Precise Predictions for Track-based Observables

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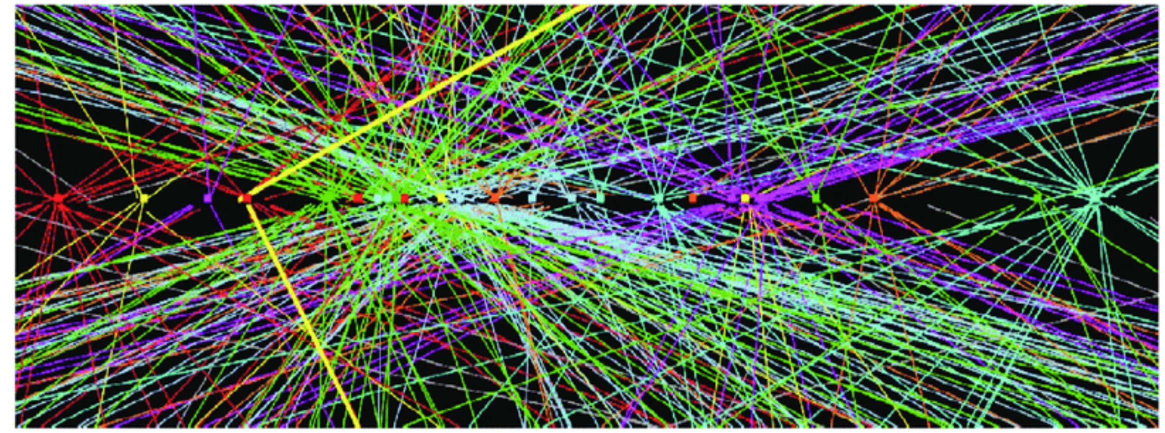
NWO

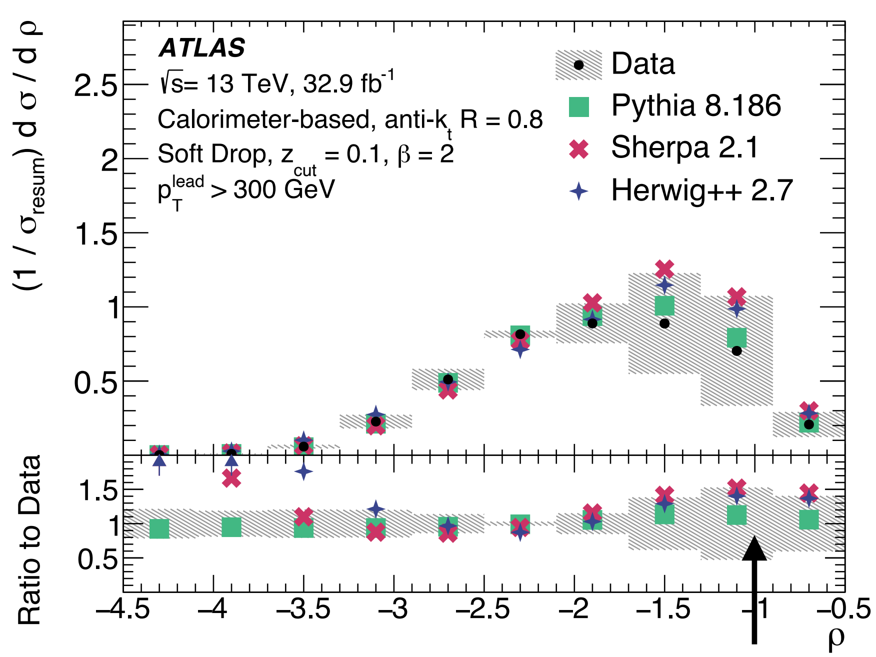
Jet Physics: From RHIC/LHC to EIC

Motivation for track-based measurements

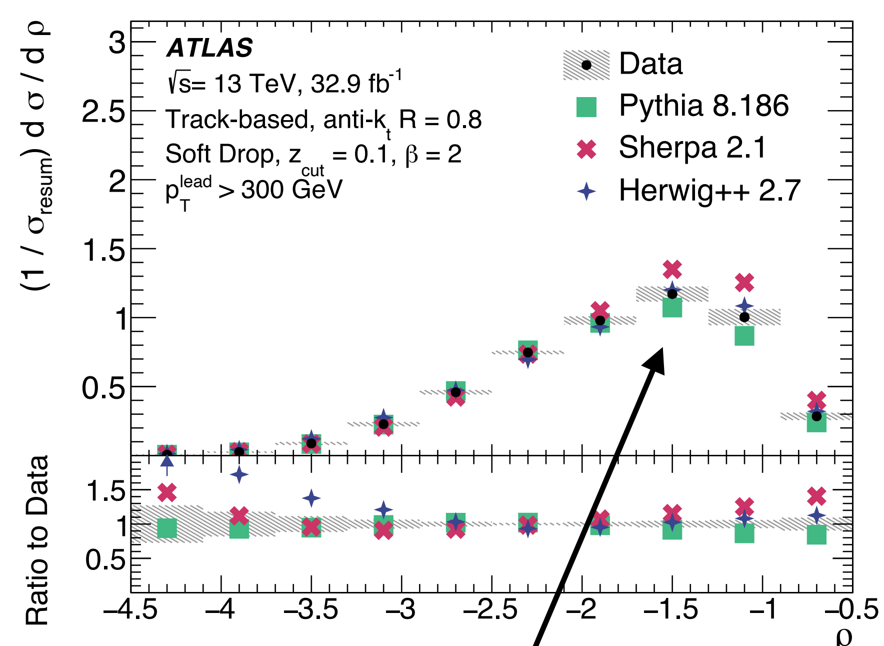
- Superior angular resolution
  → crucial for jet substructure
- Pile-up removal



- Improvement from tracks for groomed $\rho = \ln(m^2/p_T^2)$



Pile up



Difference of parton showers larger than exp. uncertainty

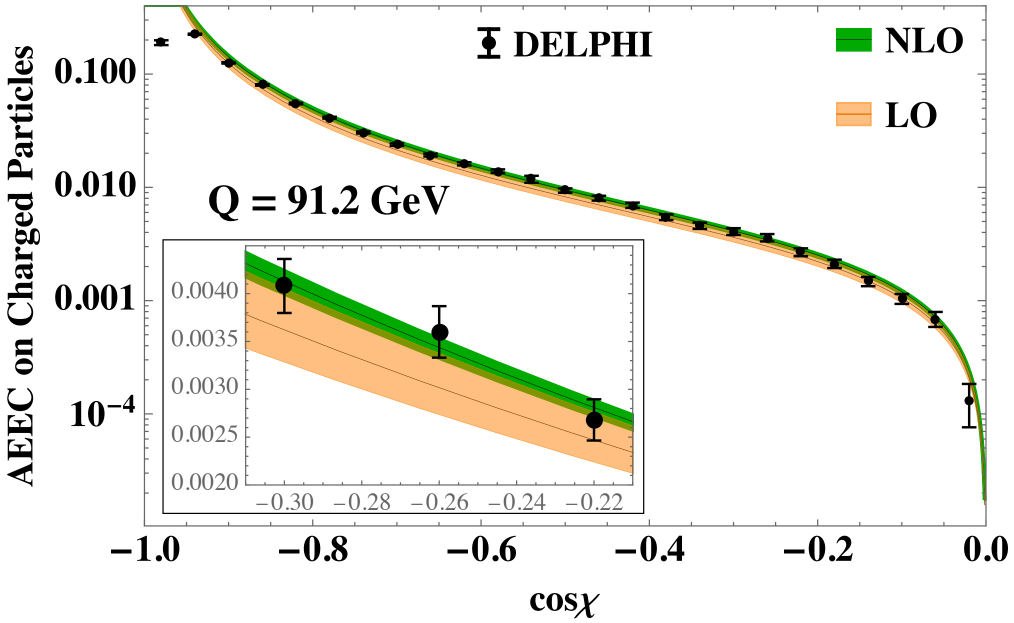
Overview and main message

- Track-based measurements are sensitive to hadronization
  → **modeled** in parton showers.

- Track functions offer **systematically** improvable framework.

- Now **extended to** $\mathcal{O}(\alpha_s^2)$
  → high precision + checks on formalism.

- Some applications:
  - Energy correlators
    [See talks by Ian Moult and Kyle Lee]
  - Azimuthal decorrelation in V+jet
    [See talk by Yang-Ting Chien]



Precision phenomenology with tracks is possible!

Outline

1. Track functions
   [arXiv:1303.6637 - Chang, Procura, Thaler, WW]

2. Track functions at order α_s^2
   [arXiv:2108.01674 - Li, Moult, Schrijnder van Velzen, WW, Zhu,
   arXiv:2201.05166 - Jaarsma, Li, Moult, WW, Zhu,
   ongoing work - Chen, Jaarsma, Li, Moult, WW, Zhu]

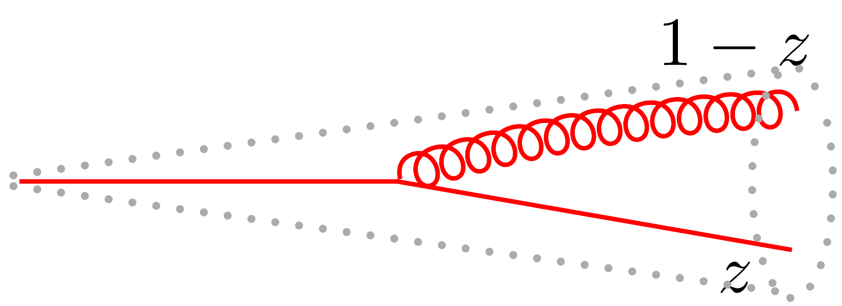
3. Energy correlators
   [arXiv:2108.01674 - Li, Moult, Schrijnder van Velzen, WW, Zhu]

4. Conclusions and outlook

1. Track functions

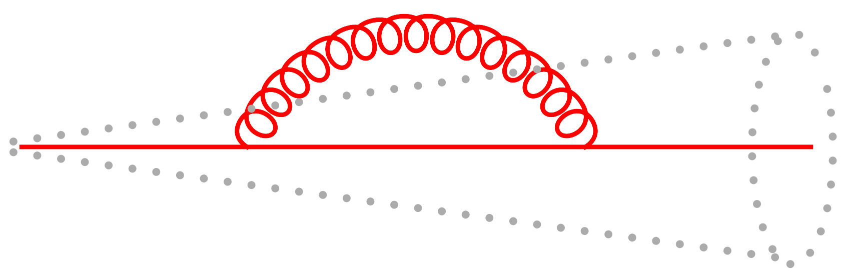
[arXiv:1303.6637 - Chang, Procura, Thaler, WW]

1. Track-based measurements are not IRC safe



track fraction:

z



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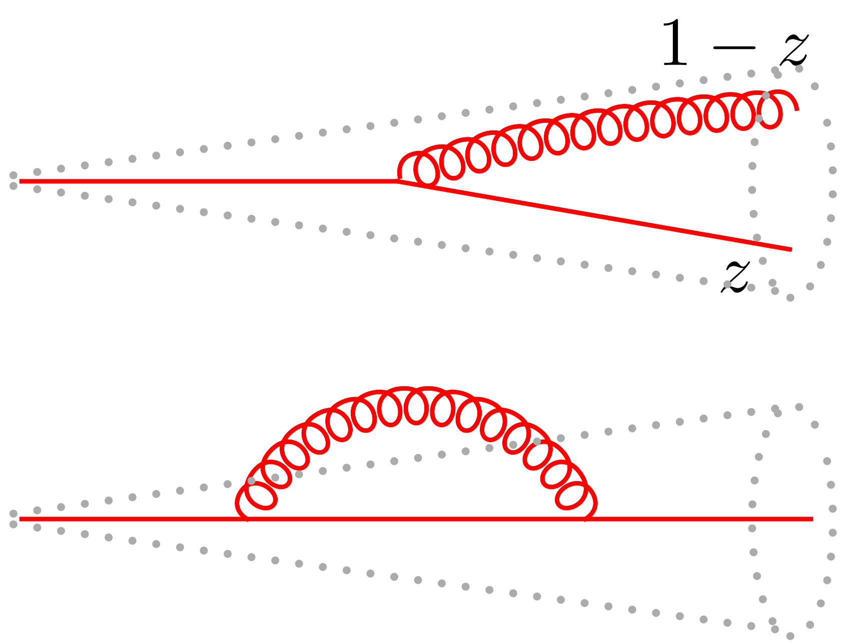
- E.g. partonic calculation of energy fraction x of tracks in a jet.
- Mismatch of collinear divergences between real and virtual.

1. Track-based measurements are not IRC safe



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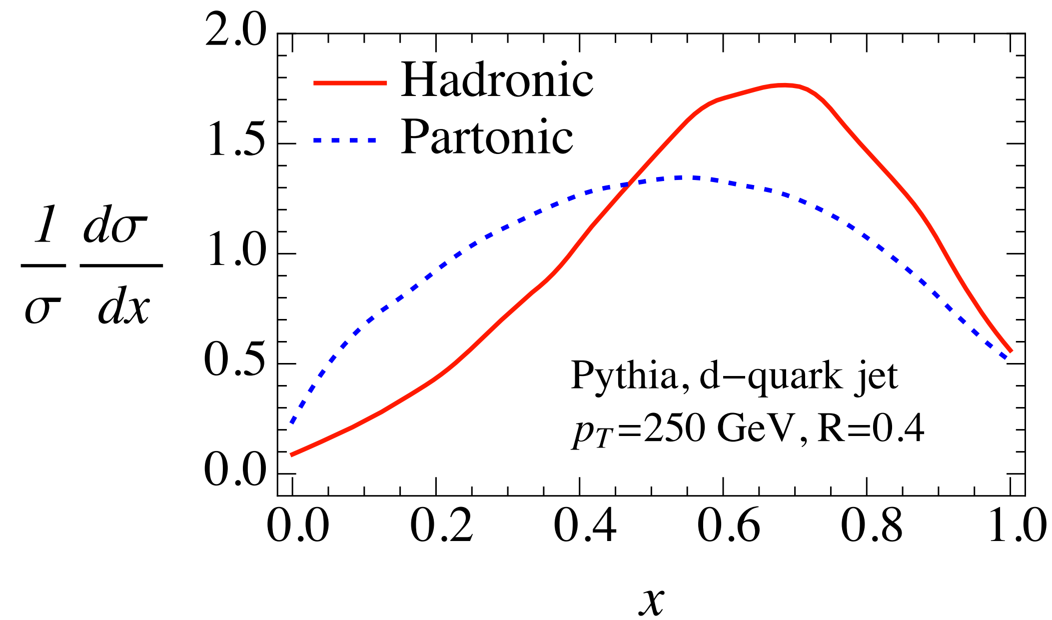
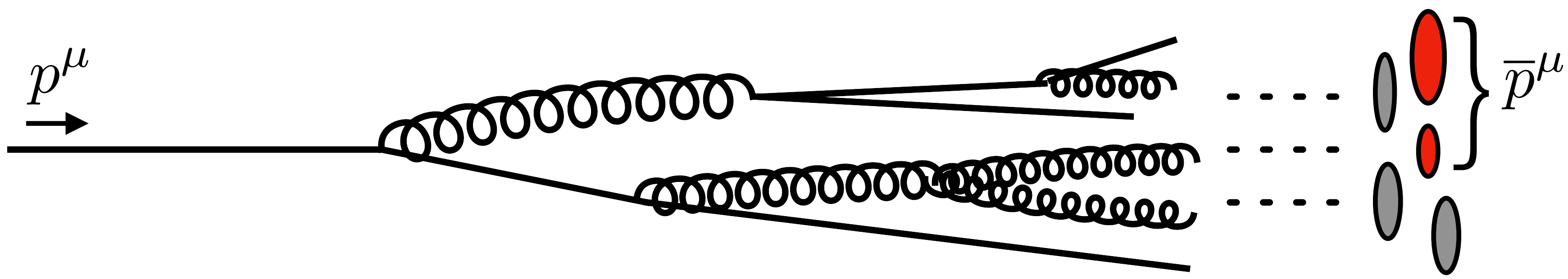


- E.g. partonic calculation of energy fraction x of tracks in a jet.
- Mismatch of collinear divergences between real and virtual.
- Large hadronization corrections in parton showers.

1. Track function



- $T_i(x, \mu)$ describes momentum fraction x of initial parton i converted to charged hadrons, i.e. $\bar{p}^\mu = x p^\mu + \mathcal{O}(\Lambda_{\mathrm{QCD}})$

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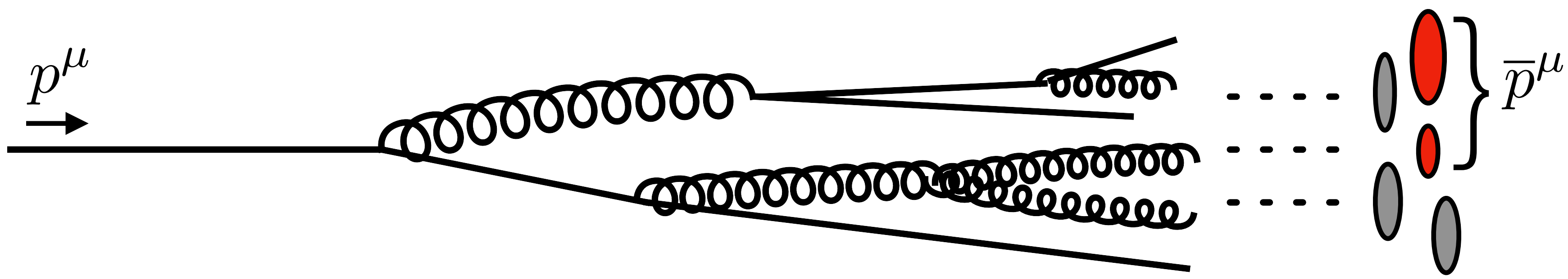

- $T_i(x, \mu)$ describes momentum fraction x of initial parton i converted to charged hadrons, i.e. $\bar{p}^\mu = x p^\mu + \mathcal{O}(\Lambda_{\mathrm{QCD}})$
- Nonperturbative, process-independent function.
- Conservation of probability: $\int_0^1 \mathrm{d}x\, T_i(x) = 1$
- Definition in light-cone gauge:

all charged hadrons in X

$$T_q(x) = \int \mathrm{d}y^+ \mathrm{d}^2 y_\perp e^{ik^- y^+/2} \sum_X \delta\Big(x - \frac{\bar{p}^-}{k^-}\Big)$$

$$\times \frac{1}{2N_c} \mathrm{tr}\Big[\frac{\gamma^-}{2} \langle 0|\psi(y^+, 0, y_\perp)|X\rangle\langle X|\bar{\psi}(0)|0\rangle\Big]$$

1. Connection to fragmentation functions

- First and second moment:

$$\int_0^1 \mathrm{d}x\, x\, T_i(x,\mu) = \sum_{\text{charged } h} \int_0^1 \mathrm{d}x\, x\, D_{i\to h}(x,\mu)$$

$$\int_0^1 \mathrm{d}x\, x^2\, T_i(x,\mu) = \sum_{\text{charged } h} \int_0^1 \mathrm{d}x\, x^2\, D_{i\to h}(x,\mu)$$

$$+ \sum_{\text{charged } h_1, h_2} \int_0^1 \mathrm{d}x_1\, \mathrm{d}x_2\, x_1\, x_2\, D_{i\to h_1 h_2}(x_1, x_2, \mu)$$

dihadron fragmentation

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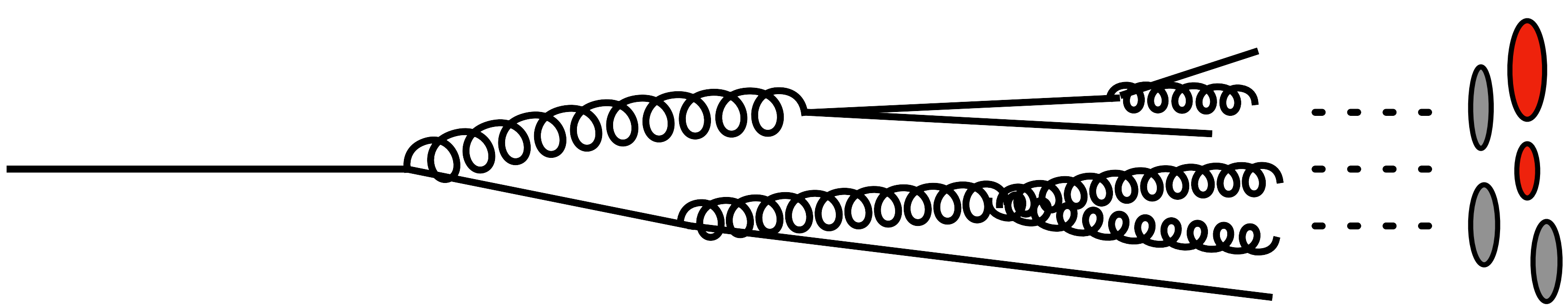
dihadron fragmentation

- Track function encodes **correlations** between hadrons:



$$(x_1+x_2)^2 = x_1^2 + x_2^2 + 2x_1x_2$$

- If no new correlations for ≥ 3 hadrons → $T_i(x)$ is Gaussian.

1. Track-based calculations

- Consider a cross section differential in observable e

$$\frac{\mathrm{d}\sigma}{\mathrm{d}e} = \sum_N \int \mathrm{d}\Pi_N \, \frac{\mathrm{d}\sigma_N}{\mathrm{d}\Pi_N} \, \delta[e - \hat{e}(\{p_i^\mu\})]$$

- At leading order, for the track-based measurement $\bar{e}$

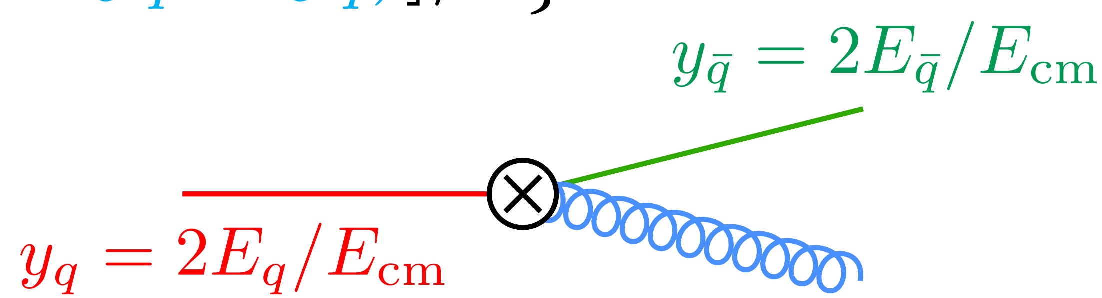
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\bar{e}} = \sum_N \int \mathrm{d}\Pi_N \, \frac{\mathrm{d}\sigma_N}{\mathrm{d}\Pi_N} \, \underbrace{\int \prod_{i=1}^{N} \mathrm{d}x_i \, T_i(x_i)}_{\text{hadronization}} \, \delta[e - \hat{e}(\{x_i p_i^\mu\})]$$

- Beyond leading order, there is a cancellation of IR divergences, $\mathrm{d}\sigma_N \to \mathrm{d}\bar{\sigma}_N$, similar to fragmentation functions/PDFs.

1. Example: track fraction in e+e-

- Cross section differential in track fraction w up to $\mathcal{O}(\alpha_s)$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}w} = \int \mathrm{d}y_q \mathrm{d}y_{\bar{q}} \, \frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}y_q \mathrm{d}y_{\bar{q}}} \int \mathrm{d}x_q \, T_q(x_q) \int \mathrm{d}x_{\bar{q}} \, T_q(x_{\bar{q}}) \int \mathrm{d}x_g \, T_g(x_g)$$
$$\times \, \delta\{w - [x_q y_q + x_{\bar{q}} y_{\bar{q}} + x_g(2 - y_q - y_{\bar{q}})]/2\}$$



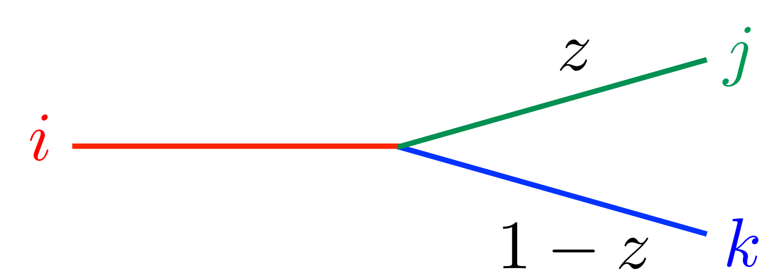
- $1/\epsilon_{\mathrm{IR}}$ pole in partonic cross section $\mathrm{d}\sigma$ cancels against pole in **partonic** track function, resulting in finite $\mathrm{d}\bar{\sigma}$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}y_q \mathrm{d}y_{\bar{q}}} = \sigma^{(0)} \Big\{ \delta(1 - y_q)\delta(1 - y_{\bar{q}})$$
$$+ \frac{\alpha_s C_F}{2\pi} \Big[-\frac{1}{\epsilon_{\mathrm{IR}}} \Big(\frac{1 + y_q^2}{1 - y_q} \Big)_+ \delta(1 - y_{\bar{q}}) + \dots \Big] \Big\}$$

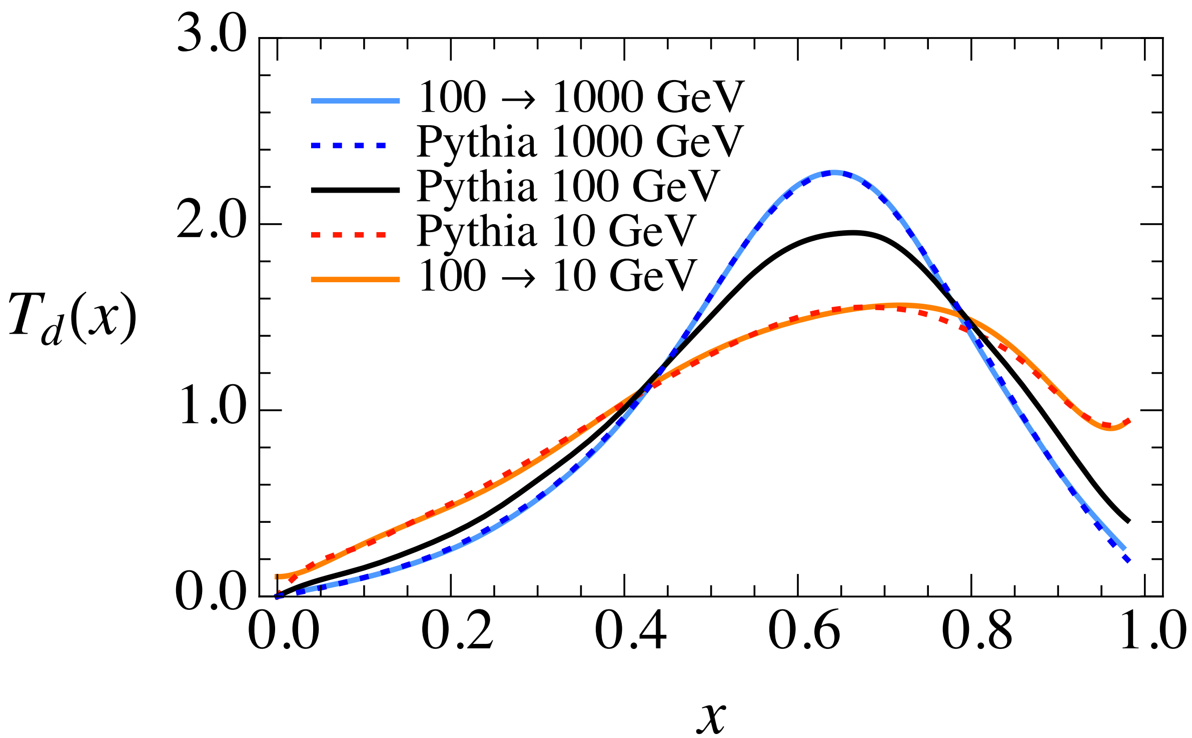
1. Track function at order α_s

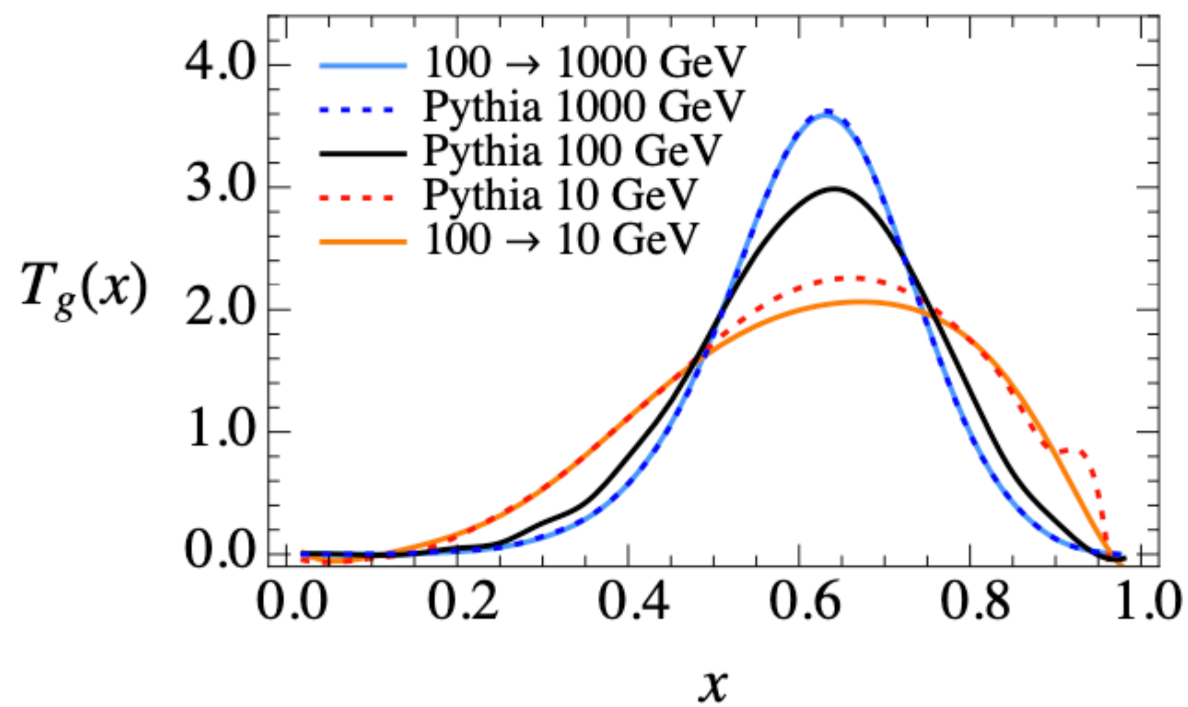
- Partonic track function is scaleless in dim. reg., but needed:
  - $1/\epsilon_{\rm IR}$ cancels against IR pole in partonic cross section $\mathrm{d}\sigma_N$
  - $1/\epsilon_{\rm UV}$ is renormalized, leads to evolution of track function.
- At order α_s:

$$T_{i,\mathrm{bare}}^{(1)}(x) = \sum_j \int \mathrm{d}z \left[\frac{\alpha_s}{4\pi}\left(\frac{1}{\epsilon_{\rm UV}} - \frac{1}{\epsilon_{\rm IR}}\right) P_{ji}(z)\right] \int \mathrm{d}x_1\, T_j^{(0)}(x_1,\mu)$$
$$\times \int \mathrm{d}x_2\, T_k^{(0)}(x_2,\mu)\,\delta\big[x - z x_1 - (1-z)x_2\big]$$



1. Track function evolution at order α_s





- **Nonlinear** evolution:

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} T_i(x,\mu) = \sum_{j,k} \int \mathrm{d}z\, \frac{\alpha_s}{2\pi} P_{ji}(z) \int \mathrm{d}x_1\, T_j(x_1,\mu) \int \mathrm{d}x_2\, T_k(x_2,\mu) \times \delta[x - z x_1 - (1-z) x_2]$$

- **Consistent** with extraction from Pythia at different energies.

2. Energy correlators

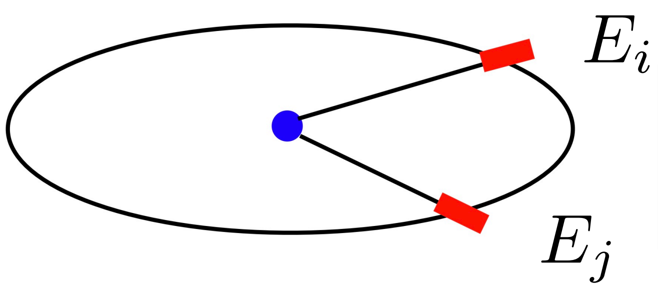
[arXiv:2108.01674 - Li, Moult, Schrijnder van Velzen, WW, Zhu]

2. Energy-Energy Correlator

- Weighted cross section in e+e- collisions

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\chi} = \sum_{i,j} \int \mathrm{d}\sigma \, \frac{E_i E_j}{Q^2} \, \delta(\cos\chi - \cos\theta_{ij})$$

[Basham, Brown, Ellis, Love]



- Tracks are essential to measure EEC at small angles.

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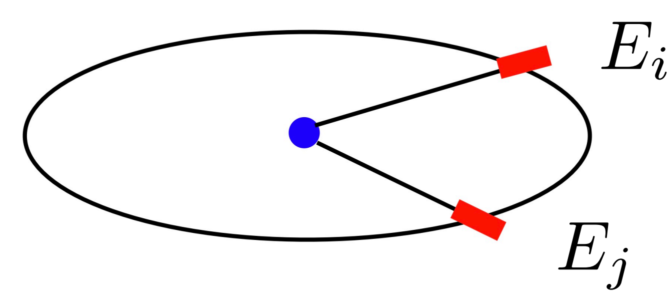


[Basham, Brown, Ellis, Love]

- Tracks are essential to measure EEC at small angles.

- Conversion to tracks is simple:

$$E_i \to \int \mathrm{d}x_i \, T_i(x_i) \, x_i E_i = T_i(1) E_i$$

[Chen, Moult, Zhang, Zhu]

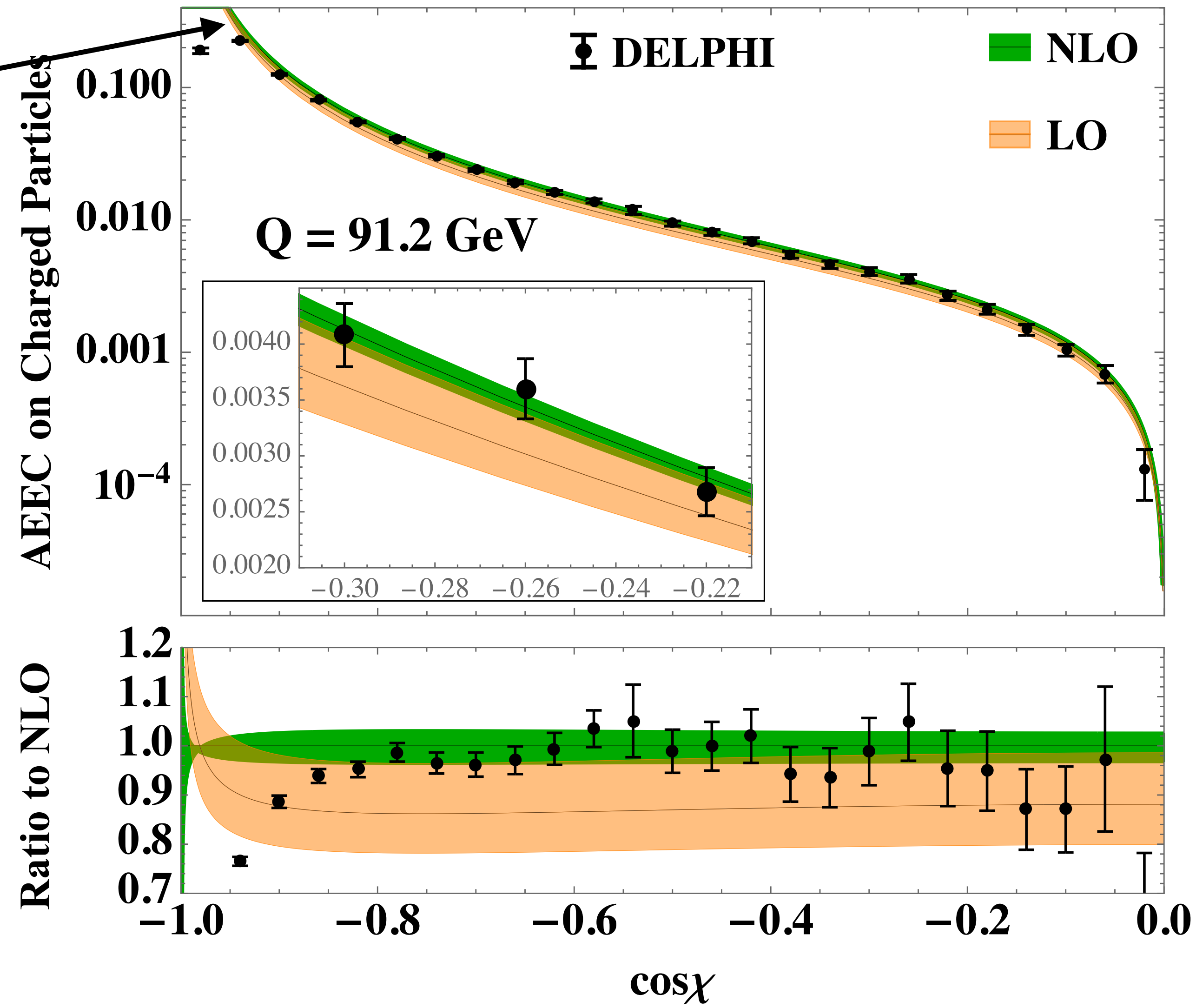
- Collinear limit $\chi = 0$ involves $T_i(2)$.

- Generalizing, *N*-point energy correlators involve at most the *N*th **moment** of track functions.

2. Results for track-based EEC

- First $\mathcal{O}(\alpha_s^2)$ result for track-based measurement:

resummation needed



$$\mathrm{AEEC}(\cos\chi) = \mathrm{EEC}(\cos\chi) - \mathrm{EEC}(-\cos\chi)$$

- Uncertainty reduced at NLO, tantalizing agreement with data.

3. Track functions at order α_s^2

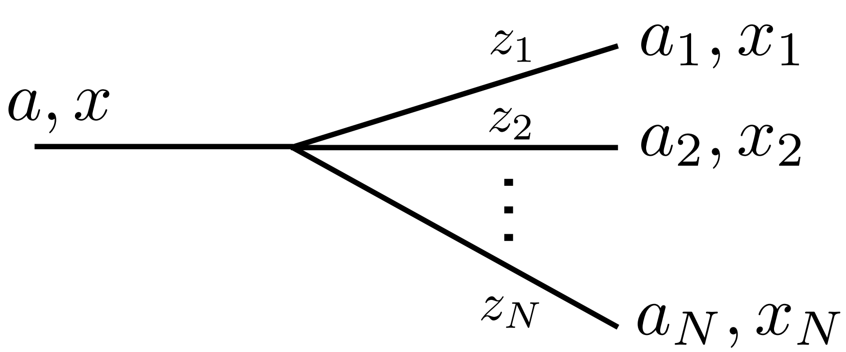
[arXiv:2108.01674 - Li, Moult, Schrijnder van Velzen, WW, Zhu,
arXiv:2201.05166 - Jaarsma, Li, Moult, WW, Zhu,
ongoing work - Chen, Jaarsma, Li, Moult, WW, Zhu]

3. Form of track function evolution

generalized splitting function

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2} T_a(x) = \sum_N \sum_{\{a_f\}} \left[\prod_{i=1}^N \int_0^1 \mathrm{d}z_i\right] \delta\Big(1 - \sum_{i=1}^N z_i\Big) P_{a\to\{a_f\}}(\{z_f\})$$

$$\times \left[\prod_{i=1}^N \int_0^1 \mathrm{d}x_i\, T_{a_i}(x_i)\right] \delta\Big(x - \sum_{i=1}^N z_i x_i\Big)$$



- At order α_s^{N-1} it includes up to $1 \to N$ splittings.

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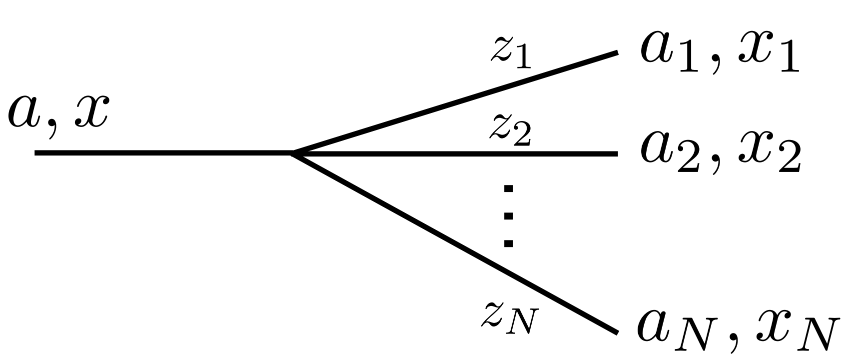
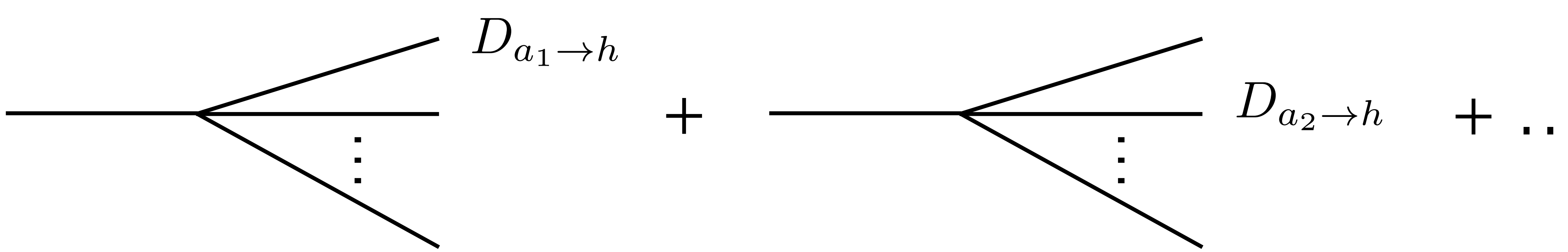


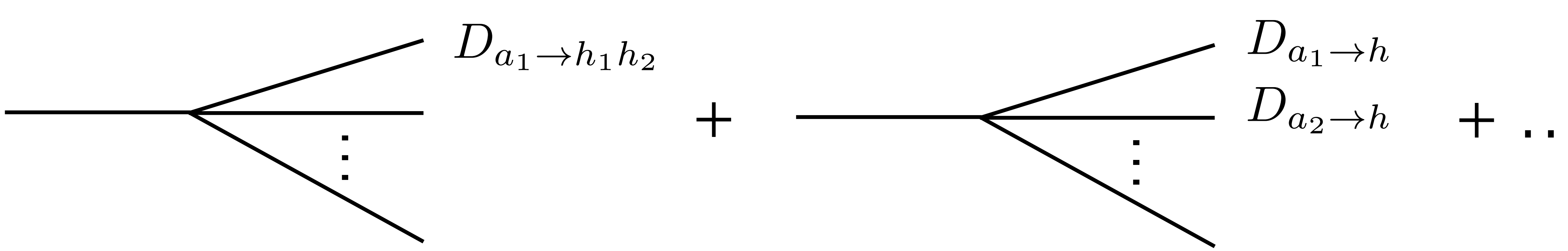
- At order α_s^{N-1} it includes up to $1 \to N$ splittings.
- Shift symmetry: $T_a(x) \to T_a(x+b),\ T_{a_i}(x) \to T_{a_i}(x+b)$ due to momentum conservation.

3. The most general collinear evolution

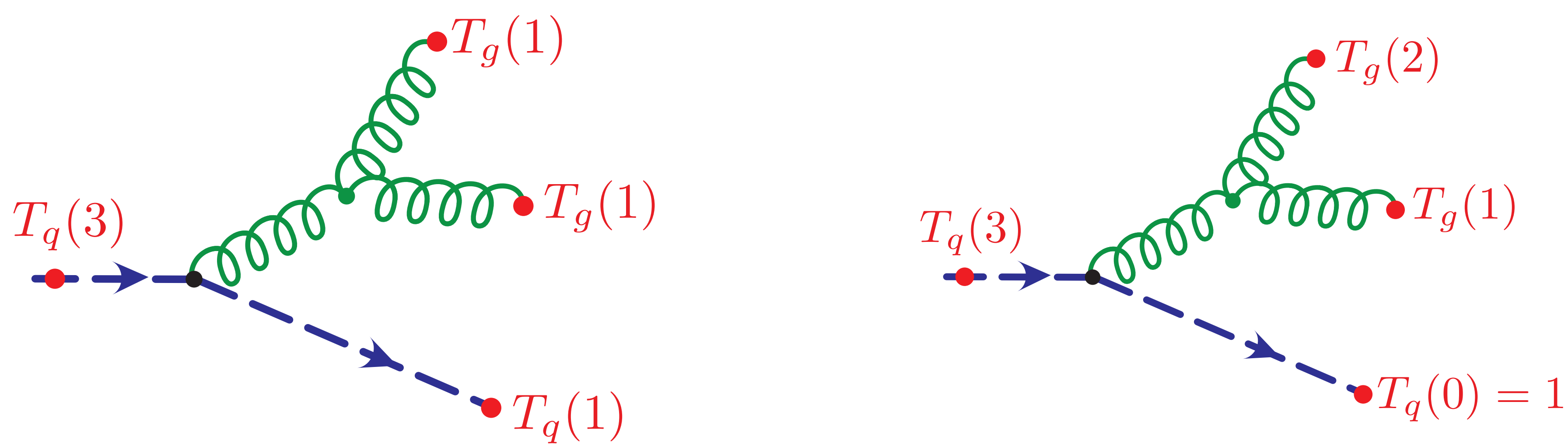
- Evolution of (multi-hadron) fragmentation functions follows from this equation, i.e. all information is in $P_{a \to \{a_f\}}(\{z_f\})$
  - Fragmentation function:



  - Dihadron fragmentation function:



3. Evolution of integer moments



- Taking **integer** moments and using multinomial expansion:

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2} T_a(n) = \sum_N \sum_{\{a_f\}} \sum_{\{m_f\}} \gamma_{a\to\{a_f\}}(\{m_f\}) \prod_{i=1}^{N} T_{a_i}(m_i)$$

  - Here $n = m_1 + \cdots + m_N$
  - For the diagonal case $N = 1, n = m_1$, the anomalous dimension $\gamma_{a\to a_1}$ is the DGLAP splitting function.

3. Shift symmetry of moment space evolution

- Shift symmetry implies invariance of the evolution under

$$T_a(0) \to T_a(0) = 1\,, \qquad T_a(1) \to T_a(1) - b\,,$$
$$T_a(2) \to T_a(2) - 2bT_a(1) + b^2\,, \quad \ldots$$

- Make manifest by using shift-invariant central moments

$$\Delta = T_q(1) - T_g(1), \quad \sigma_a(2) = T_a(2) - T_a(1)^2, \quad \ldots$$

- E.g. for one quark flavor, evolution of first two moments reads:

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2}\Delta = [-\gamma_{qq}(2) - \gamma_{gg}(2)]\Delta$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2}\begin{pmatrix}\sigma_g(2)\\ \sigma_q(2)\end{pmatrix} = \begin{pmatrix}-\gamma_{gg}(3) & -\gamma_{qg}(3)\\ -\gamma_{gq}(3) & -\gamma_{qq}(3)\end{pmatrix}\begin{pmatrix}\sigma_g(2)\\ \sigma_q(2)\end{pmatrix} + \begin{pmatrix}\gamma_g^{\Delta^2}\\ \gamma_q^{\Delta^2}\end{pmatrix}\Delta^2$$

3. Shift symmetry for pure Yang-Mills

- At all orders:

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2}T(1) = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2}\sigma(2) = -\gamma(3)\sigma(2)$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2}\sigma(3) = -\gamma(4)\sigma(3)$$

- At $\mathcal{O}(\alpha_s^2)$:

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2}\sigma(4) = -\gamma(5)\sigma(4) + [-6\gamma(3) + 8\gamma(4) - 3\gamma(5)]\sigma^2(2)$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2}\sigma(5) = -\gamma(6)\sigma(5) + [-10\gamma(3) + 10\gamma(4) - 2\gamma(6)]\sigma(2)\sigma(3)$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2}\sigma(6) = &-\gamma(7)\sigma(6) + [-15\gamma(3) + 40\gamma(4) - 60\gamma(5) + 48\gamma(6) - 15\gamma(7)]\sigma^3(2) \\ &+ [-15\gamma(3) + 20\gamma(4) - 15\gamma(5) + 12\gamma(6) - 5\gamma(7)]\sigma^2(3) + c(\sigma(2)\sigma(4) - \sigma^3(2) - \sigma^2(3))\end{aligned}$$

first time we need to calculate something for pure Yang-Mills!

3. Track function calculation at α_s^2

- Use track jet function $J_i(s, x)$ differential in invariant mass s of **all** particles and momentum fraction x of **charged** particles.
  - Calculate by integrating NLO $1 \to 2$ and LO $1 \to 3$ splitting functions [Kosower, Uwer; Ritzmann, WW]

3. Track function calculation at α_s^2

- Use track jet function $J_i(s,x)$ differential in invariant mass s of **all** particles and momentum fraction x of **charged** particles.
  - Calculate by integrating NLO $1 \to 2$ and LO $1 \to 3$ splitting functions [Kosower, Uwer; Ritzmann, WW]
  - Consistency of factorization in SCET implies same UV poles as invariant mass jet function $J_i(s)$ [Becher, Neubert; Becher, Bell]
  - After renormalization, remaining $1/\epsilon$ poles are infrared and cancel when matching onto track functions

$$J_i^{(2)} = \underbrace{T_i^{(2)}}_{\text{goal}} + \underbrace{\sum_j \mathcal{J}_{i\to jk}^{(1)} \otimes [T_j^{(1)} T_k^{(0)}]}_{\text{known from } \mathcal{O}(\alpha_s)} + \underbrace{\sum_{j,k} \mathcal{J}_{i\to jk\ell}^{(2)} \otimes [T_j^{(0)} T_k^{(0)} T_\ell^{(0)}]}_{\text{finite, not needed}}$$

3. Results for moment space evolution at α_s^2

- Central moments:

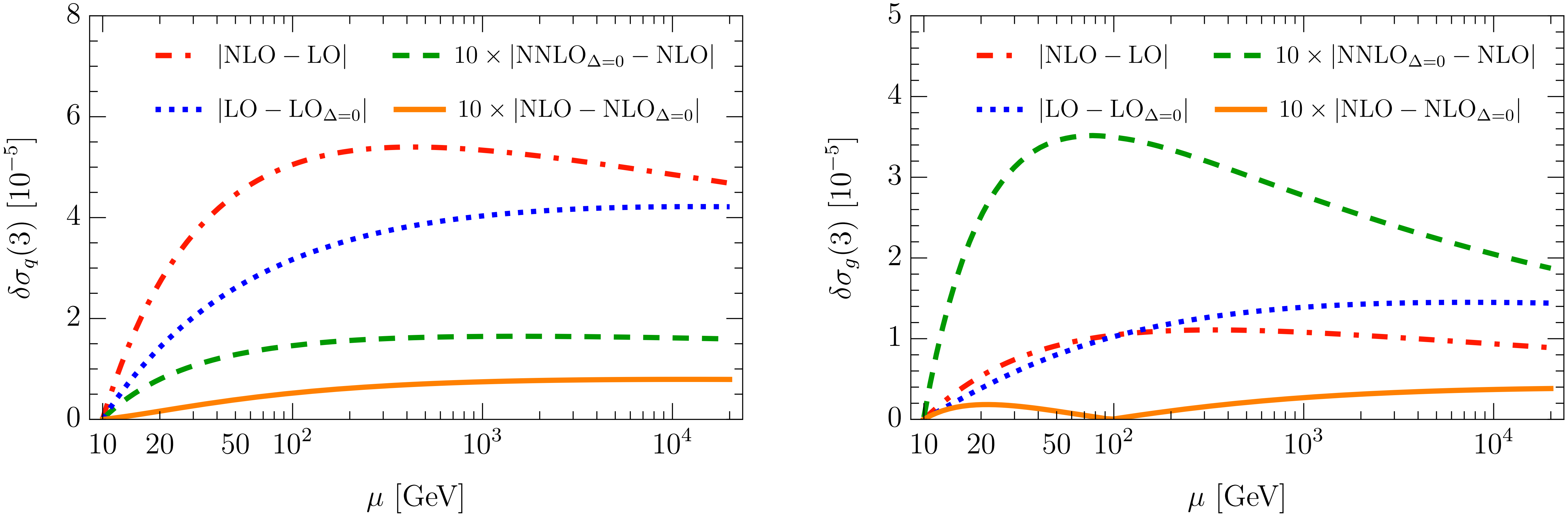
$$\left.\frac{\mathrm{d}\sigma_g(2)}{\mathrm{d}\ln\mu^2}\right|_{\alpha_s^2} = -\gamma_{gg}^{(1)}(3)\sigma_g(2) + \sum_i \Bigg\{ -\gamma_{qg}^{(1)}(3)\left(\sigma_{q_i}(2) + \sigma_{\bar{q}_i}(2) + \Delta_{q_i}^2 + \Delta_{\bar{q}_i}^2\right)$$
$$+ T_F\left[\left(\frac{12413}{1350} - \frac{52}{45}\pi^2\right)C_A + \frac{1528}{225}C_F - \frac{16}{25}n_f T_F\right]\Delta_{q_i}\Delta_{\bar{q}_i}\Bigg\}$$

$$\left.\frac{\mathrm{d}\sigma_g(3)}{\mathrm{d}\ln\mu^2}\right|_{\alpha_s^2} = -\gamma_{gg}^{(1)}(4)\sigma_g(3) + \sum_i \Bigg\{ -\gamma_{qg}^{(1)}(4)\left(\sigma_{q_i}(3) + 3\sigma_{q_i}(2)\Delta_{q_i} + \Delta_{q_i}^3\right)$$
$$+ T_F\left[\left(-\frac{638}{45} + \frac{8}{3}\pi^2\right)C_A - \frac{3803}{250}C_F\right]\sigma_g(2)\Delta_{q_i}$$
$$+ T_F\left[\left(\frac{5321}{3000} - \frac{2}{5}\pi^2\right)C_A + \frac{1523}{240}C_F - \frac{12}{25}n_f T_F\right]\left(\sigma_{q_i}(2)\Delta_{\bar{q}_i} + \Delta_{q_i}^2\Delta_{\bar{q}_i}\right) + (q \leftrightarrow \bar{q})\Bigg\}$$

- Regular moments do not make shift symmetry manifest:

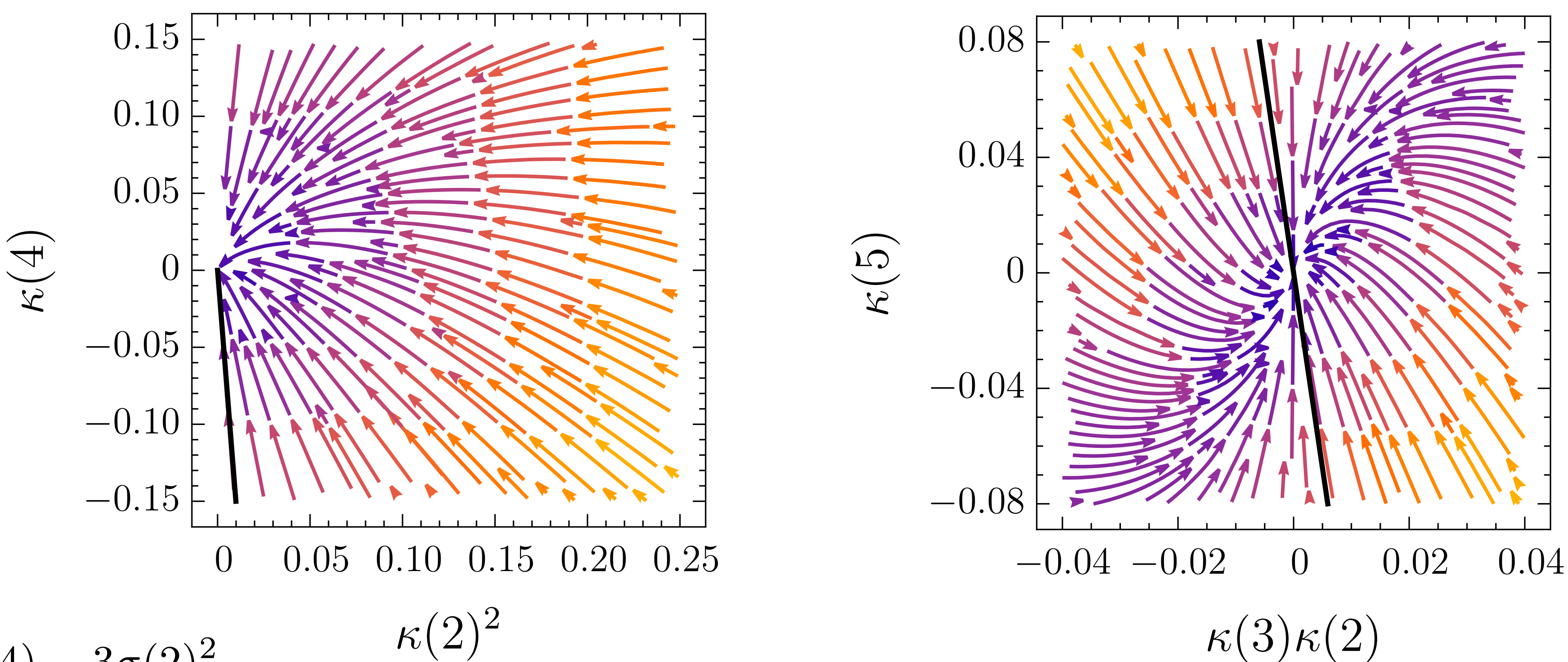
$$\left.\frac{\mathrm{d}T_g(2)}{\mathrm{d}\ln\mu^2}\right|_{\alpha_s^2} = -\gamma_{gg}^{(1)}(3)T_g(2) - \sum_i \gamma_{qg}^{(1)}(3)\left(T_{q_i}(2) + T_{\bar{q}_i}(2)\right)$$
$$+ \left[C_A^2\left(-8\zeta_3 + \frac{26}{45}\pi^2 + \frac{2158}{675}\right) - \frac{4}{9}C_A n_f T_F\right] T_g(1)T_g(1)$$
$$+ \sum_i \left[T_F\left(-\frac{299}{225}C_A - \frac{4387}{900}C_F\right)\right] T_g(1)\left(T_{q_i}(1) + T_{\bar{q}_i}(1)\right)$$
$$+ \sum_i T_F\left[\left(\frac{12413}{1350} - \frac{52}{45}\pi^2\right)C_A + \frac{1528}{225}C_F - \frac{16}{25}n_f T_F\right] T_{q_i}(1)T_{\bar{q}_i}(1)$$

3. Smallness of Δ in QCD



- Up to the third moment all nonlinear terms in the evolution involve Δ.
- Δ terms are suppressed at least one order in perturbation theory → NNLO evolution for first three moments.

3. Nonlinearities



$$\kappa(4) = \sigma(4) - 3\sigma(2)^2$$
$$\kappa(5) = \sigma(5) - 10\sigma(3)\sigma(2)$$

- Evolution of higher moments involve nonlinear terms without Δ.
- Easiest visualized for pure Yang-Mills theory.

3. Results in *x*-space

- Main challenge is disentangling singularities in momentum fractions → use sector decomposition.

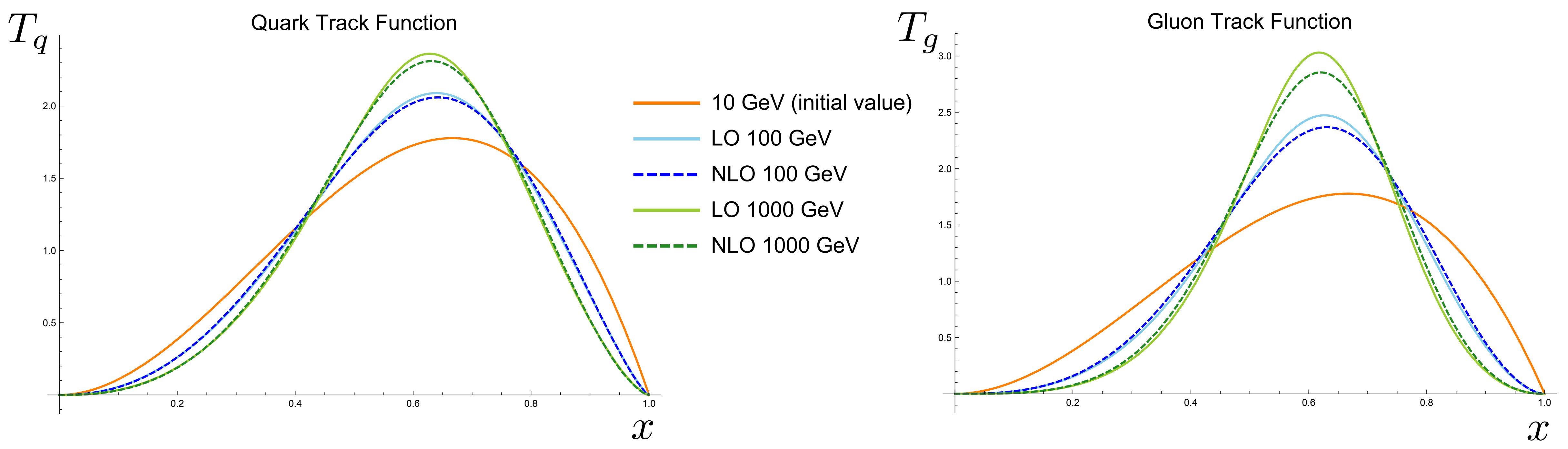
- For $\mathcal{N} = 4$ Super Yang-Mills:

$$
\begin{aligned}
\left.\frac{\mathrm{d}T(x)}{\mathrm{d}\ln\mu^2}\right|_{a^2} = & \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \int_0^1 \mathrm{d}z\ T(x_1)T(x_2)\ \delta\left(x - x_1\frac{1}{1+z} - x_2\frac{z}{1+z}\right) \\
& \times \left\{-25\zeta_3\ \delta(z) + \frac{8}{3}\pi^2\left[\frac{1}{z}\right]_+ + \frac{32\ln^2(z+1)}{z} - \frac{16\ln(z)\ln(z+1)}{z}\right\} \\
& + \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \int_0^1 \mathrm{d}x_3 \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}t\ T(x_1)T(x_2)T(x_3) \\
& \times \delta\left(x - x_1\frac{1}{1+z+zt} - x_2\frac{z}{1+z+zt} - x_3\frac{zt}{1+z+zt}\right) \\
& \times 8\left\{\frac{4\ln(1+z)}{z}\left[\frac{1}{t}\right]_+ + \left[\frac{1}{z}\right]_+\left(4\left[\frac{\ln t}{t}\right]_+ - \frac{\ln t}{1+t} - \frac{7\ln(1+t)}{t}\right)\right. \\
& \left. - \frac{\ln t}{(1+t)(1+tz)} + \frac{\ln(1+t)}{(1+t)(1+z)} + \frac{\ln(1+t)}{(1+t)(1+tz)} - \frac{\ln(1+z)}{(1+t)z} + \ldots\right\}
\end{aligned}
$$

- Checked against moment-space results.

3. Results in *x*-space for QCD

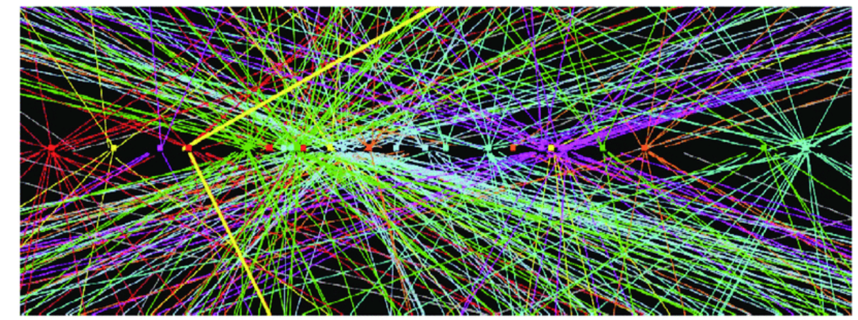
- We have results in *x*-space for full QCD, with two numerical implementations (based on moments or Fourier series)



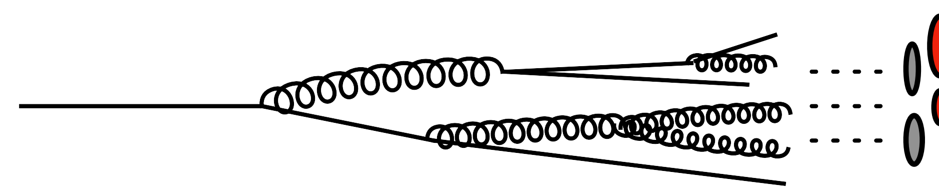
- Using toy initial condition $T_i(x, \mu = 10\mathrm{GeV}) = 12x^2(1-x)$

4. Conclusions and outlook

- Tracks are appealing because of their superior angular resolution and reduced effect of pile-up.



- Track functions offer systematically improvable framework.



$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2}\Delta = [-\gamma_{qq}(2) - \gamma_{gg}(2)]\Delta$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2}\begin{pmatrix}\sigma_g(2)\\ \sigma_q(2)\end{pmatrix} = \begin{pmatrix}-\gamma_{gg}(3) & -\gamma_{qg}(3)\\ -\gamma_{gq}(3) & -\gamma_{qq}(3)\end{pmatrix}\begin{pmatrix}\sigma_g(2)\\ \sigma_q(2)\end{pmatrix} + \begin{pmatrix}\gamma_g^{\Delta^2}\\ \gamma_q^{\Delta^2}\end{pmatrix}\Delta^2$$

- Track function evolution is most general collinear evolution equation.

- Formalism extended to α_s^2 → high precision possible!