Power Corrections from Theshold Resummation

G. Sterman, on work with Werner Vogelsang, 2208.00937

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1. Motivation

Leading-power, factorized cross sections . . . generic, leading-power, factorized form

$$p^0\,\frac{d\sigma_{AB\to\beta+X}}{d^3p} \;=\; \sum_{ab}\int dx_a dx_b\; \phi_{a/A}(x_a,\mu^2)\,\phi_{b/B}(x_b,\mu^2)$$
$$\times\;\omega_{ab\to\beta+X}\left(x_a p_A, x_b p_B, p, \mu, \alpha_s(\mu^2)\right)$$
$$+\;\text{power corrections}$$

Higher orders in ω_{ab} motivates estimates of powers. Higher twist parton distributions begin at $1/Q$ to a power. Is it one or two? This has consequences in search for precision.

But there is another potential source of powers correction, from partonic threshold (Q depends on p_T and η):

$$\hat{s} \;=\; x_a x_b S \;\to\; Q^2$$

These arise from soft radiation (k), and can be isolated in terms of logarithms in moments N:

$$(1-\beta_r\cdot k/Q)^N \;\sim\; \exp[-N\beta_r\cdot k/Q]\;\;(1+\;\mathcal{O}(1/N))$$

in terms of a process-dependent vector β_r (see below).

Threshold resummation organizes such logarithms. Threshold-resummed series exhibits renormalon power corrections ('t Hooft 1974).

Finding renormalons:

- **We will identify in resummed cross sections integrals over the running coupling:**

$$\left(\frac{N}{Q}\right)^n \int_0^\kappa d\mu\, \mu^{n-1}\, A_i\left(\alpha_s(\mu^2)\right) \tag{1}$$

  for some anomalous dimention $A(\alpha_s)$. For this discussion, a renormalon is the Landau pole, accompanied by a definite Q-dependence.

- **Recent: Caola, Ravasio, Limatola, Melnikov, Nason, "On linear power corrections in certain collider observables," JHEP 01, 093 (2022) [2108.08897]).**
  - **Cross sections: hadrons to color-singlet bosons at measured p_T, η.**
  - **Method – mass-depndence for gluon at NLO is a diagnostic. History: inspired by Beneke, Braun 1998 for DY.**
  - **The mass is a quick stand-in for a dispersive model of the running coupling, first applied for event shapes (Dokshitzer, Marchesini, Webber).**
  - **Result – the prefactor is $1/Q^2$, not $1/Q$.**

- **Our work: For eikonal approximation, identify behavior (1) for same reactions to all orders in QCD, with same qualitative result.**
  (See also Laenen, GS, Vogelsang, 2000 for DY).

2. Double-moment cross sections

Based on GS, Vogelsang (2001) for direct photon production; extended to massive bosons

Factorized cross section at fixed rapidity:

$$\frac{p_T^3 d\sigma_{AB\to\beta X}}{dp_T d\eta} = \sum_{a,b} \int_{-\frac{U}{S+T-m^2}}^{1} dx_a\, \phi_{a/A}\left(x_a, \mu^2\right) \int_{\frac{-x_a(T-m^2)-m^2}{x_a S+U-m^2}}^{1} dx_b\, \phi_{b/B}\left(x_b, \mu^2\right) \times \omega_{ab}\left(\hat{x}_T, \hat{\eta}, r, \frac{\mu^2}{\hat{s}}\right)$$

Kinematics for a massive produced boson: m, p_T, η

$$T = m^2 - \sqrt{S}\sqrt{m^2 + p_T^2}\, \mathrm{e}^{-\eta},$$
$$U = m^2 - \sqrt{S}\sqrt{m^2 + p_T^2}\, \mathrm{e}^{\eta}$$

the "new" x_T for a massive boson

$$x_T \equiv \frac{p_T + \sqrt{m^2 + p_T^2}}{\sqrt{S}} \equiv \frac{p_T + m_T}{\sqrt{S}}$$

And, partonic variables:

$$\hat{x}_T \equiv \frac{x_T}{\sqrt{x_a x_b}}, \qquad \hat{\eta} \equiv \eta - \frac{1}{2}\ln\frac{x_a}{x_b}, \qquad r \equiv \frac{p_T}{m_T}$$

Mellin/Fourier transforms at fixed p_T:

$$\Sigma_{AB\to\beta X}(N,M,p_T) \;\equiv\; \int_{-\infty}^{\infty} d\eta\, \mathrm{e}^{iM\eta} \int_0^{x^2_{T,\max}} dx_T^2\, (x_T^2)^{N-1}\, \frac{p_T^3 d\sigma_{AB\to\beta X}}{dp_T d\eta}$$

Here $x^2_{T,\max}$ is the kinematic upper limit on x_T^2, given at fixed rapidity by

$$x^2_{T,\max} \;=\; \frac{\cosh^2\eta}{(1-r)^2}\left(1-\sqrt{1-\frac{1-r^2}{\cosh^2\eta}}\right)^2\;.$$

factorize nicely:

$$\begin{aligned}
\Sigma_{AB\to\beta X}(N,M,p_T) \;&=\; \sum_{a,b}\int_0^1 dx_a\, x_a^{N+iM/2}\phi_{a/A}\left(x_a,\mu^2\right)\int_0^1 dx_b\, x_b^{N-iM/2}\phi_{b/B}\left(x_b,\mu^2\right)\\
&\times\; \int_{-\infty}^{\infty} d\hat\eta\, \mathrm{e}^{iM\hat\eta}\int_0^{\hat x^2_{T,\max}} d\hat x_T^2\;\left(\hat x_T^2\right)^{N-1}\;\omega_{ab}\left(\hat x_T,\hat\eta,r,\frac{\mu^2}{\hat s}\right)\\
&\equiv\; \sum_{a,b}\tilde\phi_{a/A}^{N+1+\frac{iM}{2}}(\mu^2)\,\tilde\phi_{b/B}^{N+1-\frac{iM}{2}}(\mu^2)\,\tilde\omega_{ab}\left(N,M,r,\frac{\mu^2}{p_T^2}\right)
\end{aligned}$$

in terms of

$$\tilde\phi^{n}_{i/H}\left(\mu^2\right) \;\equiv\; \int_0^1 dx\, x^{n-1}\,\phi_{i/H}(x,\mu^2)$$

and a moment space perturbative function $\tilde\omega\left(N,M,r,\frac{\mu^2}{p_T^2}\right)$ . . .

What the partonic factor looks like:

$$\tilde{\omega}_{ab}\left(N, M, r, \frac{\mu^2}{p_T^2}\right) \;\equiv\; \int_{-\infty}^{\infty} d\hat{\eta}\, \mathrm{e}^{iM\hat{\eta}} \int_0^{\hat{x}^2_{T,\max}} d\hat{x}_T^2\; \left(\hat{x}_T^2\right)^{N-1}\, \omega_{ab}\left(\hat{x}_T, \hat{\eta}, r, \frac{\mu^2}{\hat{s}}\right)$$

with

$$\hat{x}^2_{T,\max} \;=\; \frac{\cosh^2\hat{\eta}}{(1-r)^2}\left(1-\sqrt{1-\frac{1-r^2}{\cosh^2\hat{\eta}}}\right)^2 \;\stackrel{r\to 1}{\to}\; 1$$

Resummation logs will appear in the variable:

$$\zeta \;\equiv\; \frac{s_4}{\hat{s}} \;\equiv\; \frac{\hat{s}+\hat{t}+\hat{u}-m^2}{\hat{s}}$$

where

$$\begin{aligned}\hat{t} \;&=\; m^2 + x_a(T-m^2)\,,\\ \hat{u} \;&=\; m^2 + x_b(U-m^2)\,.\end{aligned}$$

The invariant s_4 provides a natural measure of the distance from threshold. In terms of $\hat{x}_T, \hat{\eta}$ and r we have

$$\zeta \;=\; 1 + \hat{x}_T^2\,\frac{1-r}{1+r} - \frac{2\hat{x}_T}{1+r}\,\cosh\hat{\eta}$$

which may be inverted to give

$$\hat{x}_T(\zeta) \;=\; \frac{\cosh\hat{\eta}}{1-r}\left(1 \;-\; \sqrt{1-\frac{(1-r^2)(1-\zeta)}{\cosh^2\hat{\eta}}}\right)$$

The double moments now look like:

$$\tilde{\omega}_{ab}\left(N, M, r, \frac{\mu^2}{p_T^2}\right) = \int_{-\infty}^{\infty} d\hat{\eta}\, \mathrm{e}^{iM\hat{\eta}} \frac{1+r}{\cosh\hat{\eta}} \left(\frac{\cosh\hat{\eta}}{(1-r)}\left(1-\sqrt{1-\frac{1-r^2}{\cosh^2\hat{\eta}}}\right)\right)^{2N-1}$$

$$\times \int_0^1 \frac{d\zeta}{\sqrt{1-\frac{(1-r^2)(1-\zeta)}{\cosh^2\hat{\eta}}}} \left(\frac{1-\sqrt{1-\frac{(1-r^2)(1-\zeta)}{\cosh^2\hat{\eta}}}}{1-\sqrt{1-\frac{1-r^2}{\cosh^2\hat{\eta}}}}\right)^{2N-1} \omega_{ab}\left(\hat{x}_T, \hat{\eta}, r, \frac{\mu^2}{\hat{s}}\right)$$

Sort of complex-looking, but the ζ integrand is exponentially suppressed away from $\zeta = 0$ at large N.

$$\tilde{\omega}_{ab}\left(N, M, r, \frac{\mu^2}{p_T^2}\right) = \int_{-\infty}^{\infty} d\hat{\eta}\, \mathrm{e}^{iM\hat{\eta}} \frac{1+r}{\cosh\hat{\eta}} \left(\frac{\cosh\hat{\eta}}{(1-r)}\left(1-\sqrt{1-\frac{1-r^2}{\cosh^2\hat{\eta}}}\right)\right)^{2N-1}$$

$$\times \int_0^1 \frac{d\zeta}{\sqrt{1-\frac{(1-r^2)(1-\zeta)}{\cosh^2\hat{\eta}}}} \exp\left[-(2N-1)\frac{\zeta}{2}\left(1+\frac{\cosh\hat{\eta}}{\sqrt{r^2+\sinh^2\hat{\eta}}}\right)\right]$$

$$\times\, \omega_{ab}\left(\hat{x}_T, \hat{\eta}, r, \frac{\mu^2}{\hat{s}}\right) + \ldots\,,$$

a standard Laplace transform gives the large-N behavior – effective value of N is linked in a mild way to $\hat{\eta}$. The $\hat{\eta}$ integrand still decreases rapidly for large $\hat{\eta}$, in the same manner as seen in the massless case.

Discuss large-N behavior of $\tilde{\omega}_{ab}$ below – main result here is inverse of the double transform,

$$\frac{p_T^3 \mathrm{d}\sigma_{AB\to\beta X}}{\mathrm{d}p_T \mathrm{d}\eta} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dM\, \mathrm{e}^{-iM\eta}\, \frac{1}{2\pi i}\int_{\mathcal{C}} dN\, (x_T^2)^{-N}\, \Sigma_{AB\to\beta X}(N,M,p_T)$$

contour $\mathcal{C}$ xtending into the left half-plane where the integrand is exponentially suppressed.

- when $N \geq Q/\Lambda_{\mathrm{QCD}}$ suppression as $\exp[\ln x_T(Q/\Lambda_{\mathrm{QCD}})]$.
- Below: exponentiated nonperturbative corrections proportional to $(N/Q)^2$ for $N\Lambda_{\mathrm{QCD}}/Q \leq 1$
- As long as $x_T < 1 - \Lambda_{\mathrm{QCD}}/Q$, insensitive to very large N.
- very large N behavior of the resummed exponent does not influence the result. That is, we only need to follow N to order Q/Λ.
- Summary: Can use the same technique for massive bosons as for direct photon

3. Eikonal cross sections

Near partonic threshold, the limit $\zeta \to 0$ for

$$a + b \to \beta + r\,,$$

where again, $\beta = \gamma, \gamma^*, W, Z, H$

Notation: $2N - 1 \to N$

- Eikonal approximmation for $\tilde{\omega}_{ab}(N, M)$.

- for any radiation k, hard parton p: $k^2 \ll 2p \cdot k$.

- Gluon momentum $\beta_k^\mu |k_0|/Q$, with k_0 the energy of the radiation and β_k the velocity vector for k. Contributions to s_4 suppressed by relative $\beta_k \cdot \beta_r / N$.

- Radiation in the recoil direction: $(\beta_k = \beta_r)$, factor into a partonic jet functions, Expect $\exp[-N m_{\rm jet}^2/Q^2]$: nominal contributions like N/Q^2, rather than N^2/Q^2.

- Summary: eikonal corrections are leading at least by a power of N.

The large-N transform and threshold kinematics

At large values of N in Eq. (2), the integral will be restricted to a region where $\zeta = s_4/\hat{s}$ is of order $1/N$. Relate integral over ζ to an integral over the momenta of final states.

- **Near partonic threshold, Laplace transform in $s_4/\hat{s}_{\min}$:**

$$\int_0 d\zeta \;\; \mathrm{e}^{-N\zeta} \;\rightarrow\; \int_0 \frac{ds_4}{\hat{s}_{\min}} \exp\left(-N \frac{s_4}{\hat{s}_{\min}}\right) \; ,$$

  where

$$\hat{s}_{\min} \;=\; \left(p_T^2 + m^2\right) \cosh^2 \hat{\eta} \left[1 + \sqrt{1 - \frac{1 - r^2}{\cosh^2 \hat{\eta}}} \right]^2 \; .$$

- **What is s_4 in terms of soft radiation?**

$$s_4 \;=\; (k + p_r)^2 \;=\; (k + p_r^0 \beta_r)^2 \;\sim\; 2 p_r^0 \, \beta_r \cdot k \, ,$$

  where

$$\begin{aligned} p_r^0 &= \sqrt{p_T^2 \cosh^2 \hat{\eta} + m^2 \sinh^2 \hat{\eta}} \, , \\ p_r^3 &= -\sqrt{p_T^2 + m^2} \, \sinh \hat{\eta} \, , \\ \mathrm{p}_{r,T} &= -\mathrm{p}_T \, . \end{aligned}$$

- **Conclusion: moments in terms of velocity of recoil parton, β_r, and η-dependent hard scale.**

$$\exp\left(-N \frac{s_4}{\hat{s}_{\min}}\right) \;=\; \exp\left(-N \, \frac{\beta_r \cdot k}{Q}\right) \quad \text{with} \quad Q \;\equiv\; \frac{\hat{s}_{\min}}{2 p_r^0}$$

Eikonal cross sections and Wilson lines

- **Basic factorization result:**

$$\tilde{\omega}^{(\mathrm{eik})}_{abr}(N,Q,\hat{\eta},\mu) \;=\; H_{abr}(p_T,\hat{\eta},\mu)\times\frac{\tilde{\sigma}^{(\mathrm{eik})}_{abr}(N/Q,\hat{\eta},\mu,\epsilon)}{\tilde{\phi}^{(\mathrm{eik})}_{a/a}\,(N_a,\mu,\epsilon)\,\tilde{\phi}^{(\mathrm{eik})}_{b/b}\,(N_b,\mu,\epsilon)}\,,$$

- **For 1PI cross section:**

$$N_a \;=\; N\,\frac{\beta_b\cdot\beta_r}{\beta_a\cdot\beta_b}\,,$$
$$N_b \;=\; N\,\frac{\beta_a\cdot\beta_r}{\beta_a\cdot\beta_b}\,,$$

$\beta_{a,b}$ **velocity four-vectors for incoming particles;** β_r **for colored recoil particle.**

- **Simplification at large** N**, no parton mixing in evolution.**

- **Building blocks:**

$$\Phi_{\beta}^{(R)}(\lambda_2,\lambda_1;x) \;\equiv\; \mathrm{P}\;\exp\left[-ig\int_{\lambda_1}^{\lambda_2} d\lambda\,\beta\cdot A^{(R)}(\lambda\beta+x)\right]$$

- **For example, the annihlation channel:**

$$[U_{q\bar{q}g}(x)]_{d;ji} \;\equiv\; \mathrm{T}\;\left(\left[\Phi_{\beta_g}^{(g)}(\infty,0;x)\right]_{d,e}\; u_{e,ji}^{(q\bar{q})}(x)\right)\;,$$

where "T" denotes time ordering, and where

$$u_{e,ji}^{(q\bar{q})}(x) \;\equiv\; \left[\Phi_{\beta_{\bar{q}}}^{(\bar{q})}(0,-\infty;x)\right]_{jl}\left(T_e^{(q)}\right)_{lk}\;\left[\Phi_{\beta_q}^{(q)}(0,-\infty;x)\right]_{ki}\;, \qquad (2)$$

$T_e^{(q)}$ **the SU(3) in the fundamental representation.**

- **Giving the eikonal cross section**

$$\begin{aligned}
\tilde{\sigma}_{abr}^{(\mathrm{eik})}(N/Q,\hat{\eta},\mu,\epsilon) &= \sum_X \mathrm{e}^{-N(\beta_r\cdot k_X/Q)}\left\langle 0|U_{abr}^{\dagger}(0)|X\right\rangle\langle X|U_{abr}(0)|0\rangle \\
&= \sum_X \left\langle 0|U_{abr}^{\dagger}(0)|X\right\rangle\left\langle X|\mathrm{e}^{-N(i\beta_r\cdot\partial_x/Q)}\;U_{abr}(x)|0\right\rangle_{x=0} \\
&= \left\langle 0|U_{abr}^{\dagger}(0)\;\mathrm{e}^{-N(i\beta_r\cdot\partial_x/Q)}\;U_{abr}(x)|0\right\rangle_{x=0}\,.
\end{aligned}$$

Cancellation of final-state interactions

- Special for color-singlet cross sections: outgoing Wilson changes exponential of the derivative in weight functiok into exponential of covariant derivative:

$$\mathrm{e}^{-\frac{iN\beta_r\cdot\partial_x}{Q}}\,\Phi^{(r)}_{\beta_r}(\infty,0;x) \;=\; \Phi^{(r)}_{\beta_r}(\infty,0;x)\;\mathrm{e}^{-\frac{iN\beta_r\cdot D^{(r)}(A(x))}{Q}}\,,$$

- Then final state interactions all cancel, leaving series of local operators with all β_r dependence:

$$\begin{aligned}\tilde{\sigma}^{(\mathrm{eik})}_{abr}(N/Q,\hat{\eta},\mu,\epsilon) &= \langle 0|u^{ab\dagger}(0)\;[\,\mathrm{e}^{-\frac{iN\beta_r\cdot D^{(r)}(A(0))}{Q}}\,u^{ab}(0)]|\,0\rangle \\ &= \langle 0|u^{ab\dagger}(0)\;\left[\left(1\;-\;\frac{iN\beta_r\cdot D^{(r)}(A)}{Q}+\ldots\right)u^{ab}(0)\right]|\,0\rangle \\ &= \;\sigma^{(\mathrm{eik},0)}_{abr}\;(1+\mathcal{O}(\alpha_s))\;.\end{aligned}$$

- Now all we need is a formula for $\sigma^{(\mathrm{eik})}$ that "knows about" this cancellation . . . here it's "webs" . . .

4. Web exponentiation and resummation

- In terms of web functions, the cross sections exponentiates (phase space at fixed $\beta_r \cdot k$ is symmetric)

- Graphical exponentiation and web functions:

$$\tilde{\sigma}_{abr}^{(\mathrm{eik})}(N/Q, \hat{\eta}, \mu, \epsilon) \;=\; \tilde{\sigma}_{abr}^{(\mathrm{eik},0)}\, \mathrm{e}^{E_{abr}(N/Q,\hat{\eta},\mu,\epsilon)}\,,$$

wwhere

$$\begin{aligned} E_{abr}(N/Q, \hat{\eta}, \mu, \epsilon) \;&=\; \int \frac{d^D k}{(2\pi)^D}\, \left(\mathrm{e}^{-N\frac{\beta_r \cdot k}{Q}} - 1\right)\, \theta\left(\frac{Q}{\sqrt{2}} - k^+\right)\, \theta\left(\frac{Q}{\sqrt{2}} - k^-\right) \\ &\times\;\; w_{abr}\left(\left\{\frac{\beta_i \cdot k\, \beta_j \cdot k}{\beta_i \cdot \beta_j}\right\}, k^2, \mu^2, \alpha_s(\mu^2)\right)\,. \end{aligned}$$

- Demand it vanishes at $N = 0$ (fully inclusive): virtual is all in “-1”

- To get the cross section, eikonal PDFs:

$$\tilde{\phi}_{i/i}(N, \mu, \epsilon) \;=\; \exp\left[\int_0^{\mu^2} \frac{dk_T^2}{k_T^{2(1+\epsilon)}}\, A_i\left(\alpha_s(k_T^2)\right)\, \int_0^1 dz\, \frac{z^N - 1}{1 - z}\right]\,,$$

Anomalous dimension $A_i(\alpha_s) = C_i\,(\alpha_s/\pi) + \ldots$ with $C_q = C_F = 4/3$ and $C_g = C_A = 3$.

- **The functions we're after:**

$$\tilde{\omega}^{(\mathrm{eik})}_{abr}(N, Q, \hat{\eta}, \mu) \;=\; H_{abr}(p_T, \hat{\eta}, \mu)\, \sigma^{(\mathrm{eik},0)}_{abr}\; \mathrm{e}^{\hat{E}_{abr}(N/Q,\hat{\eta},\mu)}\,,$$

where

$$\hat{E}_{abr}(N/Q, \hat{\eta}, \mu) \;=\; \int \frac{d^D k}{(2\pi)^D}\, \left(\mathrm{e}^{-N\frac{\beta_r\cdot k}{Q}} - 1\right)\, w_{abr}\left(\left\{\frac{\beta_i\cdot k\,\beta_j\cdot k}{\beta_i\cdot\beta_j}\right\}, k^2, \mu^2, \alpha_s(\mu^2)\right)$$
$$-\sum_{i=a,b}\int_0^{\mu^2}\frac{dk_T^2}{k_T^{2(1+\epsilon)}}\, A_i\left(\alpha_s(k_T^2)\right)\int_0^1 dz\,\frac{z^{N_i}-1}{1-z}$$

The cancellation of collinear singularities is in the exponent.

- w_{abr} **is the relevant "web function" for three Wilson lines. It is the exact logarithm of the cross section. (Gatheral, 1983).**

- **The web function w_{abr} has the properties (Berger, 2003; Mitov, GS, Sung, 2010)**
  - **3-eikonal irreducible**
  - **nonfactorizable color factors (e.g. like $C_A C_F$, not C_F^2 for quark Wilson lines)**
  - **see recent work by Gardi, Magnea, Laenen & collaborators.**
  - **No collinear or soft SUB-divergences**
  - **RG invariance (Polyakov (1974) Dotsenko & Vergeles (1980))**

$$\mu \frac{d}{d\mu} w_{abr} \left(\left\{ \frac{\beta_i \cdot k \, \beta_j \cdot k}{\beta_i \cdot \beta_j} \right\}, k^2, \mu^2, \alpha_s(\mu^2) \right) = 0 \, .$$

  - **Arguments invariant under rescalings of the βs.**

- **Leading powers in Q come from the range of k where $N\beta_r \cdot k/Q > 1$ and real-gluon emission is exponentially suppressed. Webs general eikonal approximation to threshold resummation.**

- **Nonleading powers in N/Q arise from the region $N\beta_r \cdot k/Q < 1$.**

5. The lowest order web and renormalons

- **Lowest-order web function for $q\bar{q} \to \beta + g$, given by the interference terms between gluon emission from each of the three Wilson lines.**
- **Useful normaliation**

$$\begin{aligned} \beta_a^\mu &= \sqrt{2}\,\delta_{\mu +}\,, \\ \beta_b^\mu &= \sqrt{2}\,\delta_{\mu -}\,, \end{aligned}$$

giving $\beta_a \cdot \beta_b = 2$ and

$$\frac{\beta_a \cdot k\, \beta_b \cdot k}{\beta_a \cdot \beta_b} = \frac{k^2 + k_T^2}{2}\,.$$

- **Expand the recoil velocity**

$$\beta_r^\mu = \frac{\beta_r \cdot \beta_b}{\beta_a \cdot \beta_b}\,\beta_a^\mu + \frac{\beta_r \cdot \beta_a}{\beta_a \cdot \beta_b}\,\beta_b^\mu + \beta_{r,T}\,.$$

- **Which gives**

$$\begin{aligned} w_{q\bar{q}g}^{(1)} &= 4\pi g^2\,\mu^{2\epsilon}\,\delta(k^2)\,\left(C_F \frac{\beta_a \cdot \beta_b}{\beta_a \cdot k \beta_b \cdot k} + \frac{C_A}{2}\,\frac{\beta_a \cdot \beta_b}{\beta_a \cdot k \beta_b \cdot k}\,\frac{\beta_{r,T} \cdot k_T}{\beta_r \cdot k}\right) \\ &\equiv C_F\,u_{q\bar{q}g}^{(1)} + C_A\,v_{q\bar{q}g}^{(1)}\,. \end{aligned}$$

See explicitly that only collinear singularities are from incoming lines

- **All this gives ...**

- **The full exponent**

$$
\begin{aligned}
\hat{E}^{(1)}_{abr}(N/Q,\hat{\eta},\mu) &= C_F \int \frac{d^4k}{(2\pi)^4}\, \left(\mathrm{e}^{-N\frac{\beta_r\cdot k}{Q}} - 1 \right) u^{(1)}_{abr}(k) \\
&+ \int_0^{\mu^2} \frac{dk_T^2}{k_T^2}\, C_F\, \frac{\alpha_s(k_T^2)}{\pi}\, \ln\left(\frac{\bar{N}\beta_{r,T}}{2}\right) \\
&+ C_A \int \frac{d^4k}{(2\pi)^4}\, \left(\mathrm{e}^{-N\frac{\beta_r\cdot k}{Q}} - 1 \right) v^{(1)}_{abr}(k) \\
&\equiv C_F\, U^{(1)}_{q\bar{q}g}(N/Q,\hat{\eta},\mu) + C_A\, V^{(1)}_{q\bar{q}g}(N/Q,\hat{\eta},\mu)\,,
\end{aligned}
$$

- **Note the runnning coupling in the subtraction.**

- **Treat C_F and C_A parts separately**

- For the C_F term, k^+ integral:

$$\int_{k_T^2/\sqrt{2}Q}^{Q/\sqrt{2}} \frac{dk^+}{2k^+} \left(\mathrm{e}^{-\frac{N}{Q}\left(\beta_r^- k^+ + \beta_r^+ \frac{k_T^2}{2k^+}\right)} - 1 \right) = K_0\left(\frac{N\beta_{r,T}k_T}{Q}\right) + \ln\left(\frac{k_T}{Q}\right) + \mathcal{O}\left(\mathrm{e}^{-N}\right) .$$

- Any k_T less than Q/N. For larger k_T, $K_0(z) \sim z^{-1/2}e^{-z}$
$\rightarrow$ leading power resummation for larger k_T

- The subracted term in the exponent

$$U^{(1)}_{q\bar{q}g}(N/Q,\hat{\eta},\mu) = 2\int_0^{Q^2} \frac{dk_T^2}{k_T^2} \frac{\alpha_s(k_T^2)}{\pi} \left[I_0\left(\frac{N\beta_{r,T}k_T}{Q}\right) K_0\left(\frac{N\beta_{r,T}k_T}{Q}\right) + \ln\left(\frac{\bar{N}\beta_{r,T}k_T}{2Q}\right)\right]$$

- Two things to notice: (1) All even powers:

- Log term in the expansion of K_0 cancels the subtraction log:

$$\begin{aligned} K_(z) &= \ln\left(\frac{ze^{\gamma}}{2}\right) + \left(\frac{z}{2}\right)^2 + \ldots \\ I_0(z) &= 1 + \left(\frac{z}{2}\right)^2 + \ldots \end{aligned} \qquad (3)$$

- Argument of $\alpha_s(k_T)$ follows from web RG invariance and lack of subdivergences – only the overall collinear singularity gives logs. This is why there aren't renormalons someplace else we didn't look.

- **For C_A term, use derivative with respect to N, which cancels the $\beta_r \cdot k$ denominator**

$$\frac{d}{dN} V^{(1)}_{q\bar{q}g}(N/Q, \hat{\eta}, \mu) \;=\; \frac{2\beta_{r,T}}{Q} \int_0^{Q^2} \frac{dk_T^2}{k_T} \frac{\alpha_s(k_T^2)}{\pi} I_1\left(\frac{N\beta_{r,T}k_T}{Q}\right) K_0\left(\frac{N\beta_{r,T}k_T}{Q}\right) .$$

- **I_1 is from the azimuthal integral of k_T.**

- **Here, finite without a collinear subtraction.**

- **$I_1(z) \sim z + \mathcal{O}(z^3)$ for small z, and replaces k_T in denominator by Q.**

- **Because exponent vanishes at $N = 0$, we can integrate the expanded form.**

- **Final result for the exponent:**

$$\hat{E}^{(1)}_{abr}(N/Q,\hat{\eta},\mu) \;=\; C_F\, U^{(1)}_{q\bar{q}g}(N/Q,\hat{\eta},\mu) \;+\; C_A\, V^{(1)}_{q\bar{q}g}(N/Q,\hat{\eta},\mu)\,,$$

- **Is**

$$U^{(1)}_{q\bar{q}g}(N/Q,\hat{\eta},\mu) \;=\; \frac{1}{2}\left(\frac{N\beta_{r,T}}{Q}\right)^2 \int_0 dk_T^2\,\frac{\alpha_s(k_T^2)}{\pi}\,\left[1-2\ln\left(\frac{\bar{N}\beta_{r,T}k_T}{2Q}\right)\right] \;+\; \mathcal{O}(N^4/Q^4)\,,$$

$$V^{(1)}_{q\bar{q}g}(N/Q,\hat{\eta},\mu) \;=\; -\,\frac{1}{2}\left(\frac{N\beta_{r,T}}{Q}\right)^2 \int_0 dk_T^2\,\frac{\alpha_s(k_T^2)}{\pi}\ln\left(\frac{\bar{N}\beta_{r,T}k_T}{Q}\right) \;+\; \mathcal{O}(N^4/Q^4)\,.$$

- **We see explicitly the absence of linear powers.**
- **And the (lowest-order) renormalon when $k_T^2=\Lambda^2$ at N^2/Q^2. A “Borel” form is:**

$$\alpha_s(k_T^2) \;=\; \int_0^\infty d\sigma\,\left(\frac{k_T^2}{\Lambda^2}\right)^{-\sigma\beta_0/4\pi}$$

$$\int dk_T^2\;\alpha_s(k_T^2) \;=\; \int_0^\infty d\sigma\frac{\Lambda^{\sigma\beta_0/2\pi}}{1-\sigma\beta_0/4\pi}$$

- **But more generally, the k_T^2 integral is just undefined at this level.**
- **An alternate approach, at NLO and fixed coupling is a gluon mass in the web:**

$$\delta(k^2) \;\to\; \delta(k^2-\lambda^2)$$

Also gives only even powers of λ.

6.Beyond lowest order i web

- In this case the web function has non-zero k^2 as well as k_T, but still no subdivergences.
- We expand the exponents in

$$\hat{E}_{abr}(N/Q,\hat{\eta},\mu) = \int \frac{d^D k}{(2\pi)^D} \left(e^{-N\frac{\beta_r \cdot k}{Q}} - 1 \right) w_{abr}\left(\left\{ \frac{\beta_i \cdot k\, \beta_j \cdot k}{\beta_i \cdot \beta_j} \right\}, k^2, \mu^2, \alpha_s(\mu^2) \right)$$

$$- \sum_{i=a,b} \int_0^{\mu^2} \frac{dk_T^2}{k_T^{2(1+\epsilon)}} A_i\left(\alpha_s(k_T^2)\right) \int_0^1 dz \frac{z^{N_i} - 1}{1-z}$$

for an expansion in $\beta_r \cdot k$ and $\ln x \sim 1 - x$.

- Because webs have no subdivergences, even one power of $\beta_r \cdot k$ makes the web collinear finite for k in β_r direction.

- This leaves dependence on only

$$\frac{\beta_a \cdot k \beta_b \cdot k}{\beta_a \cdot \beta_b} = \frac{k_T^2 + k^2}{2}$$

in a-b c.m. frame. We can do an energy integral to any order in α_s !

- **For example, the first power in $\beta_r \cdot k$ given terms where we can do the energy integrals**

$$
\begin{aligned}
\mathcal{E}_a^{[1]}(N/Q, \hat{\eta}, \mu, \epsilon) &= -\frac{N}{Q}\frac{\beta_a \cdot \beta_r}{\beta_a \cdot \beta_b}\,\Omega_{1-\epsilon}\int_0^{\kappa^2}\frac{dk_T^2\, k_T^{-2\epsilon}}{2(2\pi)^D} \\
&\quad \times \int_0^{Q^2/(\beta_a\cdot\beta_r N)^2 - k_T^2} dk^2\ G_a\left(k^2, k_T^2\right)\ \sqrt{\left(\frac{Q}{\beta_a \cdot \beta_r N}\right)^2 - k^2 - k_T^2} \\
&= -\frac{\Omega_{1-\epsilon}}{2\pi}\int_0^{\kappa^2} dk_T^2\, k_T^{-2\epsilon}\int_0^{Q^2/(\beta_a\cdot\beta_r N)^2 - k_T^2}\frac{dk^2}{4(2\pi)^{D-1}}\ G_a\left(k^2, k_T^2\right) \\
&\quad \times \left[1 - \frac{\left(k^2 + k_T^2\right)(\beta_a \cdot \beta_r)^2 N^2}{2Q^2} + \ldots\right],
\end{aligned}
$$

- **The "1" term is cancelled by the collinear subtraction for the β_a direction. We can't do the k^2 integral, but factorization requires that**

$$
\int_0^{Q^2/(\beta_a\cdot\beta_r N)^2 - k_T^2}\frac{dk^2}{4(2\pi)^{D-1}}\ G_a\left(k^2, k_T^2\right)\ =\ \frac{A_a\left(\alpha_s(k_T^2)\right)}{k_T^2}\ +\ \mathcal{A}_a(k_T^2, Q^2)\,,
$$

- **The finite, collinear-singular terms cancels and the power correction is then from the $k_T^2 + k^2$ term,.**

- **This contines at all powers of $\beta_r \cdot k$.**

7. Outlook

- **Result is general, but still purely eikonal.**

- **Extension beyond eikonal approximation should be possible (much is known about "next to eikonal": Laenen, Magnea . . .)**

- **DIS color-singlet boson production is a special case.**

- **The phenomenology remains to be studied systematically. (Say using dispersive model as in Dasgupta & Webber for DIS)**

- **Application to single-hadron is nontrivial because with two Wilson lines in the final state, the relevant cancellations will be more elaborate. Also, webs are matrices in the space of color tensors. It should be fun.**