Angular Integrals in d dimensions

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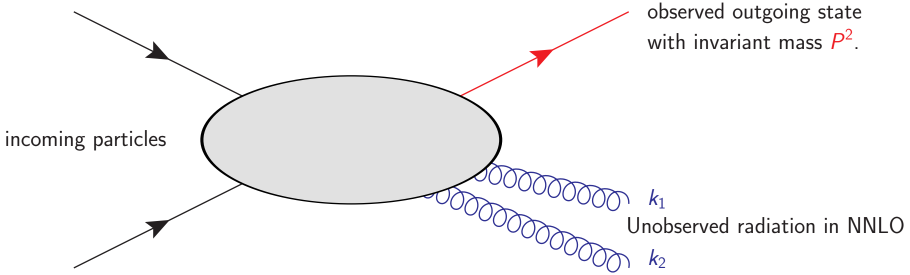
V.E.Lyubovitskij, F.Wunder, A.S.Zhevlakov, *New ideas for handling of loop and angular integrals in D-dimensions in QCD*, JHEP **06** (2021)

1. Introduction and literature review
2. Appearance in perturbative calculations
3. Analytic calculation of Van Neerven integrals
4. Properties of angular integrals
5. All order ε-expansion
6. Outlook

Introduction and literature review

What integrals are we talking about?



Two particle phase space ($d = 4 - 2\varepsilon$):

$$\int \mathrm{dPS}_{2,P} = \int \frac{\mathrm{d}^{d-1} k_1}{(2\pi)^{d-1} 2k_1^0} \int \frac{\mathrm{d}^{d-1} k_2}{(2\pi)^{d-1} 2k_2^0} (2\pi)^d \delta^d \left(P - k_1 - k_2\right)$$

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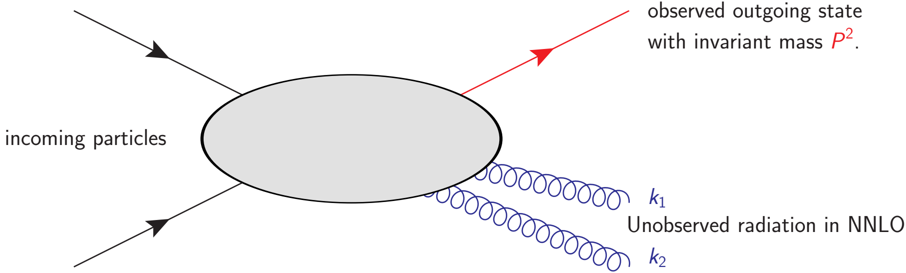

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Angular integration measure:

$$\mathrm{d}\Omega_{k_1 k_2} \equiv \mathrm{d}\theta_1 \sin^{1-2\varepsilon}\theta_1 \mathrm{d}\theta_2 \sin^{-2\varepsilon}\theta_2$$

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Typical integral:

$$\int \mathrm{d}\Omega_{k_1 k_2} \frac{1}{(a + b\cos\theta_1)^j (A + B\cos\theta_1 + C\sin\theta_1\cos\theta_2)^l}$$

The Van Neerven integral

j, l integers; a, b, A, B, C real (or complex) parameters:

$$\int \mathrm{d}\Omega_{k_1 k_2} \frac{1}{(a + b\cos\theta_1)^j (A + B\cos\theta_1 + C\sin\theta_1\cos\theta_2)^l}$$

Divided into classes:

	$a^2 = b^2$	$a^2 \neq b^2$
$A^2 = B^2 + C^2$	massless	single massive
$A^2 \neq B^2 + C^2$	single massive	double massive

A look at the literature: What is (not) known?

W.Beenakker, H.Kuijf, W.L. van Neerven, J.Smith, *QCD corrections to heavy quark production in $p\bar{p}$ collisions*, Phys.Rev.D **40** (1989)

40 QCD CORRECTIONS TO HEAVY-QUARK PRODUCTION IN $p\bar{p}$ . . . 77

where we have integrated over all angles of the gluon momentum which do not appear in $|M^R|^2$. After changing the integration variables E_2 and χ to t_1 and u_1, using

$$E_2=-\frac{t_1+u_1}{2\sqrt{s}}, \quad \cos\chi=\frac{u_1-t_1}{\sqrt{(t_1+u_1)^2-4sm^2}} , \quad \text{(B13)}$$

we find

$$s^2\frac{d^2\sigma}{dt_1du_1}=K(\tfrac{1}{2})^3\frac{S_\epsilon^2\mu^{-\epsilon}}{\Gamma(n-3)}\left[\frac{t_1u_1-sm^2}{s\mu^2}\right]^{n/2-2}\frac{s_4^{n-3}}{(s_4+m^2)^{n/2-1}}\int_0^\pi d\theta_1\sin^{n-3}\theta_1\int_0^\pi d\theta_2\sin^{n-4}\theta_2|M^R|^2 , \quad \text{(B14)}$$

which is the formula used in Sec. IV.

APPENDIX C

Here we give the angular integrals of the terms $(s')^k(s'')^l$ which arise from the partial fractioning of the square of matrix element $|M^R|^2$ in (4.3). The general expression for the angular integral is given in (4.9). The specific four-dimensional integrals for $a^2\neq b^2$ and $A^2\neq B^2+C^2$ are listed first:[25]

$$I_4^{(0,0)}=2\pi , \quad \text{(C1)}$$

$$I_4^{(-1,0)}=2\pi a , \quad \text{(C2)}$$

$$I_4^{(0,-1)}=2\pi A , \quad \text{(C3)}$$

$$I_4^{(1,0)}=\frac{\pi}{b}\ln\frac{a+b}{a-b} , \quad \text{(C4)}$$

$$I_4^{(0,1)}=\frac{\pi}{\sqrt{B^2+C^2}}\ln\left[\frac{A+\sqrt{B^2+C^2}}{A-\sqrt{B^2+C^2}}\right] , \quad \text{(C5)}$$

$$I_4^{(1,1)}=\frac{\pi}{\sqrt{X}}\ln\left[\frac{aA-bB+\sqrt{X}}{aA-bB-\sqrt{X}}\right] , \quad \text{(C6)}$$

$$I_4^{(1,2)}=\pi\left[\frac{2a(B^2+C^2)-2bAB}{(A^2-B^2-C^2)X}+\frac{b(bA-aB)}{X^{3/2}}\ln\left[\frac{aA-bB+\sqrt{X}}{aA-bB-\sqrt{X}}\right]\right] , \quad \text{(C7)}$$

$$I_4^{(2,1)}=\pi\left[\frac{2b(bA-aB)}{(a^2-b^2)X}+\frac{a(B^2+C^2)-bAB}{X^{3/2}}\ln\left[\frac{aA-bB+\sqrt{X}}{aA-bB-\sqrt{X}}\right]\right] , \quad \text{(C8)}$$

$$I_4^{(2,2)}=\pi\left[\frac{2b^2}{(a^2-b^2)X}+\frac{2(B^2+C^2)}{(A^2-B^2-C^2)X}-\frac{6b^2C^2}{X^2}\right.$$

with $X=(aA-bB)^2-(A^2-B^2-C^2)(a^2-b^2)$,

$$I_4^{(-1,-1)}=2\pi(aA+\tfrac{1}{3}bB) , \quad \text{(C10)}$$

$$I_4^{(-1,1)}=\pi\left[\frac{2bB}{B^2+C^2}+\frac{a(B^2+C^2)-bAB}{(B^2+C^2)^{3/2}}\times\ln\left[\frac{A+\sqrt{B^2+C^2}}{A-\sqrt{B^2+C^2}}\right]\right] , \quad \text{(C11)}$$

$$I_4^{(0,2)}=2\pi\frac{1}{A^2-B^2-C^2} , \quad \text{(C12)}$$

$$I_4^{(-2,0)}=2\pi(a^2+\tfrac{1}{3}b^2) , \quad \text{(C13)}$$

$$I_4^{(0,-2)}=2\pi[A^2+\tfrac{1}{3}(B^2+C^2)] , \quad \text{(C14)}$$

$$I_4^{(-2,1)}=\pi\left[\frac{4abB}{B^2+C^2}+\frac{b^2A(C^2-2B^2)}{(B^2+C^2)^2}+\left[\frac{[a(B^2+C^2)-bAB]^2}{(B^2+C^2)^{5/2}}-\frac{b^2C^2(A^2-B^2-C^2)}{2(B^2+C^2)^{5/2}}\right]\times\ln\left[\frac{A+\sqrt{B^2+C^2}}{A-\sqrt{B^2+C^2}}\right]\right] , \quad \text{(C15)}$$

$$I_4^{(-1,2)}=\pi\left[\frac{2[a(B^2+C^2)-bAB]}{(B^2+C^2)(A^2-B^2-C^2)}+\frac{bB}{(B^2+C^2)^{3/2}}\ln\left[\frac{A+\sqrt{B^2+C^2}}{A-\sqrt{B^2+C^2}}\right]\right] , \quad \text{(C16)}$$

$$I_4^{(-2,2)}=\pi\left[\frac{2b^2(B^2-C^2)}{(B^2+C^2)^2}+\frac{2[a(B^2+C^2)-bAB]^2}{(A^2-B^2-C^2)(B^2+C^2)^2}+\left[\frac{2bB[a(B^2+C^2)-bAB]}{(B^2+C^2)^{5/2}}\right.\right.$$

78 BEENAKKER, KUIJF, VAN NEERVEN, AND SMITH 40

$$I_4^{(2,-2)}=\pi\left[\frac{2(B^2-C^2)}{b^2}+\frac{2(bA-aB)^2}{b^2(a^2-b^2)}+\frac{aC^2+2B(bA-aB)}{b^3}\ln\left[\frac{a+b}{a-b}\right]\right] , \quad \text{(C18)}$$

In n dimensions[24,25] we need the following integrals, which we classify into groups where $b=-a$, $A^2=B^2+C^2$, or $b=-a$, $A^2\neq B^2+C^2$, or $a^2\neq b^2$, $A^2=B^2+C^2$, respectively.

Group 1, $b=-a$ and $A^2=B^2+C^2$:

$$\hat{I}_n^{(1,0)}=2\pi\frac{1}{a}\frac{1}{n-4} , \quad \text{(C19)}$$

$$\hat{I}_n^{(1,1)}=2\pi\frac{1}{aA}\frac{1}{n-4}F_{1,2}\left[1,1,\tfrac{1}{2}n-1,\frac{A-B}{2A}\right]=2\pi\frac{1}{aA}\frac{1}{n-4}\left[\frac{A+B}{2A}\right]^{n/2-3}\left[1+\tfrac{1}{4}(n-4)^2\mathrm{Li}_2\left[\frac{A-B}{2A}\right]+O((n-4)^3)\right] , \quad \text{(C20)}$$

$$\hat{I}_n^{(-1,1)}=2\pi\frac{a}{A}\frac{n-2}{(n-4)(n-3)}F_{1,2}\left[-1,1,\frac{1}{2}n-1,\frac{A-B}{2A}\right]=\pi\frac{a(A+B)}{A^2}\left[\frac{2}{n-4}-\frac{2B}{A+B}+O(n-4)\right] , \quad \text{(C21)}$$

$$\hat{I}_n^{(-2,1)}=2\pi\frac{a^2}{A}\frac{n}{(n-4)(n-3)}F_{1,2}\left[-2,1,\frac{1}{2}n-1,\frac{A-B}{2A}\right]=\pi\frac{a^2(A+B)^2}{A^3}\left[\frac{2}{n-4}+\frac{A^2-4AB-3B^2}{(A+B)^2}+O(n-4)\right] . \quad \text{(C22)}$$

Group 2, $b=-a$ and $A^2\neq B^2+C^2$:

$$\bar{I}_n^{(1,-1)}=\pi\frac{A+B}{a}\left[\frac{2}{n-4}-\frac{2B}{A+B}+O(n-4)\right] , \quad \text{(C23)}$$

$$\bar{I}_n^{(1,-2)}=\pi\frac{(A+B)^2}{a}\left[\frac{2}{n-4}+\frac{C^2-4AB-2B^2}{(A+B)^2}+O(n-4)\right] , \quad \text{(C24)}$$

$$\bar{I}_n^{(1,1)}=\pi\frac{1}{a(A+B)}\left\{\frac{2}{n-4}+\ln\left[\frac{(A+B)^2}{A^2-B^2-C^2}\right]+\frac{n-4}{2}\left[\ln^2\left[\frac{A-\sqrt{B^2+C^2}}{A+B}\right]-\frac{1}{2}\ln^2\left[\frac{A+\sqrt{B^2+C^2}}{A-\sqrt{B^2+C^2}}\right]+2\,\mathrm{Li}_2\left[-\frac{B+\sqrt{B^2+C^2}}{A-\sqrt{B^2+C^2}}\right]-2\,\mathrm{Li}_2\left[\frac{B-\sqrt{B^2+C^2}}{A+B}\right]\right]+O((n-4)^2)\right\} , \quad \text{(C25)}$$

$$\bar{I}_n^{(1,2)}=\pi\frac{1}{a(A+B)^2}\left\{\frac{2}{n-4}+\ln\left[\frac{(A+B)^2}{A^2-B^2-C^2}\right]+\frac{2(B^2+C^2+AB)}{A^2-B^2-C^2}+\frac{n-4}{2}\left[\ln^2\left[\frac{A-\sqrt{B^2+C^2}}{A+B}\right]-\frac{1}{2}\ln^2\left[\frac{A+\sqrt{B^2+C^2}}{A-\sqrt{B^2+C^2}}\right]+2\,\mathrm{Li}_2\left[-\frac{B+\sqrt{B^2+C^2}}{A-\sqrt{B^2+C^2}}\right]-2\,\mathrm{Li}_2\left[\frac{B-\sqrt{B^2+C^2}}{A+B}\right]-2\frac{(A+B)\sqrt{B^2+C^2}}{A^2-B^2-C^2}\ln\left[\frac{A+\sqrt{B^2+C^2}}{A-\sqrt{B^2+C^2}}\right]-2\ln\left[\frac{(A+B)^2}{A^2-B^2-C^2}\right]\right]\right.$$

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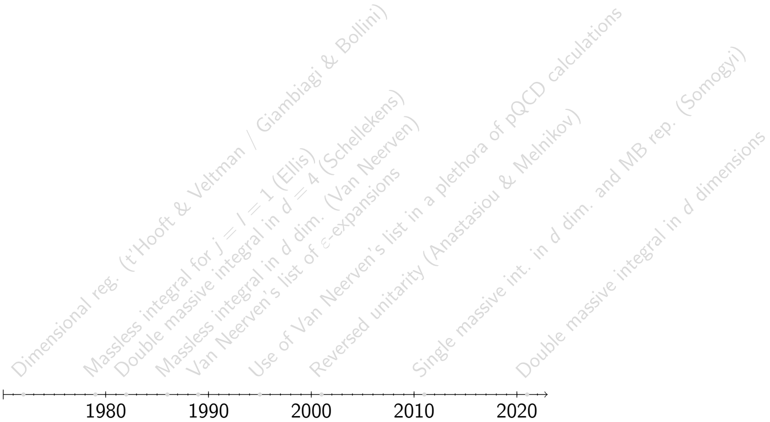

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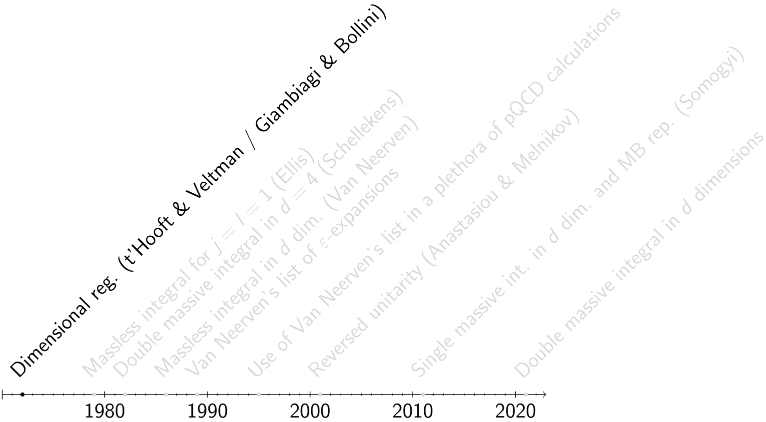

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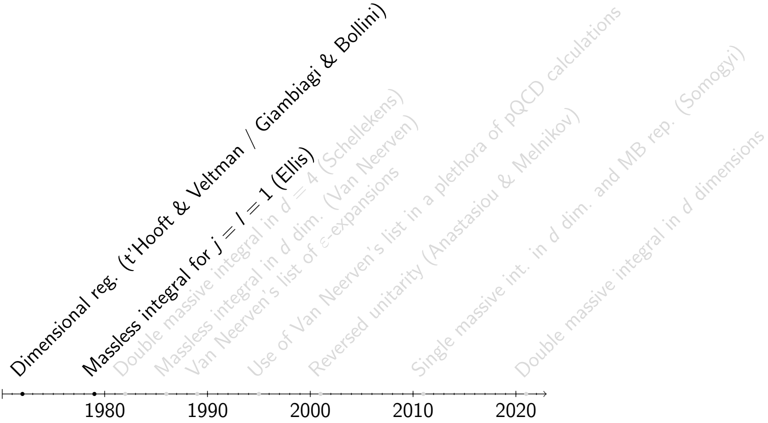

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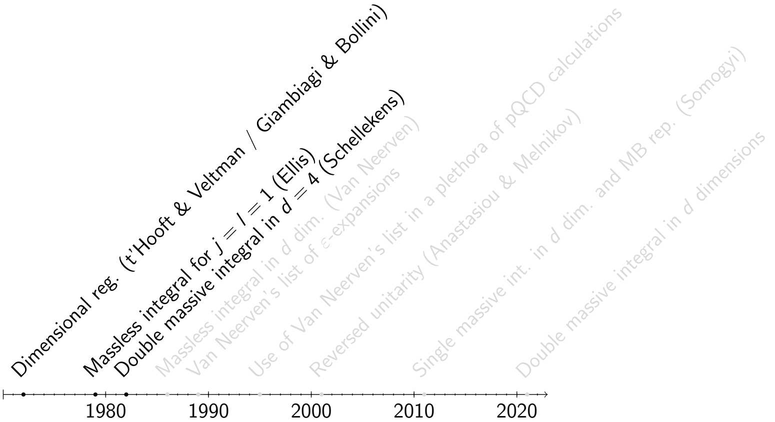

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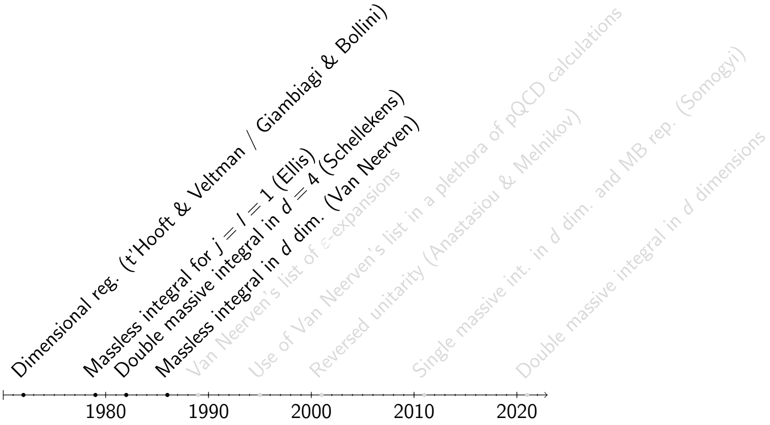

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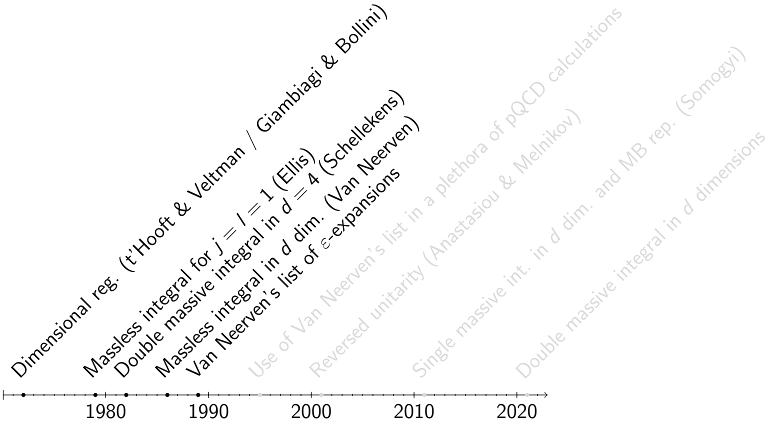

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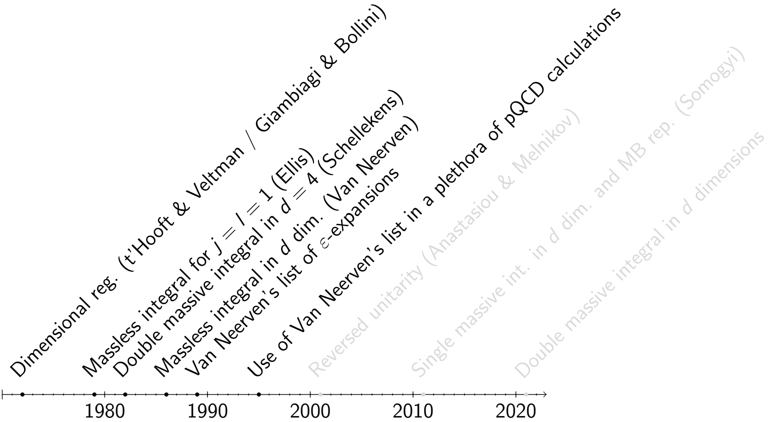

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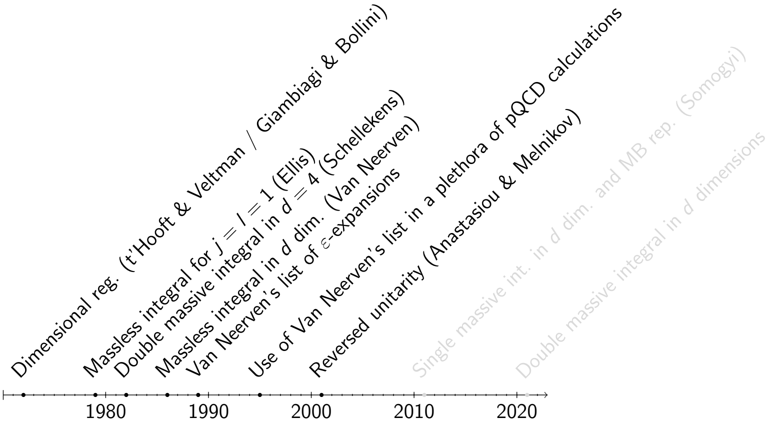

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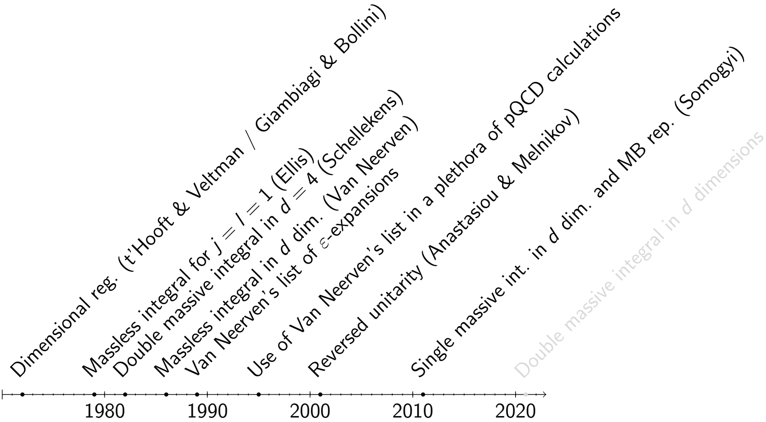

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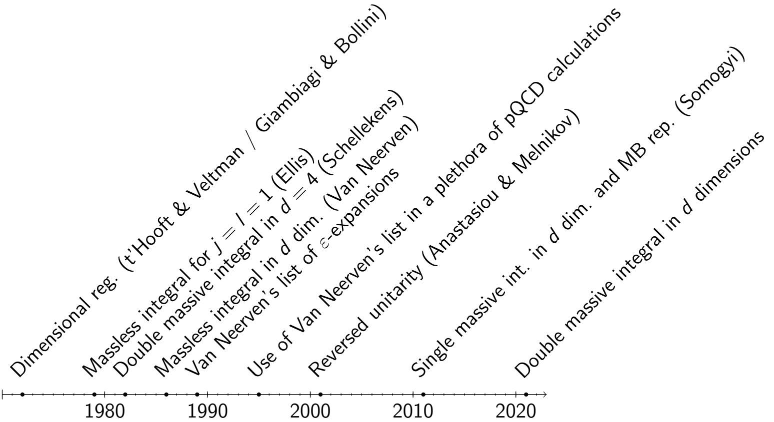

Are angular integrals relevant?

No...

- Thanks to reversed unitarity PSIs are related to loop integrals
- for loop integrals there are a lot of powerful techniques, such as integration-by-part relations, reduction to Master integrals, differential equations etc.
- From this analytic results can be obtained order by order in ε

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- direct integration simpler in particular cases
- old Van Neerven list still in use
- analytic solution for general d possible
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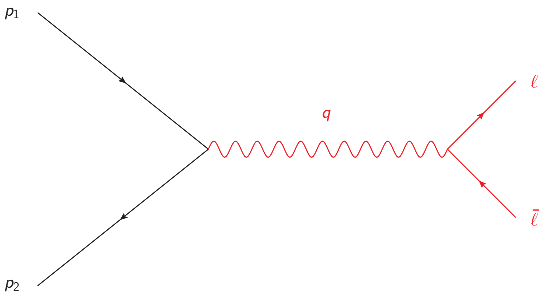
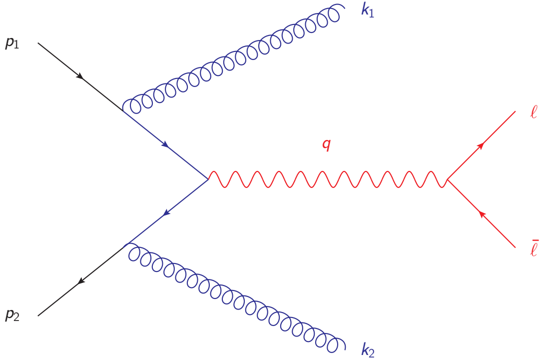
Appearance in perturbative calculations

Example: Drell-Yan



Example: Drell-Yan double real corrections



Kinematics in CMS

- In the CMS it holds

$$\vec{p}_1 + \vec{p}_2 - \vec{q} = 0$$

- The propagators have the general form

$$\frac{1}{(k_{1,2} - p)^2} = \frac{1}{p^2 - 2k_{1,2} \cdot p}$$

with external momentum p; "linear" in $k_{1,2}$

- scale all momenta by their corresponding energy component:

$$k_1 = E_k \left(1, \mathbf{k}\right), \qquad k_2 = E_k \left(1, -\mathbf{k}\right), \qquad p_i = E_i \left(1, \mathbf{v_i}\right).$$

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Propagators

- Define *scaled linear propagators* ($\vec{k}$ is a unit 3-vector)

$$\Delta_k\left(\vec{v}_i\right) = \frac{1}{1 - \vec{v}_i \cdot \vec{k}}$$

- We can express phys. propagators

$$\frac{1}{(k_{1,2} - p_i)^2 - m^2} = \frac{1}{p_i^2 - m^2 + m_k^2 - 2E_iE_k}\,\Delta_k\left(\pm\vec{\bar{v}}_i\right)$$

introducing the scaled vector

$$\vec{\bar{v}}_i = \frac{\vec{v}_i}{\frac{p_i^2 - m^2 + m_k^2}{2E_iE_k} - 1}.$$

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Propagators

In NNLO-DY the scaled momenta are

$$v_1 = \frac{p_1}{E_1} = (1, \vec{v}_1)\,, \qquad v_2 = \frac{p_2}{E_2} = (1, \vec{v}_2)\,,$$
$$v_q = \frac{q}{E_q} = (1, \vec{v}_q)\,, \qquad \bar{v}_q = (1, \vec{\bar{v}}_q)\,,$$

where

$$\vec{\bar{v}}_q = \frac{\vec{v}_q}{1 + \frac{Q^2}{2E_k E_q}} = \vec{v}_q\, \frac{t+u}{t+u-2Q^2}\,.$$

The squared amplitude depends on

$$\Delta_k\left(\vec{v}_1\right)^{n_1} \Delta_k\left(-\vec{v}_1\right)^{n_2} \Delta_k\left(\vec{v}_2\right)^{n_3} \Delta_k\left(-\vec{v}_2\right)^{n_4} \Delta_k\left(\vec{\bar{v}}_q\right)^{n_5} \Delta_k\left(-\vec{\bar{v}}_q\right)^{n_6}\,.$$

Partial Fractioning

If vectors are *linearly dependent*, i.e. $\sum \lambda_i \vec{v}_i = 0$, the number of prop. can be reduced by *partial fractioning*.

- *two-point partial fractioning*

$$\Delta_k\left(\vec{v}_1, \vec{v}_2\right) = \frac{1}{\lambda_1 + \lambda_2}\left[\lambda_2 \Delta_k\left(\vec{v}_1\right) + \lambda_1 \Delta_k\left(\vec{v}_2\right)\right]$$

- *three-point partial fractioning*

$$\Delta_k\left(\vec{v}_1, \vec{v}_2, \vec{v}_3\right) = \frac{\lambda_3 \Delta_k\left(\vec{v}_1, \vec{v}_2\right) + \lambda_2 \Delta_k\left(\vec{v}_1, \vec{v}_3\right) + \lambda_1 \Delta_k\left(\vec{v}_2, \vec{v}_3\right)}{\lambda_1 + \lambda_2 + \lambda_3}$$

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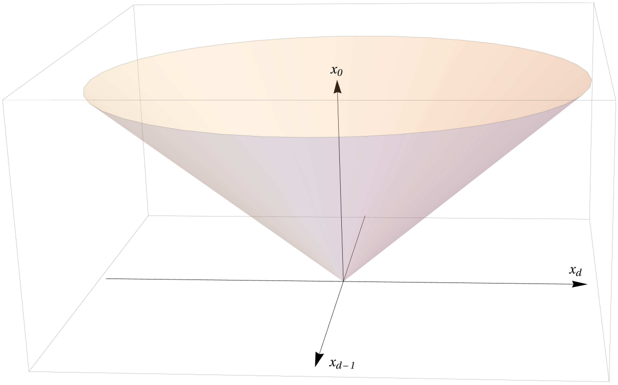
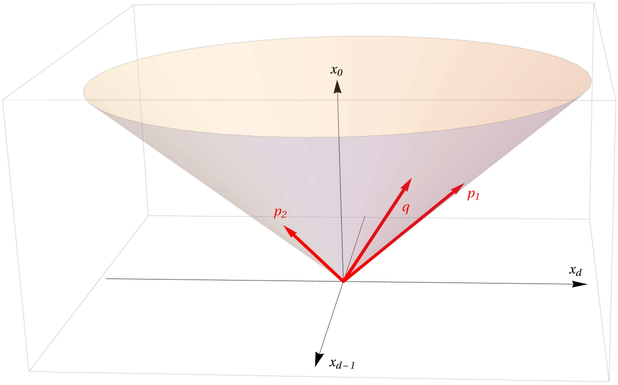
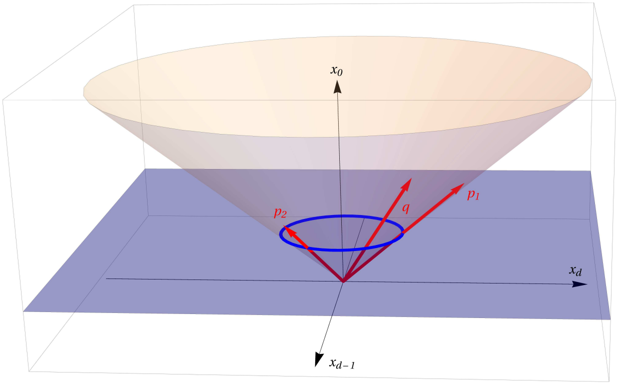
Spacetime picture



Spacetime picture



Spacetime picture



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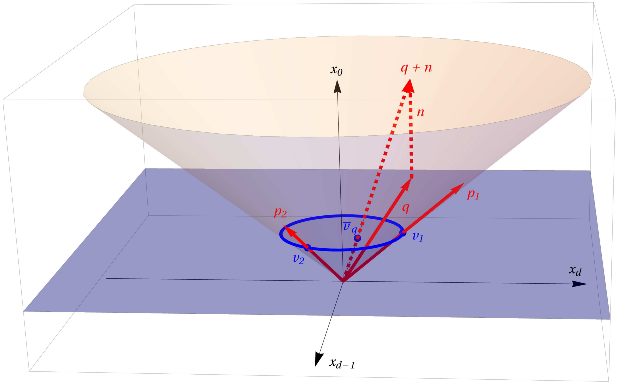

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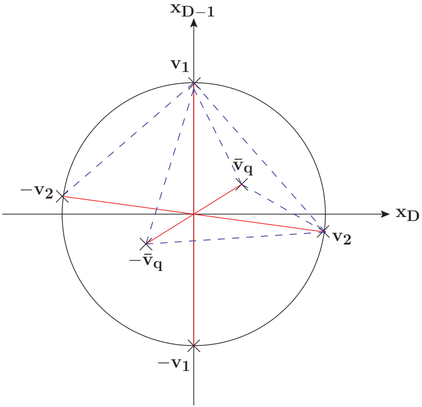

Three vectors are linearly dependent if they lie on a straight line.

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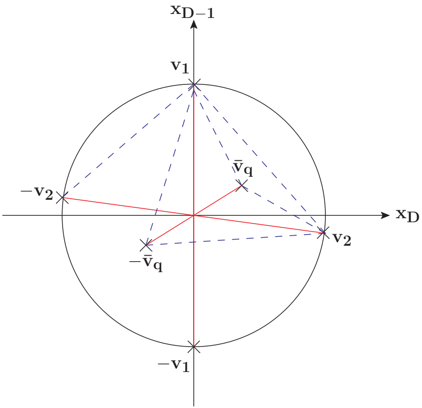

Three vectors are linearly dependent if they lie on a straight line.

Resulting integrals

$$I_{j,l}^{(0)}(u_{12}^{\pm};\varepsilon) = \int \mathrm{d}\Omega_{k_1,k_2} \Delta_k^j(\vec{v}_1)\,\Delta_k^l(\mp\vec{v}_2)$$

$$= \int \mathrm{d}\Omega_{k_1,k_2} \Delta_k^j(-\vec{v}_1)\,\Delta_k^l(\pm\vec{v}_2)\ ,$$

$$I_{j,l}^{(1)}(u_{i\bar{q}}^{\pm}, v_{\bar{q}\bar{q}};\varepsilon) = \int \mathrm{d}\Omega_{k_1,k_2}\, \Delta_k^j\left(\vec{\bar{v}}_q\right)\Delta_k^l(\mp\vec{v}_i)$$

$$= \int \mathrm{d}\Omega_{k_1,k_2}\, \Delta_k^j\left(-\vec{\bar{v}}_q\right)\Delta_k^l(\pm\vec{v}_i)\ ,$$

with $u_{12}^{\pm} = 1 \pm (1 - v_{12})\,,\quad u_{i\bar{q}}^{\pm} = 1 \pm (1 - v_{i\bar{q}}),\quad v_{ij} = v_i \cdot v_j\ .$

Analytic calculation of Van Neerven integrals

Angular integral

$$I_{j,l}^{(n)}\left(v_{12}, v_{11}, v_{22}; \varepsilon\right) = \int \mathrm{d}\Omega_{k_1 k_2} \frac{1}{(v_1 \cdot k)^j (v_2 \cdot k)^l}$$

- n is the number of non-zero masses
- integral depends on scalar product $v_{12} = v_1 \cdot v_2$ and masses $v_{11} = v_1 \cdot v_1$, $v_{22} = v_2 \cdot v_2$
- analytic function in ε, vanishing masses induce (collinear) poles at $\varepsilon = 0$

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No denominators

$$I^{(0)}(\varepsilon) = \int \mathrm{d}\Omega_{k_1 k_2} = \int_0^\pi \mathrm{d}\theta_1 \sin^{1-2\varepsilon}\theta_1 \int_0^\pi \mathrm{d}\theta_2 \sin^{-2\varepsilon}\theta_2$$

Substitution: $\cos\theta_i = 1 - 2t_i$

$$I^{(0)}(\varepsilon) = 2^{1-4\varepsilon} \int_0^1 \mathrm{d}t_1\, t_1^{-\varepsilon}(1-t_1)^{-\varepsilon} \int_0^1 \mathrm{d}t_2\, t_2^{-\frac{1}{2}-\varepsilon}(1-t_2)^{-\frac{1}{2}-\varepsilon},$$

Beta function: $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 \mathrm{d}t\, t^{x-1}(1-t)^{y-1}$.

$$I^{(0)}(\varepsilon) = \frac{2\pi}{1-2\varepsilon}$$

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One denominator, massless

Rotate s.t. $v_1 = (1, \vec{0}_{d-3}, 0, 1)$, $k = (1, \ldots, \sin\theta_1 \cos\theta_2, \cos\theta_1)$:

$$I_j^{(0)}(\varepsilon) = \int \mathrm{d}\Omega_{k_1 k_2} \frac{1}{(v_1 \cdot k)^j} = \int_0^\pi \mathrm{d}\theta_1 \frac{\sin^{1-2\varepsilon}\theta_1}{(1-\cos\theta_1)^j} \int_0^\pi \mathrm{d}\theta_2 \sin^{-2\varepsilon}\theta_2$$

Pochhammer symbol:

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = (x+n-1)\cdot\dots\cdot(x+1)\cdot x$$

$$I_j^{(0)}(\varepsilon) = \frac{2\pi}{1-2\varepsilon} \frac{(2-j-2\varepsilon)_j}{2^j(1-j-\varepsilon)_j}$$

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Now $v_1 = (1, \vec{0}_{d-3}, 0, \beta)$, hence $v_{11} = 1 - \beta^2 \neq 0$:

$$I_j^{(1)}(v_{11}, \varepsilon) = \int_0^\pi \mathrm{d}\theta_1 \frac{\sin^{1-2\varepsilon}\theta_1}{(1-\beta\cos\theta_1)^j} \int_0^\pi \mathrm{d}\theta_2 \sin^{-2\varepsilon}\theta_2$$

Gauss hypergeometric fct.:

$${}_2F_1(a,b,c,z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \mathrm{d}t\, t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}$$

$$I_j^{(1)}(v_{11}, \varepsilon) = \frac{I^{(0)}(\varepsilon)}{(1-\beta)^j} \, {}_2F_1\left(j, 1-\varepsilon, 2-2\varepsilon, -\frac{2\beta}{1-\beta}\right)$$

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Quadratic trafo:

$${}_2F_1(a,b,2b,x) = \left(1-\frac{x}{2}\right)^{-a}\,{}_2F_1\left(\frac{a}{2},\frac{a+1}{2},b+\frac{1}{2},\left(\frac{x}{2-x}\right)^2\right)$$

Alternative form:

$$I_j^{(1)}(v_{11};\varepsilon) = I^{(0)}(\varepsilon)\,{}_2F_1\left(\frac{j}{2},\frac{j+1}{2},\frac{3}{2}-\varepsilon,1-v_{11}\right)$$

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Interlude: Mellin-Barnes representation

- Mellin-Trafo: $F(z) = \int_0^\infty \frac{\mathrm{d}x}{x} x^z f(z)$
- Mellin inverse: $f(x) = \int_{-\mathrm{i}\infty+c}^{\mathrm{i}\infty+c} \frac{\mathrm{d}z}{2\pi\mathrm{i}} x^{-z} F(z)$
- for Gauss hypergeometric function:

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{}_2F_1(a,b,c,x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)} \times \int_{-i\infty}^{i\infty} \frac{\mathrm{d}z}{2\pi i} \Gamma(a+z)\Gamma(b+z)\Gamma(c-a-b-z)\Gamma(-z)(1-x)^z
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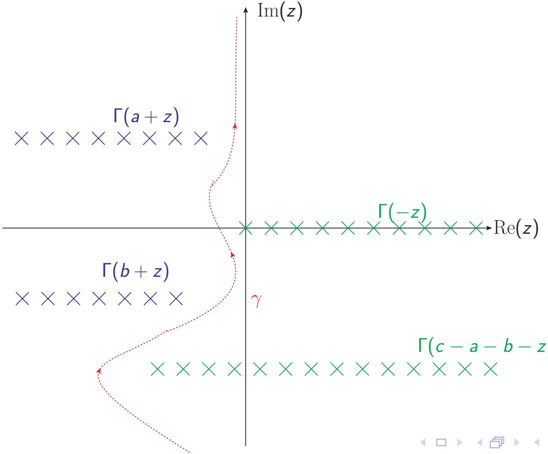
Mellin-Barnes representation

Robert Hjalmar Mellin (1854-1933)



Ernest William Barnes (1874-1953)

MB representation of one denominator integral

$$I_j^{(1)}(v_{11};\varepsilon) = \frac{I^{(0)}(\varepsilon)}{\Gamma(1-\varepsilon)} \frac{(2-j-2\varepsilon)_j}{2^j\Gamma(j)} \times \int_{-i\infty}^{i\infty} \frac{\mathrm{d}z}{2\pi i} \Gamma(j+2z)\Gamma(1-j-\varepsilon-z)\Gamma(-z) \left(\frac{v_{11}}{4}\right)^z$$

Note the simple dependence on v_{11}, which is nice for further integrations.

Two denominators, massless

$$I^{(0)}_{j,l}(v_{12};\varepsilon) = \int \mathrm{d}\Omega_{k_1 k_2} \frac{1}{(v_1 \cdot k)^j (v_2 \cdot k)^l} \longleftarrow$$

Trick: Feynman parametrization

$$= \frac{1}{B(j,l)} \int_0^1 \mathrm{d}x_1\, x_1^{j-1} \int_0^1 \mathrm{d}x_2\, x_2^{l-1}\, \delta(1 - x_1 - x_2) \times \int \mathrm{d}\Omega_{k_1 k_2} \frac{1}{((x_1 v_1 + x_2 v_2) \cdot k)^{j+l}} \,.$$

Introduce new vector $v \equiv x_1 v_1 + x_2 v_2$ with mass $v^2 = 2 x_1 x_2 v_{12}$. Remaining angular integral is $I^{(1)}_{j+l}(2 x_1 x_2 v_{12};\varepsilon)$ for which we can use its MB representation.

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$$\times \int \mathrm{d}\Omega_{k_1 k_2} \frac{1}{((x_1 v_1 + x_2 v_2)\cdot k)^{j+l}} .$$

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$$= \frac{1}{B(j,l)} \int_0^1 \mathrm{d}x_1\, x_1^{j-1} \int_0^1 \mathrm{d}x_2\, x_2^{l-1}\, \delta(1-x_1-x_2) \times \int \mathrm{d}\Omega_{k_1 k_2} \frac{1}{((x_1 v_1 + x_2 v_2)\cdot k)^{j+l}}\,.$$

Introduce new vector $v \equiv x_1 v_1 + x_2 v_2$ with mass $v^2 = 2x_1 x_2 v_{12}$.
Remaining angular integral is $I_{j+l}^{(1)}(2x_1x_2v_{12};\varepsilon)$ for which we can use its MB representation.

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- MB representation:

$$I_{j,l}^{(0)}(v_{12};\varepsilon) = \frac{I^{(0)}(\varepsilon)}{\Gamma(1-\varepsilon)} \frac{(2-j-l-2\varepsilon)_{j+l}}{2^{j+l}\Gamma(j)\Gamma(l)}$$
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$$I_{j,l}^{(0)}(v_{12};\varepsilon) = I_{j+l}^{(0)}(\varepsilon)\,\frac{(1-j-l-\varepsilon)_j}{(1-j-\varepsilon)_j}\,{}_2F_1\left(j,l,1-\varepsilon,1-\frac{v_{12}}{2}\right)$$

First calculated by Willy van Neerven (1947-2007) in 1984.

photograph courtesy DESY

Two denominators, single massive

$$I^{(1)}_{j,l}(v_{12}, v_{11}; \varepsilon) = \int \mathrm{d}\Omega_{k_1 k_2} \frac{1}{(v_1 \cdot k)^j (v_2 \cdot k)^l}\,, \qquad v_{11} \neq 0,\ v_{22} = 0$$

Feynman parametrization and MB representaion again, now with $v^2 = x_1^2 v_{11} + 2x_1 x_2 v_{12}$.

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Interlude: Binomi-Mellin-Newton integral

Question: How to "multiply out" $(a+b)^{z_1}$ as a sum of products $a^i b^j$ like we can do for $z_1 \in \mathbb{N}$?

Answer: Use MB-representation:

$$(a+b)^{z_1} = \frac{1}{\Gamma(-z_1)} \int_{-i\infty}^{i\infty} \frac{\mathrm{d}z_2}{2\pi i} a^{z_1-z_2} b^{z_2} \Gamma(-z_2)\Gamma(-z_1+z_2),$$

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- Factorize $\left(\frac{x_1^2 v_{11}}{4} + \frac{x_1 x_2 v_{12}}{2}\right)^z$ using BMN integral
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$$I_{j,l}^{(1)}(v_{12}, v_{11}; \varepsilon) = \frac{I^{(0)}(\varepsilon)}{2^l v_{12}^j} \frac{(2-j-l-2\varepsilon)_{j+l}}{(1-l-\varepsilon)_l \Gamma(j)}$$

$$\times \int_0^1 \mathrm{d}t\, t^{j-1}(1-t)^{1-j-l-2\varepsilon}(1-\tau_+ t)^{l-1+\varepsilon}(1-\tau_- t)^{l-1+\varepsilon}$$

with $\tau_\pm = 1 - (1 \pm \sqrt{1-v_{11}})/v_{12}$.

Appell function: $F_1(a,b,c,d,x,y) = \frac{1}{B(a,d-a)} \int_0^1 \mathrm{d}t\, t^{a-1}(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c}$

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First calculated by Gábor Somogyi (2011).

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- Use BNM integral to "multiply out" $(-(1 - v_{12}) - x_1^2(v_{12} - v_{11}) - x_2^2(v_{12} - v_{22}))^z$
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$$I_{j,l}(v_{12}, v_{11}, v_{22}; \varepsilon) = I^{(0)}(\varepsilon) v_{12}^{1-j-l-\varepsilon}$$
$$F_B^{(3)}\left(\frac{j}{2}, \frac{l}{2}, \frac{3-j-l}{2} - \varepsilon, \frac{j+1}{2}, \frac{l+1}{2}, \frac{1-j-l}{2} - \varepsilon, \frac{3}{2} - \varepsilon; x_1, x_2, x_3\right)$$

with $x_1 = 1 - \frac{v_{11}}{v_{12}}$, $x_2 = 1 - \frac{v_{22}}{v_{12}}$, $x_3 = 1 - v_{12}$.

Lauricella function:

$$F_B^{(3)}(a_1, a_2, a_3, b_1, b_2, b_3, c; x_1, x_2, x_3) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_m (a_2)_n (a_3)_p (b_1)_m (b_2)_n (b_3)_p}{(c)_{m+n+p}} \frac{x_1^m}{m!} \frac{x_2^n}{n!} \frac{x_3^p}{p!}$$

Interlude: Hypergeometric functions

- Gauss function: ${}_2F_1(a_1, b_1, c; x_1) = \sum_{m=0}^{\infty} \frac{(a_1)_m (b_1)_m}{(c)_m} \frac{x_1^m}{m!}$
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Two denominators, double massive: The sneaky way

Idea: Use partial fractioning to reduce double massive to single massive integral!

Remember: Three vectors on a line in the $x^0 = 1$ plane are always linearly dependent and thus admit partial fractioning.

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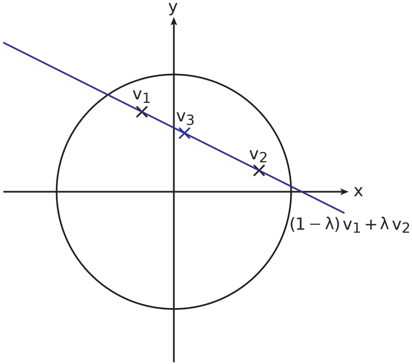
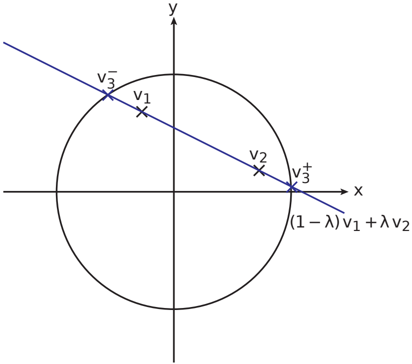
Two-point splitting



$$\Delta_k\left(\vec{v}_1, \vec{v}_2\right) = \lambda\, \Delta_k\left(\vec{v}_1, \vec{v}_3\right) + (1-\lambda)\, \Delta_k\left(\vec{v}_2, \vec{v}_3\right)\ , \text{ where}$$
$$\vec{v}_3 = (1-\lambda)\, \vec{v}_1 + \lambda\, \vec{v}_2.$$

Two-mass splitting



$$\vec{v}_3^{\pm} = \frac{v_{12} - v_{22} \pm \sqrt{v_{12}^2 - v_{11} v_{22}}}{2v_{12} - v_{11} - v_{22}} \vec{v}_1 + \frac{v_{12} - v_{11} \mp \sqrt{v_{12}^2 - v_{11} v_{22}}}{2v_{12} - v_{11} - v_{22}} \vec{v}_2$$

Two-mass splitting

Using this idea we can split the product of two massive vectors $\vec{v}_1$ and $\vec{v}_2$ into single-massive products.

$$\Delta_k^j(\vec{v}_1)\,\Delta_k^l(\vec{v}_2) = \sum_{n=0}^{j-1}\binom{l-1+n}{l-1}\lambda_\pm^l(1-\lambda_\pm)^n\Delta_k^{j-n}(v_1)\,\Delta_k^{l+n}(\vec{v}_3^{\pm}) + \sum_{n=0}^{l-1}\binom{j-1+n}{j-1}\lambda_\pm^n(1-\lambda_\pm)^j\Delta_k^{l-n}(\vec{v}_2)\,\Delta_k^{j+n}(\vec{v}_3^{\pm})$$

Two denominators, double massive: The sneaky way

$$I^{(2)}_{j,l}(v_{12}, v_{11}, v_{22}; \varepsilon) = \sum_{n=0}^{j-1} \binom{l-1+n}{l-1} \lambda_\pm^l (1-\lambda_\pm)^n I^{(1)}_{j-n,l+n}(v_{13}^\pm, v_{11}; \varepsilon)$$
$$+ \sum_{n=0}^{l-1} \binom{j-1+n}{j-1} \lambda_\pm^n (1-\lambda_\pm)^j I^{(1)}_{l-n,j+n}(v_{23}^\pm, v_{22}; \varepsilon)\,,$$

where $\lambda_\pm = \dfrac{v_{12} - v_{11} \pm \sqrt{v_{12}^2 - v_{11}v_{22}}}{2v_{12} - v_{11} - v_{22}}$, $v_{13}^\pm = (1-\lambda_\pm)v_{11} + \lambda_\pm v_{12}$, $v_{23}^\pm = (1-\lambda_\pm)v_{12} + \lambda_\pm v_{22}$.

Properties of angular integrals

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There are many non-trivial relations within the family of angular integrals, including:

(a) Partial differential equations
(b) Dimensional shift identities
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Partial differential equations

Introducing "light-cone coordinates" $v_\pm = \frac{1}{2}(v_{11} \pm v_{22})$:

$$\left(\frac{\partial^2}{\partial v_+^2} - \frac{\partial^2}{\partial v_-^2} - \frac{\partial^2}{\partial v_{12}^2}\right) I_{j,l} = 0$$

- two dimensional homogenous wave equation with "time" v_+ and "speed of light" $c = 1$
- "light-cone" at vanishing Gram determinant $0 = v_{11}v_{22} - v_{12}^2$.
- independent of ε, i.e. satisfied by all orders of ε-expansion

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$$\left(\frac{\partial^2}{\partial v_+^2} - \frac{\partial^2}{\partial v_-^2} - \frac{\partial^2}{\partial v_{12}^2} \right) I_{j,l} = 0$$

- two dimensional homogenous wave equation with "time" v_+ and "speed of light" $c = 1$
- "light-cone" at vanishing Gram determinant $0 = v_{11}v_{22} - v_{12}^2$.
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Dimensional shift identities

Recall integration measure: $\mathrm{d}\theta_1 \sin^{1-2\varepsilon}\theta_1 \theta_2 \sin^{-2\varepsilon}\theta_2$.
Additional factors of $\sin^2\theta_1 \sin^2\theta_2$ corresponding to a dimensional shift $\varepsilon \to \varepsilon - 1$ can be expressed in terms of propagators.

$$I_{j,l}(\varepsilon-1) = \int \mathrm{d}\Omega_{k_1 k_2} \sin^2\theta_1 \sin^2\theta_2 \, \Delta_k^j(\vec{v}_1)\, \Delta_k^l(\vec{v}_2)$$

$$= \frac{1}{\Delta_{12}} \Big[(v_{11}v_{22} - v_{12}^2) I_{j,l}(\varepsilon) - (1-v_{11}) I_{j,l-2}(\varepsilon) - (1-v_{22}) I_{j-2,l}(\varepsilon)$$

$$+ 2(v_{12}-v_{11}) I_{j,l-1}(\varepsilon) + 2(v_{12}-v_{22}) I_{j-1,l}(\varepsilon) + 2(1-v_{12}) I_{j-1,l-1}(\varepsilon) \Big]$$

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&= \frac{1}{\Delta_{12}} \Big[(v_{11}v_{22} - v_{12}^2) I_{j,l}(\varepsilon) - (1 - v_{11}) I_{j,l-2}(\varepsilon) - (1 - v_{22}) I_{j-2,l}(\varepsilon) \\
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$$

Dimensional shift identities

There is also a *second dimensional shift identity* with $\varepsilon \to \varepsilon + 1$:

$$I_{j,l}(\varepsilon+1) = \frac{j+l-1+2\varepsilon}{1+2\varepsilon} I_{j,l}(\varepsilon) - \frac{j}{1+2\varepsilon} I_{j+1,l}(\varepsilon) - \frac{l}{1+2\varepsilon} I_{j,l+1}(\varepsilon)$$

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Recursion relations

- relating integrals with different j and l
- derived by integrating by parts w.r.t. θ_1 and θ_2 and expressing everything in terms of propagators again
- reduction to only 3 *master integrals* $I_{0,0}$, $I_{1,0}$ and $I_{1,1}$ possible
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Recursion relations

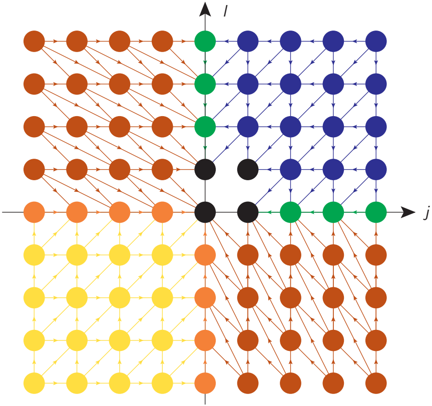
Using integration-by-parts w.r.t. θ_1 and θ_2 we obtain the relations (ii) and (iii) in addition to (i) derived from differentiation w.r.t. v_{12}:

$$
\begin{aligned}
\text{(i)}:\ 0 &= (j-l)\,(1-v_{12})\,I_{j,l} - j\,(1-v_{11})\,I_{j+1,l-1} + l\,(1-v_{22})\,I_{j-1,l+1} \\
&\quad + j\,(v_{12}-v_{11})\,I_{j+1,l} - l\,(v_{12}-v_{22})\,I_{j,l+1} \\
\text{(ii)}:\ 0 &= (j+l-1+2\varepsilon)\,I_{j,l} - (2j+l+2\varepsilon)\,I_{j+1,l} + v_{11}\,(j+1)\,I_{j+2,l} \\
&\quad - l\,I_{j,l+1} + l\,v_{12}\,I_{j+1,l+1} \\
\text{(iii)}:\ 0 &= (j+l-1+2\varepsilon)\,I_{j,l} - (2l+j+2\varepsilon)\,I_{j,l+1} + v_{22}\,(l+1)\,I_{j,l+2} \\
&\quad - j\,I_{j+1,l} + j\,v_{12}\,I_{j+1,l+1}
\end{aligned}
$$

Recursion relations

(1) Solve the recursion relations in 5 different regions on j-l grid

(2) Reduce everything to master integrals $I_{0,0}$, $I_{1,0}$, $I_{0,1}$, $I_{1,1}$



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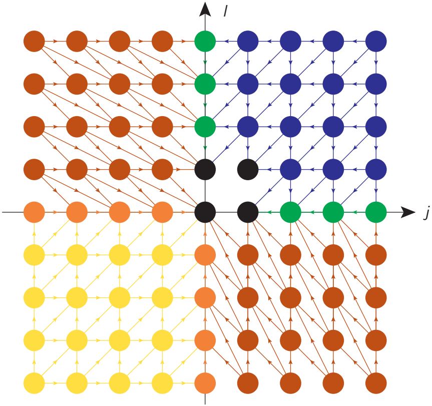

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Due to recursion relations ε-expansion is only necessary for non-trivial master integrals:

(1) Massive integral with one denominator $I_1^{(1)}(v_{11};\varepsilon)$

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(3) Single massive integral with two denominators $I_{1,1}^{(1)}(v_{12},v_{11};\varepsilon)$

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Massive integral with one denominator

All-order ε-expansion of

$$I_1^{(1)}(v_{11};\varepsilon) = \frac{2\pi}{1-2\varepsilon}\frac{1}{1-\sqrt{1-v_{11}}}\,{}_2F_1\left(1, 1-\varepsilon, 2-2\varepsilon, 1-\frac{1+\sqrt{1-v_{11}}}{1-\sqrt{1-v_{11}}}\right)$$

involves *Nielsen polylogarithms*:

$$S_{n,p}(x) \equiv \frac{(-1)^{n+p-1}}{(n-1)!p!}\int_0^1 \frac{\mathrm{d}t}{t} \log^{n-1} t \log^p(1-xt)$$

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Massive integral with one denominator

Start with integral representation

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{}_2F_1(1,1-\varepsilon,2-2\varepsilon,x) = (1-x)^{-\varepsilon}{}_2F_1(1-\varepsilon,1-2\varepsilon,2-2\varepsilon,x)
$$
$$
= (1-x)^{-\varepsilon}(1-2\varepsilon)\int_0^1 \mathrm{d}t\, t^{-2\varepsilon}(1-xt)^{-1+\varepsilon}\,.
$$

After partial integration:

$$
\int_0^1 \mathrm{d}t\, t^{-2\varepsilon}(1-xt)^{-1+\varepsilon} =
$$
$$
-\frac{(1-x)^{\varepsilon}}{x\varepsilon} - \int_0^1 \frac{\mathrm{d}t}{t}\, t^{-2\varepsilon}(1-xt)^{\varepsilon}
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$$

$$
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$$

Add and subtract 1 in the integral:

$$
\int_0^1 \mathrm{d}t\, t^{-2\varepsilon}(1-xt)^{-1+\varepsilon} =
$$

$$
-\frac{(1-x)^{\varepsilon}}{x\varepsilon} - \frac{2}{x}\int_0^1 \frac{\mathrm{d}t}{t}\, t^{-2\varepsilon}\left[\Big((1-xt)^{\varepsilon} - 1\Big) + 1\right]
$$

Massive integral with one denominator

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$$
= (1-x)^{-\varepsilon}(1-2\varepsilon)\int_0^1 \mathrm{d}t\, t^{-2\varepsilon}(1-xt)^{-1+\varepsilon}\,.
$$

Split off singularity at $t=0$:

$$
\int_0^1 \mathrm{d}t\, t^{-2\varepsilon}(1-xt)^{-1+\varepsilon} =
$$
$$
-\frac{(1-x)^{\varepsilon}}{x\varepsilon} - \frac{2}{x}\int_0^1 \frac{\mathrm{d}t}{t}\, t^{-2\varepsilon}\Big((1-xt)^{\varepsilon}-1\Big) - \frac{2}{x}\int_0^1 \frac{\mathrm{d}t}{t}\, t^{-2\varepsilon}
$$

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$$

Expand in ε:

$$
\int_0^1 \mathrm{d}t\, t^{-2\varepsilon}(1-xt)^{-1+\varepsilon} = -\frac{(1-x)^{\varepsilon}}{x\varepsilon} - \frac{2}{x}\int_0^1 \frac{\mathrm{d}t}{t}\, t^{-2\varepsilon}
$$
$$
- \frac{2}{x}\sum_{n=0}^{\infty}\frac{(-2\varepsilon)^n}{n!}\sum_{m=1}^{\infty}\frac{\varepsilon^m}{m!}\int_0^1 \frac{\mathrm{d}t}{t}\,\log^n t\,\log^m(1-xt)
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= (1-x)^{-\varepsilon}(1-2\varepsilon)\int_0^1 \mathrm{d}t\, t^{-2\varepsilon}(1-xt)^{-1+\varepsilon}\,.
$$

Identify integrals as Nielsen polylogarithms:

$$
\int_0^1 \mathrm{d}t\, t^{-2\varepsilon}(1-xt)^{-1+\varepsilon} = -\frac{(1-x)^{\varepsilon}}{x\varepsilon} + \frac{1}{x\varepsilon}
$$
$$
-\frac{2}{x}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}(-1)^m 2^n \varepsilon^{n+m} S_{n+1,m}(x)
$$

Massive integral with one denominator

$$I_1^{(1)}(v_{11};\varepsilon) = \frac{\pi}{\sqrt{1-v_{11}}}\left(\frac{1+\sqrt{1-v_{11}}}{1-\sqrt{1-v_{11}}}\right)^{-\varepsilon} \times \sum_{N=0}^{\infty}\sum_{m=0}^{N} 2^{N-m}(-1)^{m+1} S_{N-m,m+1}\left(1-\frac{1+\sqrt{1-v_{11}}}{1-\sqrt{1-v_{11}}}\right)\varepsilon^N$$

Up to order ε:

$$I_1^{(1)}(v_{11};\varepsilon) = \frac{\pi}{\sqrt{1-v_{11}}}\left(\frac{1+\sqrt{1-v_{11}}}{1-\sqrt{1-v_{11}}}\right)^{-\varepsilon}\left[\log\left(\frac{1+\sqrt{1-v_{11}}}{1-\sqrt{1-v_{11}}}\right) + \frac{\varepsilon}{2}\log^2\left(\frac{1+\sqrt{1-v_{11}}}{1-\sqrt{1-v_{11}}}\right) - 2\varepsilon\,\mathrm{Li}_2\left(1-\frac{1+\sqrt{1-v_{11}}}{1-\sqrt{1-v_{11}}}\right) + \mathcal{O}(\varepsilon^2)\right]$$

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Massless integral with two denominators

Start again with hypergeometric representation

$$I_{1,1}^{(0)}(v_{12}, v_{11}; \varepsilon) = -\frac{\pi}{\varepsilon}\, {}_2F_1\left(1, 1, 1-\varepsilon, 1-\frac{v_{12}}{2}\right) .$$

For all-order ε-expansion use again integral representation and Nielsen polylogarithms.

Massless integral with two denominators

$$I_{1,1}^{(0)}(v_{12},\varepsilon) = \pi \left(\frac{v_{12}}{2}\right)^{-1-\varepsilon} \times \left[-\frac{1}{\varepsilon} + \sum_{N=1}^{\infty} \sum_{m=1}^{N} (-1)^m S_{N-m+1,m}\left(1 - \frac{v_{12}}{2}\right) \varepsilon^N\right]$$

Up to order ε:

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Single-massive integral with two denominators

Start with hypergeometric representation

$$I_{1,1}^{(1)}(v_{12}, v_{11}; \varepsilon) = -\frac{\pi}{\varepsilon v_{12}} \left(\frac{v_{11}}{v_{12}^2}\right)^{\varepsilon} F_1\left(-2\varepsilon, -\varepsilon, -\varepsilon, 1-2\varepsilon, \omega_+, \omega_-\right) ,$$

$$\omega_\pm = \frac{\tau_\pm}{1-\tau_\pm} = 1 - \frac{v_{12}}{1 \pm \sqrt{1-v_{11}}}$$

Appell function has integral representation

$$F_1(-2\varepsilon, -\varepsilon, -\varepsilon, 1-2\varepsilon, x, y) = -2\varepsilon \int_0^1 \mathrm{d}t\, t^{-1-2\varepsilon}(1-xt)^{\varepsilon}(1-yt)^{\varepsilon}$$

Admits same strategy for ε-expansion as previously.

Single-massive integral with two denominators

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Single-massive integral with two denominators

We arrive at

$$F_1(\dots) = 1 - 2\varepsilon \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{(-2)^n}{n!m!} \varepsilon^{m+n} \int_0^1 \frac{\mathrm{d}t}{t} \log^n t \log^m ((1-xt)(1-yt)) \ .$$

Introduce *Double Nielsen Polylogarithms*

$$S_{n,p_1,p_2}(x,y) \equiv \frac{(-1)^{n+p_1+p_2-1}}{(n-1)!p_1!p_2!} \int_0^1 \frac{\mathrm{d}t}{t} \log^{n-1} t \log^{p_1}(1-xt) \log^{p_2}(1-yt)$$

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Single-massive integral with two denominators

$$I_{1,1}^{(1)}(v_{12}, v_{11}, \varepsilon) = \frac{\pi}{v_{12}} \left(\frac{v_{11}}{v_{12}^2}\right)^{\varepsilon} \times \left[-\frac{1}{\varepsilon} + \sum_{N=1}^{\infty} \sum_{m=1}^{N} (-1)^m 2^{N-m+1} \sum_{k=0}^{m} S_{N-m+1,m-k,k}(\tau_+, \tau_-)\, \varepsilon^N\right]$$

Up to order ε

$$I_{1,1}^{(1)}(v_{12}, v_{11}, \varepsilon) = \frac{\pi}{v_{12}} \left(\frac{v_{11}}{v_{12}^2}\right)^{\varepsilon} \left[-\frac{1}{\varepsilon} - 2\varepsilon \left(\mathrm{Li}_2\left(1 - \frac{v_{12}}{1+\sqrt{1-v_{11}}}\right) + \mathrm{Li}_2\left(1 - \frac{v_{12}}{1-\sqrt{1-v_{11}}}\right)\right) + \mathcal{O}(\varepsilon^2)\right]$$

Double-massive integral with two denominators

We use two-mass splitting

$$I_{1,1}^{(2)}(v_{12}, v_{11}, v_{22}; \varepsilon) = \frac{1}{\sqrt{X}} \left[v_{13} I_{1,1}^{(1)}(v_{13}, v_{11}; \varepsilon) - v_{23} I_{1,1}^{(1)}(v_{23}, v_{22}; \varepsilon) \right]$$

with

$$\begin{aligned} X &= v_{12}^2 - v_{11} v_{22} \\ v_{13} &= \frac{v_{11}\left(v_{22} + \sqrt{X}\right) - v_{12}\left(v_{12} + \sqrt{X}\right)}{v_{11} + v_{22} - 2v_{12}}, \\ v_{23} &= \frac{v_{22}\left(v_{11} - \sqrt{X}\right) - v_{12}\left(v_{12} - \sqrt{X}\right)}{v_{11} + v_{22} - 2v_{12}} \end{aligned}$$

Double-massive integral with two denominators

$$
\begin{aligned}
I_{1,1}^{(2)}(v_{12}, v_{11}, v_{22}; \varepsilon) = \frac{\pi}{\sqrt{X}} \Bigg[& \log\left(\frac{v_{12}+\sqrt{X}}{v_{12}-\sqrt{X}}\right) \\
& -\varepsilon\Bigg(\frac{1}{2}\log^2\frac{v_{11}}{v_{13}^2} - \frac{1}{2}\log^2\frac{v_{22}}{v_{23}^2} \\
& +2\,\mathrm{Li}_2\left(1-\frac{v_{13}}{1-\sqrt{1-v_{11}}}\right) + 2\,\mathrm{Li}_2\left(1-\frac{v_{13}}{1+\sqrt{1-v_{11}}}\right) \\
& -2\,\mathrm{Li}_2\left(1-\frac{v_{23}}{1-\sqrt{1-v_{22}}}\right) - 2\,\mathrm{Li}_2\left(1-\frac{v_{23}}{1+\sqrt{1-v_{22}}}\right)\Bigg)\Bigg] \\
& + \mathcal{O}(\varepsilon^2)
\end{aligned}
$$

$$\Omega(\beta_i,\beta_j,\tau_{ij}) = P_{ij}\int_0^\pi \frac{d\theta_\gamma}{\sin^{2\epsilon-1}\theta_\gamma}\int_0^\pi \frac{d\phi_\gamma}{\sin^{2\epsilon}\phi_\gamma}\times \left[\frac{1}{(1-\beta_i\cos\theta_\gamma)(1-\beta_j\cos\theta_\gamma\cos\chi_{ij}-\beta_j\sin\theta_\gamma\cos\phi_\gamma\sin\chi_{ij})}\right], \quad \text{(D.12)}$$

where $\cos\chi_{ij} = 2\tau_{ij}-1$, $\sin\chi_{ij} = \sqrt{1-\cos^2\chi_{ij}}$ and $P_{ij} = (1-\beta_i\beta_j(2\tau_{ij}-1))/2\pi$.

Before matching β_i, β_j and τ_{ij} to the cases we have, consider $\Omega(\beta_i,\beta_j,\tau_{ij})$. For $\beta_i \neq 1$ and $\beta_j \neq 1$, the result to $\mathcal{O}(\epsilon)$ is not known in the literature. This is needed for isolating the collinear logs, since they arise from the $\mathcal{O}(\epsilon)$ part of the angular integrals multiplied by the $1/\epsilon$ from the r_{soft}.

However, through [52], we were able to get an expression for $\Omega(\beta_i,\beta_j,\tau_{ij})$. The result is

$$\Omega_{ij} = \frac{\pi P_{ij}}{2C_{ij}}\left\{\ln\left(\frac{v_{ij}+C_{ij}}{v_{ij}-C_{ij}}\right) + \epsilon\left[-\ln\left(\frac{1-C_{ii}}{1+C_{ii}}\right)\ln\left(\frac{R_{ij}+S_{ij}}{R_{ij}-S_{ij}}\right) + \left(\sum_{a,b=1}^{4}\left[-1+2\left(\delta_{a2}+\delta_{a3}\right)\right]\left[1-2\left(\delta_{b3}+\delta_{b4}\right)\right]G(r_{ij}^{(a)},r_{ij}^{(b)},1)\right)\right]\right\}. \quad \text{(D.13)}$$

The functions $G(a,b,1)$ are generalised polylogarithms of weight 2, and for our parameters a and b the following representation holds

$$G(a,b,1) = \mathrm{Li}_2\left(\frac{b-1}{b-a}\right) - \mathrm{Li}_2\left(\frac{b}{b-a}\right) + \ln\left(1-\frac{1}{b}\right)\ln\left(\frac{1-a}{b-a}\right),$$

$$G(a,a,1) = \frac{1}{2}\ln^2\left(1-\frac{1}{a}\right), \quad \text{(D.14)}$$

G.Isidori, S.Nabeebaccus and R.Zwicky, *QED corrections in $\bar{B}\to\bar{K}\ell^+\ell^-$ at the double-differential level*, JHEP **12** (2020)

Outlook

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- **Mathematics:** Understanding of multivariable hypergeometric functions, integral representations and generalized logarithms
- **Physics:** Where can we bring the results to good use?
- **Generalization:** Three denominator angular integral and beyond

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Thank You for Your Attention!

Backup-Slides

Partial differential equations

$$I_{j,l}(v_{12}, v_{11}, v_{22}; \varepsilon) = \int \mathrm{d}\Omega_{k_1 k_2} \Delta_k^j(v_1) \Delta_k^l(v_2) \, .$$

It is convenient to choose coordinates such that

$$\Delta_k(\vec{v}_1) = \frac{1}{1 - \vec{v}_1 \cdot \vec{k}} = \frac{1}{1 - \beta_1 \cos\theta_1} \, ,$$

$$\Delta_k(\vec{v}_2) = \frac{1}{1 - \vec{v}_2 \cdot \vec{k}} = \frac{1}{1 - \beta_2 \cos\vartheta \cos\theta_1 - \beta_2 \sin\vartheta \sin\theta_1 \cos\theta_2} \, ,$$

Take derivatives of $\Delta_k(\vec{v}_1)$ and $\Delta_k(\vec{v}_1)$ w.r.t. to β_1, β_2 and ϑ and express the result in terms of propagators.

Partial differential equations

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Partial differential equations

We obtain e.g.:

$$\begin{aligned}\frac{\partial}{\partial\beta_1} I_{j,l} &= \int \mathrm{d}\Omega_{k_1 k_2}\, \frac{\partial}{\partial\beta_1} \Delta_k^j(\vec{v}_1)\, \Delta_k^l(\vec{v}_2) \\ &= \frac{j}{\beta_1} \int \mathrm{d}\Omega_{k_1 k_2}\, (1 - \Delta_k^{-1}(\vec{v}_1))\, \Delta_k^{j-1}(v_1)\, \Delta_k^l(v_2)\, \Delta_k^2(v_1) \\ &= \frac{j}{\beta_1}\left(I_{j+1,l} - I_{j,l}\right).\end{aligned}$$

Analogous for $\frac{\partial}{\partial\beta_2}$ and $\frac{\partial}{\partial\vartheta}$. Translate to coordinate independent variables v_{12}, v_{11}, v_{22}, e.g. $\frac{\partial}{\partial v_{11}} = -\frac{1}{2\beta_1}\frac{\partial}{\partial\beta_1} - \frac{\cot\vartheta}{2\beta_1^2}\frac{\partial}{\partial\vartheta}$.

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Partial differential equations

Set of resulting equations, $\Delta_{12} = (1 - v_{11})(1 - v_{22}) - (1 - v_{12})^2$:

$$\begin{aligned}
\frac{\partial}{\partial v_{12}} I_{j,l} &= \frac{l}{\Delta_{12}} \left[(v_{22} - v_{12}) I_{j,l+1} - (1 - v_{12}) I_{j,l} + (1 - v_{22}) I_{j-1,l+1} \right] \\
&= \frac{j}{\Delta_{12}} \left[(v_{11} - v_{12}) I_{j+1,l} - (1 - v_{12}) I_{j,l} + (1 - v_{11}) I_{j+1,l-1} \right], \\
\frac{\partial}{\partial v_{11}} I_{j,l} &= \frac{j}{2\Delta_{12}} \left[(v_{22} - v_{12}) I_{j+1,l} + (1 - v_{22}) I_{j,l} - (1 - v_{12}) I_{j+1,l-1} \right], \\
\frac{\partial}{\partial v_{22}} I_{j,l} &= \frac{l}{2\Delta_{12}} \left[(v_{11} - v_{12}) I_{j,l+1} + (1 - v_{11}) I_{j,l} - (1 - v_{12}) I_{j-1,l+1} \right].
\end{aligned}$$

Partial differential equations

Combine to system of 1st order PDEs:

$$2l \frac{\partial}{\partial v_{11}} I_{j,l+1} = j \frac{\partial}{\partial v_{12}} I_{j+1,l} \,,$$
$$2j \frac{\partial}{\partial v_{22}} I_{j+1,l} = l \frac{\partial}{\partial v_{12}} I_{j,l+1} \,.$$

Corresponding 2nd order PDE:

$$\left(\frac{\partial^2}{\partial v_{12}^2} - 4 \frac{\partial^2}{\partial v_{11} \partial v_{22}} \right) I_{j,l}(v_{12}, v_{11}, v_{22}; \varepsilon) = 0$$

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Interlude: Binomi-Mellin-Newton integral

Question: How to "multiply out" $(a+b)^{z_1}$ as a sum of products $a^i b^j$ like we can do for $z_1 \in \mathbb{N}$?

Answer: Use MB-representation:

$$(a+b)^{z_1} = \frac{1}{\Gamma(-z_1)} \int_{-i\infty}^{i\infty} \frac{\mathrm{d}z_2}{2\pi i}\, a^{z_1-z_2} b^{z_2} \Gamma(-z_2)\Gamma(-z_1+z_2)\,,$$

Compare: Cahen-Mellin integral

$$\mathrm{e}^{-z_1} = \int_{-i\infty}^{i\infty} \frac{\mathrm{d}z_2}{2\pi i}\, z_1^{z_2} \Gamma(-z_2)\,.$$

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Double Nielsen Polylogarithms

$$\mathrm{Li}_{m_1,\ldots,m_k} = \sum_{0<n_1<\cdots<n_k} \frac{z_1^{n_1} z_2^{n_2} \ldots z_k^{n_k}}{n_1^{m_1} n_2^{m_2} \ldots n_k^{m_k}}$$

$$S_{n,p_1,p_2}(x,y) = \sum_{\vec{z}\in\vec{x}_{p_1}\sqcup\!\sqcup\vec{y}_{p_2}} \mathrm{Li}_{1,1,\ldots,1,n+1}\left(\frac{z_{p_1+p_2}}{z_{p_1+p_2-1}}, \ldots, \frac{z_2}{z_1}, z_1\right)$$

$$S_{n,1,1}(x,y) = \sum_{\vec{z}\in x\sqcup\!\sqcup y} \mathrm{Li}_{1,n+1}\left(\frac{z_2}{z_1}, z_1\right) = \mathrm{Li}_{1,n+1}\left(\frac{y}{x}, x\right) + \mathrm{Li}_{1,n+1}\left(\frac{x}{y}, y\right)$$