THE ENERGY-MOMENTUM TENSOR IN QCD

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The Energy-Momentum Tensor

Energy Density Momentum Density

$$T^{\mu\nu} = \begin{array}{cccc} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{array}$$

shear forces

pressure

Energy Flux Momentum Flux

$$\langle N(p')|T^{\mu\nu}_{q,g}|N(p)\rangle$$

- Where does the spin of the proton come from?
- What are the mechanical properties (pressure, shear forces) inside the proton ?
- What is the origin of the proton mass?

Outline

- Definition of the QCD Energy-Momentum Tensor (EMT)
- Angular momentum density: definition and interpretation
- Parametrization of the proton off-forward matrix elements of the EMT: form factors
- D-term: calculation using t-channel dispersion relations
- Forward matrix elements and proton mass decomposition

Canonical Energy Momentum Tensor

Emmy Noether (1882-1935)

If a system has a continuous symmetry property,
then there are corresponding quantities whose values are conserved in time

Translation invariance $\longrightarrow$ Conservation of the canonical EMT $T_C^{\mu\nu}(x)$

Lorentz invariance $\longrightarrow$ Conservation of the generalized Angular Momentum (AM) density $J_C^{\mu\alpha\beta}(x)$

$$J_C^{\mu\alpha\beta}(x) = L_C^{\mu\alpha\beta} + S_C^{\mu\alpha\beta} \qquad L_C^{\mu\alpha\beta}(x) = x^\alpha T_C^{\mu\beta}(x) - x^\beta T_C^{\mu\alpha}(x)$$

Space components: $J_C^i(x) = \frac{1}{2}\epsilon^{ijk} J_C^{0jk}(x)$

$$\vec{J}_C = \underbrace{\vec{L}_C}_{\text{Orbital AM}} + \underbrace{\vec{S}_C}_{\text{Spin}}$$

$T_C^{\mu\nu}$ is in general neither gauge-invariant nor symmetric

Belinfante improved EMT

$$T^{\mu\nu}_{\text{Bel}}(x) = T^{\mu\nu}_C(x) + \partial_\lambda G^{\lambda\mu\nu}(x)$$

Belinfante generalized AM

$$J^{\mu\alpha\beta}_{\text{Bel}}(x) = J^{\mu\alpha\beta}_C(x) + \partial_\lambda[x^\alpha G^{\lambda\mu\beta}(x) - x^\beta G^{\lambda\mu\alpha}(x)]$$

with the super-potential

$$G^{\lambda\mu\nu}(x) = \frac{1}{2}[S^{\lambda\mu\nu}_C(x) + S^{\mu\nu\lambda}_C(x) + S^{\nu\mu\lambda}_C(x)] = -G^{\mu\lambda\nu}(x)$$

$$\Downarrow$$

$$J^{\mu\alpha\beta}_{\text{Bel}}(x) = x^\alpha T^{\mu\beta}_{\text{Bel}}(x) - x^\beta T^{\mu\alpha}_{\text{Bel}}(x)$$

Belinfante, Rosenfeld (1940)

Canonical ↔ Belinfante

Canonical ⟷ Belinfante

in general not symmetric

$$T_C^{[\mu\nu]}(x) = -\partial_\alpha S^{\alpha\mu\nu}(x) \neq 0$$

$$[\mu\nu] = \mu\nu - \nu\mu$$

symmetric

$$T_{\mathrm{Bel}}^{[\mu\nu]}(x) = 0$$

Canonical $\longleftrightarrow$ Belinfante

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symmetric

$$T_{\mathrm{Bel}}^{[\mu\nu]}(x) = 0$$

clear distinction between OAM and spin
at the density level

$$J_C^{\mu\alpha\beta}(x) = L_C^{\mu\alpha\beta}(x) + S_C^{\mu\alpha\beta}(x)$$

$$L_C^{\mu\alpha\beta}(x) = x^\alpha T_C^{\mu\beta}(x) - x^\beta T_C^{\mu\alpha}(x)$$

purely OAM density

$$J_{\mathrm{Bel}}^{\mu\alpha\beta}(x) = x^\alpha T_{\mathrm{Bel}}^{\mu\beta}(x) - x^\beta T_{\mathrm{Bel}}^{\mu\alpha}(x)$$

Canonical ⟷ Belinfante

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purely OAM density

$$J_{\text{Bel}}^{\mu\alpha\beta}(x) = x^\alpha T_{\text{Bel}}^{\mu\beta}(x) - x^\beta T_{\text{Bel}}^{\mu\alpha}(x)$$

The definition at the density level is modified, while the total charge does not change

$$\int T_C^{\mu\nu}(x)\,dx = \int T_{\text{Bel}}^{\mu\nu}(x)\,dx$$

$$\int J_C^{\mu\alpha\beta}(x)\,dx = \int J_{\text{Bel}}^{\mu\alpha\beta}(x)\,dx$$

Kinetic EMT in QCD

Ji, 1997

$$T^{\mu\nu}_{\text{kin}}(x) = T^{\mu\nu}_{\text{kin},q}(x) + T^{\mu\nu}_{\text{kin},g}$$

Gluon contribution: $T^{\mu\nu}_{\text{kin},g}(x) = T^{\mu\nu}_{\text{Bel},g}(x)$

Quark contribution: $T^{\mu\nu}_{\text{kin},q}(x) = \frac{1}{2}\bar{\psi}(x)\gamma^\mu i\overleftrightarrow{D}^\nu\psi(x)$ $\quad D^\mu = \partial^\mu + igA^\mu$

$$\frac{1}{2}T^{\{\mu\nu\}}_{\text{kin},q}(x) = T^{\mu\nu}_{\text{Bel},q}(x) \qquad \frac{1}{2}T^{[\mu\nu]}_{\text{kin},q}(x) = -\partial_\lambda S^{\lambda\mu\nu}_q(x)$$

$$S^{\lambda\mu\nu}_q(x) = \frac{1}{2}\epsilon^{\lambda\mu\nu\alpha}\bar{\psi}(x)\gamma_\alpha\gamma_5\psi(x)$$

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Kinetic generalized AM

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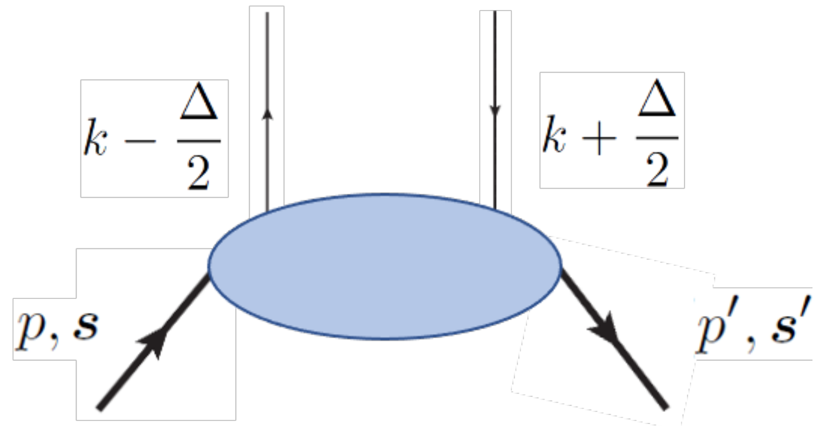
$$\downarrow \qquad\qquad \downarrow$$

$$\vec{J}_{\text{kin},q}(x) = \vec{L}_{\text{kin},q}(x) + \vec{S}_q(x) \qquad \vec{J}_{\text{Bel},q}(x) = \vec{J}_{\text{kin},q}(x) + \vec{M}_q(x)$$ different distributions

$$\int \vec{J}_{\text{Bel},q}(x)\,dx = \int \vec{J}_{\text{kin},q}(x)\,dx$$ equal total charge

Form Factors of the EMT



$$\Delta = p' - p$$

$$t = \Delta^2$$

$$P = \frac{p + p'}{2}$$

Belinfante-Rosenfeld EMT

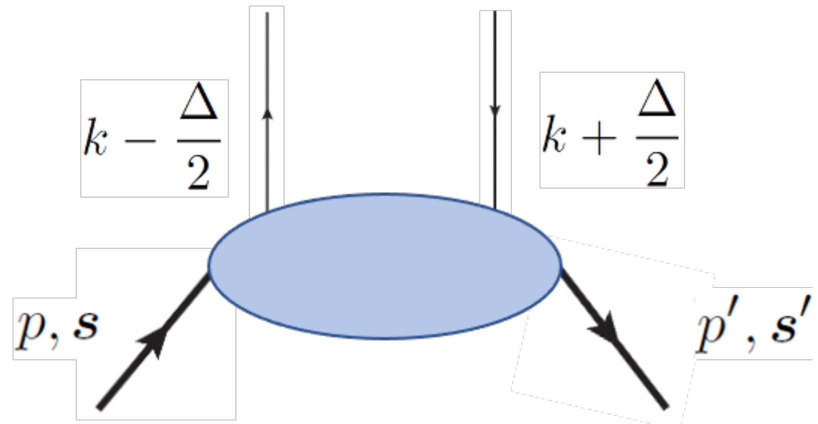
$$\langle p', s' | T^{\mu\nu}_{\mathrm{Bel},a}(0) | p, s\rangle = \bar{u}(p', s')\, \Gamma^{\mu\nu}_{\mathrm{Bel},a}(P, \Delta)\, u(p, s)$$

$$\Gamma^{\mu\nu}_{\mathrm{Bel},a}(P, \Delta) = \frac{P^{\{\mu}\gamma^{\nu\}}}{2} A_a(t) + \frac{P^{\{\mu} i\sigma^{\nu\}\Delta}}{4M} B_a(t) + \frac{\Delta^\mu \Delta^\nu - g^{\mu\nu}\Delta^2}{4M} D_a(t) + M g^{\mu\nu} \bar{C}_a(t)$$

$(a = q, g)$

Bakker et al. (2004)

Form Factors of the EMT



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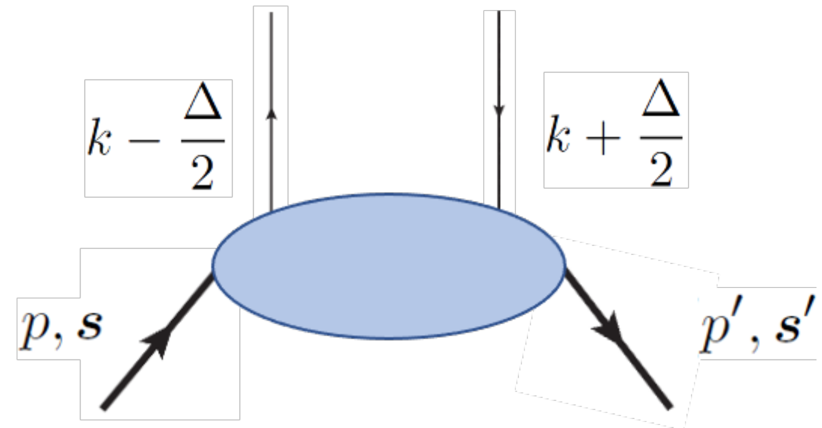
$$\Gamma^{\mu\nu}_{\mathrm{kin},a}(P,\Delta) = \Gamma^{\mu\nu}_{\mathrm{Bel},a}(P,\Delta) + \frac{P^{[\mu}\gamma^{\nu]}}{2} C_a(t)$$

$$C_g(t) = 0$$

Ji (1997)

Form Factors of the EMT



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$(a = q, g)$

Bakker et al. (2004)

Kinetic EMT

$$\langle p', S' | T^{\mu\nu}_{\text{kin},a}(0) | p, S \rangle = \bar{u}(p', S')\, \Gamma^{\mu\nu}_{\text{kin},a}(P, \Delta)\, u(p, S)$$

$$\Gamma^{\mu\nu}_{\text{kin},a}(P, \Delta) = \Gamma^{\mu\nu}_{\text{Bel},a}(P, \Delta) + \frac{P^{[\mu}\gamma^{\nu]}}{2} C_a(t) \qquad C_g(t) = 0$$

Angular momentum relation

$$J^a_z = \frac{1}{2}(A_a + B_a)$$

$$J^q_z = L^q_{\text{kin},z} + S^q_{\text{kin},z}$$

$$L^q_{\text{kin},z} = \frac{1}{2}(A_q + B_q + C_q)$$

$$S^q_{\text{kin},z} = -\frac{1}{2} C_q$$

Ji (1997)

Form Factors of the quark spin operator

$$S_q^{\lambda\mu\nu}(x) = \frac{1}{2}\epsilon^{\lambda\mu\nu\alpha}\bar{\psi}(x)\gamma_\alpha\gamma_5\psi(x)$$

$$\langle p', S'|S_q^{\lambda\mu\nu}(0)|p, S\rangle = \frac{1}{2}\epsilon^{\lambda\mu\nu\alpha}\bar{u}(p', s')\Big[\gamma_\alpha\gamma_5\, G_A^q(t) + \frac{\Delta_\alpha\gamma_5}{2M}\, G_P^q(t)\Big]u(p, s)$$

$G_A^q(t)$ axial form factor

$G_P^q(t)$ pseudoscalar form factor

QCD equation of motion $\quad \frac{1}{2}T_{\mathrm{kin},q}^{[\mu\nu]}(x) = -\partial_\lambda S_q^{\lambda\mu\nu}(x)$

$$\Downarrow$$

$$C_q(t) = -G_A^q(t)$$

Lorcé, Mantovani, Pasquini, Phys. Lett. B776, 38 (2018)

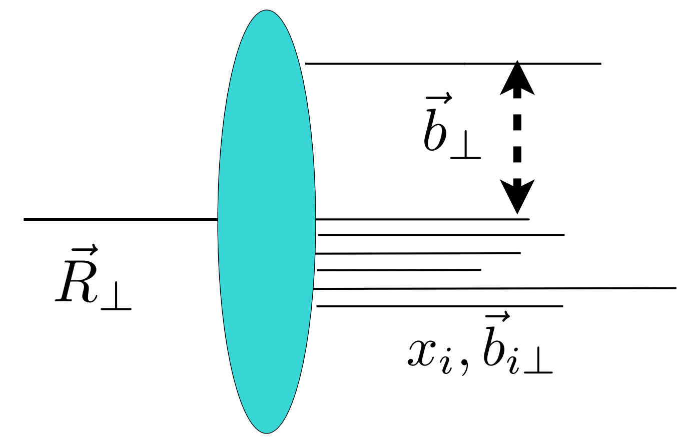
AM Distribution in the impact parameter space

Drell-Yan frame

$$\Delta^+ = 0 \quad \vec{\Delta}_\perp \neq 0$$

impact parameter space

$$\vec{\Delta}_\perp \xleftrightarrow{\text{FT}} \vec{b}_\perp$$



$$J_z(\vec{b}_\perp) = J_z^{\text{naive}}(\vec{b}_\perp) + J_z^{\text{corr}}(\vec{b}_\perp)$$

$$J_z^{\text{naive}}(\vec{b}_\perp) = \int \frac{d\vec{\Delta}_\perp}{(2\pi)^2} e^{-i\vec{\Delta}_\perp\cdot\vec{b}_\perp} J_z(t = -\vec{\Delta}_\perp^2)$$

$$\frac{1}{2}(A(t) + B(t))$$

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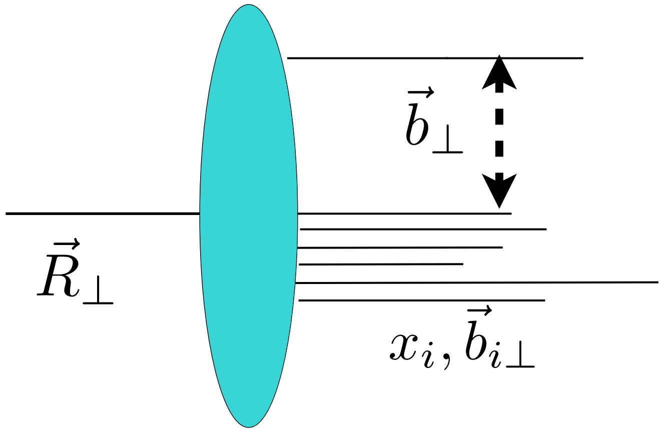
$$\Delta^+ = 0 \quad \vec{\Delta}_\perp \neq 0$$

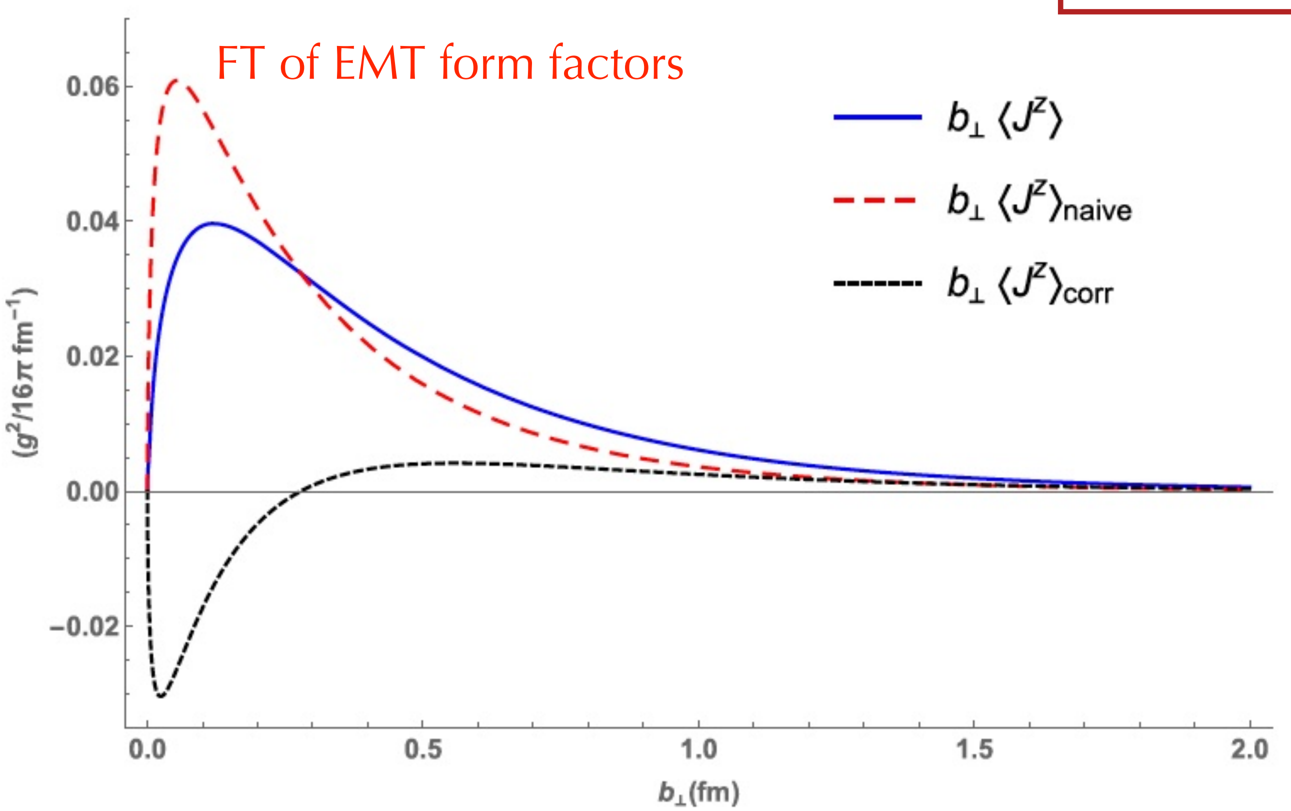
impact parameter space

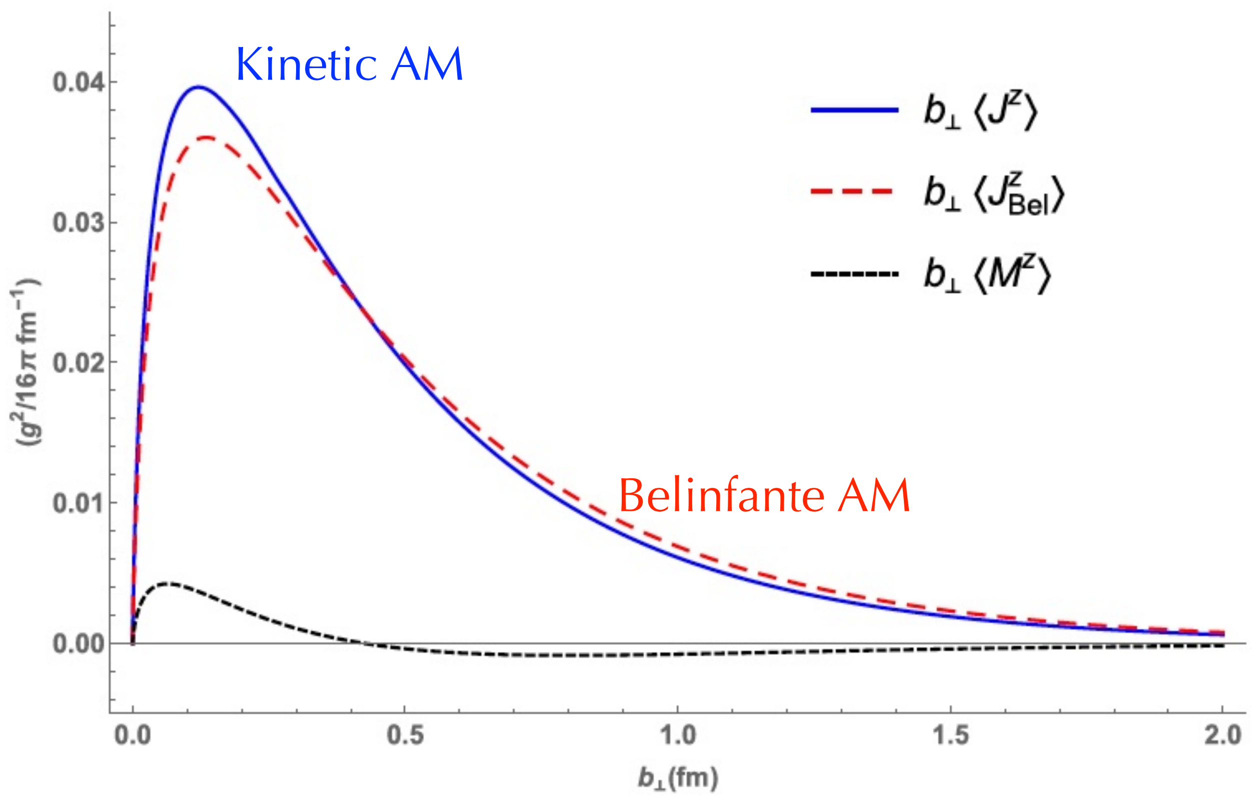
$$\vec{\Delta}_\perp \overset{\text{FT}}{\longleftrightarrow} \vec{b}_\perp$$



$$J_z(\vec{b}_\perp) = J_z^{\text{naive}}(\vec{b}_\perp) + J_z^{\text{corr}}(\vec{b}_\perp) \qquad J_z^{\text{naive}}(\vec{b}_\perp) = \int \frac{d\vec{\Delta}_\perp}{(2\pi)^2} e^{-i\vec{\Delta}_\perp \cdot \vec{b}_\perp} \underbrace{J_z(t = -\vec{\Delta}_\perp^2)}_{\frac{1}{2}(A(t)+B(t))}$$

Scalar diquark model

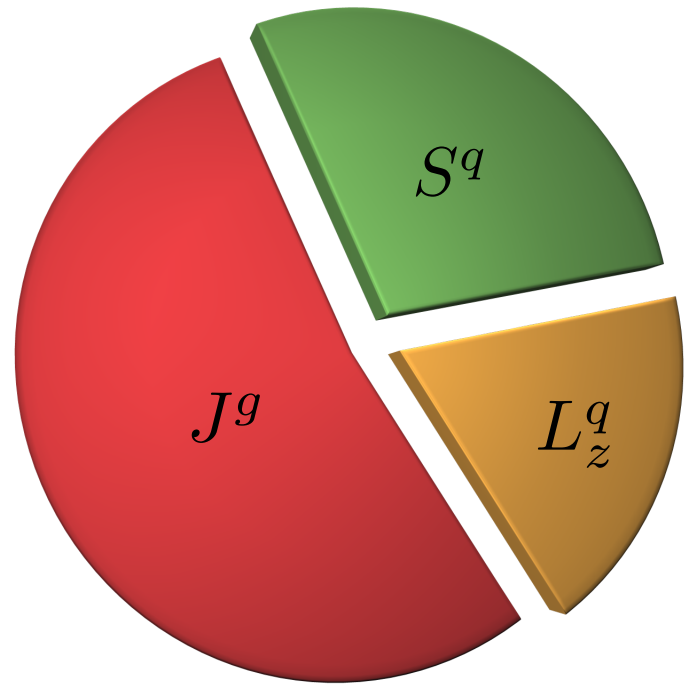




vanishing gauge field $\longrightarrow$ kinetic OAM = canonical OAM

Lorcé, Mantovani, Pasquini, Phys. Lett. B776, 38 (2018)

Link to generalized parton distributions



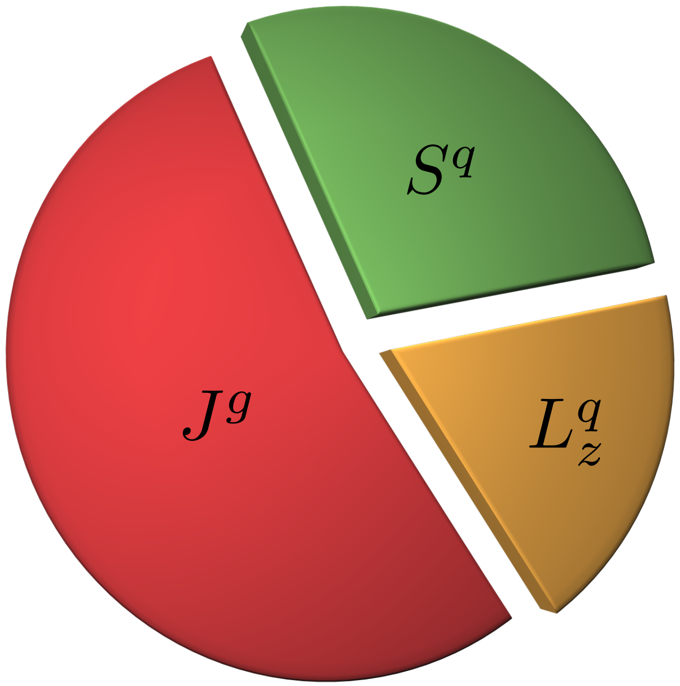
Ji's sum rule: $$J^{q,g}(t=0) = \frac{1}{2}\int_{-1}^{1} dx\, x\, \left(H^{q,g}(x,\xi,0) + E^{q,g}(x,\xi,0)\right) \qquad \left(\xi = -\frac{\Delta^+}{2P^+}\right)$$

at $\xi = 0$ unpolarized PDF — not directly accessible

$$J^q = L^q + S^q \qquad\qquad S^q = \frac{1}{2}\int_0^1 dx\, g_1(x)$$

- Requires extrapolation at t=0
- Requires spanning x at fixed values of ξ ($\xi = 0$ is the most convenient)
- $J^q(x) \neq \frac{1}{2}x(H^q(x,0,0) + E^q(x,0,0)$ contribution from surface term

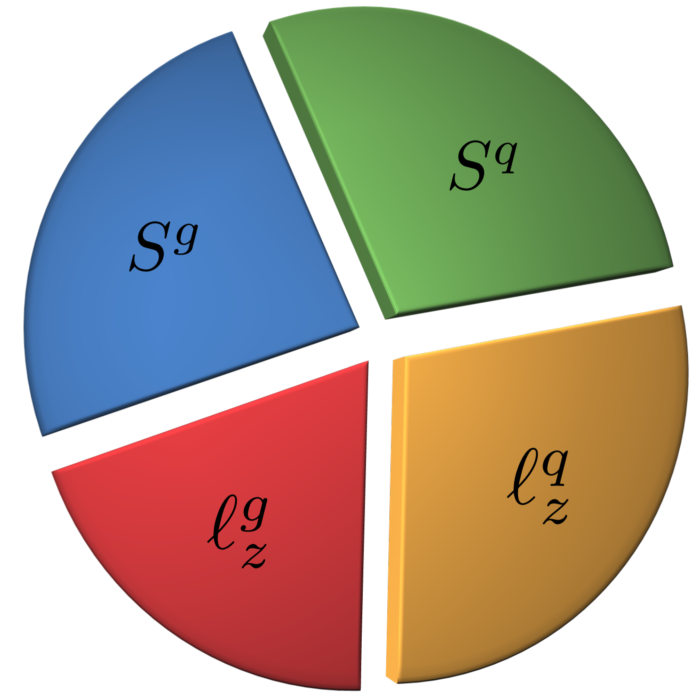
Ji (kinetic EMT) Sum Rule



$$\frac{1}{2} = \underbrace{S^q(\mu) + L_z^q(\mu)}_{J^q} + J^g(\mu)$$

- each term is gauge invariant
- frame independent
- it works also for the transverse AM in the infinite momentum frame
- J^q and J^g can be obtained from moments of GPDs

Jaffe-Manohar (canonical EMT) Sum Rule



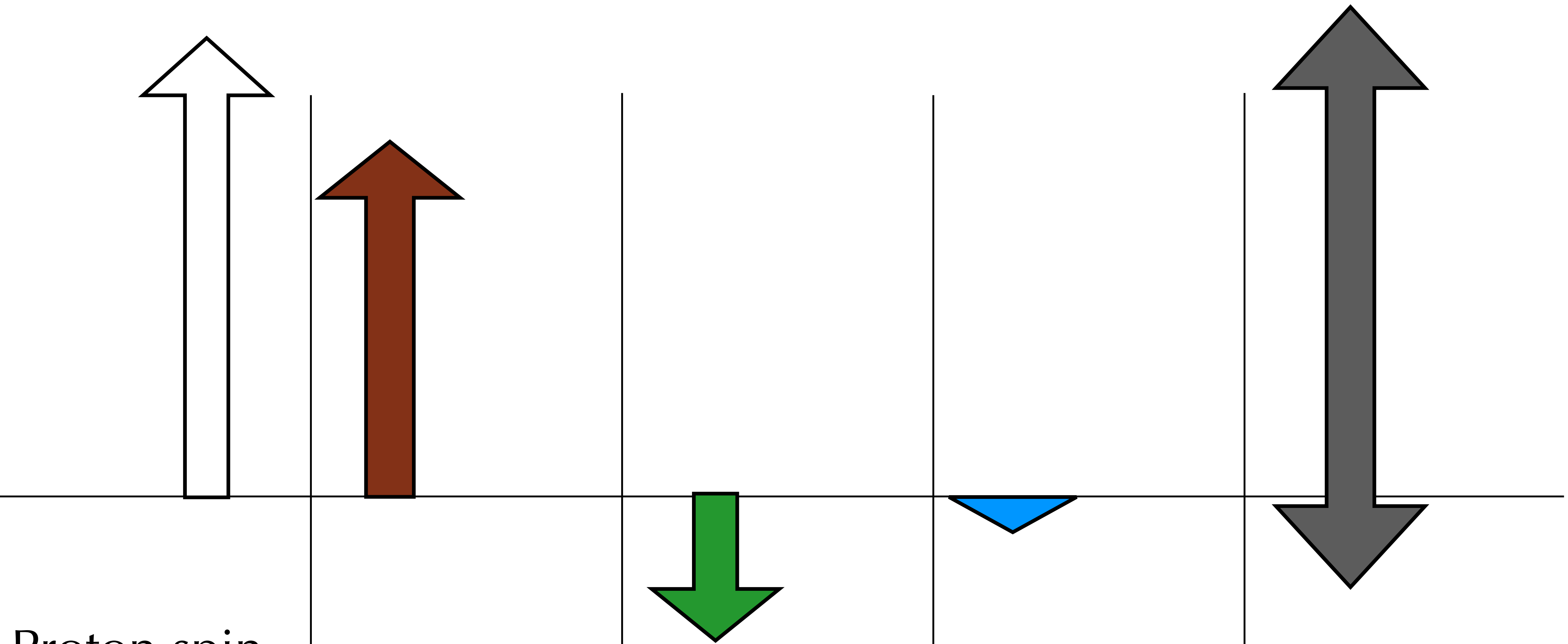
$$\frac{1}{2} = S^q(\mu) + \ell_z^q(\mu) + \ell_z^g(\mu) + S^g(\mu)$$

- ℓ_z^q, ℓ_z^g, S^g are gauge dependent, BUT measurable
- S^q, S^g can be obtained from pol. PDFs
- ℓ_z^q, ℓ_z^g can be obtained from twist-3 GPDs and Wigner distributions
- simple partonic interpretation in the IMF

$$\ell_z^q - l_z^{q,\mathrm{pot}} = L_z^q \longrightarrow \ell_z^g + S^g + l_z^{q,\mathrm{pot}} = J^g$$

Y. Guo, X. Ji, K. Shiells, NPB 96 (2021) 115440; X. Ji, F. Yuan, Y. Zhao, Nature Rev. Phys. 3 (2021) 1

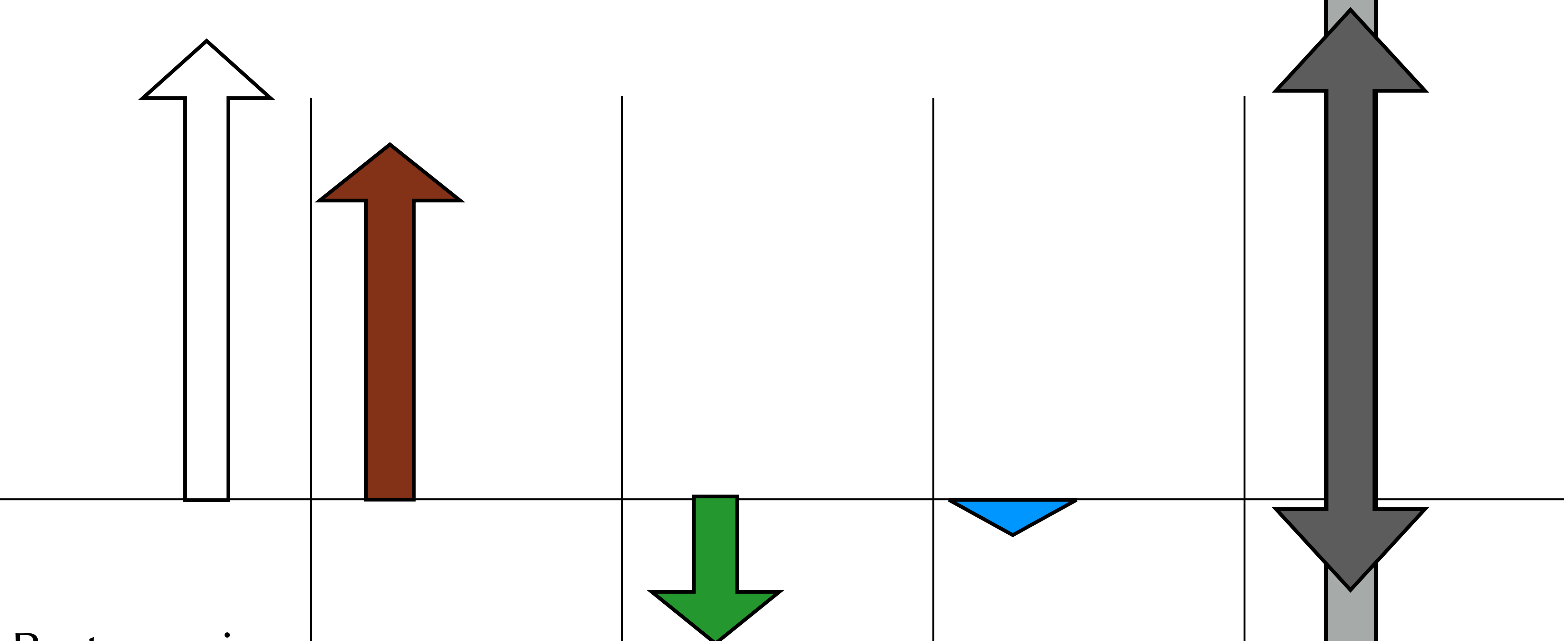
Status of Spin Sum Rule



Proton spin
(Q^2=10 GeV^2)

$u+\bar{u}$		$d+\bar{d}$		$s+\bar{s}$		g		
spin (%)	OAM	spin (%)	OAM	spin (%)	OAM	spin (%)	OAM	
81 ± 1		-43 ± 1		-11 ± 2		40 ± 60		DSSV14
79 ± 5		-45 ± 5		12 ± 8		3 ± 324		NNPDF

Status of Spin Sum Rule



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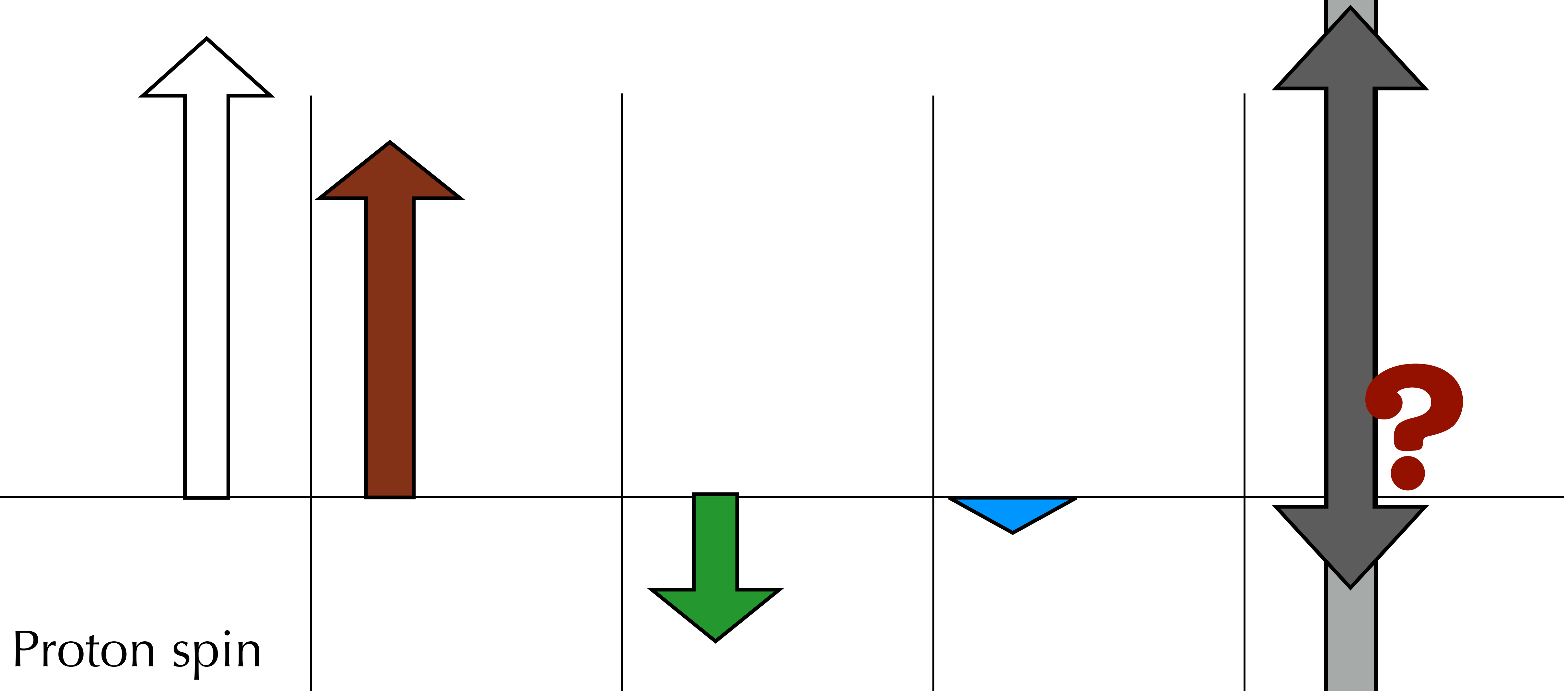

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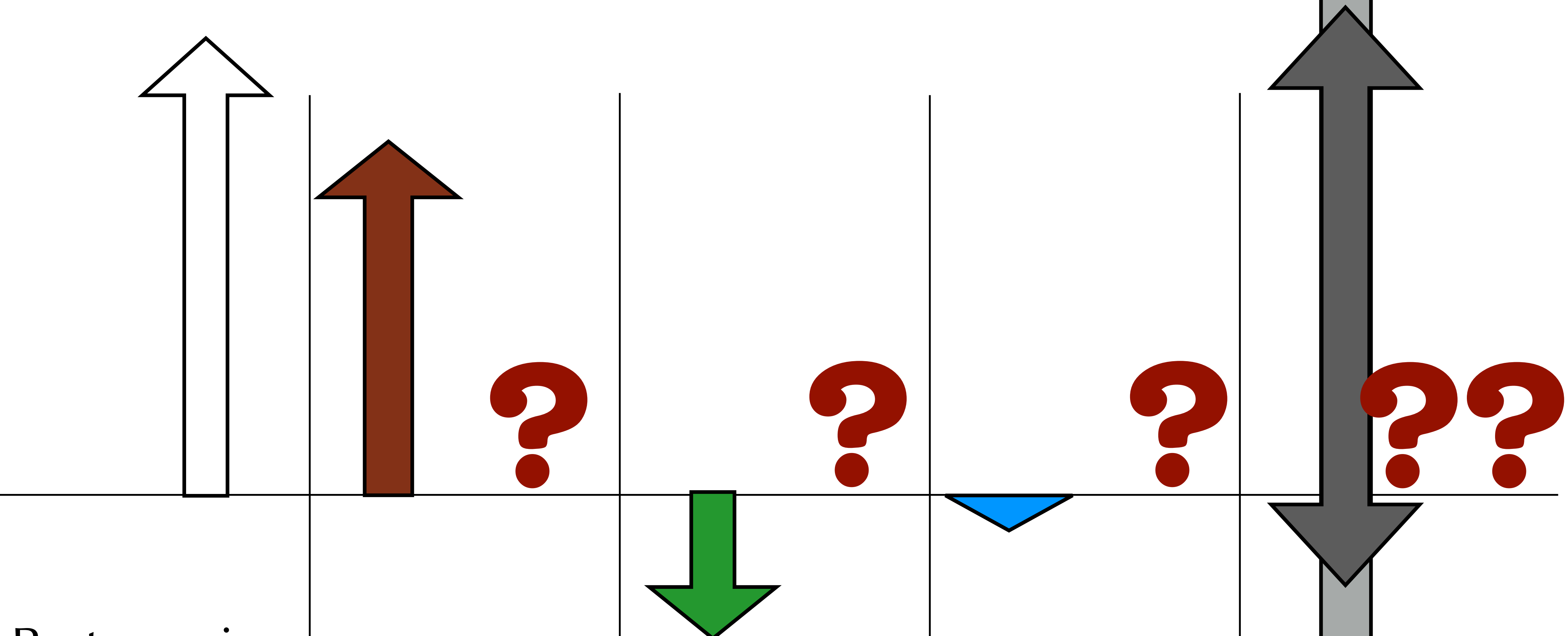

Proton spin
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81 ± 1		-43 ± 1		-11 ± 2		40 ± 60		DSSV14
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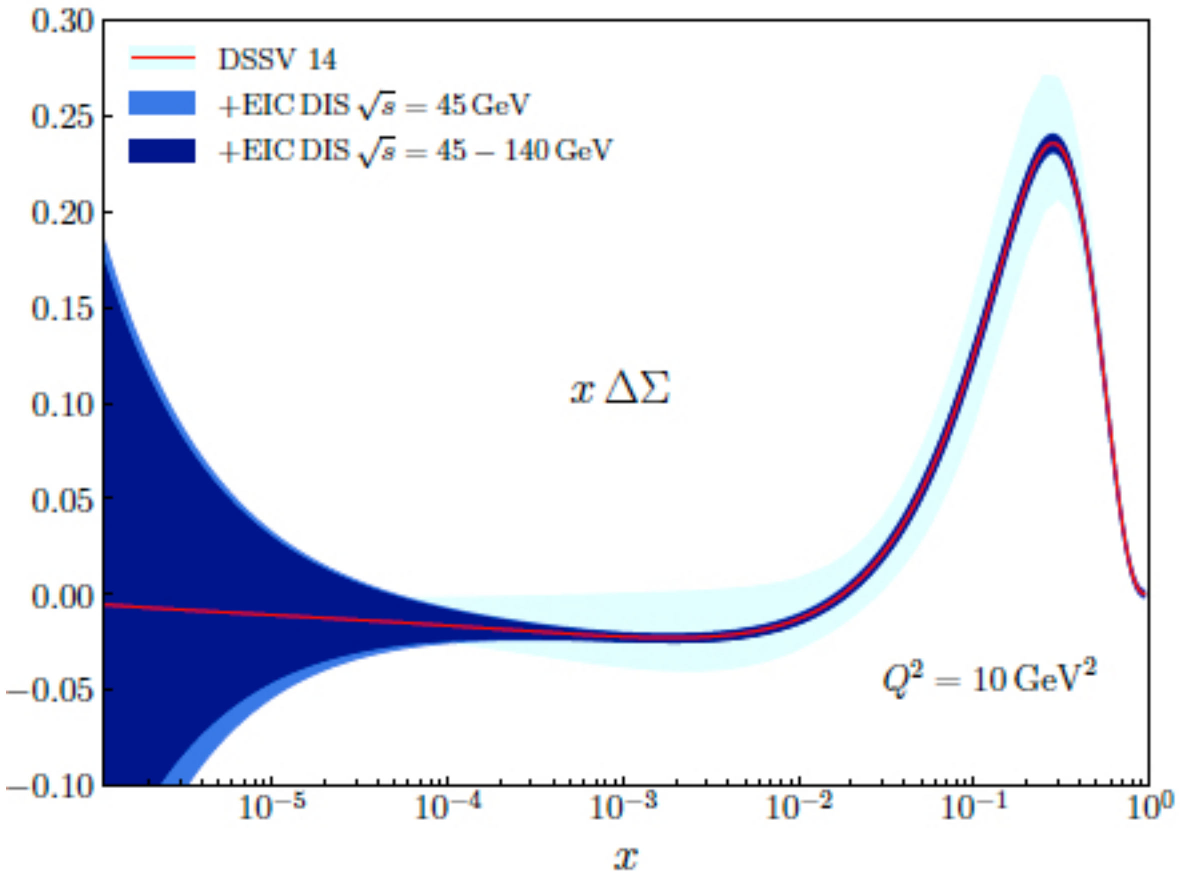
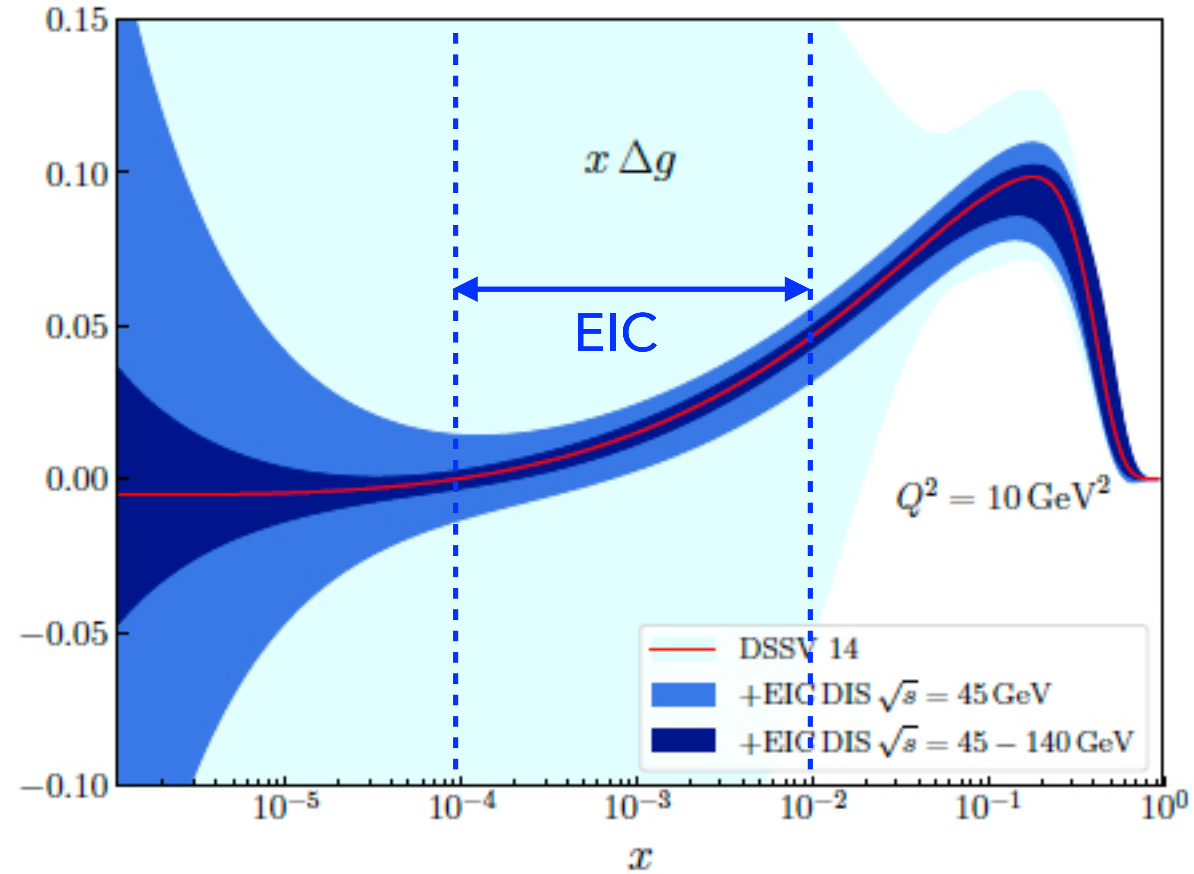
Impact of future EIC for quark and gluon spin contributions

Quark Spin



Gluon Spin



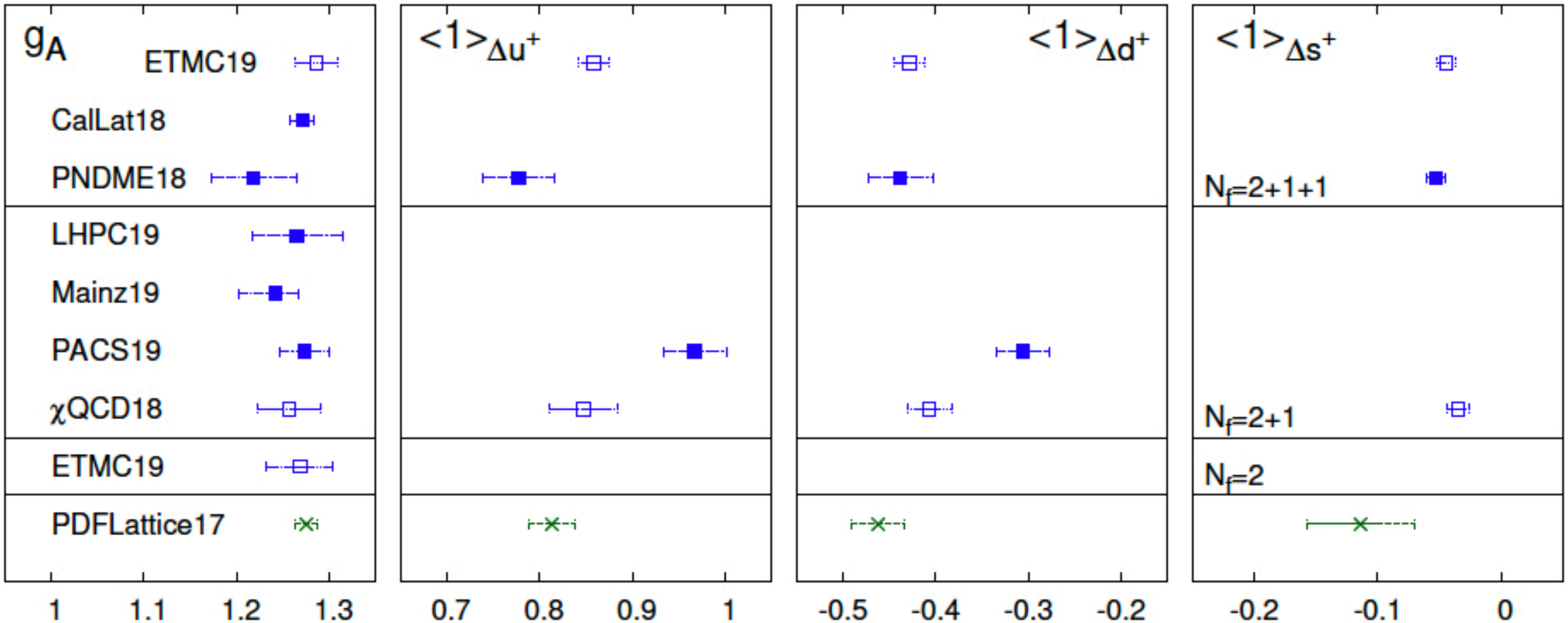
EIC Yellow Report: arXiv: 2103.05419

We are constantly improving the knowledge of the contributions to the spin of the nucleon

However the details on the flavor and sea contributions are still sketchy

Comparison with Lattice calculations

Quark spin $Q^2 = 4\ \mathrm{GeV}^2$



PDFLattice17: *average of phenomenological extractions: JAM15, NNPDFpol1.1, DSSV08 from the community white paper, Prog.Part.Nucl.Phys. 100 (2018) 107*

Overall fair agreement between lattice calculations and phenomenological fits

The uncertainties of the two have comparable size

Lattice QCD results could provide useful inputs to global fits of polarized PDFs

Orbital Angular momentum of the proton from available GPD measurements

$$J^{q,g} = \frac{1}{2}\int_{-1}^{1} dx\, x\, \left(H^{q,g}(x,\xi,0) + E^{q,g}(x,\xi,0)\right) \qquad L^q = J^q - S^q$$

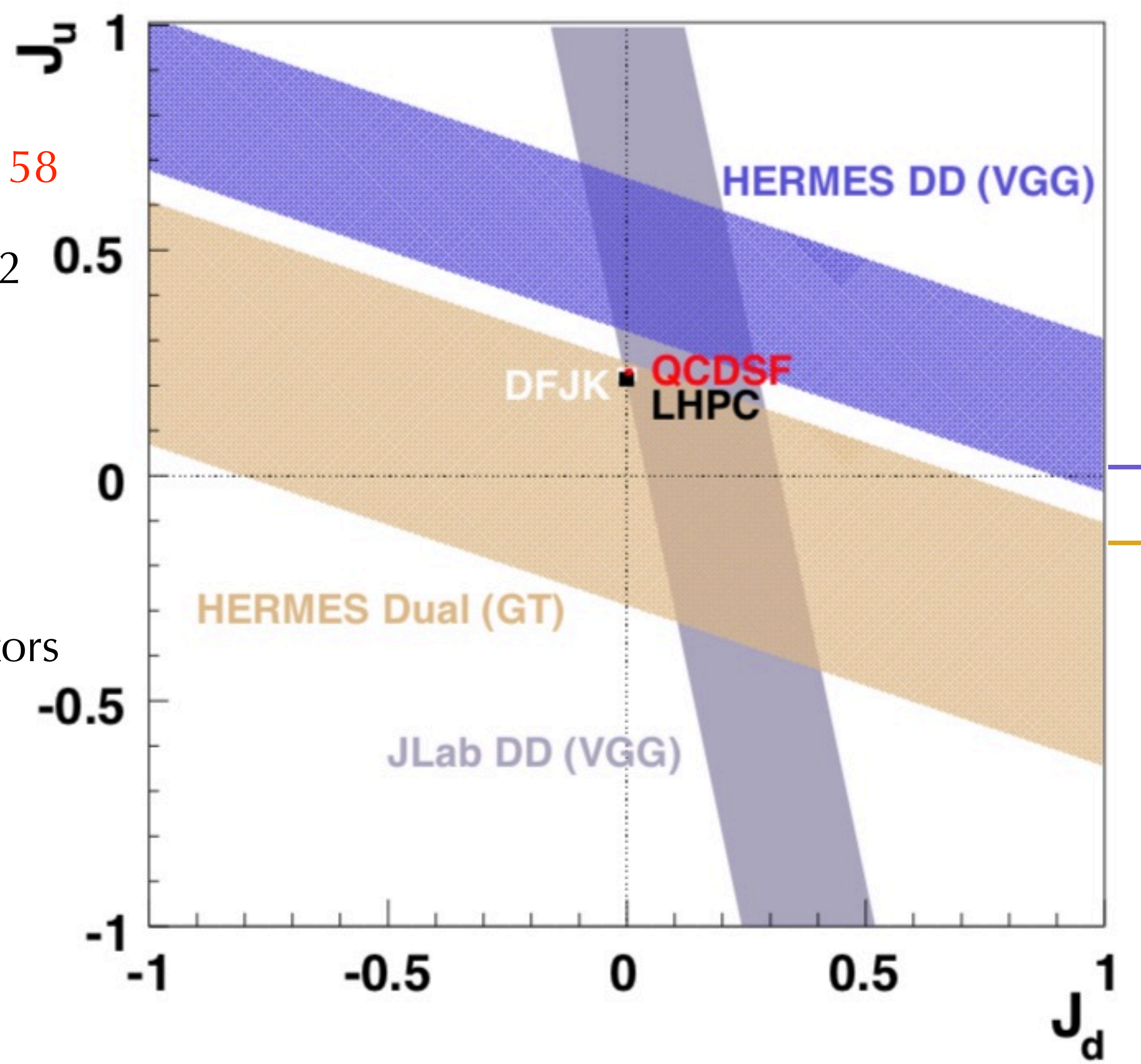
Lattice results

QCDSF: PoS (Lattice 2007) 158

LHPC: PRD77 (2008) 094502

GPDs extracted from form factors

DFJK, EPJC39 (2005) 1



extractions from HERMES data using two different models

JLab Hall A, Phys. Rev. Lett. 99 (2007) 242501

Hermes Coll., JHEP 06 (2008) 066

Improved accuracy with JLab12 and future EIC measurements!

The D-term form factor

$$\Gamma^{\mu\nu}_{\mathrm{Bel},a}(P,\Delta) = \frac{P^{\{\mu}\gamma^{\nu\}}}{2} A_a(t) + \frac{P^{\{\mu} i\sigma^{\nu\}\Delta}}{4M} B_a(t) + \frac{\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2}{4M} D_a(t) + M g^{\mu\nu} \bar{C}_a(t)$$

$A_q(0) + A_g(0) = 1$ quarks and gluons carry 100% of the nucleon momentum

$B_q(0) + B_g(0) = 0$ nucleon spin due to quarks + gluons: $J_q + J_g = \frac{1}{2}$

$\bar{C}_q(0) + \bar{C}_g(0) = 0$ total EMT is conserved

$D_q(0) + D_g(0) = D = \frac{4}{5} d_1$ D-term: unconstrained!

``last global unknown property"

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``last global unknown property"

D-term form factor from t-channel dispersion relations

BP, Polyakov, Vanderhaeghen, PLB 739 (2014) 133

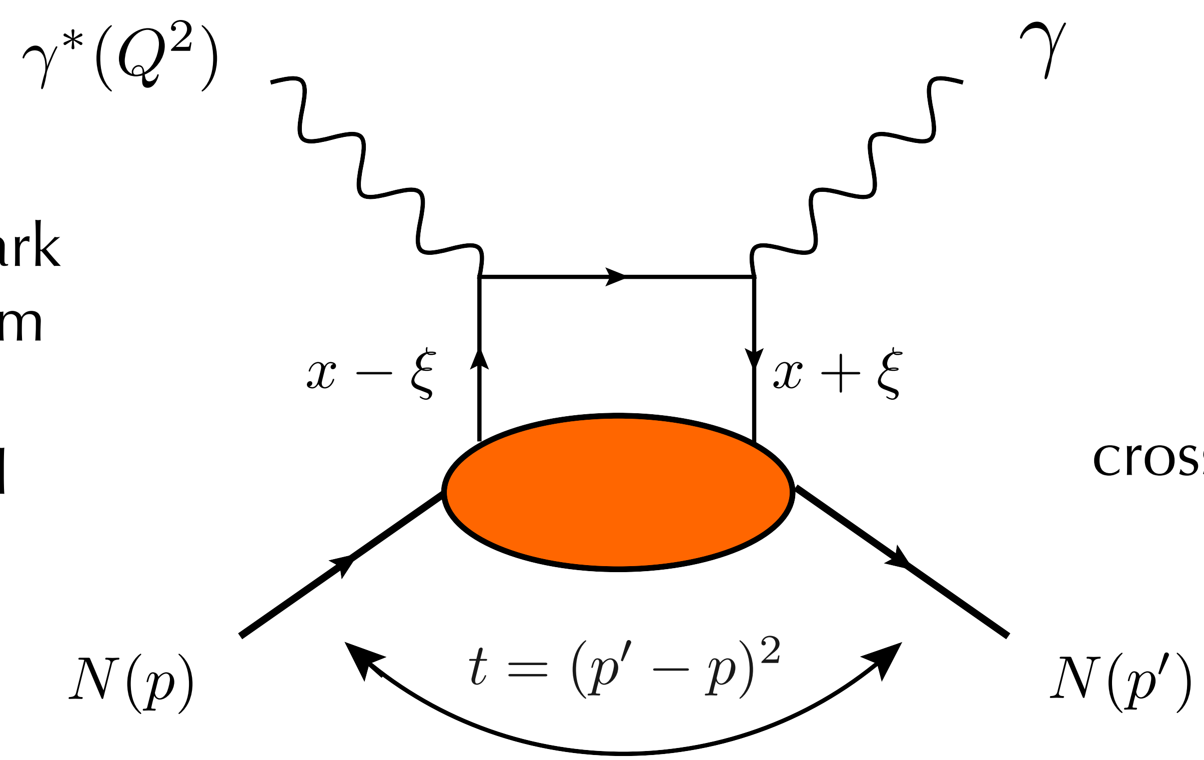
- DR representation based on analyticity and unitarity
- Use input from existing data of other processes $\longrightarrow$ reduced model dependence

DVCS at leading twist

x : average fraction of quark longitudinal momentum

ξ : fraction of longitudinal momentum transfer

t : nucleon momentum transfer



$t \ll \nu,\ Q^2$

$\downarrow$

crossing symmetric variable

$$\nu = \frac{s-u}{4M_N}$$

DVCS tensor at twist 2:

$$T^{\mu\nu} = \sum_{i=1}^{4} A_i(\nu, t, Q^2)\, O_i^{\mu\nu}$$

unpolarized quark

$$A_1 = \mathcal{H} + \mathcal{E}$$

$$A_2 = \mathcal{E}$$

long. polarized quark

$$A_3 = \tilde{\mathcal{H}}$$

$$A_4 = \tilde{\mathcal{E}}$$

Compton form factors: $\mathcal{F} = \int_0^1 \mathrm{d}x\, F^+(x,\xi,t,Q^2)\left[\frac{1}{x-\xi+i\epsilon} + \frac{1}{x+\xi-i\epsilon}\right]$ $\quad F = \left\{H,\ E,\ \tilde{H},\ \tilde{E}\right\}$

$\downarrow$

singlet GPDs: $F^+(x,\xi,t) = F(x,\xi,t) - F(-x,\xi,t)$

Dispersion Relations for DVCS amplitudes

- s-channel subtracted DRs:

$$\mathrm{Re}\, A_2(\nu, t, Q^2) = \Delta(t, Q^2) + \frac{2}{\pi}\nu^2 \mathcal{P} \int_{\nu_0}^{\infty} \mathrm{Im}\, A_2(\nu', t, Q^2) \frac{\mathrm{d}\nu'}{\nu'(\nu'^2 - \nu^2)}$$

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- t-channel DRs for subtraction function

$$\Delta(t, Q^2) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} \mathrm{d}t' \frac{\mathrm{Im}_t\, A_2(0, t', Q^2)}{t' - t} + \frac{1}{\pi}\int_{-\infty}^{-a} \mathrm{d}t' \frac{\mathrm{Im}_t\, A_2(0, t', Q^2)}{t' - t}$$

$$\downarrow$$

$$-a = -2(m_\pi^2 + 2M_N m_\pi) - Q^2$$

Dispersion Relations for DVCS amplitudes

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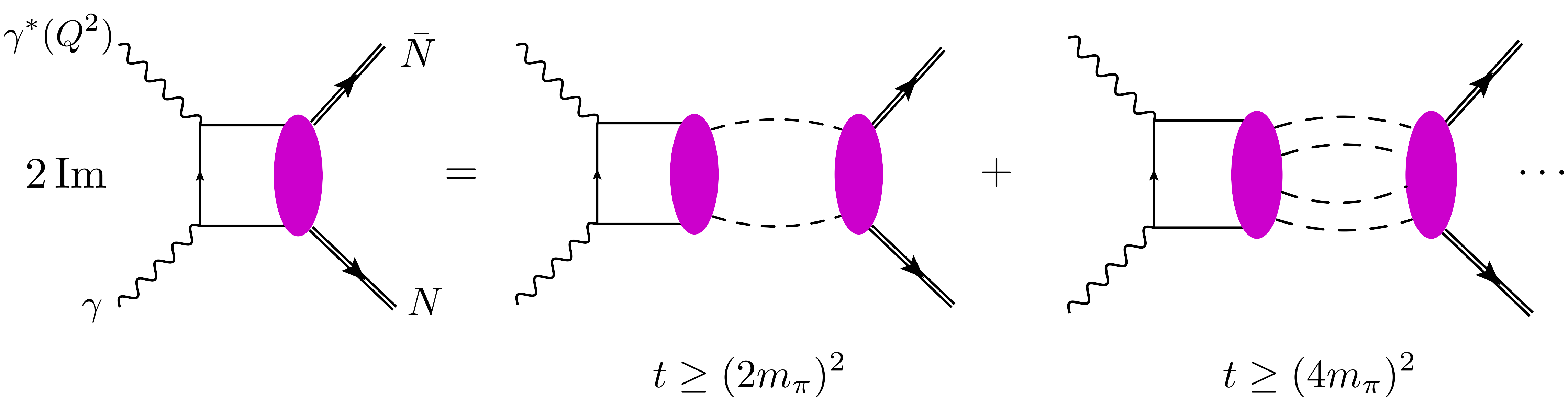
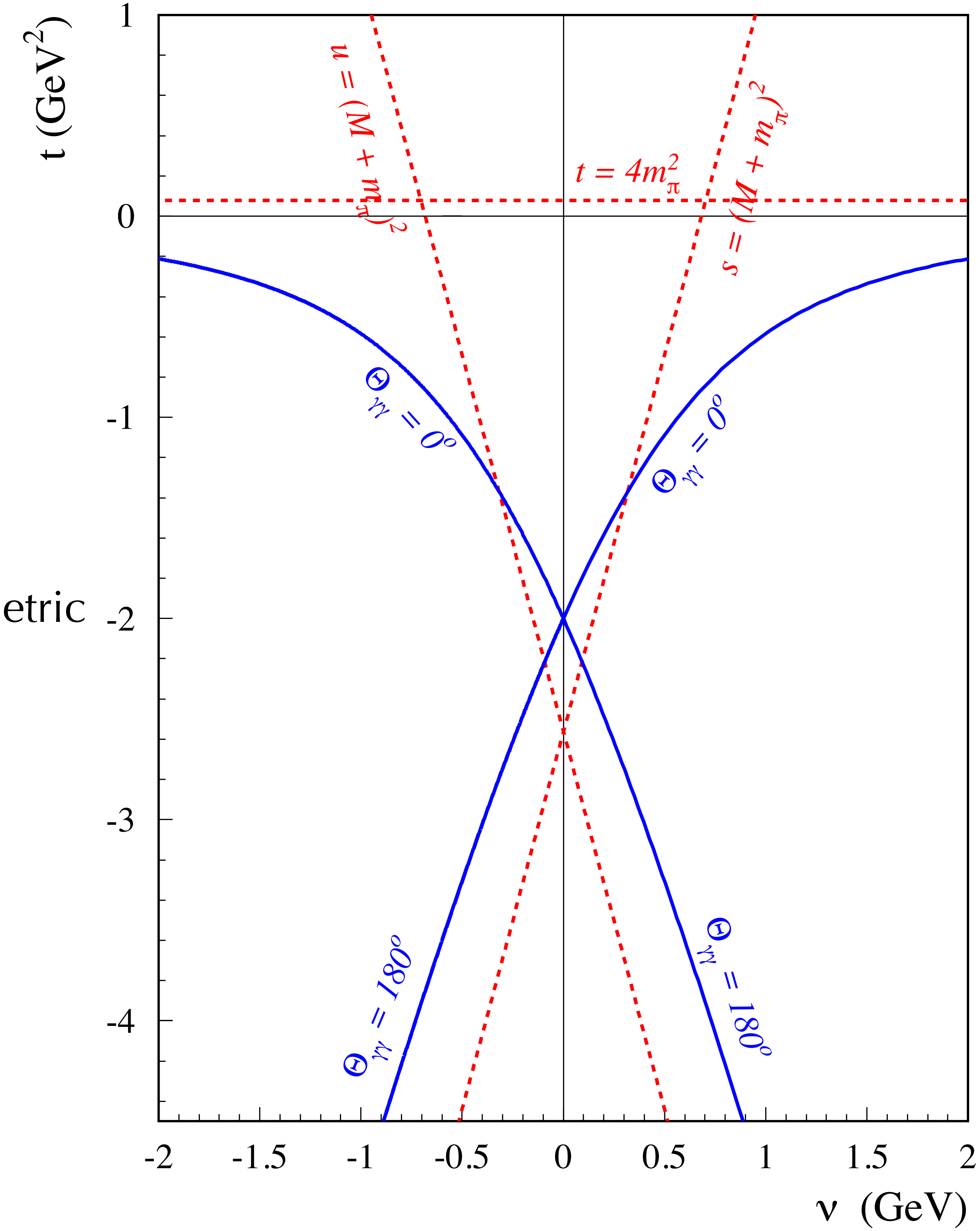
Unitarity relation in t-channel



Fixed

$Q^2 = -2\ \mathrm{GeV}^2$

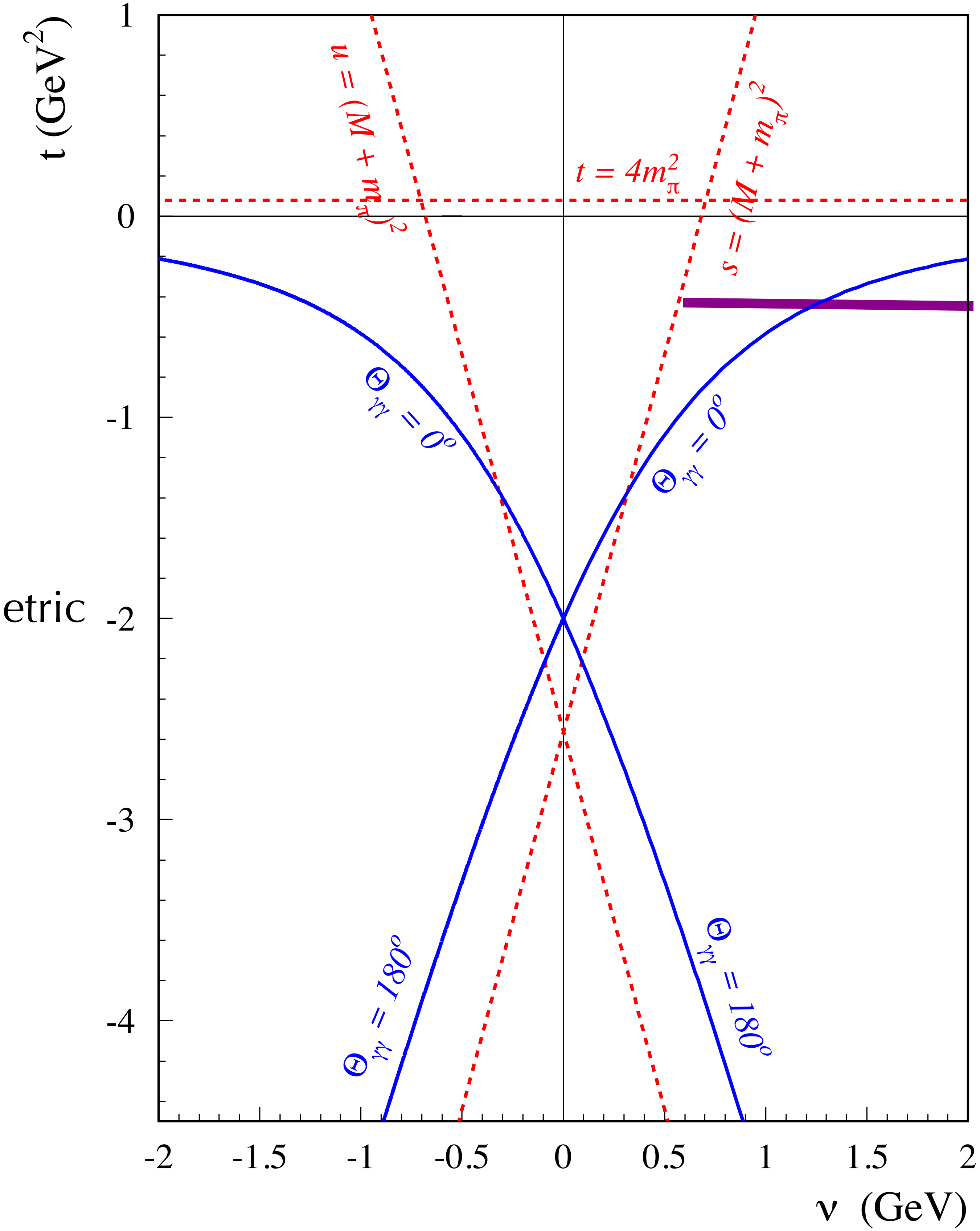


$\nu = \frac{s-u}{4M_N}$

crossing symmetric variable

Fixed

$Q^2 = -2\ \mathrm{GeV}^2$



DRs in the s-channel
at fixed t

$$\nu = \frac{s-u}{4M_N}$$

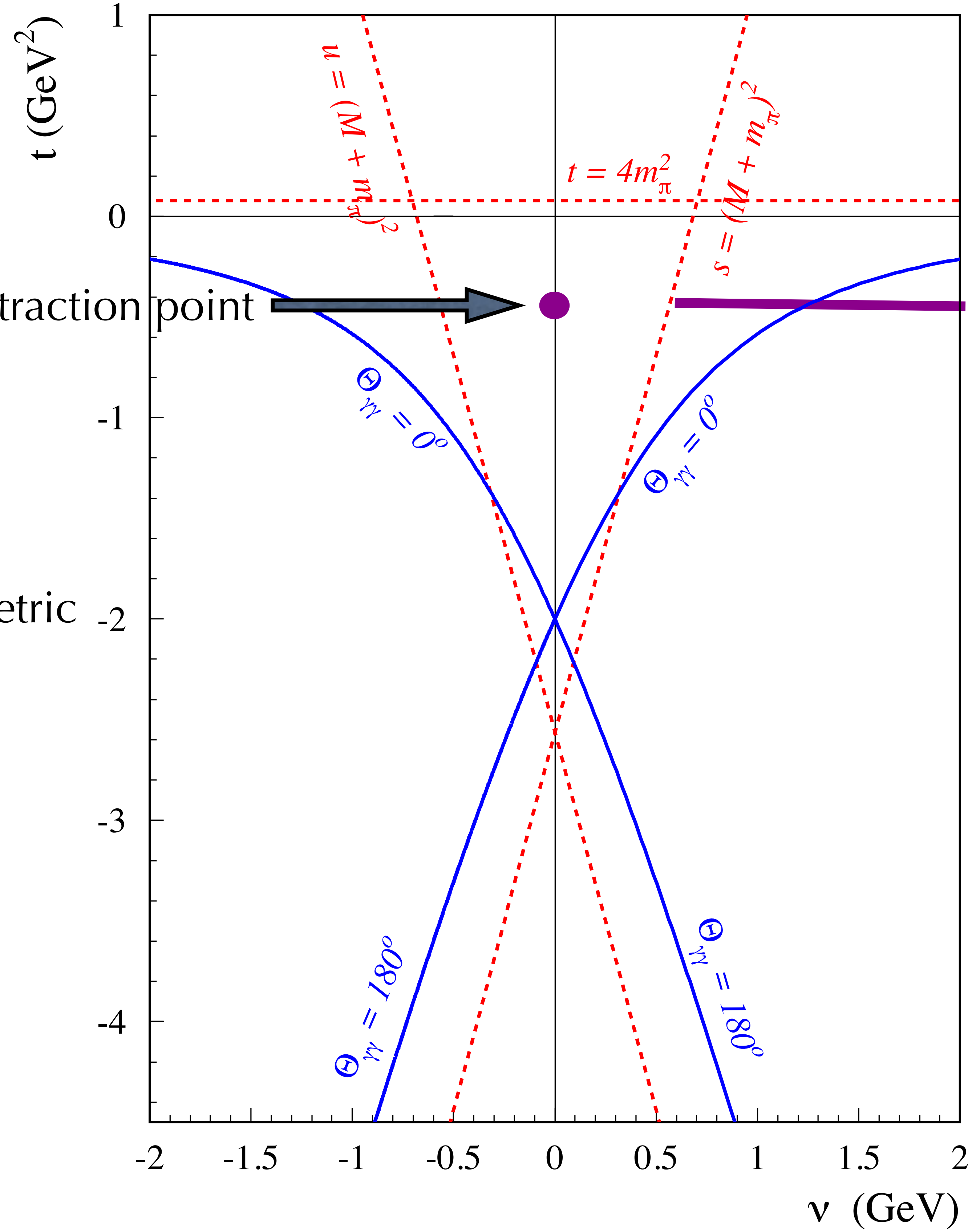
crossing symmetric
variable

Fixed

$Q^2 = -2\ \text{GeV}^2$

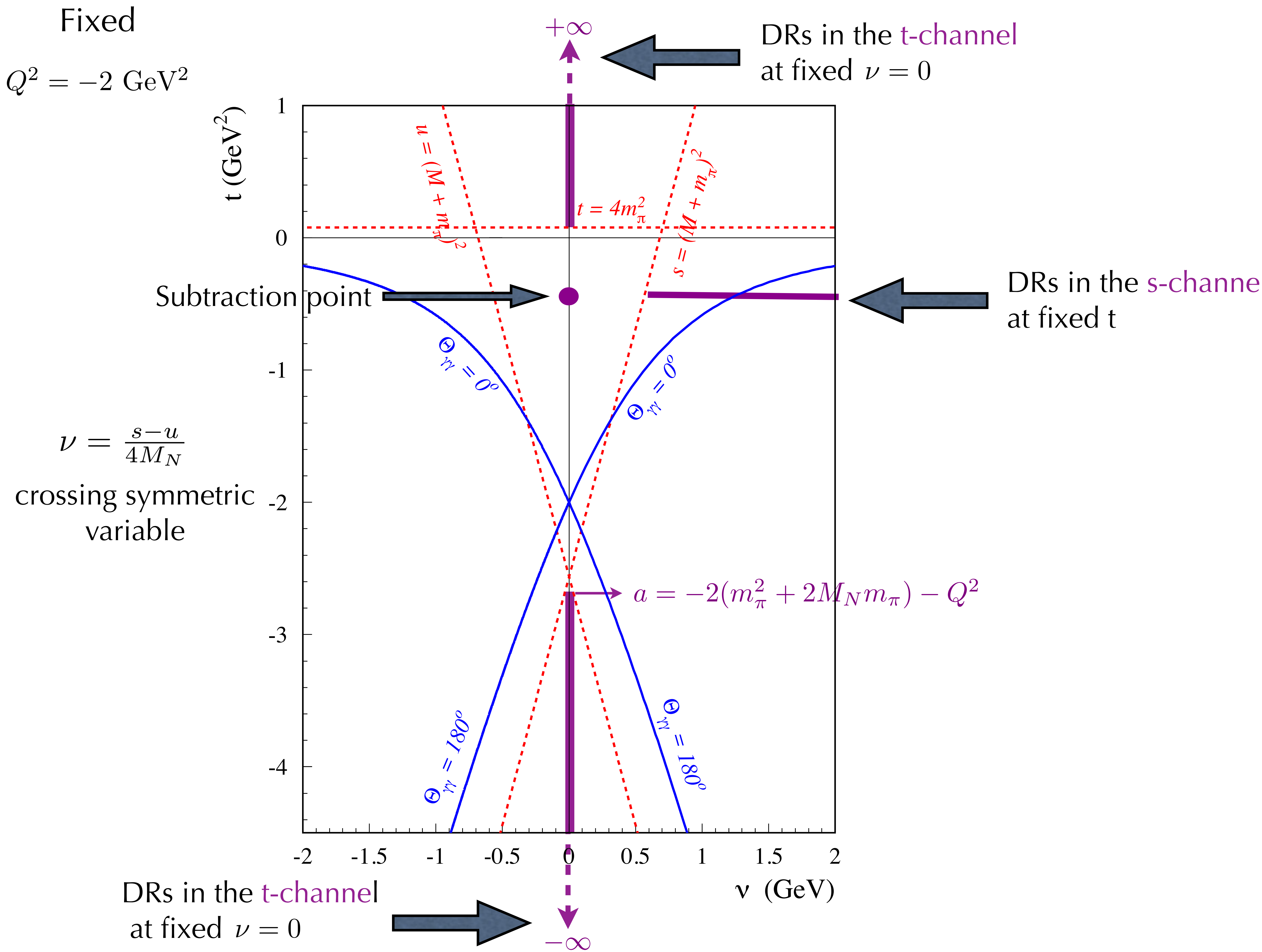
Subtraction point

DRs in the s-channel at fixed t

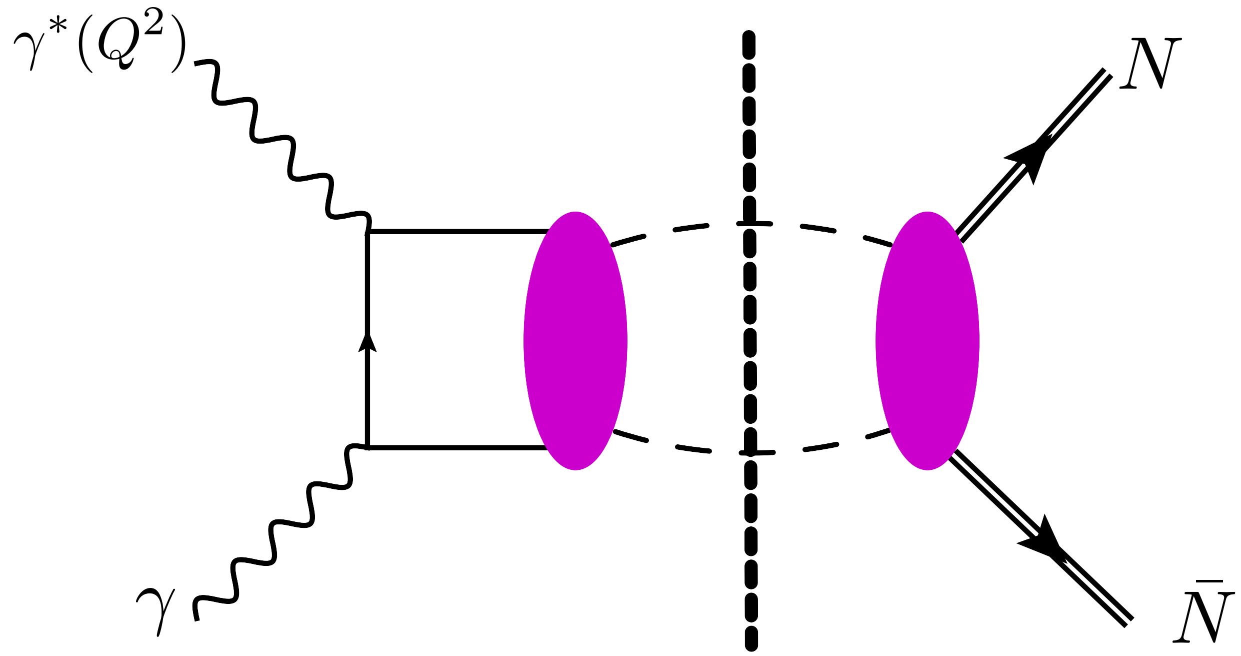


$\nu = \frac{s-u}{4M_N}$

crossing symmetric variable


Fixed
$Q^2 = -2\ \mathrm{GeV}^2$
$+\infty$
DRs in the t-channel
at fixed $\nu = 0$
t (GeV^2)
$u = (M + m_\pi)^2$
$t = 4m_\pi^2$
$s = (M + m_\pi)^2$
Subtraction point
DRs in the s-channel
at fixed t
$\Theta_{\gamma\gamma} = 0^o$
$\Theta_{\gamma\gamma} = 0^o$
$\nu = \frac{s-u}{4M_N}$
crossing symmetric
variable
$a = -2(m_\pi^2 + 2M_N m_\pi) - Q^2$
$\Theta_{\gamma\gamma} = 180^o$
$\Theta_{\gamma\gamma} = 180^o$
1
0
-1
-2
-3
-4
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
ν (GeV)
DRs in the t-channel
at fixed $\nu = 0$
$-\infty$

Unitarity Relations in the t-channel

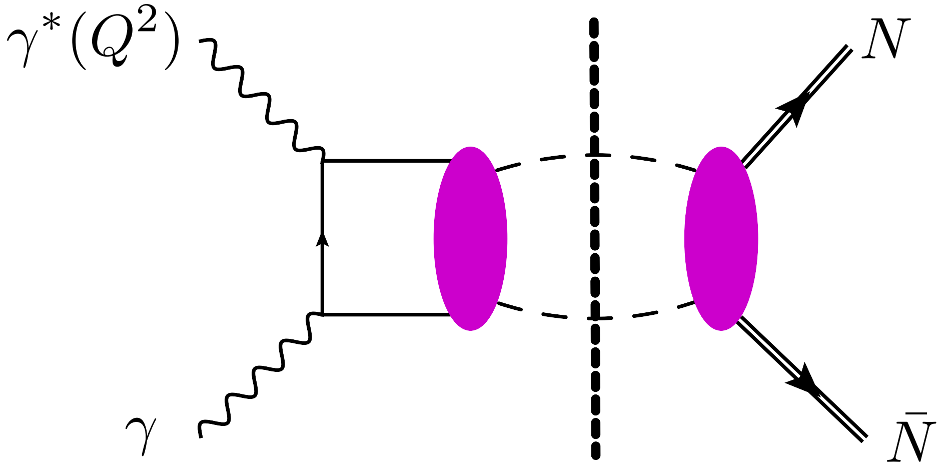


- Charge conjugation

- Partial wave expansion with $\nu = 0 \to \theta_t = 90^o$

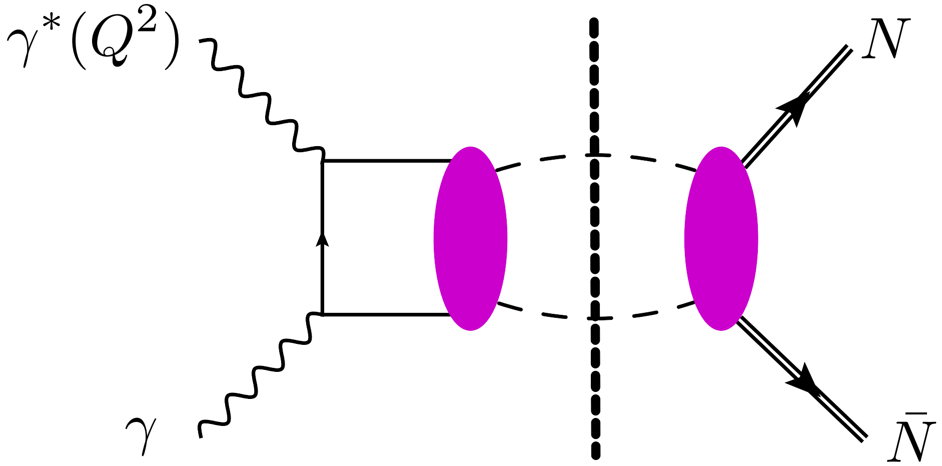
two-pion intermediate state with

$I = 0 \qquad J = 0, 2, \cdots$



Two-pion intermediate states with $I = 0$ and $J = 0,\, 2$

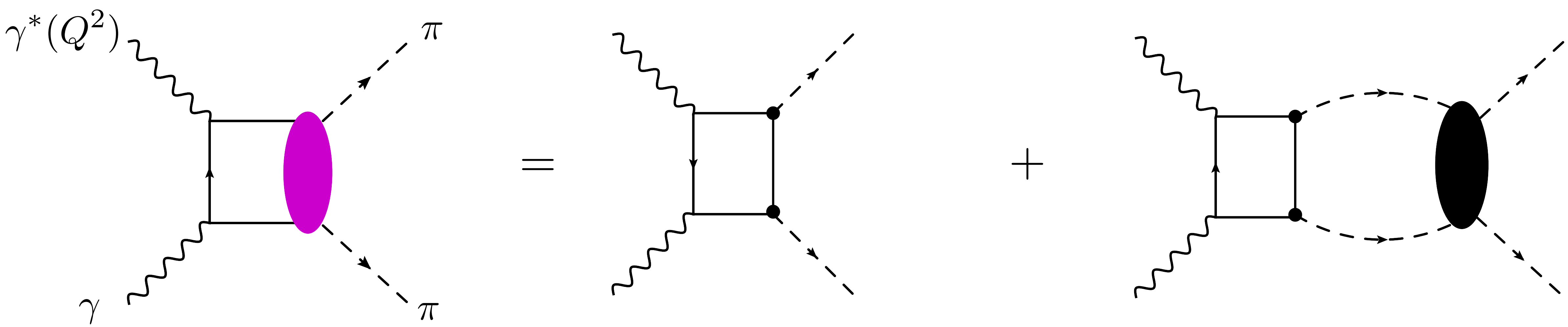
$$\Delta(t, Q^2) = \sum_{\{n\,\mathrm{odd}\}} d_n(t, Q^2) \longrightarrow \text{DRs for } d_1(t, Q^2)$$

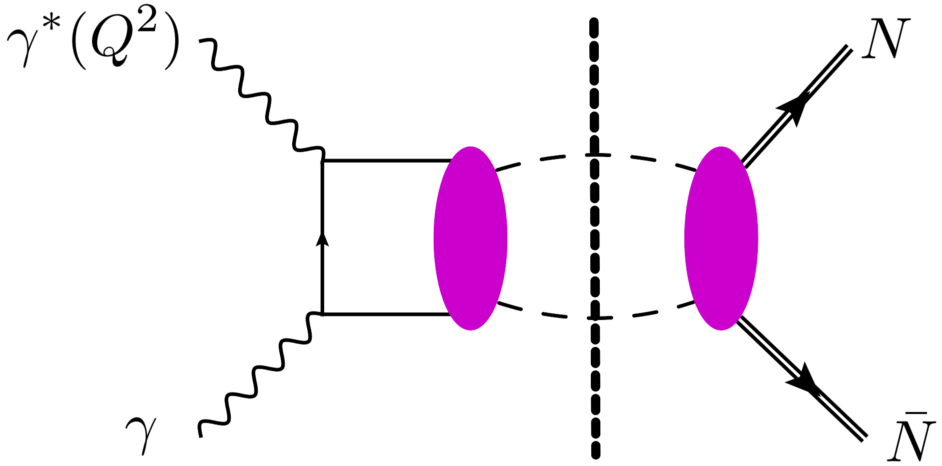


Two-pion intermediate states with $I = 0$ and $J = 0,\ 2$

$\Delta(t, Q^2) = \sum_{\{n\,\mathrm{odd}\}} d_n(t, Q^2)$ ⟶ DRs for $d_1(t, Q^2)$

- $\gamma^*\gamma \to \pi\pi$: two-pion generalized distribution amplitudes ⟶ inputs $\begin{cases} \text{pion singlet PDF} \\ \pi\pi \text{ phase-shifts} \end{cases}$





Two-pion intermediate states with $I = 0$ and $J = 0,\, 2$

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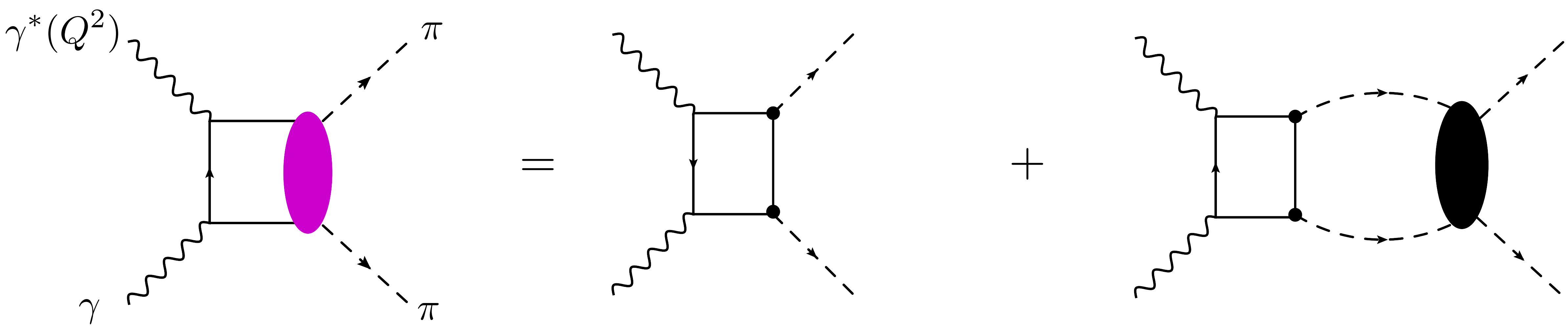
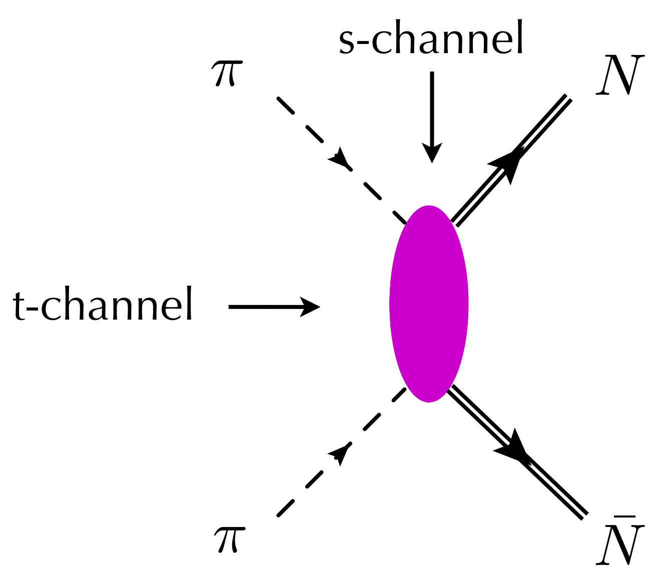
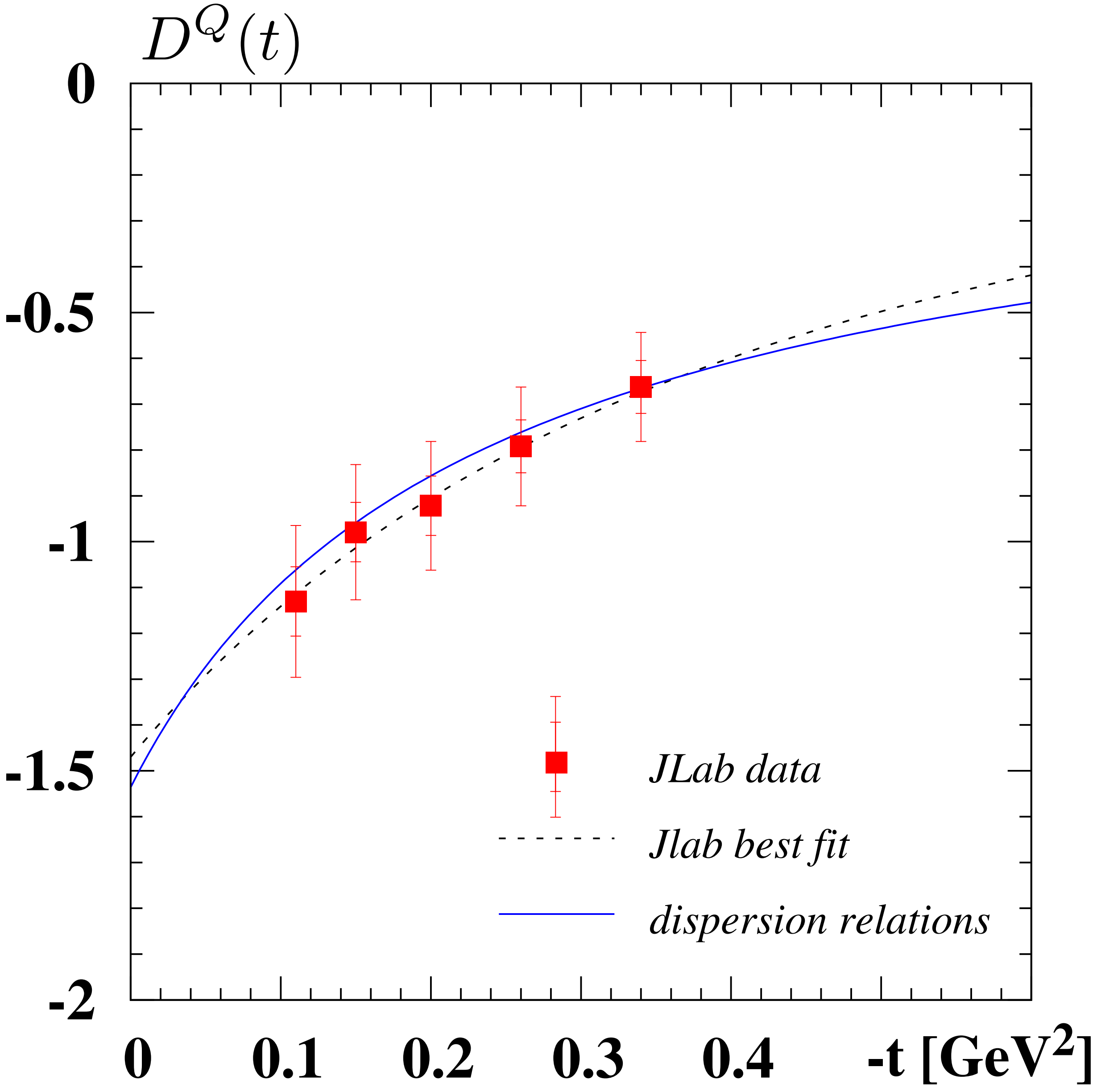


- $\pi\pi \to N\bar{N}$: analytical continuation of s-channel partial-wave helicity amplitudes

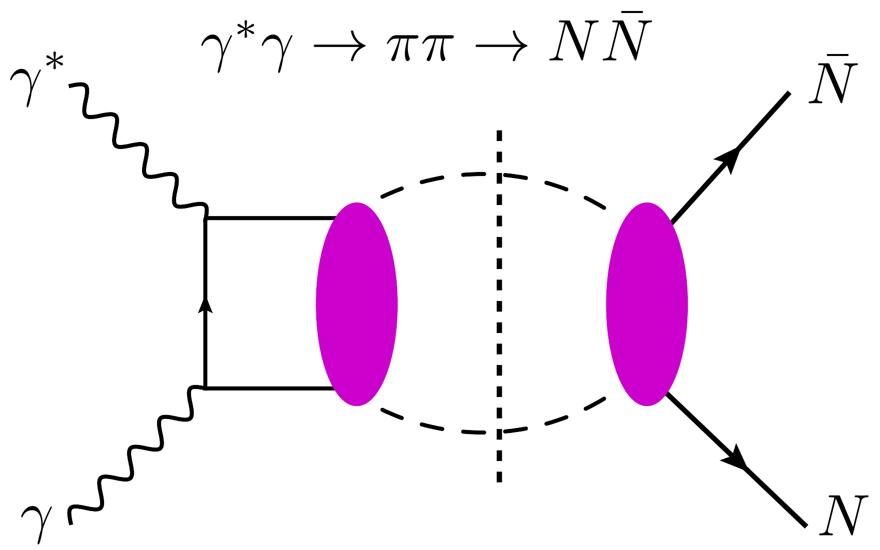
$\longrightarrow$ input $\pi\pi$ phase-shifts



D-term form factor and radial pressure distribution



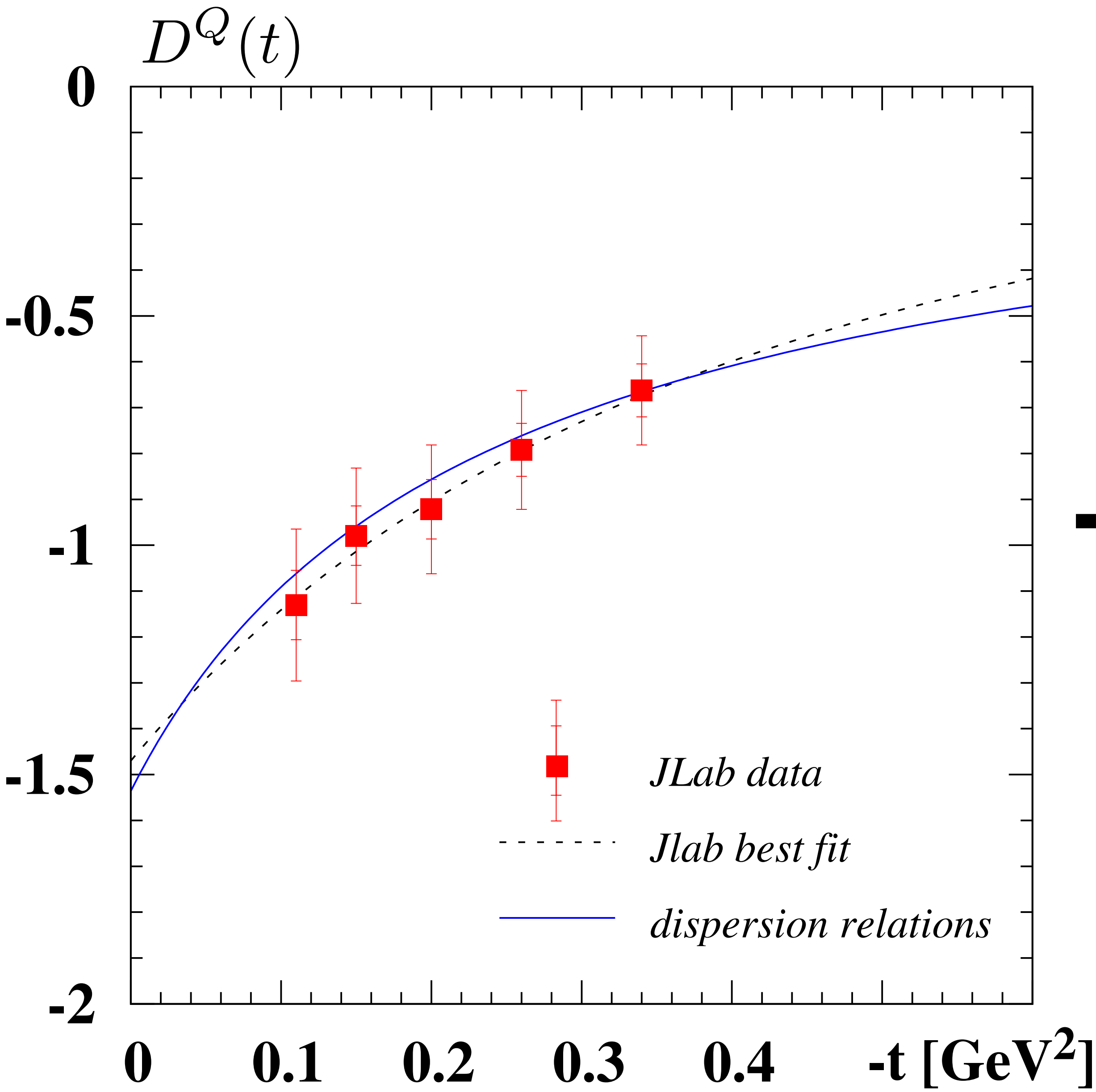
D-term from t-channel dispersion relations



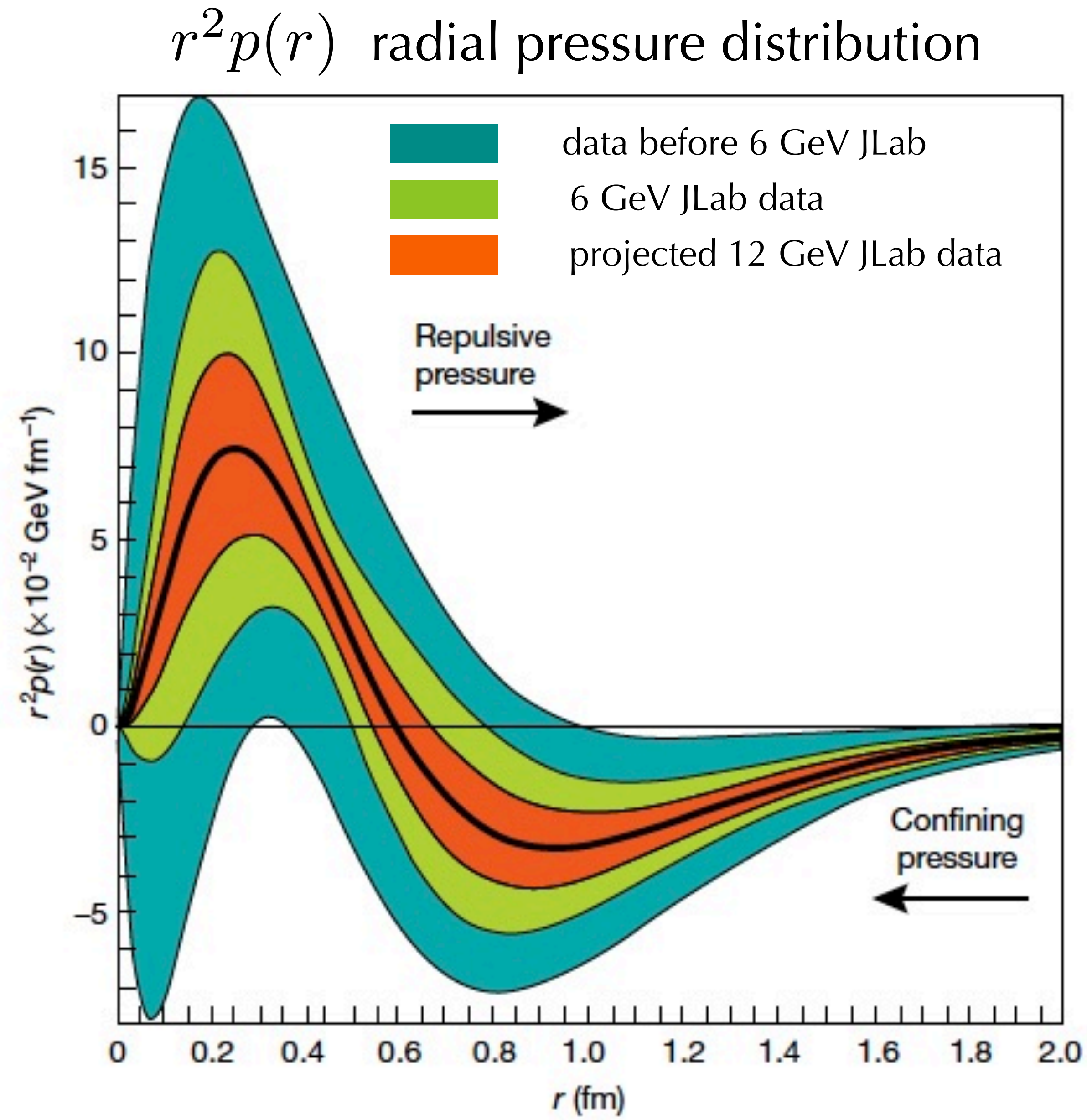
BP, Polyakov, Vanderhaeghen, PLB 739 (2014) 133

Girod, Elouadrhiri, Burkert,Nature 557 (2018) 7705

D-term form factor and radial pressure distribution



FT



$$r^2 p(r) = \frac{1}{3}\frac{d}{dr} r^2 \frac{d}{dr}\tilde{D}(r)$$

$$\int_0^\infty \mathrm{d}r\, r^2\, p(r) = 0$$

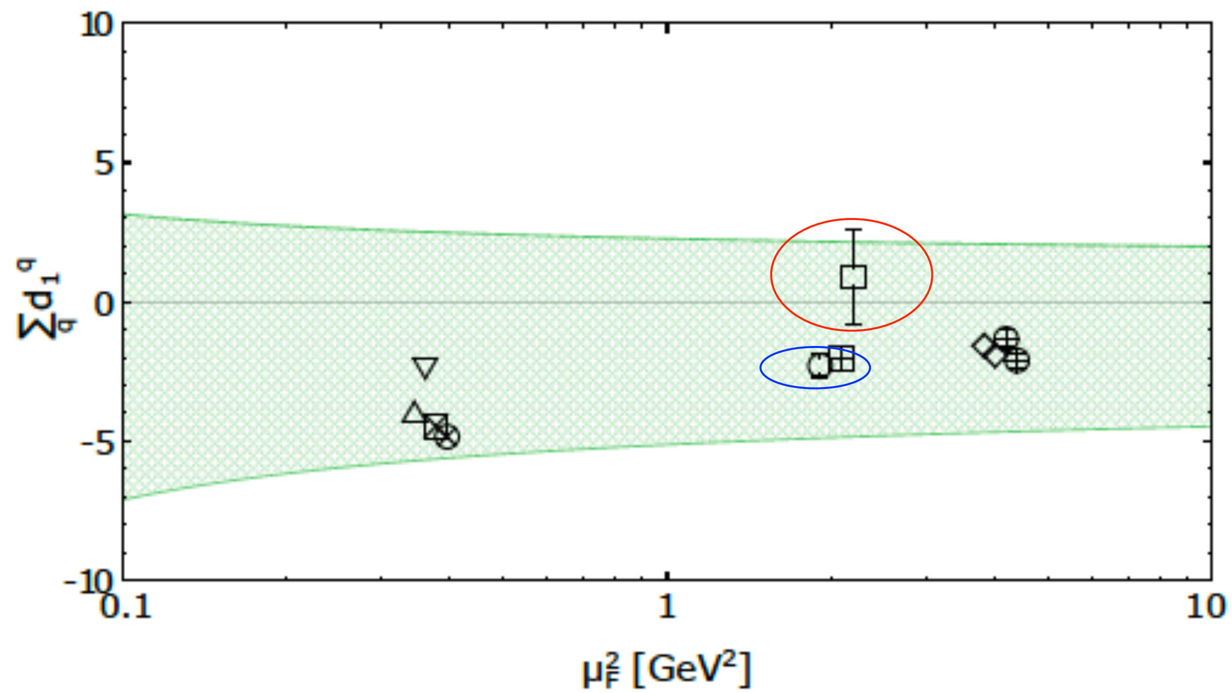
BP, Polyakov, Vanderhaeghen, PLB 739 (2014) 133

Girod, Elouadrhiri, Burkert,Nature 557 (2018) 7705

Necessary to verify model assumptions in the exp extraction with more data coming from JLab, COMPASS and the future EIC, EIcC

Kumericki, Nature 570 (2019) 7759; Dutrieux et al, Eur. Phys. J. C81 (2021) 4

global fit to DVCS data
with artificial neural networks



CLAS data, with fixed param., Girod et al.

CLAS data, with neural networks Kumericki

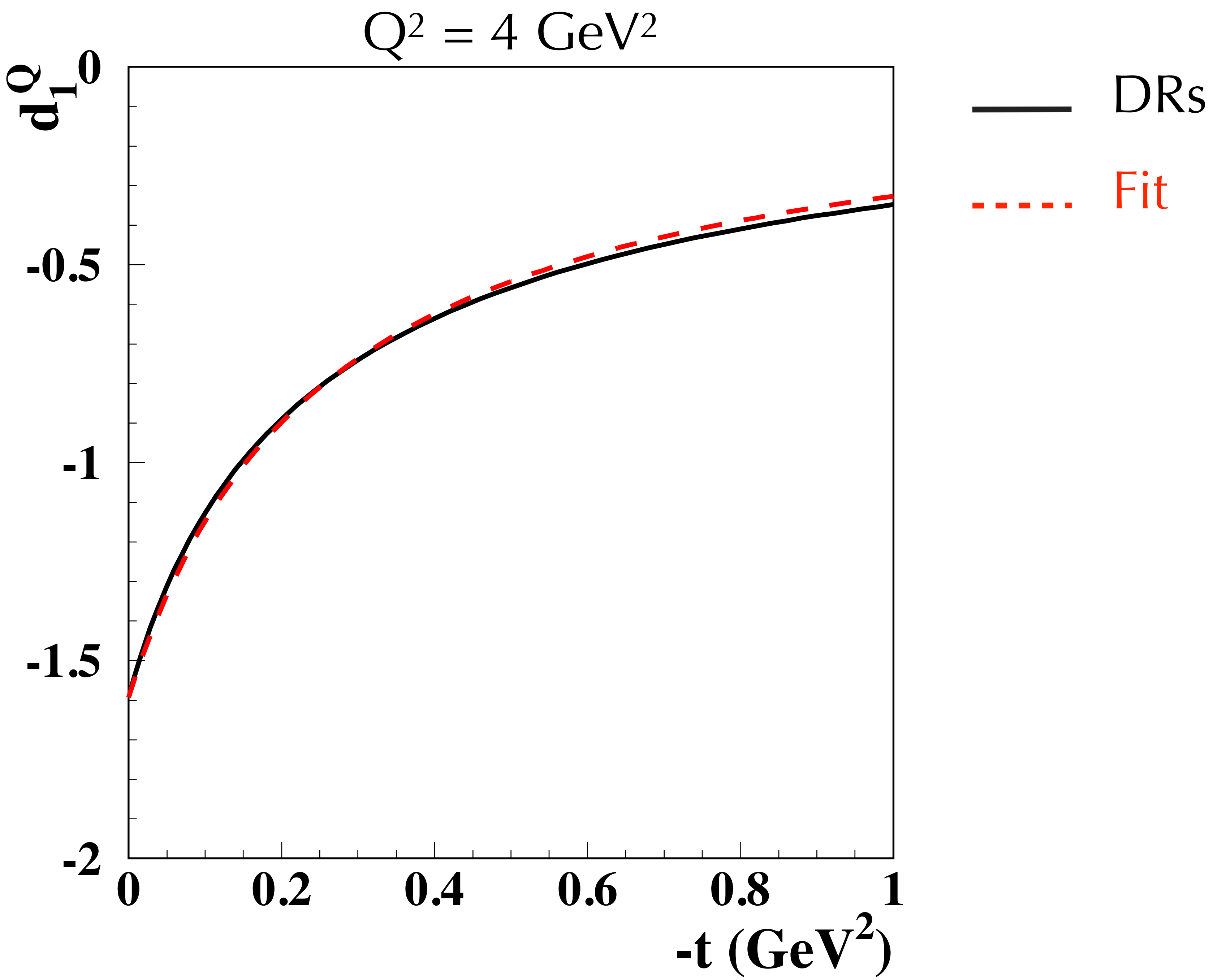
$$\sum_q d_1^q < 0$$

in all model calculations
for a stable proton

Marker in Fig. 3	$\sum_q d_1^q(\mu_F^2)$	μ_F^2 in GeV^2	# of flavours	Type
○	$-2.30 \pm 0.16 \pm 0.37$	2.0	3	from experimental data
□	0.88 ± 1.69	2.2	2	from experimental data
◇	-1.59	4	2	t-channel saturated model
	-1.92	4	2	t-channel saturated model
△	-4	0.36	3	χQSM
▽	-2.35	0.36	2	χQSM
⊠	-4.48	0.36	2	Skyrme model
⊞	-2.02	2	3	LFWF model
⊗	-4.85	0.36	2	χQSM
⊕	-1.34 ± 0.31	4	2	lattice QCD ($\overline{\mathrm{MS}}$)
	-2.11 ± 0.27	4	2	lattice QCD ($\overline{\mathrm{MS}}$)

D-term form factor: t-dependence

Q= u + d

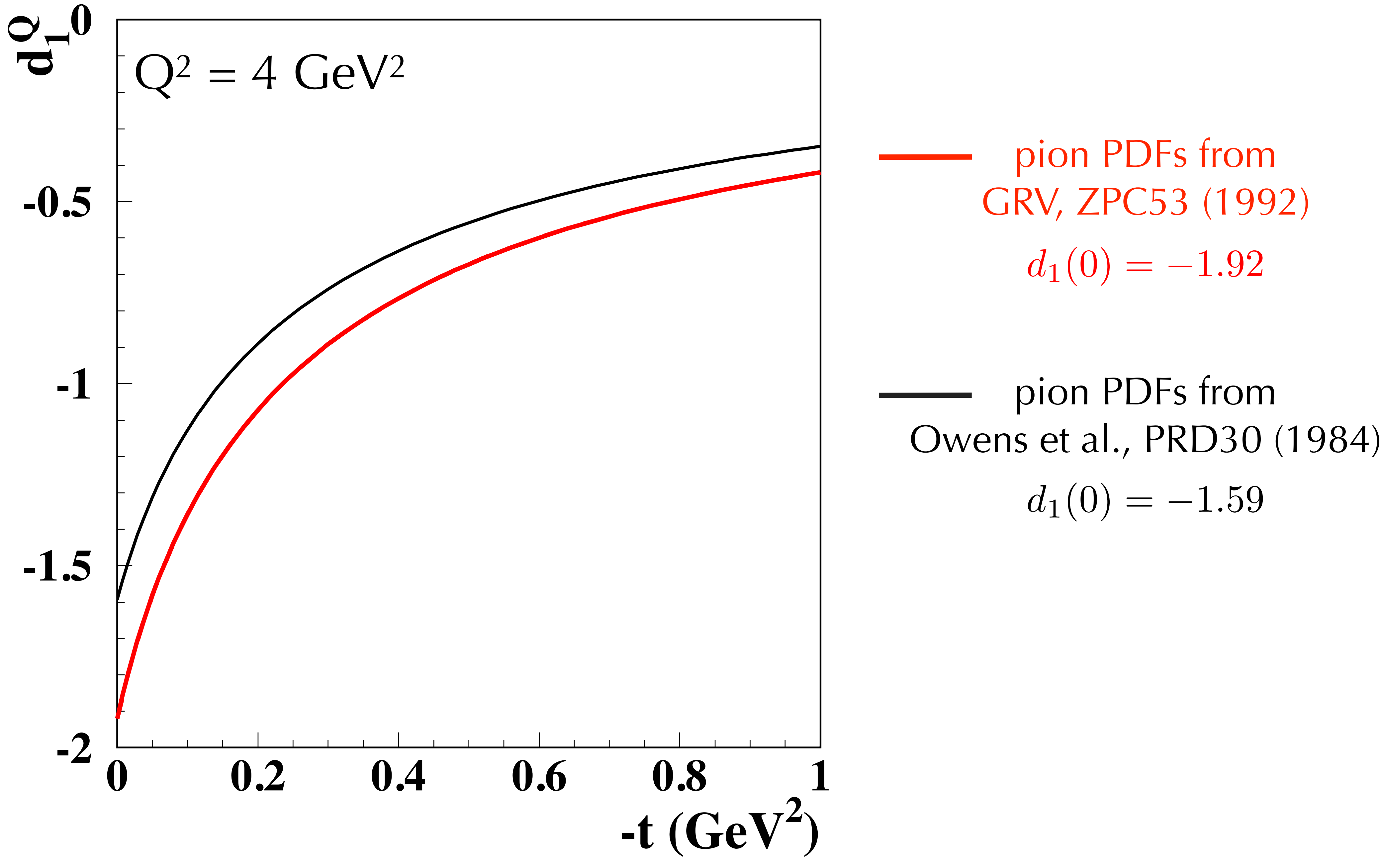


Fit: $F^Q(t) = \dfrac{d_1^Q(0)}{[1 - t/(\alpha M_D^2)]^\alpha}$ with $M_D = 0.487\,\mathrm{GeV}$

$\alpha = 0.841$

D-term form factor: dependence on pion PDFs

Q= u + d



χQSM	Skyrme model	Effective LFWFs
$d_1^Q(0) = -2.35$	$d_1^Q(0) = -4.48$	$d_1^Q(0) = -2.01$
Schweitzer et al., (2007)	*Schweitzer et al., (2007)*	*Mueller and Hwang, (2014)*

EMT and the proton mass

- Forward matrix element of total EMT

$$\langle T^{\mu\nu} \rangle \equiv \langle p | T^{\mu\nu} | p \rangle = 2p^\mu p^\nu$$

Proton mass

$$n \, \langle T^{\mu}{}_{\mu} \rangle = M$$

Trace decomposition

$$n \, \langle T^{00} \rangle = M$$

Energy decomposition

$$\left. \frac{\langle H_{\text{QCD}} \rangle}{\langle p | p \rangle} \right|_{\vec{p}=0} = M$$

QCD Hamiltonian

$$H_{\text{QCD}} = \int d^3x \mathcal{H}_{QCD} = \int d^3x \, T^{00}$$

EMT and the proton mass

- Forward matrix element of total EMT

$$\langle T^{\mu\nu}\rangle \equiv \langle p|T^{\mu\nu}|p\rangle = 2p^\mu p^\nu$$

Proton mass

$$n\,\langle T^\mu{}_\mu\rangle = M \qquad n\,\langle T^{00}\rangle = M \qquad \frac{\langle H_{\mathrm{QCD}}\rangle}{\langle p|p\rangle}\Big|_{\vec{p}=0} = M$$

Trace decomposition

Energy decomposition

QCD Hamiltonian

$$H_{\mathrm{QCD}} = \int d^3x \mathcal{H}_{QCD} = \int d^3x\, T^{00}$$

- Forward matrix element quark and gluon contributions

$$\langle T^{\mu\nu}_{i,R}\rangle = 2p^\mu p^\nu\, A_i(0) + 2M^2 g^{\mu\nu}\bar{C}_i(0)$$

$$A_q(0) + A_g(0) = 1 \qquad \bar{C}_q(0) + \bar{C}_g(0) = 0$$

- in forward limit, matrix elements of EMT fully determined by two form factors

$\longrightarrow$ any mass sum rule for the proton related to at most two independent numbers

Trace decomposition

(Hatta, Rajan, Tanaka, JHEP 12, 008 (2018) / Tanaka, JHEP 01, 120 (2019))

$$M = \bar{M}_q + \bar{M}_g = n\Big(\langle (T_{q,R})^\mu{}_\mu\rangle + \langle (T_{g,R})^\mu{}_\mu\rangle\Big)$$

- Total EMT not renormalized, but individual terms $T_i^{\mu\nu}$ require (extra) renormalization

$$T^{\mu\nu} = T_q^{\mu\nu} + T_g^{\mu\nu} \qquad T_q^{\mu\nu} = \frac{i}{4}\bar{\psi}\gamma^{\{\mu}\overleftrightarrow{D}^{\nu\}}\psi \qquad T_g^{\mu\nu} = -F^{\mu\alpha}F_\alpha^\nu + \frac{g^{\mu\nu}}{4}F^2$$

- Trace anomaly of EMT: $T^\mu{}_\mu = (m\bar{\psi}\psi)_R + \gamma_m (m\bar{\psi}\psi)_R + \frac{\beta}{2g}(F^2)_R$

(Adler, Collins, Duncan, 1977 / Nielsen, 1977 / Collins, Duncan, Joglekar, 1977 / ...)

$$T^\mu{}_\mu = (T_{q,R})^\mu{}_\mu + (T_{g,R})^\mu{}_\mu \quad \begin{cases} (T_{q,R})^\mu{}_\mu = (1+y)(m\bar{\psi}\psi)_R + x(F^2)_R \\ (T_{g,R})^\mu{}_\mu = (\gamma_m - y)(m\bar{\psi}\psi)_R + \left(\frac{\beta}{2g} - x\right)(F^2)_R \end{cases}$$

Trace decomposition

(Hatta, Rajan, Tanaka, JHEP 12, 008 (2018) / Tanaka, JHEP 01, 120 (2019))

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- x and y related to finite parts of renormalization constants $\longrightarrow$ choose a scheme

D1 scheme: $x = 0,\ y = \gamma_m$ $\quad (T_{q,R})^{\mu}{}_{\mu} = (1+\gamma_m)(m\bar{\psi}\psi)_R \qquad (T_{g,R})^{\mu}{}_{\mu} = \dfrac{\beta}{2g}(F^2)_R$

D2 scheme: $x = y = 0$ $\quad (T_{q,R})^{\mu}{}_{\mu} = (m\bar{\psi}\psi)_R \qquad (T_{g,R})^{\mu}{}_{\mu} = \gamma_m(m\bar{\psi}\psi)_R + \dfrac{\beta}{2g}(F^2)_R$

Two-term energy decomposition

Lorcé, EPJC 78, 120 (2018)

$$M = U_q + U_g = n\left(\langle T^{00}_{q,R}\rangle + \langle T^{00}_{g,R}\rangle\right)$$

- Relation to EMT form factors for energy and trace decomposition

$$U_i = M\left(A_i(0) + \bar{C}_i(0)\right)$$

$$\bar{M}_i = M\left(A_i(0) + 4\bar{C}_i(0)\right)$$

- $U_i \neq \bar{M}_i$: mixture of energy and pressure ($\sim \bar{C}_i(0)$) terms
- $U_q + U_g = \bar{M}_q + \bar{M}_g$ because $\bar{C}_q + \bar{C}_g = 0$

- Energy and trace decompositions have one independent term

4 term energy decomposition of Ji

Ji, PRL 74, 1071 (1995); PRD 52, 271 (1995); Ji et al., NPB 971 (2021) 115537

- Sum rule based on decomposition of T^{00} into traceless and trace part

$$T^{\mu\nu} = \bar{T}^{\mu\nu} + \hat{T}^{\mu\nu} \qquad \hat{T}^{\mu\nu} = \frac{1}{4} g^{\mu\nu} T^{\alpha}{}_{\alpha} \qquad \bar{T}^{\mu\nu} = T^{\mu\nu} - \hat{T}^{\mu\nu}$$

traceless trace

$$\mathcal{H}_T \equiv (\bar{T}^{00})_R = (\mathcal{H}_q + \mathcal{H}_g) + \frac{3}{4}\mathcal{H}_m \qquad \mathcal{H}_S = \frac{1}{4} g_{\alpha\beta} (T^{\alpha\beta})_R = \mathcal{H}_a + \frac{1}{4}\mathcal{H}_m$$

tensor (traceless) part scalar (trace) part

- 4 term decomposition $\mathcal{H} = \mathcal{H}_S + \mathcal{H}_T = (\mathcal{H}_q + \mathcal{H}_g) + \mathcal{H}_m + \mathcal{H}_a$

$$\mathcal{H}_a = \frac{1}{4}\left[\frac{\beta}{2g}(F^2)_R + \gamma_m (\bar{\psi} m \psi)_R\right]$$ anomaly energy

$$\mathcal{H}_m = (\bar{\psi} m \psi)_R$$ mass energy

$$(\mathcal{H}_q + \mathcal{H}_g) = \lim_{\epsilon \to 0} \left(\psi^\dagger \vec{\alpha} \cdot i\vec{D}\psi + \frac{2-2\epsilon}{4-2\epsilon}\vec{E}^2 + \frac{2}{4-2\epsilon}\vec{B}^2 \right)_R$$ quark potential and kinetic energy + gluon energy

Formulas from Ji et al., NPB 971 (2021) 115537, which differ from the original papers

3 term energy decomposition

Rodini, Metz, BP, JHEP 09, 67 (2020); PRD 102, 111 (2020); Lorcé, Metz, BP, Rodini, JHEP 11,121 (2021)

$$M = \widetilde{\mathcal{H}}_q + \widetilde{\mathcal{H}}_g + \mathcal{H}_m$$

$$\widetilde{\mathcal{H}}_q = (\psi^\dagger \vec{\alpha} \cdot i\vec{D}\psi)_R \qquad \widetilde{\mathcal{H}}_g = \frac{(\vec{E}^2 + \vec{B}^2)_R}{2} \qquad \mathcal{H}_m = (\bar{\psi} m \psi)_R$$

3 term energy decomposition

Rodini, Metz, BP, JHEP 09, 67 (2020); PRD 102, 111 (2020); Lorcé, Metz, BP, Rodini, JHEP 11,121 (2021)

$$M = \widetilde{\mathcal{H}}_q + \widetilde{\mathcal{H}}_g + \mathcal{H}_m$$

$$\widetilde{\mathcal{H}}_q = (\psi^\dagger \vec{\alpha} \cdot i\vec{D}\psi)_R \qquad \widetilde{\mathcal{H}}_g = \frac{(\vec{E}^2 + \vec{B}^2)_R}{2} \qquad \mathcal{H}_m = (\bar{\psi} m \psi)_R$$

- Comparison with two-term energy decomposition

$$M = U_q + U_g = n\left(\langle T^{00}_{q,R}\rangle + \langle T^{00}_{g,R}\rangle\right) \qquad T^{00}_{q,R} = \widetilde{\mathcal{H}}_q + H_m \qquad T^{00}_{g,R} = \widetilde{\mathcal{H}}_g$$

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$$(\mathcal{H}_q + \mathcal{H}_g) + \mathcal{H}_m + \mathcal{H}_a = \widetilde{\mathcal{H}}_q + \widetilde{\mathcal{H}}_g \cancel{-\mathcal{H}_a} + \mathcal{H}_m \cancel{+\mathcal{H}_a}$$

Overview: comparison of sum rules

- Trace decomposition (in D2 scheme)

$$M = \bar{M}_q + \bar{M}_g = n\Big(\Big\langle (m\bar{\psi}\psi)_R \Big\rangle + \Big\langle \gamma_m (m\bar{\psi}\psi)_R + \frac{\beta}{2g}(F^2)_R \Big\rangle\Big)$$

-two scale-independent terms

- Two-term energy decomposition

$$M = U_q + U_g = n\Big(\Big\langle (m\bar{\psi}\psi)_R + (\psi^\dagger i\vec{D}\cdot\vec{\alpha}\psi)_R \Big\rangle + \Big\langle \frac{1}{2}(E^2 + B^2)_R \Big\rangle\Big)$$

-two scale-dependent terms

- Three-term energy decomposition

$$M = M_q + M_m + M_g = n\Big(\Big\langle (m\bar{\psi}\psi)_R \Big\rangle + \Big\langle (\psi^\dagger i\vec{D}\cdot\vec{\alpha}\psi)_R \Big\rangle + \Big\langle \frac{1}{2}(E^2 + B^2)_R \Big\rangle\Big)$$

-one scale-independent term and two scale-dependent terms

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-one scale-independent term and two scale-dependent terms

- Relation between matrix elements (virial theorem $\Big\langle \sum_i T^{ii} \Big\rangle = 0$)

$$\Big\langle (\psi^\dagger i\vec{D}\cdot\vec{\alpha}\psi)_R + \frac{1}{2}(E^2 + B^2)_R \Big\rangle = \Big\langle \gamma_m (\bar{\psi}\psi)_R + \frac{\beta}{2g}(F^2)_R \Big\rangle$$

- one can speak about contribution from trace anomaly or parton energies

- a sum rule with contributions from trace anomaly and parton energies does not appear naturally

Numerical results

Two independent inputs

- First input: parton momentum fractions a_i, related to traceless parton operators

$$\frac{3}{2}M^2\, a_q = \langle \bar{T}^{00}_{q,R}\rangle \qquad \frac{3}{2}M^2\, a_g = \langle \bar{T}^{00}_{g,R}\rangle \qquad (a_q + a_g = 1)$$

- Second input: quark mass term

$$2M^2\, b = (1+\gamma_m)\,\langle (m\bar{\psi}\psi)_R\rangle \qquad 2M^2\,(1-b) = \frac{\beta}{2g}\,\langle (F^2)_R\rangle$$

- direct input on trace anomaly (from experiment and/or LQCD) would be useful

- Three-term sum rule in terms of a_i and b

$$M_q = \frac{3}{4}M\, a_q + \frac{1}{4}M\left(\frac{(y-3)b}{1+\gamma_m} + x(1-b)\frac{2g}{\beta}\right)$$

$$M_m = M\frac{b}{1+\gamma_m}$$

$$M_g = \frac{3}{4}M\, a_g + \frac{1}{4}M\left[\frac{(\gamma_m - y)b}{1+\gamma_m} + \left(1 - x\frac{2g}{\beta}\right)(1-b)\right]$$

- Momentum fractions from CT18NNLO parametrization (at $\mu = 2$ GeV)

$$a_q = 0.586 \pm 0.013 \quad a_g = 1 - a_q = 0.414 \pm 0.013$$

- Quark mass term from sigma terms

$$\sigma_u + \sigma_d = \sigma_{\pi N} = \frac{\langle p|\hat{m}(\bar{u}u + \bar{d}d)|p\rangle}{2M} \qquad \sigma_s = \frac{\langle p|\hat{m}_s\, \bar{s}s|p\rangle}{2M} \qquad \sigma_c = \frac{\langle p|\hat{m}_c\, \bar{c}c|p\rangle}{2M}$$

- Scenario A: sigma terms from phenomenology

(Alarcon et al, 2011, 2012 / Hoferichter et al, 2015)

$$\sigma_{\pi N}\big|_{\mathrm{ChPT}} = (59 \pm 7)\,\mathrm{MeV} \quad \sigma_s\big|_{\mathrm{ChPT}} = (16 \pm 80)\,\mathrm{MeV}$$

- Scenario B: sigma terms from lattice QCD

(Alexandrou et al, 2019)

$$\sigma_{\pi N}\big|_{\mathrm{LQCD}} = (41.6 \pm 3.8)\,\mathrm{MeV} \quad \sigma_s\big|_{\mathrm{LQCD}} = (39.8 \pm 5.5)\,\mathrm{MeV}$$

$$\sigma_c\big|_{\mathrm{LQCD}} = (10.7 \pm 22)\,\mathrm{MeV}$$

main difference between scenarios: including or not σ_c

Scheme dependence for 3-term sum rule (at $\mu = 2$ GeV)

		MS	$\overline{\text{MS}}_1$	$\overline{\text{MS}}_2$	D1	D2
Scenario A	M_q	0.309 ± 0.044	0.194 ± 0.033	0.178 ± 0.032	0.362 ± 0.045	0.357 ± 0.051
	M_m	0.075 ± 0.080	0.075 ± 0.080	0.075 ± 0.080	0.075 ± 0.080	0.075 ± 0.080
	M_g	0.555 ± 0.036	0.669 ± 0.047	0.686 ± 0.048	0.502 ± 0.035	0.507 ± 0.029
Scenario B	M_q	0.234 ± 0.006	0.135 ± 0.003	0.120 ± 0.003	0.286 ± 0.006	0.272 ± 0.008
	M_m	0.187 ± 0.023	0.187 ± 0.023	0.187 ± 0.023	0.187 ± 0.023	0.187 ± 0.023
	M_g	0.517 ± 0.017	0.617 ± 0.020	0.631 ± 0.020	0.465 ± 0.017	0.479 ± 0.015

- considerable numerical scheme dependence

- qualitatively, similar results for other sum rules

- scheme dependence is not a new phenomenon

Results for trace decomposition

		$O(\alpha_s^1)$	$O(\alpha_s^2)$	$O(\alpha_s^3)$
Scenario A	$\bar{M}_q$	-0.113 ± 0.102	-0.120 ± 0.105	-0.115 ± 0.107
	$\bar{M}_g$	1.051 ± 0.102	1.057 ± 0.105	1.053 ± 0.107
Scenario B	$\bar{M}_q$	0.032 ± 0.030	0.030 ± 0.031	0.035 ± 0.030
	$\bar{M}_g$	0.906 ± 0.030	0.908 ± 0.030	0.903 ± 0.030

- perturbative expansion very stable (applies for all sum rules, and for all schemes)
- $\bar{M}_q$ can become negative

Results for two-term energy decomposition

		$O(\alpha_s^1)$	$O(\alpha_s^2)$	$O(\alpha_s^3)$
Scenario A	U_q	0.384 ± 0.035	0.383 ± 0.036	0.384 ± 0.036
	U_g	0.554 ± 0.035	0.556 ± 0.036	0.555 ± 0.036
Scenario B	U_q	0.420 ± 0.016	0.420 ± 0.017	0.421 ± 0.017
	U_g	0.518 ± 0.016	0.518 ± 0.017	0.517 ± 0.017

- very roughly, quark and gluon energies contribute equally to proton mass
- in $\overline{\text{MS}}$ scheme, contribution from gluon energy somewhat larger

Results for three-term energy decomposition

		$O(\alpha_s^1)$	$O(\alpha_s^2)$	$O(\alpha_s^3)$
Scenario A	M_q	0.311 ± 0.043	0.310 ± 0.043	0.309 ± 0.044
	M_m	0.073 ± 0.080	0.073 ± 0.079	0.074 ± 0.080
	M_g	0.554 ± 0.035	0.556 ± 0.036	0.555 ± 0.036
Scenario B	M_q	0.237 ± 0.006	0.235 ± 0.006	0.234 ± 0.006
	M_m	0.183 ± 0.023	0.184 ± 0.022	0.187 ± 0.023
	M_g	0.518 ± 0.016	0.518 ± 0.017	0.517 ± 0.017

- $M_g = U_g \rightarrow$ discussion for gluon part like for two-term energy decomposition

- M_q dominates over M_m, but feature less significant if σ_c included

- contribution of M_m is $\sim 8\%$ for Scenario A, $\sim 20\%$ for Scenario B

$\longrightarrow$ much larger than $\sim 1\%$ which is frequently attributed to Higgs mechanism

Summary

- Key information about the nucleon structure encoded in the EMT

- Definition of the EMT is not unique:

super potential terms do not play any role at the level of integrated quantities, while play a crucial role at the level of distributions

- Dispersion Relations for DVCS amplitudes

constraints from analyticity, crossing, built in

- D-term from t-channel Dispersion Relations

D-term $\longrightarrow$ two-pion correlated state with I=0, J=0, 2

model independent representation
with input from two-pion GDAs and pion-nucleon scattering

- Several sum rules for proton mass exist that are based on the EMT (T^{00} or $T^{\mu}{}_{\mu}$)

We arrived at a sum rule with three terms which have a ``clean" interpretation

All the sum rules rely on the same inputs: two independent form factors of the EMT