | 0.2441 | 0.2812 | -0.2606 | 0.0001 |
| $\gamma_{2,u}$ | -0.5078 | -0.1533 | -0.7526 | 1.0 | -0.9409 | 0.2779 | 0.2276 | 0.2266 | -0.1860 | 0.0 |
| $\gamma_{3,u}$ | 0.4839 | -0.0346 | 0.7154 | -0.9409 | 1.0 | -0.1738 | -0.1829 | -0.1327 | 0.1488 | 0.0 |
| $a_d$ | 0.4069 | 0.3596 | 0.2231 | 0.2779 | -0.1738 | 1.0 | 0.7209 | 0.9697 | -0.6529 | 0.0001 |
| $b_d$ | 0.3591 | 0.2958 | 0.2441 | 0.2276 | -0.1829 | 0.7209 | 1.0 | 0.7681 | -0.9786 | -0.0001 |
| $\gamma_{1,d}$ | 0.4344 | 0.3748 | 0.2812 | 0.2266 | -0.1327 | 0.9697 | 0.7681 | 1.0 | -0.7454 | 0.0002 |
| $\gamma_{2,d}$ | -0.3475 | -0.2748 | -0.2606 | -0.1860 | 0.1488 | -0.6529 | -0.9786 | -0.7454 | 1.0 | -0.0002 |
| $\gamma_{3,d}$ | 0.0001 | 0.0001 | 0.0001 | 0.0 | 0.0 | 0.0001 | -0.0001 | 0.0002 | -0.0002 | 1.0 |
| $a_{us}$ | -0.0683 | -0.0081 | -0.2094 | 0.3881 | -0.3206 | 0.2266 | 0.1502 | 0.2000 | -0.1293 | 0.0 |
| $b_{us}$ | -0.3508 | -0.3089 | -0.3462 | 0.0906 | -0.0537 | -0.1045 | -0.2000 | -0.2241 | 0.2798 | 0.0 |
| $\gamma_{-1,us}$ | 0.2296 | 0.1387 | 0.3367 | -0.4043 | 0.3474 | -0.1171 | -0.1127 | -0.0810 | 0.0767 | 0.0 |
| $\gamma_{1,us}$ | -0.4853 | -0.4119 | -0.3844 | -0.0365 | 0.0064 | -0.4380 | -0.3592 | -0.4957 | 0.3771 | -0.0001 |
| $A_{us}$ | 0.0506 | 0.0807 | -0.0949 | 0.3198 | -0.2560 | 0.2527 | 0.1648 | 0.2350 | -0.1509 | 0.0 |
| $a_{ds}$ | -0.0759 | -0.0443 | -0.0951 | 0.0263 | -0.0382 | -0.2565 | -0.2541 | -0.2666 | 0.2380 | 0.0 |
| $b_{bs}$ | 0.0452 | -0.0197 | 0.0345 | -0.0589 | 0.0683 | -0.2084 | 0.0190 | -0.1841 | -0.0522 | 0.0 |
| $\gamma_{1,ds}$ | -0.0492 | -0.0809 | 0.0101 | -0.1791 | 0.1309 | -0.5576 | -0.2029 | -0.4584 | 0.0946 | 0.0 |
| $A_{ds}$ | -0.1980 | -0.1262 | -0.2349 | 0.1526 | -0.1428 | -0.1113 | -0.2167 | -0.1739 | 0.2407 | 0.0 |
| $a_{ss}$ | -0.2034 | -0.1285 | -0.2362 | 0.2328 | -0.2080 | 0.0960 | 0.1596 | 0.0661 | -0.1054 | 0.0 |
| $b_{ss}$ | -0.1186 | -0.0480 | -0.1532 | 0.1549 | -0.1536 | 0.0486 | 0.1508 | 0.0267 | -0.1161 | 0.0 |
| $A_{ss}$ | -0.1013 | -0.0411 | -0.1458 | 0.1802 | -0.1625 | 0.1216 | 0.1678 | 0.0924 | -0.1196 | 0.0 |
| $a_g$ | 0.0046 | -0.0374 | 0.1109 | -0.1934 | 0.1653 | -0.0288 | -0.0122 | 0.0053 | 0.0059 | 0.0 |
| $b_g$ | 0.2662 | 0.3141 | 0.1579 | -0.0050 | -0.0207 | 0.0973 | 0.0870 | 0.0646 | -0.0666 | 0.0 |
| $\gamma_{1,g}$ | 0.2008 | 0.2274 | 0.0706 | 0.0876 | -0.0835 | 0.0919 | 0.0574 | 0.0493 | -0.0364 | 0.0 |
| $\alpha_s^{(n_f=3)}(\mu_0)$ | 0.1083 | -0.0607 | 0.0848 | -0.0250 | 0.0765 | 0.0763 | -0.0306 | 0.0725 | 0.0243 | 0.0 |
| $m_c(m_c)$ | -0.0006 | 0.0170 | -0.0104 | 0.0206 | -0.0201 | -0.0123 | -0.0161 | -0.0114 | 0.0108 | 0.0 |
| $m_b(m_b)$ | 0.0661 | 0.0554 | 0.0605 | -0.0367 | 0.0287 | -0.0116 | 0.0029 | -0.0074 | -0.0051 | 0.0 |
| $m_t(m_t)$ | -0.1339 | -0.2170 | -0.0816 | 0.0081 | 0.0250 | -0.0616 | -0.0813 | -0.0491 | 0.0736 | 0.0 |

| | $a_{us}$ | $b_{us}$ | $\gamma_{-1,us}$ | $\gamma_{1,us}$ | $A_{us}$ | $a_{ds}$ | $b_{bs}$ | $\gamma_{1,ds}$ | $A_{ds}$ | $a_{ss}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $a_u$ | -0.0683 | -0.3508 | 0.2296 | -0.4853 | 0.0506 | -0.0759 | 0.0452 | -0.0492 | -0.1980 | -0.2034 |
| $b_u$ | -0.0081 | -0.3089 | 0.1387 | -0.4119 | 0.0807 | -0.0443 | -0.0197 | -0.0809 | -0.1262 | -0.1285 |
| $\gamma_{1,u}$ | -0.2094 | -0.3462 | 0.3367 | -0.3844 | -0.0949 | -0.0951 | 0.0345 | 0.0101 | -0.2349 | -0.2362 |
| $\gamma_{2,u}$ | 0.3881 | 0.0906 | -0.4043 | -0.0365 | 0.3198 | 0.0263 | -0.0589 | -0.1791 | 0.1526 | 0.2328 |
| $\gamma_{3,u}$ | -0.3206 | -0.0537 | 0.3474 | 0.0064 | -0.2560 | -0.0382 | 0.0683 | 0.1309 | -0.1428 | -0.2080 |
| $a_d$ | 0.2266 | -0.1045 | -0.1171 | -0.4380 | 0.2527 | -0.2565 | -0.2084 | -0.5576 | -0.1113 | 0.0960 |
| $b_d$ | 0.1502 | -0.2000 | -0.1127 | -0.3592 | 0.1648 | -0.2541 | 0.0190 | -0.2029 | -0.2167 | 0.1596 |
| $\gamma_{1,d}$ | 0.2000 | -0.2241 | -0.0810 | -0.4957 | 0.2350 | -0.2666 | -0.1841 | -0.4584 | -0.1739 | 0.0661 |
| $\gamma_{2,d}$ | -0.1293 | 0.2798 | 0.0767 | 0.3771 | -0.1509 | 0.2380 | -0.0522 | 0.0946 | 0.2407 | -0.1054 |
| $\gamma_{3,d}$ | 0.0 | 0.0 | 0.0 | -0.0001 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| $a_{us}$ | 1.0 | -0.3156 | -0.8947 | -0.5310 | 0.9719 | 0.2849 | 0.0241 | -0.0470 | 0.2983 | 0.4131 |
| $b_{us}$ | -0.3156 | 1.0 | 0.1372 | 0.8258 | -0.3995 | 0.0467 | -0.0221 | -0.1190 | 0.1856 | 0.0291 |
| $\gamma_{-1,us}$ | -0.8947 | 0.1372 | 1.0 | 0.2611 | -0.7829 | -0.1695 | 0.0156 | 0.0501 | -0.2117 | -0.7191 |
| $\gamma_{1,us}$ | -0.5310 | 0.8258 | 0.2611 | 1.0 | -0.6479 | 0.0086 | 0.0076 | 0.1460 | 0.0781 | -0.0010 |
| $A_{us}$ | 0.9719 | -0.3995 | -0.7829 | -0.6479 | 1.0 | 0.2983 | 0.0515 | -0.0404 | 0.3055 | 0.2811 |
| $a_{ds}$ | 0.2849 | 0.0467 | -0.1695 | 0.0086 | 0.2983 | 1.0 | -0.1608 | 0.0719 | 0.9152 | -0.2941 |
| $b_{bs}$ | 0.0241 | -0.0221 | 0.0156 | 0.0076 | 0.0515 | -0.1608 | 1.0 | 0.7834 | -0.3022 | -0.0390 |
| $\gamma_{1,ds}$ | -0.0470 | -0.1190 | 0.0501 | 0.1460 | -0.0404 | 0.0719 | 0.7834 | 1.0 | -0.1838 | -0.1373 |
| $A_{ds}$ | 0.2983 | 0.1856 | -0.2117 | 0.0781 | 0.3055 | 0.9152 | -0.3022 | -0.1838 | 1.0 | -0.1833 |
| $a_{ss}$ | 0.4131 | 0.0291 | -0.7191 | -0.0010 | 0.2811 | -0.2941 | -0.0390 | -0.1373 | -0.1833 | 1.0 |
| $b_{ss}$ | 0.2197 | 0.0643 | -0.4479 | 0.1286 | 0.1193 | -0.1579 | -0.0260 | 0.0169 | -0.0896 | 0.6522 |
| $A_{ss}$ | 0.3627 | 0.0261 | -0.6319 | 0.0102 | 0.2412 | -0.2688 | -0.0180 | -0.0960 | -0.1797 | 0.9280 |
| $a_g$ | -0.2570 | 0.0001 | 0.2196 | 0.0039 | -0.2493 | -0.2190 | -0.0454 | -0.1031 | -0.2571 | 0.0626 |
| $b_g$ | -0.1419 | 0.1266 | 0.0694 | 0.2648 | -0.1715 | -0.0515 | 0.0917 | 0.2130 | -0.0469 | -0.0092 |
| $\gamma_{1,g}$ | -0.0241 | 0.0332 | -0.0226 | 0.1296 | -0.0489 | -0.0137 | 0.0503 | 0.1409 | 0.0022 | -0.0279 |
| $\alpha_s^{(n_f=3)}(\mu_0)$ | 0.0954 | -0.2866 | -0.0341 | -0.3493 | 0.1110 | -0.0604 | -0.1265 | -0.1811 | -0.1330 | -0.0841 |
| $m_c(m_c)$ | 0.0704 | -0.0093 | -0.0033 | -0.0462 | 0.1182 | 0.0849 | 0.0547 | 0.0413 | 0.1193 | -0.0728 |
| $m_b(m_b)$ | -0.0183 | -0.0132 | 0.0044 | 0.0209 | -0.0298 | -0.0006 | 0.0332 | 0.0695 | -0.0432 | -0.0159 |
| $m_t(m_t)$ | 0.0641 | -0.1841 | -0.0408 | -0.2635 | 0.0755 | -0.0573 | -0.1067 | -0.2003 | -0.0869 | 0.0169 |

| | $b_{ss}$ | $A_{ss}$ | $a_g$ | $b_g$ | $\gamma_{1,g}$ | $\alpha_s^{(n_f=3)}(\mu_0)$ | $m_c(m_c)$ | $m_b(m_b)$ | $m_t(m_t)$ |
|---|---|---|---|---|---|---|---|---|---|
| $a_u$ | -0.1186 | -0.1013 | 0.0046 | 0.2662 | 0.2008 | 0.1083 | -0.0006 | 0.0661 | -0.1339 |
| $b_u$ | -0.0480 | -0.0411 | -0.0374 | 0.3141 | 0.2274 | -0.0607 | 0.0170 | 0.0554 | -0.2170 |
| $\gamma_{1,u}$ | -0.1532 | -0.1458 | 0.1109 | 0.1579 | 0.0706 | 0.0848 | -0.0104 | 0.0605 | -0.0816 |
| $\gamma_{2,u}$ | 0.1549 | 0.1802 | -0.1934 | -0.0050 | 0.0876 | -0.0250 | 0.0206 | -0.0367 | 0.0081 |
| $\gamma_{3,u}$ | -0.1536 | -0.1625 | 0.1653 | -0.0207 | -0.0835 | 0.0765 | -0.0201 | 0.0287 | 0.0250 |
| $a_d$ | 0.0486 | 0.1216 | -0.0288 | 0.0973 | 0.0919 | 0.0763 | -0.0123 | -0.0116 | -0.0616 |
| $b_d$ | 0.1508 | 0.1678 | -0.0122 | 0.0870 | 0.0574 | -0.0306 | -0.0161 | 0.0029 | -0.0813 |
| $\gamma_{1,d}$ | 0.0267 | 0.0924 | 0.0053 | 0.0646 | 0.0493 | 0.0725 | -0.0114 | -0.0074 | -0.0491 |
| $\gamma_{2,d}$ | -0.1161 | -0.1196 | 0.0059 | -0.0666 | -0.0364 | 0.0243 | 0.0108 | -0.0051 | 0.0736 |
| $\gamma_{3,d}$ | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| $a_{us}$ | 0.2197 | 0.3627 | -0.2570 | -0.1419 | -0.0241 | 0.0954 | 0.0704 | -0.0183 | 0.0641 |
| $b_{us}$ | 0.0643 | 0.0261 | 0.0001 | 0.1266 | 0.0332 | -0.2866 | -0.0093 | -0.0132 | -0.1841 |
| $\gamma_{-1,us}$ | -0.4479 | -0.6319 | 0.2197 | 0.0694 | -0.0226 | -0.0341 | -0.0034 | 0.0044 | -0.0408 |
| $\gamma_{1,us}$ | 0.1286 | 0.0102 | 0.0039 | 0.2648 | 0.1296 | -0.3493 | -0.0462 | 0.0209 | -0.2635 |
| $A_{us}$ | 0.1193 | 0.2412 | -0.2493 | -0.1715 | -0.0489 | 0.1110 | 0.1182 | -0.0298 | 0.0755 |
| $a_{ds}$ | -0.1579 | -0.2688 | -0.2190 | -0.0515 | -0.0137 | -0.0604 | 0.0849 | -0.0006 | -0.0573 |
| $b_{bs}$ | -0.0260 | -0.0180 | -0.0454 | 0.0917 | 0.0503 | -0.1265 | 0.0547 | 0.0332 | -0.1067 |
| $\gamma_{1,ds}$ | 0.0169 | -0.0960 | -0.1031 | 0.2130 | 0.1409 | -0.1811 | 0.0413 | 0.0695 | -0.2003 |
| $A_{ds}$ | -0.0896 | -0.1797 | -0.2571 | -0.0469 | 0.0022 | -0.1330 | 0.1193 | -0.0432 | -0.0869 |
| $a_{ss}$ | 0.6522 | 0.9280 | 0.0626 | -0.0092 | -0.0279 | -0.0841 | -0.0728 | -0.0159 | 0.0169 |
| $b_{ss}$ | 1.0 | 0.6427 | -0.0179 | 0.1967 | 0.1164 | -0.2390 | -0.0965 | 0.0169 | -0.1675 |
| $A_{ss}$ | 0.6427 | 1.0 | -0.0211 | 0.1403 | 0.0997 | -0.1385 | 0.0216 | 0.0072 | -0.1109 |
| $a_g$ | -0.0179 | -0.0211 | 1.0 | -0.5279 | -0.8046 | 0.1838 | -0.2829 | 0.0076 | 0.3310 |
| $b_g$ | 0.1967 | 0.1403 | -0.5279 | 1.0 | 0.8837 | -0.5124 | 0.1438 | 0.1255 | -0.7275 |
| $\gamma_{1,g}$ | 0.1164 | 0.0997 | -0.8046 | 0.8837 | 1.0 | -0.2511 | 0.1829 | 0.0814 | -0.5180 |
| $\alpha_s^{(n_f=3)}(\mu_0)$ | -0.2390 | -0.1385 | 0.1838 | -0.5124 | -0.2511 | 1.0 | -0.1048 | 0.0423 | 0.6924 |
| $m_c(m_c)$ | -0.0965 | 0.0216 | -0.2829 | 0.1438 | 0.1829 | -0.1048 | 1.0 | 0.0328 | -0.1577 |
| $m_b(m_b)$ | 0.0169 | 0.0072 | 0.0076 | 0.1255 | 0.0814 | 0.0423 | 0.0328 | 1.0 | -0.0900 |
| $m_t(m_t)$ | -0.1675 | -0.1109 | 0.3310 | -0.7275 | -0.5180 | 0.6924 | -0.1577 | -0.0900 | 1.0 |

# *Determinations of $\alpha_s$ from DIS*

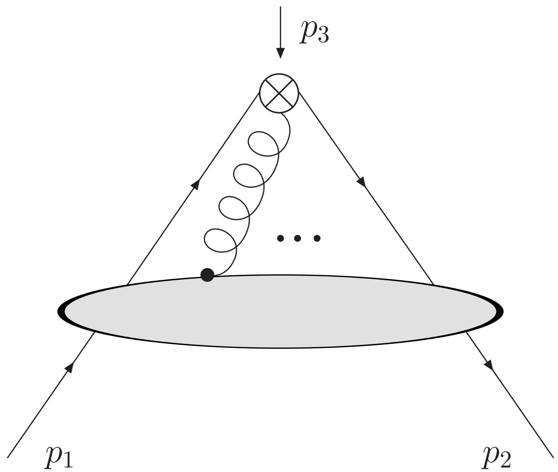
- Correlation of errors among different data DIS sets
- Target mass corrections (powers of nucleon mass $M_N^2/Q^2$)
- Variants with no higher twist give larger $\alpha_s$ values Alekhin, Blümlein, S.M. ‘17




- Theoretical uncertainty of $\alpha_s$ at NNLO from DIS data $\gtrsim \mathcal{O}(1\ldots2)\%$

# *Operator matrix elements*

# Operator matrix elements




- Quark operator of spin-$N$ and twist two
  $O^{\psi}_{\{\mu_1,\dots,\mu_N\}} = \overline{\psi}\,\gamma_{\{\mu_1} D_{\mu_2} \dots D_{\mu_N\}}\,\psi$
- $N$ covariant derivatives
  $D_{\mu,ij} = \partial_\mu \delta_{ij} + ig_s\,(t_a)_{ij}\,A^a_\mu$
  sandwiched between quark fields $\psi$, $\overline{\psi}$
- Evaluation of operators in matrix elements $A^{\psi\psi}$ with external quark states

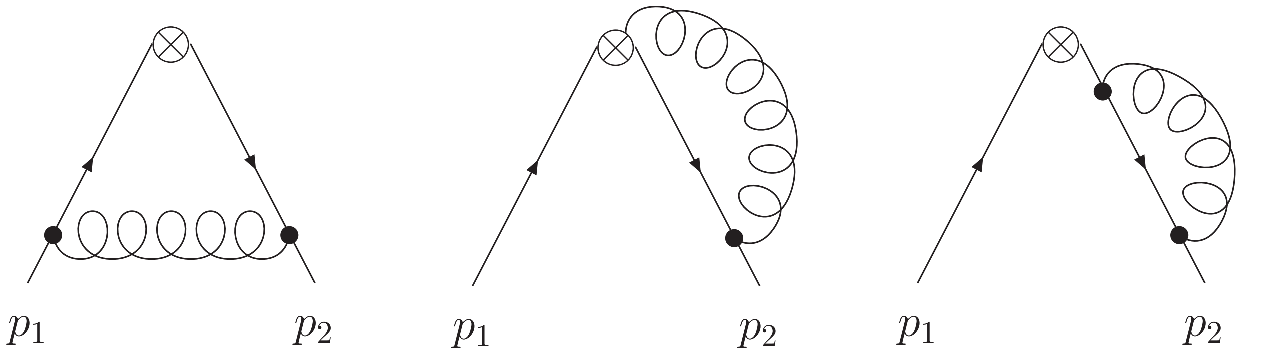
$$A^{\psi\psi}_{\{\mu_1,\dots,\mu_N\}} = \langle \psi(p_1) | O^{\psi}_{\{\mu_1,\dots,\mu_N\}}(p_3) | \overline{\psi}(p_2) \rangle$$

- Anomalous dimensions $\gamma(\alpha_s, N)$ govern scale dependence of renormalized operators

$$\frac{d}{d\ln\mu^2} O^{\rm ren} = -\gamma\, O^{\rm ren} \qquad\qquad \gamma(N) = -\int_0^1 dx\, x^{N-1} P(x)$$

- Zero-momentum transfer through operator reduces problem to computation of propagator-type diagrams

# One-loop computation




- Computation of loop integral in $D = 4 - 2\epsilon$ dimensions and expansion in $\epsilon$
  - anomalous dimension $\gamma(N)$ from ultraviolet divergence

$$\Delta^{\mu_1} \ldots \Delta^{\mu_N} \langle \psi(p_1) | O^{\psi}_{\{\mu_1,\ldots,\mu_N\}}(0) | \overline{\psi}(-p_1) \rangle =$$

$$= \quad 1 + \frac{\alpha_s}{4\pi} \frac{1}{\epsilon} \gamma^{(0)}(N) + \mathcal{O}(\alpha_s \epsilon^0) + \mathcal{O}(\alpha_s^2)$$

$$= \quad 1 + \frac{\alpha_s}{4\pi} \frac{1}{\epsilon} \left\{ C_F \left( 4S_1(N) + \frac{2}{N+1} - \frac{2}{N} - 3 \right) \right\} + \mathcal{O}(\alpha_s \epsilon^0) + \mathcal{O}(\alpha_s^2)$$

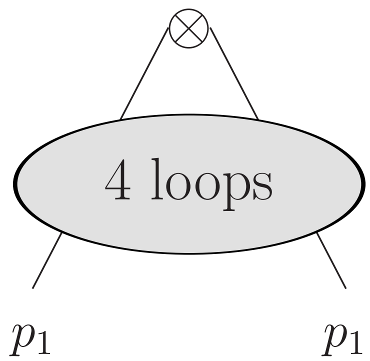
- One-loop result with harmonic sum $S_1(N) = \sum_{i=1}^{N} \frac{1}{i}$
- Details in *The Theory of Quark and Gluon Interactions*
  F.J. Yndurain

# *Four-loop computation*

- Feynman diagrams for operator matrix elements generated up to four loops with **Qgraf** Nogueira '91
- Parametric reduction of four-loop massless propagator diagrams with integration-by-parts identities encoded in **Forcer** Ruijl, Ueda, Vermaseren '17
- Symbolic manipulations with **Form** Vermaseren '00; Kuipers, Ueda, Vermaseren, Vollinga '12 and multi-threaded version **TForm** Tentyukov, Vermaseren '07
- Diagrams of same topology and color factor combined to meta diagrams
- Non-singlet anomalous dimension
  - 1 one-, 7 two-, 53 three- and 650 four-loop meta diagrams for $\gamma_{\rm ns}^{\pm}$
  - 1 three- and 29 four-loop meta diagrams for $\gamma_{\rm ns}^{\rm s}$

## *Fixed Mellin moments*

- Computation of anomalous dimensions $\gamma(N)$ for Mellin moments mostly up to $N = 18$
  - sometimes higher for complicated topologies ($N = 19$, $N = 20, \ldots$)
  - much higher for "easy" topologies, e.g., $n_f$-dependent ($N \simeq 80, \ldots$)



# *Analytic reconstruction (I)*

- Sufficiently many Mellin moments allow for reconstruction of analytic all-$N$ expressions through solution of Diophantine equations

  Lenstra, Lenstra, Lovász '82

- Anomalous dimensions $\gamma(N)$ given by harmonic sums up to weight $7$

$$S_{\pm m_1,\dots,m_k}(N) = \sum_{i=1}^{N} \frac{(\pm 1)^i}{i^{m_1}} S_{m_2,\dots,m_k}(i)$$

  - $2 \cdot 3^{w-1}$ sums at weight $w$

## *Conformal symmetry and integrability*

- Leading order diagonal splitting functions in QCD $P_{ii}^{(0)}(x)$ subject to Gribov-Lipatov reciprocity
  - invariant under mapping $x \to \dfrac{1}{x}$
- Reciprocity relation realized in $N = 4$ SYM theory (conformal theory)
  - anomalous dimensions with sums of uniform transcendentality
  - only harmonic sums of weight $w = 2l - 1$ at $l$-loops

# *Analytic reconstruction (II)*

## *The $N$-space approach*

- Reciprocity relation (RR) in $N$-space imply

$$\gamma(N) = \gamma_{\rm u}\,(N + \gamma(N) - \beta(\alpha_s))$$

- RR constraints for $\gamma_{\rm u}$ reduce number of $2^{w-1}$ sums at weight $w$ for $\gamma_{\rm u}$
  - additional denominators with powers $1/(N+1)$ give $2^{w+1}-1$ objects ($255$ at weight $7$ )
- Large-$n_c$ limit only needs harmonic sums with positive index
  - weight $w$ RR sums given by Fibonacci number $F(w)$
  - total number of unknowns (including powers $1/(N+1)$) amount to $F(w+4)-2$ ($87$ at $w=7$)
- Additional $46$ constraints from large-$x$/small-$x$ ($N \to \infty$/$N \to 0$) limit
- Solution becomes feasible with $18$ Mellin moments for $\gamma_{\rm ns}^{\pm}$

# Large-x behavior

## The large $x$-limit: $x \to 1$

- Structure of diagonal splitting functions $P_{\rm ii}$ (for $\rm i = q, g$) at large $x$

$$P_{ii}^{(n-1)}(x) \;=\; \frac{A_{n,\rm i}}{(1-x)_+} + B_{n,\rm i}\,\delta(1-x) + C_{n,\rm i}\,\ln(1-x) + D_{n,\rm i}$$

- Cusp anomalous dimension $A_{n,\rm i}$ (known from $1/\epsilon^2$-poles of QCD form factor)
  Large-$n_c$ (Henn, Lee, Smirnov, Smirnov, Steinhauser '16; S. M., Ruijl, Ueda, Vermaseren, Vogt '17); $n_f$ terms (Grozin '18; Henn, Peraro, Stahlhofen, Wasser '19); $n_f^2$ terms (Davies, Ruijl, Ueda, Vermaseren, Vogt '16; Lee, Smirnov, Smirnov, Steinhauser '17); $n_f^3$ terms (Gracey '94; Beneke, Braun, '95);
  quartic colour factors (Lee, Smirnov, Smirnov, Steinhauser '19; Henn, Korchemsky, Mistlberger '19)
- virtual anomalous dimension $B_{n,\rm i}$ (parts related to $1/\epsilon$-poles of QCD form factor)
- subleading coefficients $C_{n,\rm i}$, $D_{n,\rm i}$ known from lower order cusp anomalous dimension (S.M., Vermaseren, Vogt '04, Dokshitzer, Marchesini, Salam '05)

# *Small-x behavior (I)*

## *The small $x$-limit: $x \to 0$*

- Structure of non-singlet splitting functions $P_{\rm ns}^{\pm}$ at small $x$
  - double-logarithmic contributions with very large coefficients
  - resummation for $P_{\rm ns}^{+}$ to leading logarithmic (LL) accuracy in Mellin-$N$ space
    Kirschner, Lipatov '83

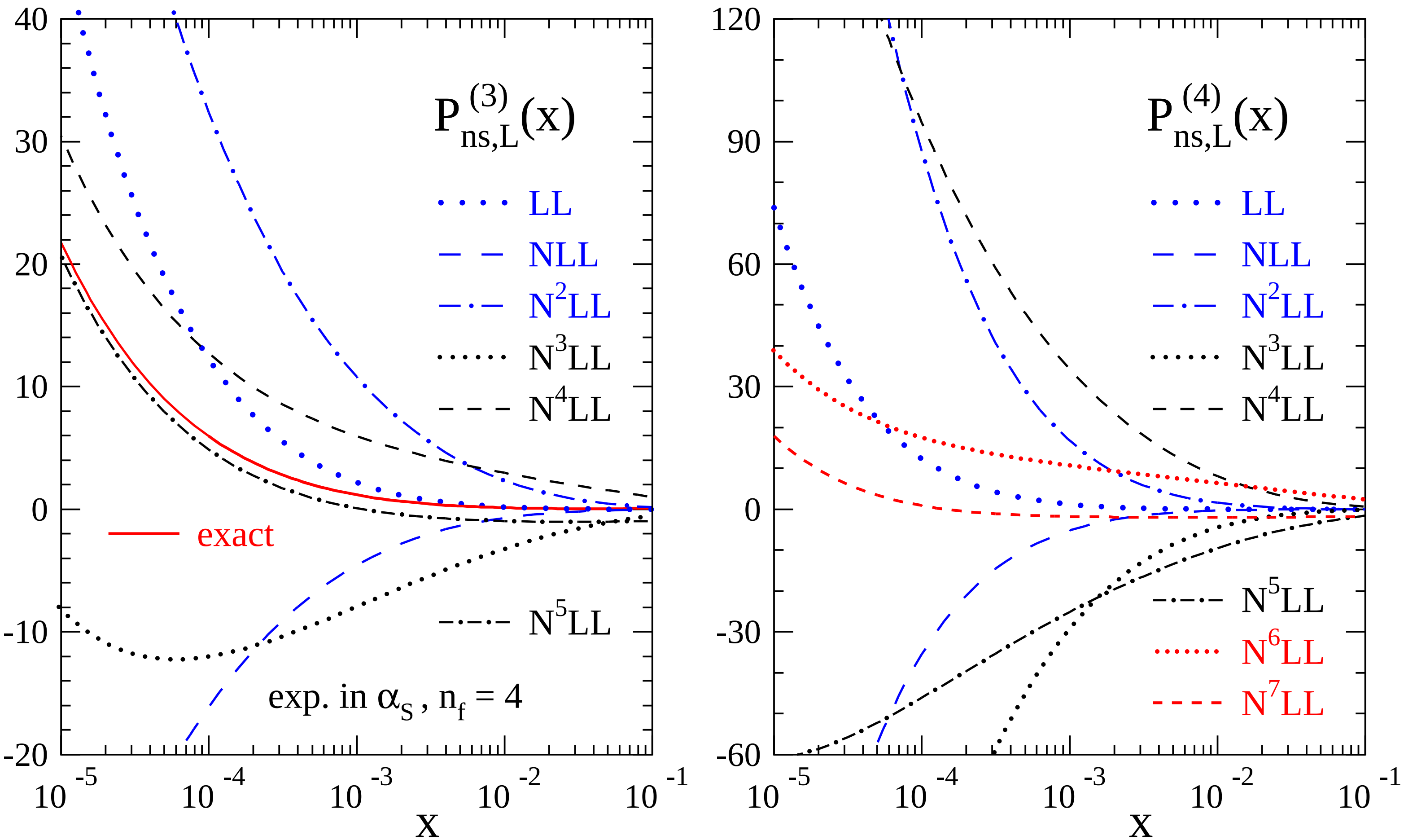
$$P^{+}_{\rm ns,LL}(N,\alpha_s) \;=\; \frac{N}{2}\left\{1-\left(1-\frac{2\alpha_s C_F}{\pi N^2}\right)^{1/2}\right\}$$

- Large-$n_c$ limit with intriguing structure
  Velizhanin '14

$$P^{+}_{\rm ns}(N,\alpha_s)\,\left(P^{+}_{\rm ns}(N,\alpha_s)-N+\beta(\alpha_s)/\alpha_s\right) \;=\; O(1)$$

  - Laurent expansion about $N=0$
- Exploit structure of the (unfactorized) structure functions in dimensional regularization
- Resummation in terms of modified Bessel functions to N$^7$LL accuracy
  Davies, Kom, S.M., Vogt '22

# *Small-x behavior (II)*

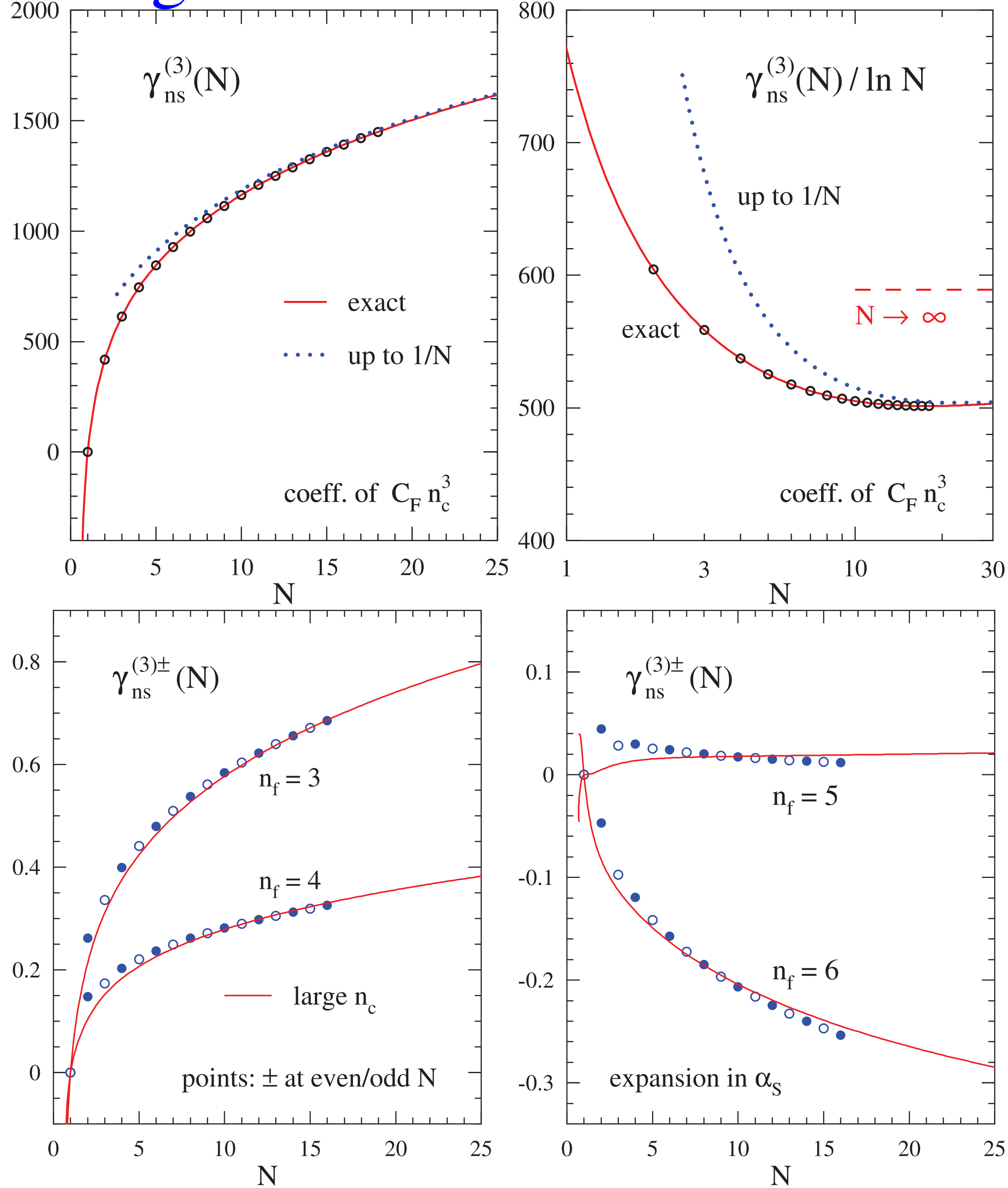



- Splitting functions $P_{\rm ns}^{(3),+}$ (left) and $P_{\rm ns}^{(4),+}$ (right) Davies, Kom, S.M., Vogt '22
  - small-$x$ approximations to the four-flavour splitting functions $P_{\rm ns,\,L}^{(n)}(x)$ in the large-$n_c$ limit
  - predictions up to N$^7$LL

# *Four-loop non-singlet Mellin moments*

- Top:
  $n_f^0$ part of anomalous dimensions $\gamma_{\rm ns}^{(3)\pm}(N)$ in large-$n_c$ limit and large-$N$ expansion

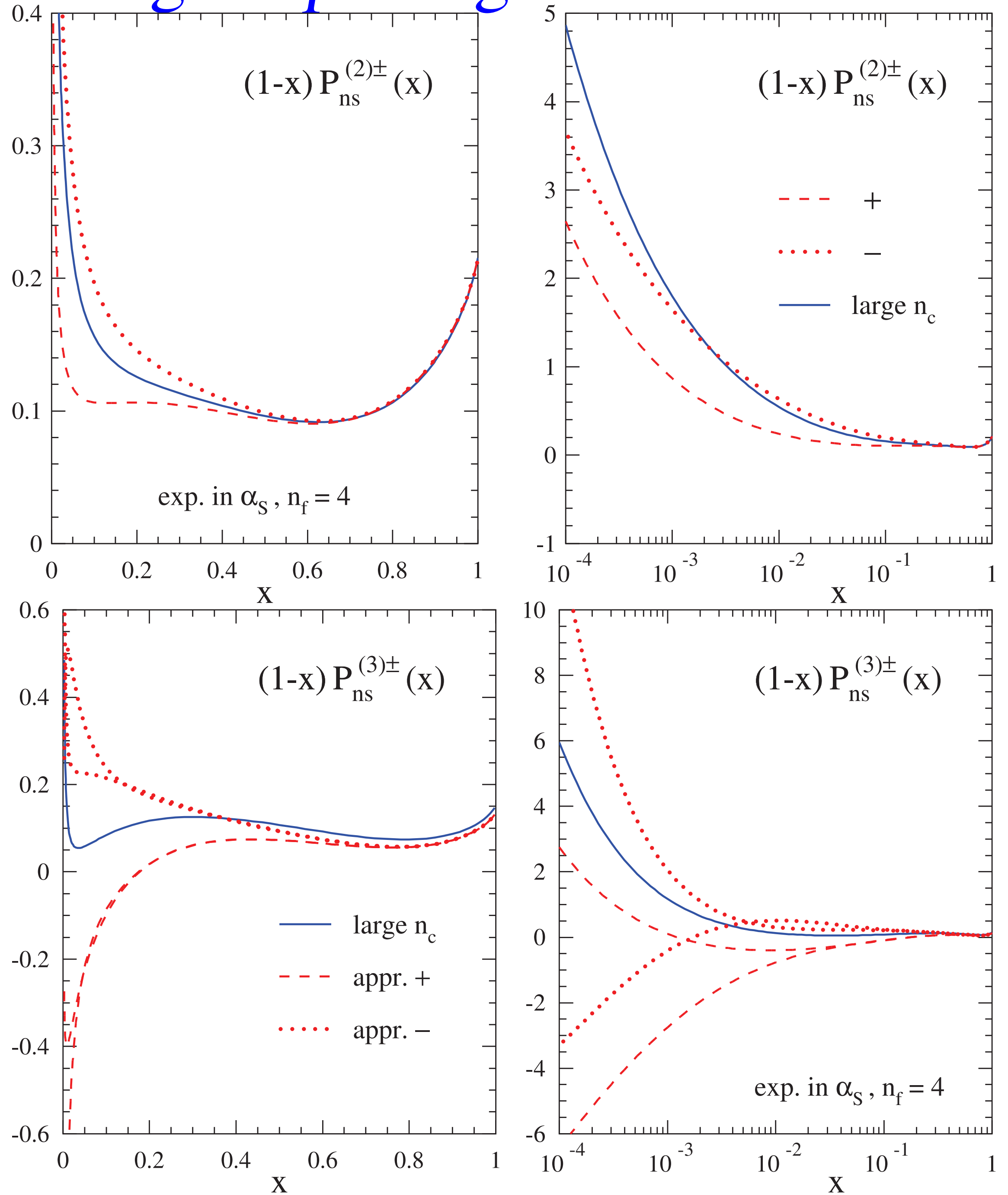
- Bottom: results for even-$N$ ( $\gamma_{\rm ns}^{(3)+}(N)$ ) and odd-$N$ ( $\gamma_{\rm ns}^{(3)-}(N)$ ) in large-$n_c$ limit for $n_f = 3, \dots 6$



# *Four-loop non-singlet splitting functions*

- Top:
  three-loop $P_{\rm ns}^{(2)\pm}(x)$
  and large-$n_c$ limit
  with $n_f = 4$

- Bottom:
  four-loop $P_{\rm ns}^{(3)\pm}(x)$
  and uncertainty bands
  beyond large-$n_c$ limit
  with $n_f = 4$

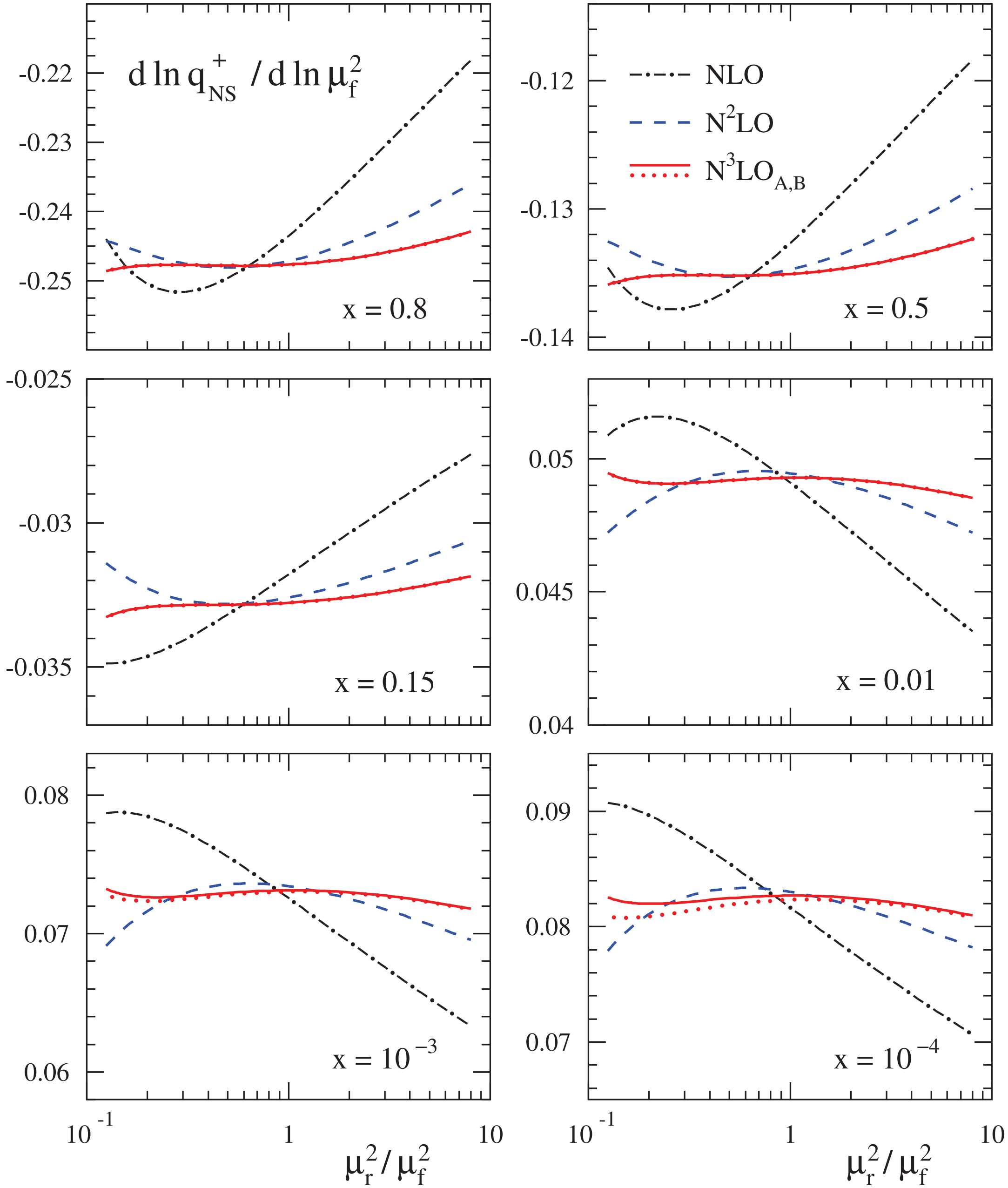


# *Scale stability of evolution*

- Renormalization scale dependence of evolution kernel $d \ln q^+_{\rm ns} / d \ln \mu_f^2$
  - non-singlet shape

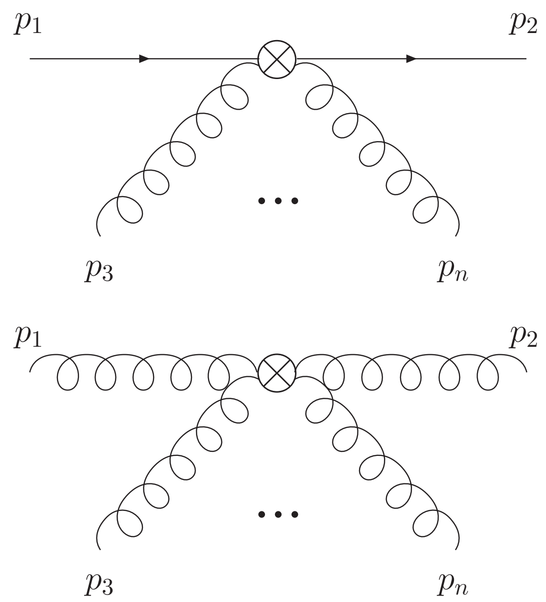
$$x q^{\,+}_{\rm ns}(x, \mu_0^2) \;=\; x^{\,0.5}(1-x)^3$$

- NLO, NNLO and N$^3$LO predictions
  - remaining uncertainty of four-loop splitting function $P^{(3)+}_{\rm ns}$ almost invisible



## *Singlet*

# *Operator matrix elements*




- Singlet operators of spin-$N$ and twist two

$$O^{\mathrm{q}}_{\{\mu_1,\ldots,\mu_N\}} = \overline{\psi}\,\gamma_{\{\mu_1} D_{\mu_2} \ldots D_{\mu_N\}}\,\psi\ ,$$

$$O^{\mathrm{g}}_{\{\mu_1,\ldots,\mu_N\}} = F_{\nu\{\mu_1} D_{\mu_2} \cdots D_{\mu_{N-1}} F_{\mu_N\}}{}^{\nu}$$

- Quartic Casimir terms at four loops are effectively 'leading-order'
    - $d_{xy}^{(4)} \equiv d_x^{abcd} d_y^{abcd}$ for representations labels $x, y$ with generators $T_r^a$

$$d_r^{abcd} = \frac{1}{6} \mathrm{Tr}\,(\,T_r^a\, T_r^b\, T_r^c\, T_r^d \,+\, \text{five } bcd \text{ permutations}\,)$$

    - anomalous dimensions fulfil relation for $\mathcal{N} = 1$ supersymmetry

$$\gamma^{(3)}_{\mathrm{qq}}(N) + \gamma^{(3)}_{\mathrm{gq}}(N) - \gamma^{(3)}_{\mathrm{qg}}(N) - \gamma^{(3)}_{\mathrm{gg}}(N) \overset{Q}{=} 0$$

- Eigenvalues of singlet splitting functions (conjectured to be) composed of reciprocity-respecting sums
    - quartic Casimir terms fulfil stronger condition Belitsky, Müller, Schäfer '99

$$\gamma^{(0)}_{\mathrm{qg}}(N)\,\gamma^{(3)}_{\mathrm{gq}}(N) \overset{Q}{=} \gamma^{(0)}_{\mathrm{gq}}(N)\,\gamma^{(3)}_{\mathrm{qg}}(N)$$

# Analytic results

- Reconstruction of analytic all-$N$ expressions for $\zeta_5$ terms from solution of Diophantine equations
  - example for $\gamma_{\rm gg}^{(3)}$ with $\eta = \frac{1}{N} - \frac{1}{N+1}$ and $\nu = \frac{1}{N-1} - \frac{1}{N+2}$

$$\gamma_{\rm gg}^{(3)}(N)\Big|_{\zeta_5\, d_{AA}^{(4)}/n_A} = \frac{64}{3}\Big(30\,(12\,\eta^2 - 4\,\nu^2 - S_1(4\,S_1 + 8\,\eta - 8\,\nu - 11) - 7\nu) + 188\,\eta - \frac{751}{3} - \frac{1}{6}\,N\,(N+1)\Big)$$

- Recall large-$N$ limit of anomalous dimensions

$$\gamma_{\rm ii}^{(k)}(N)\Big|_{N\to\infty} = A_{n,\rm i}\,\ln(N) + \mathcal{O}(\text{const}_N)$$

- Terms $S_1(N)^2 \sim \ln(N)^2$ and $N(N+1)$ proportional to $\zeta_5$ must be compensated in large-$N$ limit

# Universal anomalous dimension

- Universal anomalous dimension $\gamma_{\rm uni}$ in $N = 4$ SYM to three loops
  Kotikov, Lipatov, Onishchenko, Velizhanin '04
  - One-loop example: $\gamma_{\rm uni}^{(0)}(N) = 4n_c S_1$ emerges from

$$\gamma_{\rm qq}^{(0)}(N) = C_F \left( -3 + 2\frac{1}{N+1} - 2\frac{1}{N} + 4S_1 \right) \quad \text{or}$$

$$\gamma_{\rm gg}^{(0)}(N) = C_A \left( -\frac{11}{3} - \frac{4}{N-1} - \frac{4}{N+1} + \frac{4}{N+2} + \frac{4}{N} + 4S_1 \right) + \frac{2}{3} n_f$$

- Starting at four loops wrapping corrections to complement asymptotic Bethe ansatz
  - four-loop Bajnok, Janik, Lukowski '08, five-loop Lukowski, Rej, Velizhanin '09, six-loop [...], ...
- $\gamma(N)^{\rm wrap,(4)} \simeq S_1(N)^2 \, f^{\rm wrap}(N)$

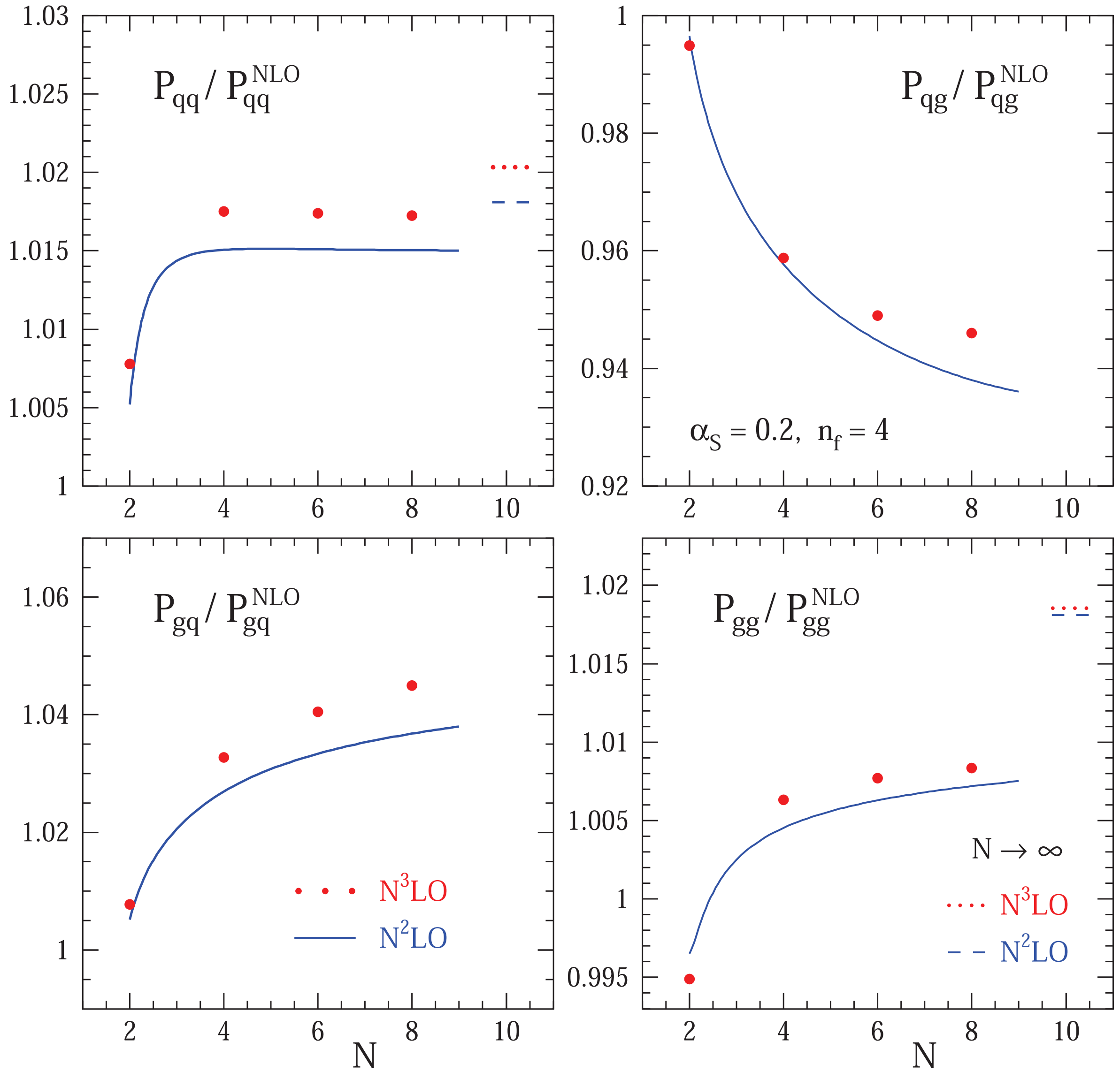
$$f^{\rm wrap}(N) \;=\; 5\zeta_5 - 2S_{-5} + 4S_{-2}\zeta_3 - 4S_{-2,-3} + 8S_{-2,-2,1} + 4S_{3,-2} - 4S_{4,1} + 2S_5$$

- Three-loop Wilson coefficient $c_{\rm ns}^{(3)}(N)$ S.M., Vermaseren, Vogt '05
  - $c_{\rm ns}^{(3)}(N) \simeq C_F \left( C_F - \frac{C_A}{2} \right)^2 \{ N(N+1) \, f^{\rm wrap}(N) \}$
- Non-planar part of $\gamma_{\rm uni}$ in $N = 4$ SYM at four loops Kniehl, Velizhanin '21

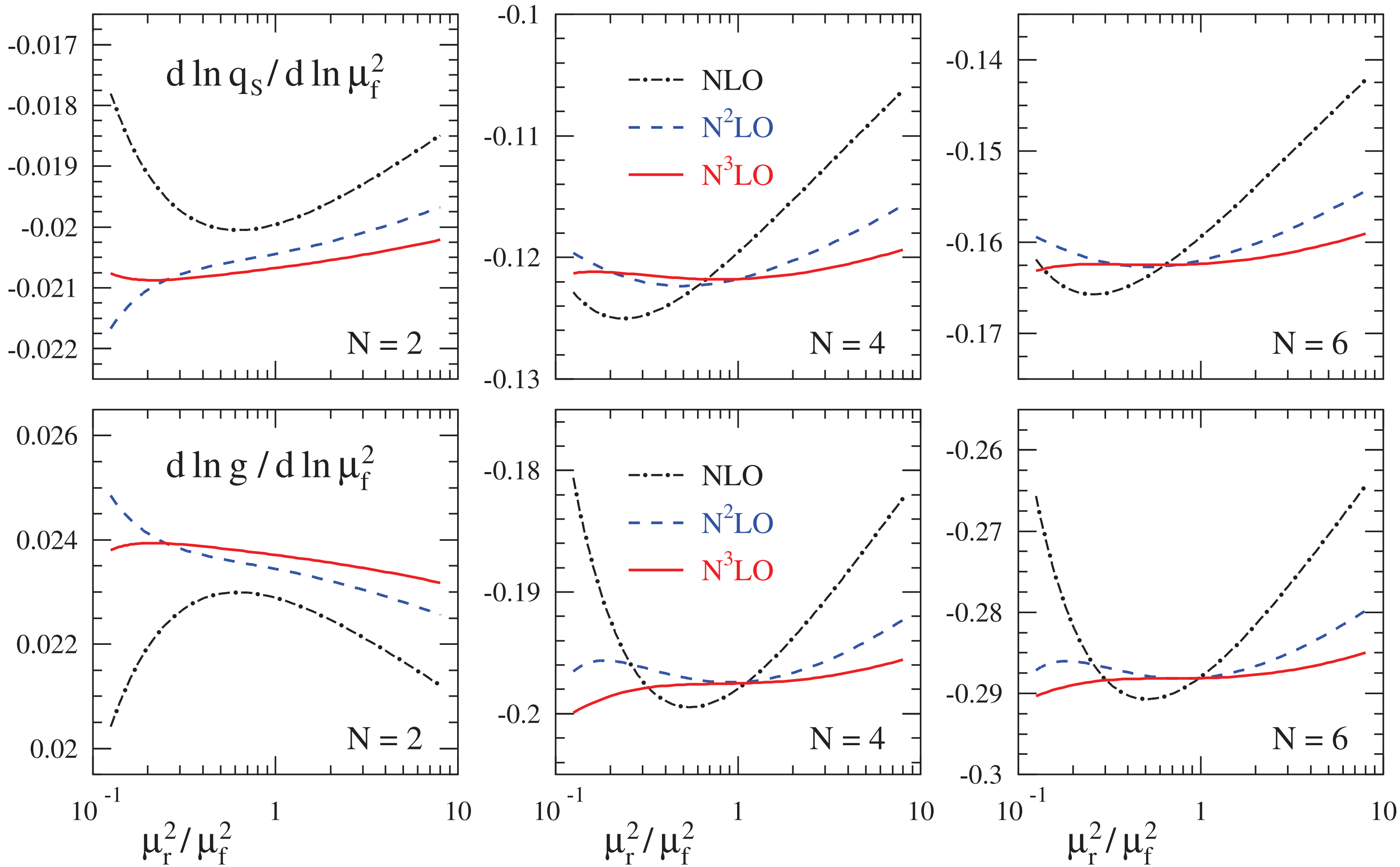
# *Four-loop singlet Mellin moments*

- Singlet moments at NNLO (lines) and $N^3$LO (even-$N$ points) normalized to NLO results
  - $\alpha_s(\mu_f) = 0.2$ and $n_f = 4$

- Large-$N$ limits in $qq$- and $gg$-channel



# Scale stability of singlet evolution




- Renormalization-scale dependence of singlet PDFs $d\ln q_{\rm s}^{\pm}/d\ln\mu_f^2$ and $d\ln g^{\pm}/d\ln\mu_f^2$ at $N = 2,\ 4,$ and $6$ using NLO, NNLO and N$^3$LO predictions with $\alpha_s(\mu_f) = 0.2$ and $n_f = 4$

## *Five-loop Mellin moments*

# *Five-loop Mellin moments*

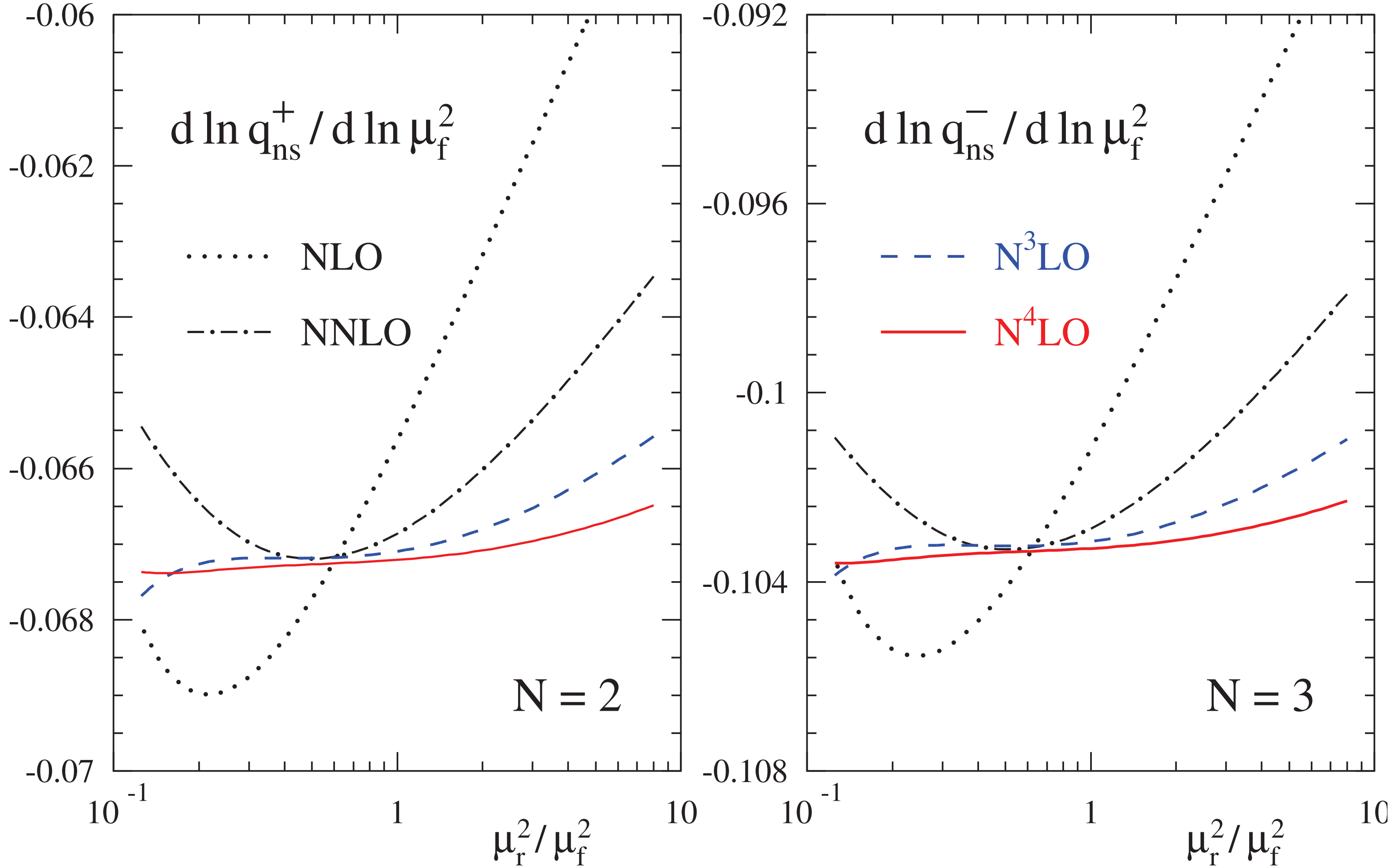
- Moments $N = 2$ and $N = 3$ for nonsinglet anomalous dimensions $\gamma_{\rm ns}^{\pm}$
- Implementation by Herzog, Ruijl '17 of local $R^*$ operation Chetyrkin, Tkachov '82; Chetyrkin, Smirnov '84 for reduction of five-loop self-energy diagrams to four-loop ones computed with **Forcer** Ruijl, Ueda, Vermaseren '17

$$
\begin{aligned}
\gamma_{\rm ns}^{(4)+}(N=2) \; = \;& \\
& C_F^5 \left[ \frac{9306376}{19683} - \frac{802784}{729}\zeta_3 - \frac{557440}{81}\zeta_5 + \frac{12544}{9}\zeta_3^2 + 8512\zeta_7 \right] \\
& - C_A C_F^4 \left[ \frac{81862744}{19683} - \frac{1600592}{243}\zeta_3 + \frac{59840}{81}\zeta_4 - \frac{142240}{27}\zeta_5 + 3072\zeta_3^2 - \frac{35200}{9}\zeta_6 + 19936\zeta_7 \right] \\
& + C_A^2 C_F^3 \left[ \frac{63340406}{6561} - \frac{1003192}{243}\zeta_3 - \frac{229472}{81}\zeta_4 + \frac{61696}{27}\zeta_5 + \frac{30976}{9}\zeta_3^2 - \frac{35200}{9}\zeta_6 + 15680\zeta_7 \right] \\
& - C_A^3 C_F^2 \left[ \frac{220224724}{19683} + \frac{4115536}{729}\zeta_3 - \frac{170968}{27}\zeta_4 - \frac{3640624}{243}\zeta_5 + \frac{70400}{27}\zeta_3^2 + \frac{123200}{27}\zeta_6 + \frac{331856}{27}\zeta_7 \right] \\
& + C_A^4 C_F \left[ \frac{266532611}{39366} + \frac{2588144}{729}\zeta_3 - \frac{221920}{81}\zeta_4 - \frac{3102208}{243}\zeta_5 + \frac{74912}{81}\zeta_3^2 + \frac{334400}{81}\zeta_6 + \frac{178976}{27}\zeta_7 \right] \\
& - \frac{d_{AA}^{(4)}}{N_A} C_F \left[ \frac{15344}{81} - \frac{12064}{27}\zeta_3 - \frac{704}{3}\zeta_4 + \frac{58400}{27}\zeta_5 - \frac{6016}{3}\zeta_3^2 - \frac{19040}{9}\zeta_7 \right] \\
& + \frac{d_{FA}^{(4)}}{N_F} C_F \left[ \frac{23968}{81} - \frac{733504}{81}\zeta_3 + \frac{176320}{81}\zeta_5 + \frac{6400}{3}\zeta_3^2 + \frac{77056}{9}\zeta_7 \right] \\
& - \frac{d_{FA}^{(4)}}{N_F} C_A \left[ \frac{82768}{81} - \frac{555520}{81}\zeta_3 + \frac{10912}{9}\zeta_4 - \frac{1292960}{81}\zeta_5 + \frac{84352}{27}\zeta_3^2 + \frac{140800}{27}\zeta_6 + 12768\zeta_7 \right] \\
& + n_f C_F^4 \left[ \frac{1824964}{19683} - \frac{463520}{243}\zeta_3 + \frac{21248}{81}\zeta_4 - \frac{16480}{81}\zeta_5 + \frac{6656}{9}\zeta_3^2 - \frac{6400}{9}\zeta_6 + \frac{8960}{3}\zeta_7 \right] \\
& - n_f C_A C_F^3 \left[ \frac{3375082}{6561} - \frac{420068}{243}\zeta_3 - \frac{48256}{81}\zeta_4 + \frac{458032}{81}\zeta_5 + \frac{3968}{3}\zeta_3^2 - \frac{8000}{3}\zeta_6 + \frac{4480}{3}\zeta_7 \right] \\
& + n_f C_A^2 C_F^2 \left[ \frac{15291499}{13122} + \frac{1561600}{243}\zeta_3 - \frac{114536}{27}\zeta_4 - \frac{252544}{243}\zeta_5 + \frac{24896}{27}\zeta_3^2 + \frac{13600}{27}\zeta_6 + \frac{11200}{27}\zeta_7 \right] \\
& - n_f C_A^3 C_F \left[ \frac{48846580}{19683} + \frac{4314308}{729}\zeta_3 - \frac{274768}{81}\zeta_4 - \frac{1389080}{243}\zeta_5 + \frac{27808}{81}\zeta_3^2 + \frac{184000}{81}\zeta_6 + \frac{39088}{27}\zeta_7 \right] \\
& + n_f \frac{d_{FA}^{(4)}}{N_F} \left[ \frac{22096}{27} + \frac{43712}{81}\zeta_3 - \frac{512}{9}\zeta_4 - \frac{217280}{81}\zeta_5 - \frac{25088}{27}\zeta_3^2 + \frac{25600}{27}\zeta_6 - 2464\zeta_7 \right] \\
& - n_f C_F \frac{d_{FF}^{(4)}}{N_F} \left[ \frac{170752}{81} - \frac{328832}{81}\zeta_3 + \frac{650240}{81}\zeta_5 - \frac{8192}{9}\zeta_3^2 - \frac{35840}{9}\zeta_7 \right] \\
& + n_f C_A \frac{d_{FF}^{(4)}}{N_F} \left[ \frac{207824}{81} + \frac{251392}{81}\zeta_3 - \frac{5632}{9}\zeta_4 - \frac{522880}{81}\zeta_5 + \frac{15872}{27}\zeta_3^2 + \frac{70400}{27}\zeta_6 - \frac{29120}{9}\zeta_7 \right] \\
& + n_f^2 C_F^3 \left[ \frac{1082297}{6561} - \frac{145792}{243}\zeta_3 + \frac{1072}{81}\zeta_4 + \frac{55552}{81}\zeta_5 + \frac{1792}{9}\zeta_3^2 - \frac{3200}{9}\zeta_6 \right] \\
& + n_f^2 C_A C_F^2 \left[ \frac{332254}{2187} - \frac{85016}{243}\zeta_3 + \frac{20752}{27}\zeta_4 - \frac{28544}{81}\zeta_5 - \frac{13952}{27}\zeta_3^2 + \frac{1600}{27}\zeta_6 \right] \\
& + n_f^2 C_A^2 C_F \left[ \frac{631400}{6561} + \frac{214268}{243}\zeta_3 - 784\zeta_4 - \frac{53344}{243}\zeta_5 + \frac{25472}{81}\zeta_3^2 + \frac{22400}{81}\zeta_6 \right] \\
& - n_f^2 \frac{d_{FF}^{(4)}}{N_F} \left[ \frac{43744}{81} - \frac{35648}{81}\zeta_3 - \frac{1792}{9}\zeta_4 - \frac{52480}{81}\zeta_5 + \frac{2048}{27}\zeta_3^2 + \frac{12800}{27}\zeta_6 \right] \\
& + n_f^3 C_F^2 \left[ \frac{265510}{19683} + \frac{11872}{729}\zeta_3 - \frac{128}{3}\zeta_4 + \frac{512}{27}\zeta_5 \right] \\
& + n_f^3 C_A C_F \left[ \frac{168677}{19683} + \frac{11872}{729}\zeta_3 + \frac{2752}{81}\zeta_4 - \frac{4096}{81}\zeta_5 \right] - n_f^4 C_F \left[ \frac{5504}{19683} + \frac{1024}{729}\zeta_3 - \frac{128}{81}\zeta_4 \right]
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{\rm ns}^{(4)-}(N=3) \; = \;& \\
& C_F^5 \left[ \frac{81472935625}{80621568} + \frac{99382175}{23328}\zeta_3 - \frac{3395975}{162}\zeta_5 - \frac{9650}{9}\zeta_3^2 + \frac{34685}{2}\zeta_7 \right] \\
& - C_A C_F^4 \left[ \frac{286028134219}{80621568} - \frac{23916529}{7776}\zeta_3 - 4490\zeta_3^2 + \frac{134090}{81}\zeta_4 - \frac{2468075}{108}\zeta_5 - \frac{55000}{9}\zeta_6 + \frac{155155}{4}\zeta_7 \right] \\
& + C_A^2 C_F^3 \left[ \frac{20173099267}{3359232} - \frac{15401281}{864}\zeta_3 + \frac{732787}{1296}\zeta_4 + \frac{1972075}{216}\zeta_5 - \frac{63830}{9}\zeta_3^2 - \frac{79750}{9}\zeta_6 + \frac{139895}{4}\zeta_7 \right] \\
& - C_A^3 C_F^2 \left[ \frac{166662991819}{20155392} - \frac{36397493}{2916}\zeta_3 - \frac{103763}{54}\zeta_4 + \frac{30994565}{3888}\zeta_5 - \frac{133990}{27}\zeta_3^2 - \frac{72875}{54}\zeta_6 + \frac{2127335}{108}\zeta_7 \right] \\
& + C_A^4 C_F \left[ \frac{75932079965}{10077696} - \frac{27693563}{23328}\zeta_3 - \frac{1791229}{1296}\zeta_4 - \frac{9417425}{1944}\zeta_5 - \frac{96700}{81}\zeta_3^2 + \frac{163625}{81}\zeta_6 + \frac{199640}{27}\zeta_7 \right] \\
& - \frac{d_{AA}^{(4)}}{N_A} C_F \left[ \frac{81725}{162} - \frac{33505}{18}\zeta_3 - \frac{1100}{3}\zeta_4 + \frac{52025}{18}\zeta_5 - \frac{7000}{3}\zeta_3^2 - \frac{48125}{36}\zeta_7 \right] \\
& - \frac{d_{FA}^{(4)}}{N_F} C_F \left[ \frac{231575}{36} + \frac{6351445}{324}\zeta_3 - \frac{2927225}{162}\zeta_5 + \frac{23210}{3}\zeta_3^2 - \frac{200410}{9}\zeta_7 \right] \\
& + \frac{d_{FA}^{(4)}}{N_F} C_A \left[ \frac{165871}{54} + \frac{1816625}{162}\zeta_3 - \frac{41800}{9}\zeta_4 - \frac{4456145}{162}\zeta_5 + \frac{196880}{27}\zeta_3^2 + \frac{200750}{27}\zeta_6 - \frac{7525}{4}\zeta_7 \right] \\
& + n_f C_F^4 \left[ \frac{1776521549}{40310784} - \frac{1332919}{486}\zeta_3 + \frac{5000}{9}\zeta_3^2 + \frac{33290}{81}\zeta_4 - \frac{30325}{81}\zeta_5 - \frac{10000}{9}\zeta_6 + \frac{14000}{3}\zeta_7 \right] \\
& - n_f C_A C_F^3 \left[ \frac{3737356319}{3359232} - \frac{2327111}{432}\zeta_3 + \frac{1280}{3}\zeta_3^2 + \frac{262069}{648}\zeta_4 + \frac{1693715}{162}\zeta_5 - \frac{14000}{3}\zeta_6 + \frac{7000}{3}\zeta_7 \right] \\
& + n_f C_A^2 C_F^2 \left[ \frac{5637513931}{3359232} + \frac{2711207}{486}\zeta_3 - \frac{5020}{27}\zeta_3^2 - \frac{457499}{108}\zeta_4 + \frac{508820}{243}\zeta_5 - \frac{20375}{27}\zeta_6 + \frac{50155}{108}\zeta_7 \right] \\
& - n_f C_A^3 C_F \left[ \frac{8766012215}{2519424} + \frac{45697231}{5832}\zeta_3 + \frac{1195}{81}\zeta_3^2 - \frac{2848403}{648}\zeta_4 - \frac{1808870}{243}\zeta_5 + \frac{222250}{81}\zeta_6 + \frac{250915}{108}\zeta_7 \right] \\
& - n_f C_F \frac{d_{FF}^{(4)}}{N_F} \left[ \frac{24385}{27} - \frac{334010}{81}\zeta_3 - \frac{8480}{9}\zeta_3^2 + \frac{1622600}{81}\zeta_5 - \frac{135380}{9}\zeta_7 \right] \\
& + n_f \frac{d_{FA}^{(4)}}{N_F} \left[ \frac{297889}{162} + \frac{154970}{81}\zeta_3 - \frac{62600}{27}\zeta_3^2 + \frac{3700}{9}\zeta_4 - \frac{122780}{81}\zeta_5 - \frac{36500}{27}\zeta_6 - 910\zeta_7 \right] \\
& + n_f C_A \frac{d_{FF}^{(4)}}{N_F} \left[ \frac{241835}{162} + \frac{333487}{81}\zeta_3 + \frac{30560}{27}\zeta_3^2 - 10780/9\,\zeta_4 - \frac{316900}{81}\zeta_5 + \frac{110000}{27}\zeta_6 - \frac{71960}{9}\zeta_7 \right] \\
& + n_f^2 C_F^3 \left[ \frac{512848319}{1679616} - \frac{57109}{54}\zeta_3 + \frac{2800}{9}\zeta_3^2 + \frac{9118}{81}\zeta_4 + \frac{86440}{81}\zeta_5 - \frac{5000}{9}\zeta_6 \right] \\
& + n_f^2 C_A C_F^2 \left[ \frac{1080083}{5832} - \frac{296729}{972}\zeta_3 - \frac{21800}{27}\zeta_3^2 + \frac{56327}{54}\zeta_4 - \frac{42860}{81}\zeta_5 + \frac{2500}{27}\zeta_6 \right] \\
& + n_f^2 C_A^2 C_F \left[ \frac{61747877}{419904} + \frac{2496811}{1944}\zeta_3 + \frac{39800}{81}\zeta_3^2 - \frac{3503}{3}\zeta_4 - \frac{88990}{243}\zeta_5 + \frac{35000}{81}\zeta_6 \right] \\
& - n_f^2 \frac{d_{FF}^{(4)}}{N_F} \left[ \frac{19435}{27} - \frac{53366}{81}\zeta_3 + \frac{3200}{27}\zeta_3^2 - \frac{3160}{9}\zeta_4 - \frac{70000}{81}\zeta_5 + \frac{20000}{27}\zeta_6 \right] \\
& + n_f^3 C_F^2 \left[ \frac{28758139}{1259712} + \frac{21673}{729}\zeta_3 - \frac{610}{9}\zeta_4 + \frac{800}{27}\zeta_5 \right] \\
& + n_f^3 C_A C_F \left[ \frac{13729181}{1259712} + \frac{14947}{729}\zeta_3 + \frac{4390}{81}\zeta_4 - \frac{6400}{81}\zeta_5 \right] - n_f^4 C_F \left[ \frac{259993}{629856} + \frac{1660}{729}\zeta_3 - \frac{200}{81}\zeta_4 \right]
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{\rm ns}^{(4)\,\rm v}(N=3) \; = \; & \gamma_{\rm ns}^{(4)-}(N=3) \\
& + n_f \frac{d_{abc}d^{abc}}{N_F} \Big\{ C_F^2 \left[ \frac{79906955}{46656} + \frac{246955}{54}\zeta_3 - \frac{504550}{81}\zeta_5 \right] \\
& \quad - C_A C_F \left[ \frac{9797321}{3888} - \frac{475655}{54}\zeta_3 + \frac{17600}{9}\zeta_4 + \frac{516950}{81}\zeta_5 - \frac{500}{9}\zeta_3^2 + \frac{2800}{9}\zeta_7 \right] \\
& \quad + C_A^2 \left[ \frac{166535}{486} - \frac{1783913}{324}\zeta_3 + \frac{5555}{9}\zeta_4 + \frac{507515}{81}\zeta_5 - \frac{2035}{27}\zeta_3^2 - \frac{5500}{27}\zeta_6 - \frac{2765}{18}\zeta_7 \right] \\
& \quad + n_f C_A \left[ \frac{285985}{3888} + \frac{41954}{81}\zeta_3 + \frac{160}{27}\zeta_3^2 - \frac{1010}{9}\zeta_4 - \frac{56480}{81}\zeta_5 + \frac{1000}{27}\zeta_6 \right] \\
& \quad + n_f C_F \left[ \frac{1098323}{3888} - \frac{49720}{81}\zeta_3 + \frac{3200}{9}\zeta_4 \right] - n_f^2 \left[ \frac{21823}{1944} \right] \Big\}
\end{aligned}
$$

# *Scale stability of evolution*




- Renormalization-scale dependence of $d \ln q_{\rm ns}^{\pm} / d \ln \mu_f^2$ at $N = 2$ and $N = 3$ using NLO, NNLO, N$^3$LO and N$^4$LO predictions with $\alpha_s(\mu_f) = 0.2$ and $n_f = 4$

# *Coefficient functions at four loops*

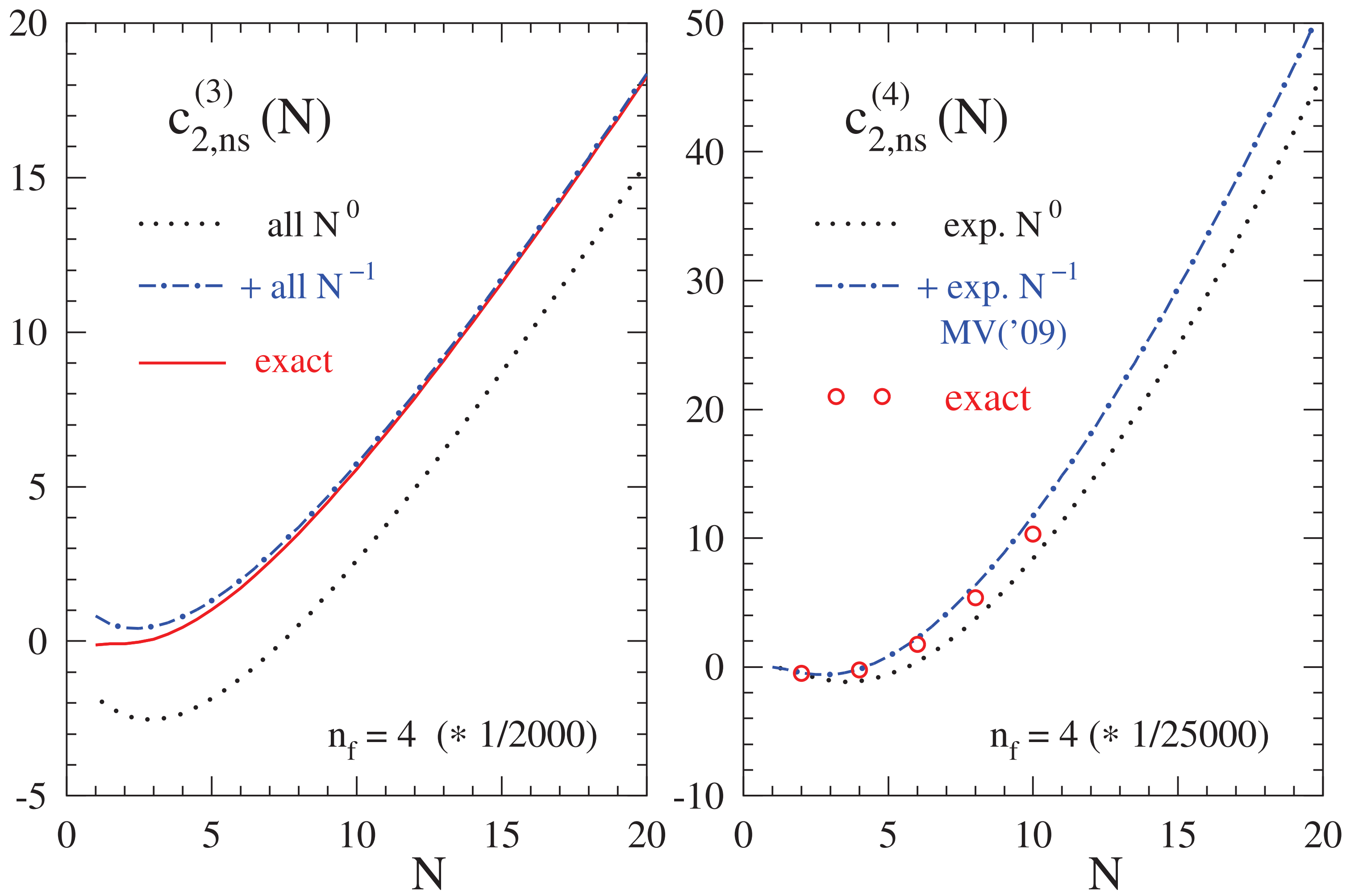
# *Four-loop non-singlet Mellin moments*

- Perturbative expansion of non-singlet coefficient functions
  - Mellin moments $N = 2,\ 4,\ 6,\ 8,\ 10,\ 12,\ 14$ of $C_{2,\mathrm{ns}}$ and $C_{L,\mathrm{ns}}$ (moments $N = 12,\ 14$ in limit of large $n_c$)
  - Mellin moments $N = 1,\ 3,\ 5,\ 7,\ 9,\ 11,\ 13,\ 15$ of $C_{3,-}$ (moments $N = 11,\ 13,\ 15$ in limit of large $n_c$)
- Numerical results for $C_{2,\mathrm{ns}}(N, n_f)$

S.M., Ruijl, Ueda, Vermaseren, Vogt *to appear*

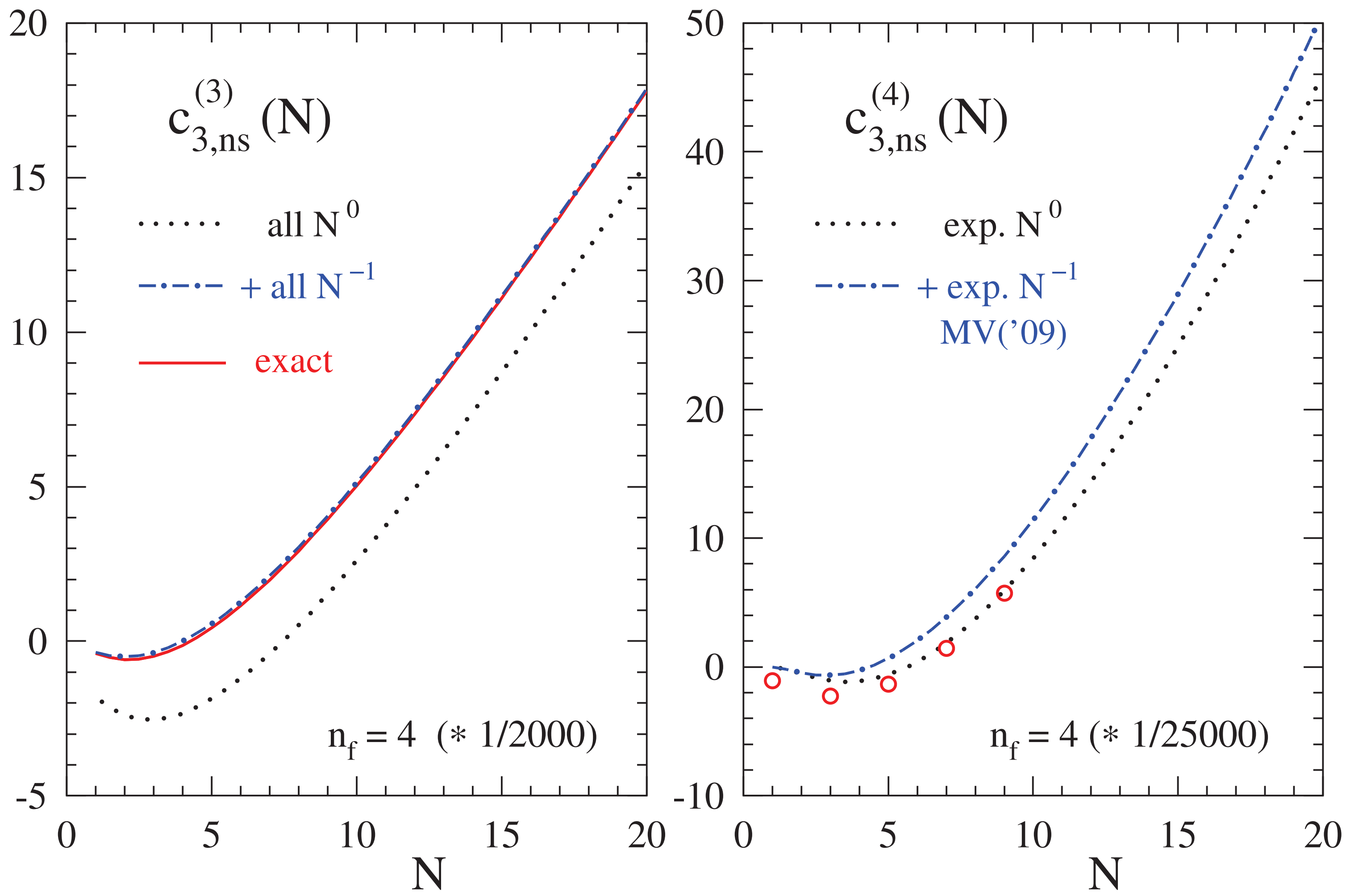
$$
\begin{aligned}
C_{2,\mathrm{ns}}(2,4) &= 1 + 0.0354\ \alpha_s - 0.0231\ \alpha_s^2 - 0.0613\ \alpha_s^3 - 0.4746\ \alpha_s^4\ , \\
C_{2,\mathrm{ns}}(4,4) &= 1 + 0.4828\ \alpha_s + 0.4711\ \alpha_s^2 + 0.4727\ \alpha_s^3 - 0.2458\ \alpha_s^4\ , \\
C_{2,\mathrm{ns}}(6,4) &= 1 + 0.8894\ \alpha_s + 1.2054\ \alpha_s^2 + 1.7572\ \alpha_s^3 + 1.7748\ \alpha_s^4\ , \\
C_{2,\mathrm{ns}}(8,4) &= 1 + 1.2358\ \alpha_s + 2.0208\ \alpha_s^2 + 3.5294\ \alpha_s^3 + 5.3921\ \alpha_s^4\ , \\
C_{2,\mathrm{ns}}(10,4) &= 1 + 1.5359\ \alpha_s + 2.8608\ \alpha_s^2 + 5.6244\ \alpha_s^3 + 10.324\ \alpha_s^4\ .
\end{aligned}
$$

# *Four-loop non-singlet Mellin moments*



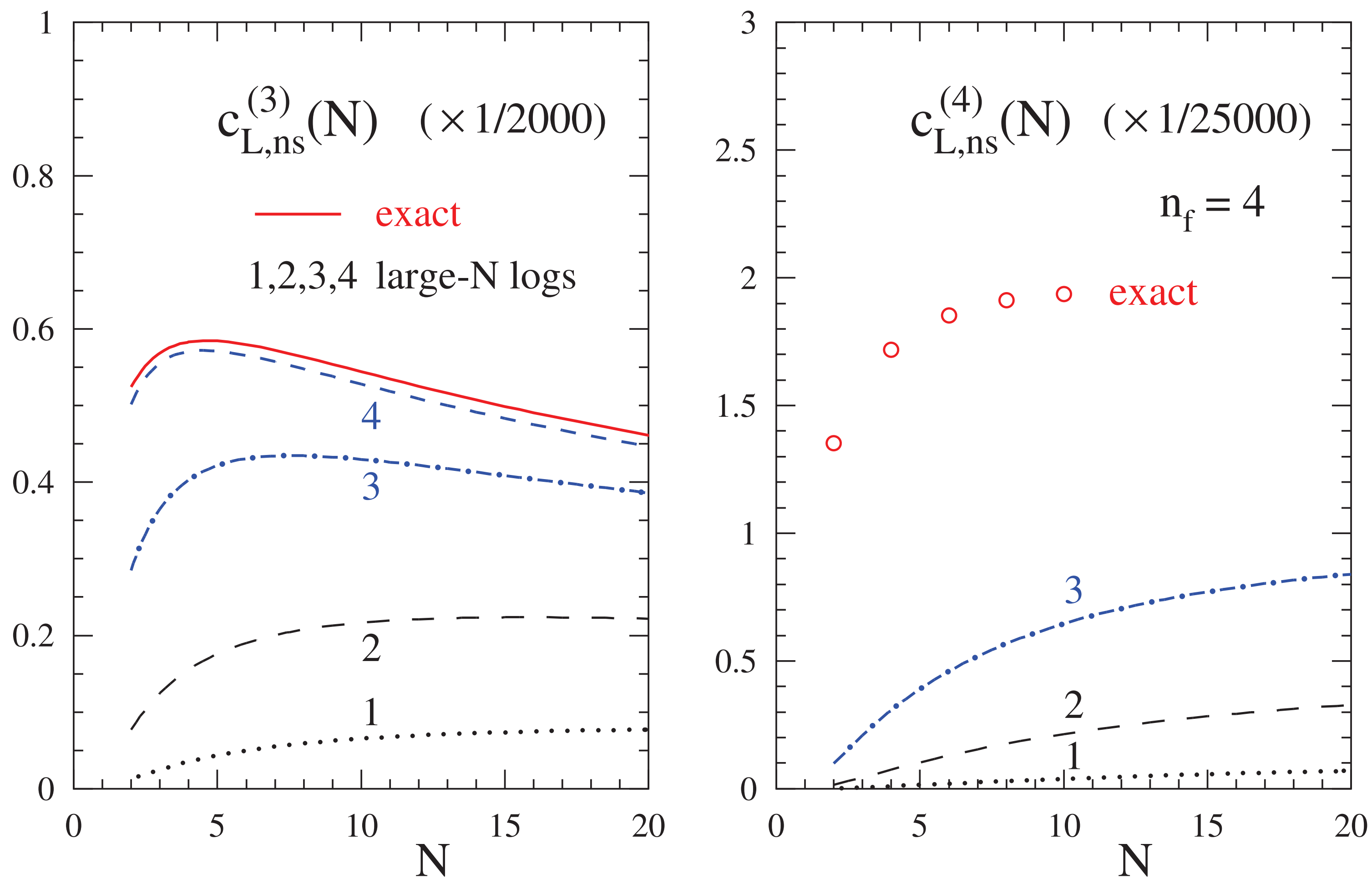

- Exact results for $c^{(3)}_{2,\mathrm{ns}}$ (N$^3$LO) at $n_f = 4$ (rescaled by $2000 \simeq (4\pi)^3$)
- Moments for $c^{(4)}_{2,\mathrm{ns}}$ (N$^4$LO) at $n_f = 4$ (rescaled by $25000 \simeq (4\pi)^4$)
- Comparison with contributions provided by large-$N$ resummations

# *Four-loop non-singlet Mellin moments*



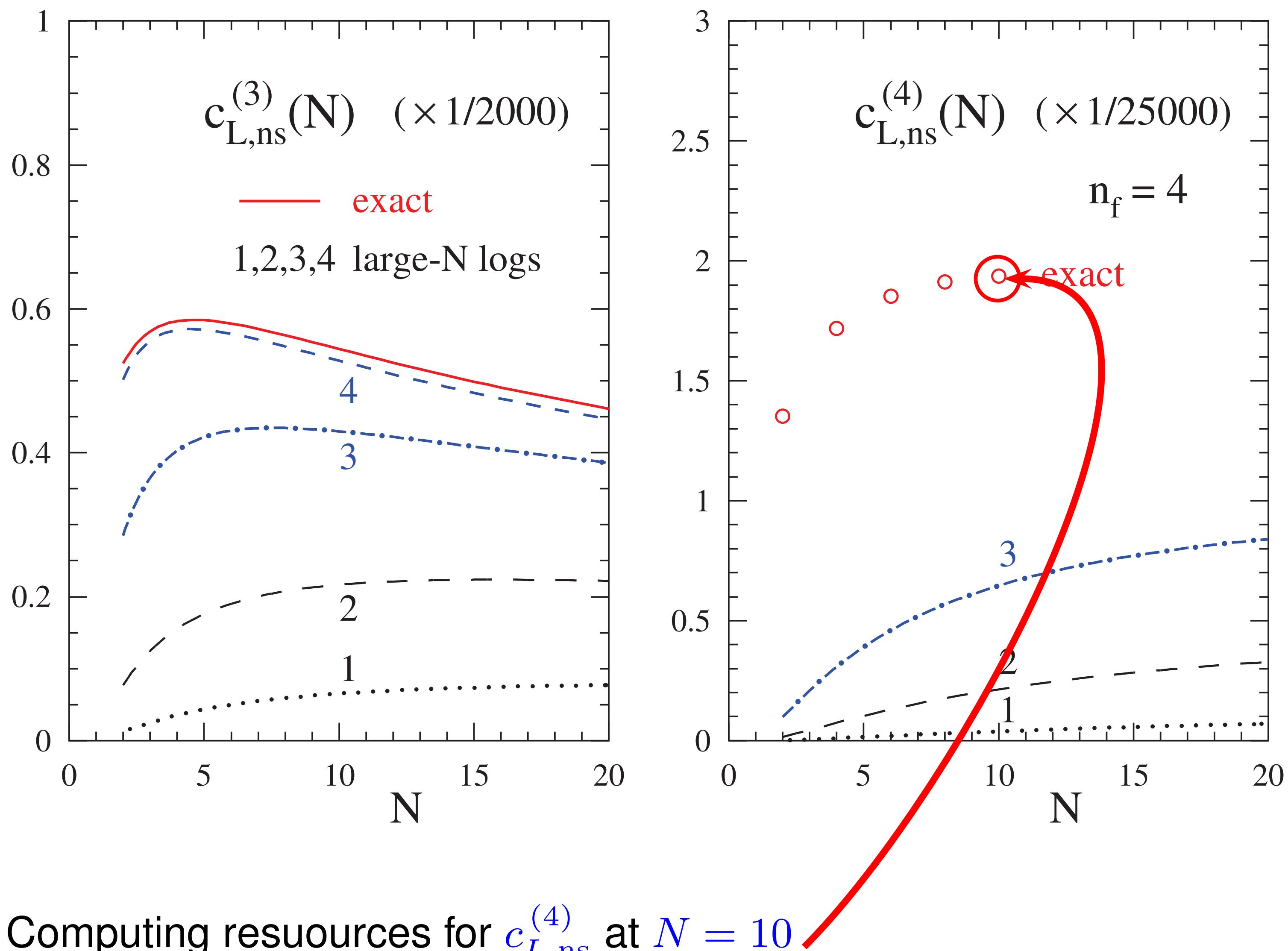

- Exact results for $c^{(3)}_{3,ns}$ (N$^3$LO) at $n_f = 4$ (rescaled by $2000 \simeq (4\pi)^3$)
- Moments for $c^{(4)}_{3,ns}$ (N$^4$LO) at $n_f = 4$ (rescaled by $25000 \simeq (4\pi)^4$)
- Comparison with contributions provided by large-$N$ resummations

# *Four-loop non-singlet Mellin moments*




- Exact results for $c^{(3)}_{L,\mathrm{ns}}$ (N$^3$LO) and moments for $c^{(4)}_{3,\mathrm{ns}}$ (N$^4$LO) at $n_f = 4$
- Tower of logarithms $\ln^4(N)/N$, . . . , $\ln(N)/N$ at N$^3$LO
- Tower of logarithms $\ln^6(N)/N$, . . . , $\ln^4(N)/N$ at N$^4$LO

# *Four-loop non-singlet Mellin moments*




- Computing resuources for $c^{(4)}_{L,ns}$ at $N = 10$
  - single core CPU time $\mathcal{O}(800.000)$h (**TForm** speed-up is $\mathcal{O}(10)$h)
  - $\mathcal{O}(20)$ TByte of disk space at intermediate stages of computation

# Threshold resummation

- Coefficient function in large $x$-limit have large logarithms at n$^{\rm th}$-order

$$\alpha_s^n \, \frac{\ln^{2n-1}(1-x)}{(1-x)_+} \longleftrightarrow \alpha_s^n \, \ln^{2n}(N)$$

- Threshold resummation in Mellin space

$$C^N \;=\; (1 + \alpha_s\, g_{01} + \alpha_s^2\, g_{02} + \ldots) \cdot \exp\left(G^N\right) \;+\; \mathcal{O}(N^{-1} \ln^n N)$$

- Control over logarithms $\ln(N)$ with $\lambda = \beta_0 \alpha_s \ln(N)$ to N$^k$LL accuracy

$$G^N \;=\; \ln(N) g_1(\lambda) + g_2(\lambda) + \alpha_s g_3(\lambda) + \alpha_s^2 g_4(\lambda) + \alpha_s^3 g_5(\lambda) + \ldots$$

  - $g_1(\lambda)$: LL Sterman '87; Appell, Mackenzie, Sterman '88
  - $g_2(\lambda)$: NLL Catani Trenatdue '89
  - $g_3(\lambda)$: NNLL or N$^2$LL Vogt '00; Catani, Grazzini, de Florian, Nason '03
  - $g_4(\lambda)$: N$^3$LL S.M., Vermaseren, Vogt '05
  - $g_5(\lambda)$: N$^4$LL Das, S.M., Vogt '19

- Resummed $G^N$ predicts fixed orders in perturbation theory
  - generating functional for towers of large logarithms

# DIS coefficient functions at four loops

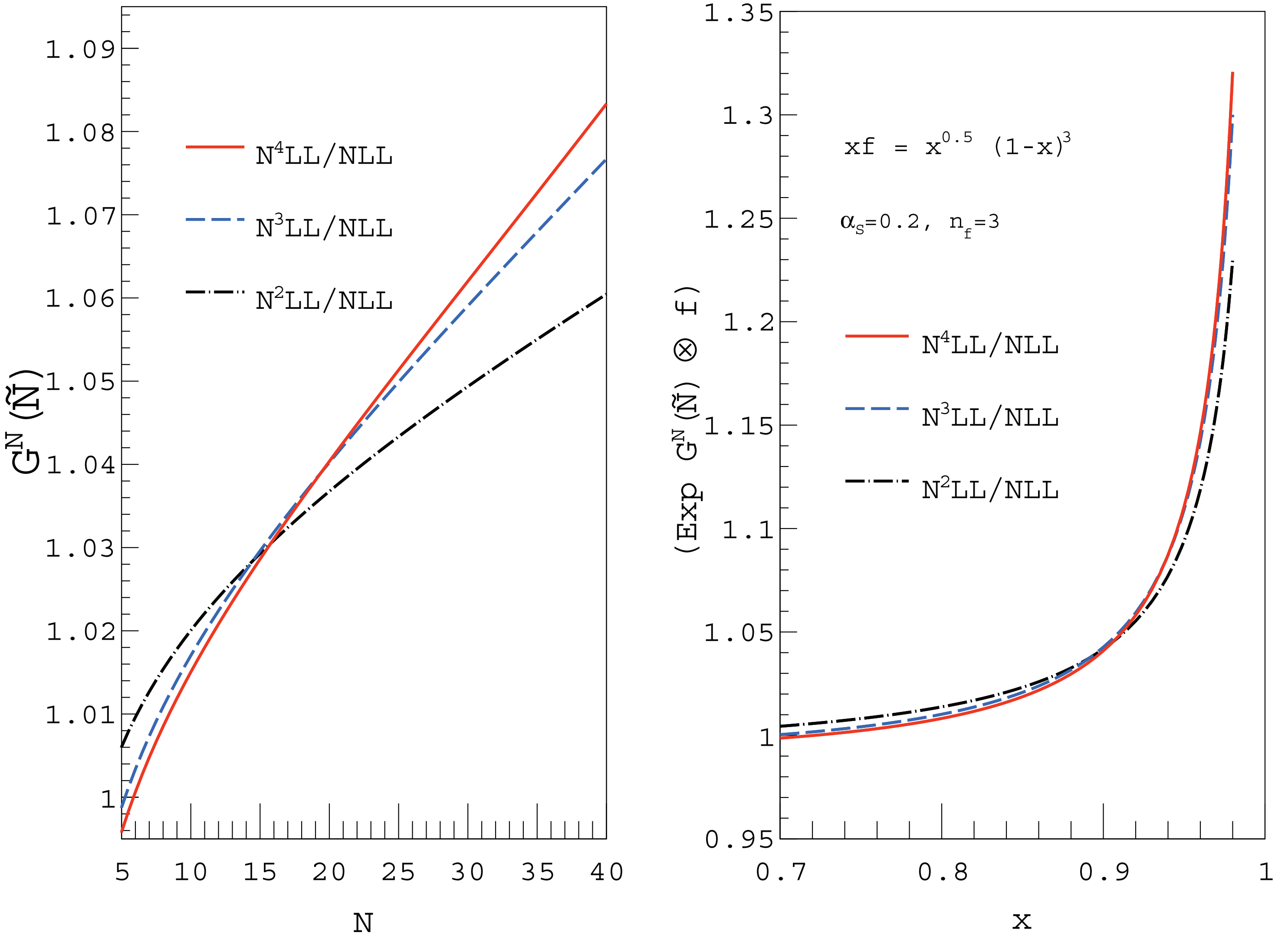
## Result

- Four-loop coefficient function $c_{2,\mathrm{q}}^{(4)}$ known $\frac{\ln^7(1-x)}{(1-x)_+}, \ldots, \frac{1}{(1-x)_+}$
- New result for $\frac{1}{(1-x)_+}$ term
  - best estimate (using partial large-$n_c$ information)

$$
\begin{aligned}
c_{2,\mathrm{q}}^{(4)}\Big|_{\frac{1}{(1-x)_+}} &= (3.874 \pm 0.010) \cdot 10^4 + (-3.496490 \pm 0.000003) \cdot 10^4\, n_f \\
&\quad + 2062.715\, n_f^2 - 12.08488\, n_f^3 + 47.55183\, n_f\, fl_{11}
\end{aligned}
$$

- Based on results for
  - Quark and gluon form factors at four loops in QCD
    Lee, von Manteuffel, Schabinger, Smirnov, Smirnov, Steinhauser ‘22
  - eikonal anomalous dimensions Dixon, Magnea, Sterman ‘08
  - Mellin moments of DIS structure functions at four loops

# Numerical results for DIS




- Left: Resummed exponent $G^N$ normalized to NLL for DIS plotted successively up to N$^4$LL for $\alpha_s = 0.2$ and $n_f = 3$
- Right: Resummed series convoluted with typical shape for a quark distribution $xf = x^{0.5}(1-x)^3$ up to N$^4$LL

# *Summary*

- Experimental precision of $\lesssim 1\%$ motivates computations at higher order in perturbative QCD
  - hard scattering in forward and off-forward kinematics
  - theoretical predictions at NNLO in QCD nowadays standard
- Push for theory results at N$^3$LO and N$^4$LO
  - evolutions equations and inclusive cross sections
  - massive use of computer algebra
- Novel structural insights into QCD from integrabilty and conformal symmetry
  - Key parts of QCD inherited from $N = 4$ Super Yang-Mills theory
  - Conformal symmetry in parts of QCD evolution equations
- Precision studies of hadron structure
  - great prospects for DIS at future colliders