Anomalous diffusion in QCD matter

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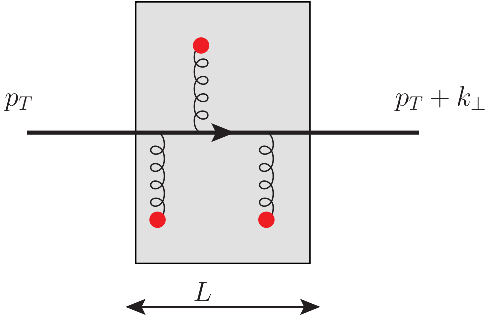
Ref: 2109.12041, 2203.xxxx

Transverse momentum broadening in QCD

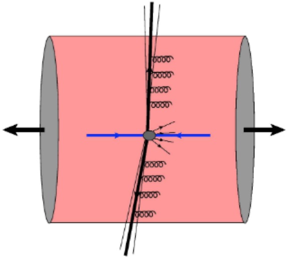
- Physical system: a highly energetic parton propagating through a dense QCD medium.
- We compute the tranverse momentum distribution $\mathcal{P}(k_\perp)$ of the outgoing parton.

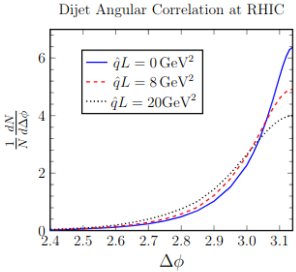


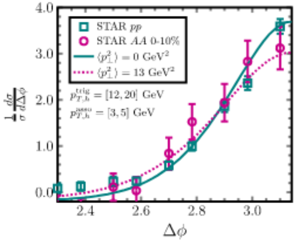
- Dense QCD medium: multiple scatterings.

Why is this problem interesting?

- "Hot QCD": Dijet azimuthal angular distributions in heavy-ion collisions: access to the TMB and the medium properties.
- Ex: studies by Mueller, Wu, Xiao, Feng 1604.04250 & Chen, Qin, Wei, Xiao, Zhang 1607.01932.





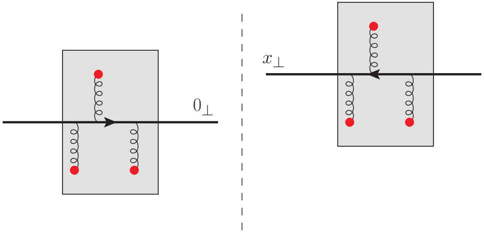


- "Cold QCD": fast probe of gluon distribution in large nuclei $L \propto A^{1/3} \gg 1$ at small-x.

TMB at tree level (1/2)

- The dipole S-matrix:

$$\mathcal{S}(\boldsymbol{x}_\perp) = \langle V^\dagger(\boldsymbol{x}_\perp) V(\boldsymbol{0}_\perp)\rangle\,, \qquad \text{with} \quad V(\boldsymbol{x}_\perp) = \mathcal{P} e^{ig\int_{-\infty}^{\infty} dx^+ A^-(x^+, \boldsymbol{x}_\perp)}$$



- Assuming independent multiple interactions, $\mathcal{S}(\boldsymbol{x}_\perp)$ exponentiates:

$$\mathcal{S}(\boldsymbol{x}_\perp) = \exp\left(-\frac{1}{4}\hat{q}(1/\boldsymbol{x}_\perp^2, L) L \boldsymbol{x}_\perp^2\right)$$

- We take this formula as our definition of $\hat{q}$.

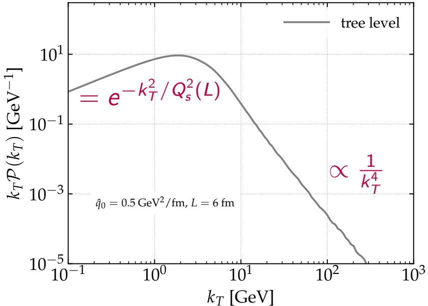
TMB at tree level (2/2)

- Fourier transform of the dipole S-matrix

$$\mathcal{P}(\boldsymbol{k}_\perp) = \int d^2\boldsymbol{x}_\perp e^{-i\boldsymbol{k}_\perp \boldsymbol{x}_\perp} e^{-\frac{1}{4}\hat{q}(1/\boldsymbol{x}_\perp^2)L\boldsymbol{x}_\perp^2}$$

- LO $\hat{q}$ from the Fourier transform of the collision rate $\gamma(\boldsymbol{q}_\perp) \sim g^4 n/\boldsymbol{q}_\perp^4$:

$$\hat{q}^{(0)}(\boldsymbol{x}_\perp) = \frac{4C_R}{\boldsymbol{x}_\perp^2}\int_{\boldsymbol{q}_\perp}(1-e^{i\boldsymbol{q}_\perp \boldsymbol{x}_\perp})\gamma(\boldsymbol{q}_\perp) = \hat{q}_0 \ln\frac{1}{\boldsymbol{x}_\perp^2\mu^2} + \mathcal{O}(\boldsymbol{x}_\perp^2\mu^2), \qquad \hat{q}_0 \propto \alpha_s^2 n$$



For an analytic expression, see e.g. Barata, Mehtar-Tani, Soto-Ontoso, Tywoniuk 2009.13667 .

The saturation momentum Q_s

- Emergent scale in the dipole S-matrix $\mathcal{S}$:

$$\mathcal{S}(\boldsymbol{x}_\perp^2 = 1/Q_s^2(L)) \equiv e^{-1/4} \Leftrightarrow \hat{q}(L, Q_s^2(L))L = Q_s^2(L)$$

- Transition between the unitarity bound $\mathcal{S} \sim 1$ and the dilute regime $\mathcal{S} \ll 1$.

- At tree-level, one finds

$$Q_s^2(L) \simeq \hat{q}_0 L \ln(\hat{q}_0 L/\mu^2)$$

- Approximate **linear** scaling with L.

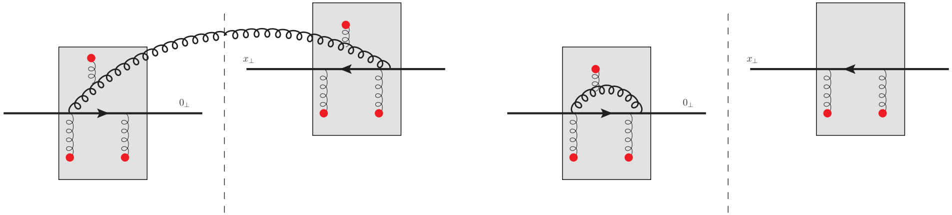
Outline

- The TMB distribution beyond leading order in the DLA.
- Extended geometric scaling.
- Universality of TMB distribution for large system sizes.
- Phenomenological applications.

Transverse momentum broadening in the double logarithmic approximation

TMB at one loop in a dense QCD medium (1/2)

- Computation at one-loop in $\alpha_s(p_T) \ll 1$, but to all-orders in $\alpha_s n$.



- Schematically one finds

$$\mathcal{P}(\boldsymbol{k}_\perp, L) = \mathcal{P}^{(0)}(\boldsymbol{k}_\perp, L) + \alpha_s \mathcal{P}^{(1)}(\boldsymbol{k}_\perp, L) + \mathcal{O}(\alpha_s^2)$$

with the NLO distribution given by Liou, Mueller, Wu, 1304.7677, Blaizot, Mehtar-Tani, 1403.2323

$$\alpha_s \mathcal{P}^{(1)}(\boldsymbol{k}_\perp, L) = 2\alpha_s N_c \Re\mathfrak{e} \int \frac{d\omega}{\omega^3} \int_0^L dt_2 \int_0^{t_2} dt_1 \int_{\boldsymbol{q}_{1\perp}, \boldsymbol{q}_{2\perp}} \mathcal{P}^{(0)}(\boldsymbol{k}_\perp - \boldsymbol{q}_{2\perp}, L - t_2)$$
$$\times \mathcal{K}(\boldsymbol{q}_{2\perp} - \boldsymbol{q}_{1\perp}, t_2, t_1) \mathcal{P}^{(0)}(\boldsymbol{q}_{1\perp}, t_1)$$

TMB at one loop in a dense QCD medium (2/2)

- The kernel $\mathcal{K}(\boldsymbol{l}_\perp, t_2, t_1)$ involves the medium 3-point function $S^{(3)}$.

$$\mathcal{K}(\boldsymbol{l}_\perp, t_2, t_1) \equiv \int_{\boldsymbol{q}_\perp, \boldsymbol{q}'_\perp} (\boldsymbol{q}_\perp \cdot \boldsymbol{q}'_\perp) \left[\tilde{S}^{(3)}(\boldsymbol{q}_\perp, \boldsymbol{q}'_\perp, \boldsymbol{l}_\perp + \boldsymbol{q}'_\perp; t_2, t_1) - \tilde{S}^{(3)}(\boldsymbol{q}_\perp, \boldsymbol{q}'_\perp, \boldsymbol{l}_\perp; t_2, t_1)\right]$$

- $\tilde{S}_3 \simeq$ correlator of three in-medium propagators $(\boldsymbol{x}_{2\perp}|\mathcal{G}(t_2, t_1)|\boldsymbol{x}_{1\perp})$.
- In the "harmonic approximation" with $\hat{q}^{(0)} \simeq \hat{q}_0$:

$$\tilde{S}^{(3)}(\boldsymbol{q}_\perp, \boldsymbol{q}'_\perp, \boldsymbol{l}_\perp; \tau = t_2 - t_1) = \frac{16\pi}{3\hat{q}_0\tau} \exp\left\{-\frac{4[\boldsymbol{l}_\perp - (\boldsymbol{q}_\perp - \boldsymbol{q}'_\perp)/2]^2}{3\hat{q}_0\tau}\right\} \frac{2\pi i}{\omega\Omega\sinh(\Omega\tau)}$$
$$\times \exp\left\{-i\frac{(\boldsymbol{q}_\perp + \boldsymbol{q}'_\perp)^2}{4\omega\Omega\coth(\Omega\tau/2)} - i\frac{(\boldsymbol{q}_\perp - \boldsymbol{q}'_\perp)^2}{4\omega\Omega\tanh(\Omega\tau/2)}\right\}$$

with $\Omega^2 = i\hat{q}_0/(2\omega)$ and $\tau = t_2 - t_1$.

Double logarithmic enhancement (1/2)

- To have an idea of the order of magnitude, let's consider the average $\langle \boldsymbol{k}_\perp^2 \rangle$ from $\mathcal{P}^{(1)}(\boldsymbol{k}_\perp)$:

$$\langle \boldsymbol{k}_\perp^2 \rangle \sim \frac{\alpha_s N_c}{\pi} \int_{\tau_0}^{L} \frac{d\tau'}{\tau'} \int_{Q_s^2(\tau)}^{p_\perp^2} \frac{d\boldsymbol{q}_\perp^2}{\boldsymbol{q}_\perp^2} \hat{q}_0$$

- Double logarithmic enhancement, but unlike DGLAP or BFKL double log, non-linear saturation bound: $Q_s^2(\tau) \simeq \hat{q}_0 \tau$.

- Constrains the emission to be triggered by a single scattering.

Double logarithmic enhancement (2/2)

- Quenching parameter gets a τ and $\boldsymbol{k}_\perp$ dependence from radiative corrections:

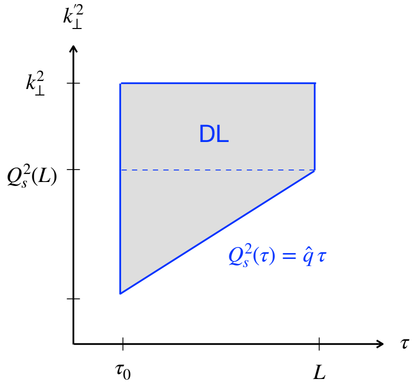
$$\hat{q}^{(1)}(\tau, \boldsymbol{k}_\perp^2) = \bar{\alpha}_s \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau'} \int_{Q_s^2(\tau)}^{\boldsymbol{k}_\perp^2} \frac{d\boldsymbol{q}_\perp^2}{\boldsymbol{q}_\perp^2} \hat{q}^{(0)} .$$

- Two distincts regimes:
  - $p_\perp^2 \leq Q_s^2(L)$

$$\hat{q}^{(1)} = \bar{\alpha}_s \hat{q}_0 \frac{1}{2} \ln^2\left(\frac{L}{\tau_0}\right)$$

  - $p_\perp^2 \geq Q_s^2(L)$ Blaizot, Dominguez, 1901.01448

$$\hat{q}^{(1)} = \bar{\alpha}_s \hat{q}_0 \left[\ln\left(\frac{\boldsymbol{k}_\perp^2}{\mu^2}\right) \ln\left(\frac{L}{\tau_0}\right) - \frac{1}{2}\ln^2\left(\frac{L}{\tau_0}\right)\right]$$



Resummation of the leading radiative corrections

- Resummation to all orders via the evolution equation

$$\frac{\partial \hat{q}(\tau, \boldsymbol{k}_\perp^2)}{\partial \tau} = \int_{Q_s^2(\tau)}^{\boldsymbol{k}_\perp^2} \frac{d\boldsymbol{k}_\perp'^2}{\boldsymbol{k}_\perp'^2} \bar{\alpha}_s(\boldsymbol{k}_\perp'^2)\ \hat{q}(\tau, \boldsymbol{k}_\perp'^2)$$

with $Q_s^2(\tau) \equiv \hat{q}(\tau, Q_s^2(\tau))\tau$.

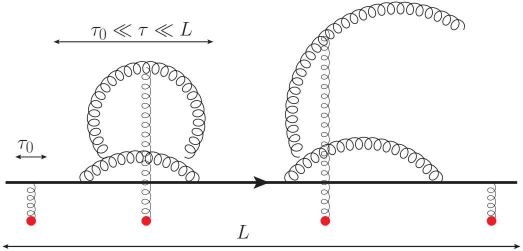
- Exponentiation of the double logarithmic corrections.

$$\mathcal{P}(\boldsymbol{k}_\perp) = \int \mathrm{d}^2\boldsymbol{x}_\perp\, e^{-i\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\perp}\ \exp\left[-\frac{1}{4}\left(\hat{q}^{(0)} + \alpha_s\hat{q}^{(1)} + \ldots\right) L\,\boldsymbol{x}_\perp^2\right]$$

cf Liou, Mueller, Wu, 1304.7677, Blaizot, Mehtar-Tani, 1403.2323, Iancu 1403.1996

Physical picture: non-locality vs quasi-locality

- Quantum corrections are non-local: logarithmic phase space for gluon fluctuations.



- However, at DLA, the exponentiation resums disconnected "towers" of radiative corrections, which can be seen as quasi-local.

Extended geometric scaling and Levy flights

Reminder: extended geometric scaling for gluon distribution

- Dipole S-matrix $S_\tau(\boldsymbol{r}_\perp) = \langle V^\dagger(\boldsymbol{r}_\perp)V(\boldsymbol{0}_\perp)\rangle_\tau$ related to the inclusive DIS cross-section at small-x:

$$\sigma(\tau = \ln(1/x), Q^2) \propto 2\int_0^1 dz \int d^2\boldsymbol{r}_\perp |\psi(z, \boldsymbol{r}_\perp, Q^2)|^2 (1 - S_\tau(\boldsymbol{r}_\perp))$$

- Energy (τ) dependence of $S_\tau(\boldsymbol{r}_\perp)$ determined from the BK equation.
- For $Q^2 \sim 1/\boldsymbol{r}_\perp^2 \ll Q_s^4/\Lambda_{\mathrm{QCD}}^2$, the dipole S-matrix satisfies extended geometric scaling

$$S_\tau(\boldsymbol{r}_\perp) \sim f\left(\frac{1}{\boldsymbol{r}_\perp^2\, Q_s^2(\tau)}\right)$$

cf Stasto, Golec-Biernat, Kwiecinski 0007192, Iancu, Itakura, McLerran 203137, etc

- A similar property holds when $S(L, \boldsymbol{x}_\perp)$ satisfies our non-linear DLA evolution equation with saturation boundary.

Asymptotic limit of TMB at fixed coupling

- In terms of logarithmic variables, $Y = \ln(L/\tau_0)$, $\rho = \ln(\boldsymbol{k}_\perp^2/\mu^2)$, $\rho_s = \ln(Q_s^2/\mu^2)$,

$$\frac{\partial \hat{q}(Y,\rho)}{\partial Y} = \bar{\alpha}_s \int_{\rho_s(Y)}^{\rho} d\rho' \hat{q}(Y,\rho')$$

- Let's look for scaling solution $\hat{q}(Y,\rho) = f(x = \rho - \rho_s(Y)$:

$$-\dot{\rho}_s f''(x) + [\dot{\rho}_s - 1]\, f'(x) - \bar{\alpha}_s f(x) = 0$$

- For physical initial conditions, the unique solution to this problem is $\dot{\rho}_s = c$ and

$$f(x) = e^{\beta x}(1 + \beta x)$$

with $\beta = (c-1)/(2c)$ and $c = 1 + 2\sqrt{\bar{\alpha}_s + \bar{\alpha}_s^2} + 2\bar{\alpha}_s$.

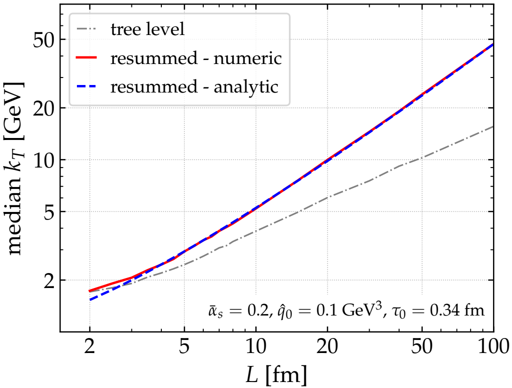
$\Longrightarrow$ **extended geometric scaling** for $x \ll \rho_s(Y)$ or $\boldsymbol{k}_\perp^2 \ll Q_s^4/\mu^2$. PC, Mehtar-Tani 2109.12041

Anomalous diffusion

- $\dot{\rho}_s = c \Longrightarrow \rho_s(Y) = cY$
- The median of the distribution scales like

$$\mathcal{M} \sim L^{1/2+\sqrt{\bar{\alpha}_s}}$$

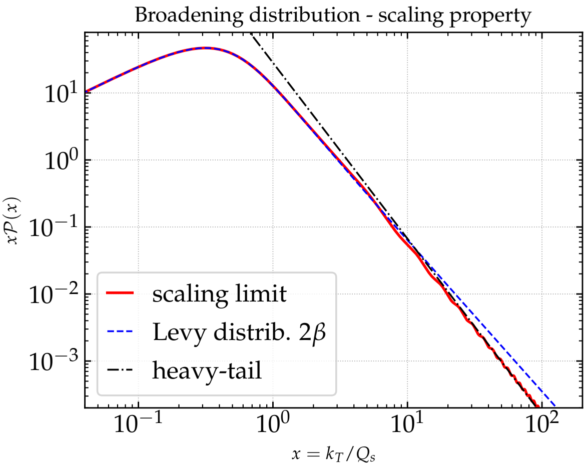
- $\Rightarrow$ super-diffusive behaviour. NLO corrections yields super-diffusion in momentum space.



Lévy-type distribution

- $\hat{q} \simeq e^{\beta x}$ at large k_T.
- Fourier transform of the "stretched" exponential $\exp(-[...]\boldsymbol{x}_\perp^\gamma)$ with $\gamma \simeq 2 + 2\sqrt{\bar{\alpha}_s} > 2$
- Heavy tailed distribution

$$\mathcal{P}(\boldsymbol{k}_\perp) \propto \frac{1}{k_T^{4-2\sqrt{\bar{\alpha}_s}}}$$



PC, Mehtar-Tani 2109.12041

Physical picture of quantum corrections

- Leading order: random walk in momentum space. Each collision transfers a typical momentum μ with Gaussian probability.
- With radiative corrections, one can still interpret the TMB distribution as a result of local random walk, but the "jump" probability distribution is of Lévy type.
- The non-linearity and self-similarity of overlapping multiple gluon radiations result in long rare steps which extends over a large range of transverse momenta.

Brownian motion Levy flight

Sub-asymptotic deviations and universality

Beyond the asymptotic limit

- We have determined the limit $Y \to \infty$ of the TMB distribution.
- What about the sub-asymptotic corrections?
- Are they universal?
- Can they be used to realistic values of $Y = \ln(L/\tau_0)$?

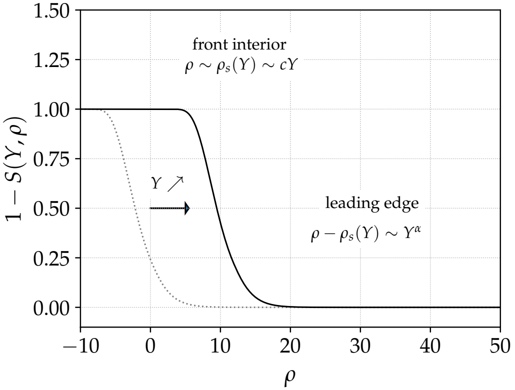
Wave front propagation into unstable state

- We borrow techniques from front propagation into unstable sate.
  Ebert, van Saarloos, 0003181
- Similar to the traveling wave interpretation of the solutions to BK.
  Munier, Peschanski, 0310357 - Beuf, 1008.0498
- Typical example: Fisher-Kolmogoroff-Petrovsky-Piscounoff eq.

$$\partial_t \phi = \partial_x^2 \phi + \phi - \phi^k$$

- Universality of the wave-front velocity $\dot{\rho}_s$:

$$\dot{\rho}_s = c + \frac{b}{Y} + \frac{d}{Y^2} + \ldots$$



Front interior vs leading edge expansion (1/2)

- Diffusive deviation from the asymptotic limit, with we consider.

$$\hat{q}(Y,\rho) = \hat{q}_0 e^{\rho_s(Y)-Y} e^{\beta x}\ Y^{\alpha} G\left(\frac{x}{Y^{\alpha}}\right)$$
$$\dot{\rho}_s(Y) = c + \dot{\sigma}_s(Y)$$

- Diffusion power characteristics of the universality class of the evolution equation.
- Homogeneity conditions fix the power α.
- $\alpha = 1/2$ for fixed coupling, $\alpha = 1/6$ for running coupling

Front interior vs leading edge expansion (2/2)

- We then write two types of expansion: front interior and leading edge.

$$\hat{q}(Y,\rho) = \hat{q}_0 e^{\rho_s(Y)-Y} e^{\beta x} \sum_{n\geq 0} \frac{1}{Y^{n\alpha}} f_n(x)$$

$$\hat{q}(Y,\rho) = \hat{q}_0 e^{\rho_s(Y)-Y} e^{\beta x} \sum_{n\geq -1} \frac{1}{Y^{n\alpha}} G_n\left(\frac{x}{Y^\alpha}\right)$$

- The matching of the front interior and leading edge, and the boundary condition enable to fix all the constants, and in particular those of the development of the wave front velocity $\dot{\rho}_s$.

Universality and non-linearity

- The asymptotic limit is universal = does not depend on the (tree-level) initial condition.
- We demonstrate that the pre-asymptotic form of $\hat{q}(Y,\rho)$ and $\rho_s(Y)$ are also universal.
- However, they depend on the shape of the equation of motion, in particular on the non-linearities:

$$\frac{\partial \hat{q}}{\partial Y} = \bar{\alpha}_s \int_{\rho_s(Y)}^{\rho} d\rho' \; \hat{q}(Y,\rho') \qquad \frac{\partial \hat{q}}{\partial Y} = \bar{\alpha}_s \int_{Y}^{\rho} d\rho' \; \hat{q}(Y,\rho')$$

Effects of non-linearities

$$\frac{\partial \hat{q}}{\partial Y} = \bar{\alpha}_s \int_Y^\rho d\rho' \; \hat{q}(Y, \rho')$$

- Analytic solutions exist, ex:

$$\rho_s(Y) = Y + \ln\left(\frac{\mathrm{I}_1(2\sqrt{\bar{\alpha}_s} Y)}{\sqrt{\bar{\alpha}_s} Y}\right)$$

- At large Y,

$$\rho_s(Y) = (1+2\sqrt{\bar{\alpha}_s})Y - \frac{3}{2}\ln(Y) + \mathcal{O}(Y^{-1})$$

$$\frac{\partial \hat{q}}{\partial Y} = \bar{\alpha}_s \int_{\rho_s(Y)}^\rho d\rho' \; \hat{q}(Y, \rho')$$

- No analytic solutions.
- From the wave front analysis, we get (with $c \simeq 1 + 2\sqrt{\bar{\alpha}_s}$)

$$\rho_s(Y) = cY - \frac{3c}{(1+c)}\ln(Y) - \frac{6c\sqrt{2\pi(c-1)}}{(1+c)^2}\frac{1}{\sqrt{Y}} + \mathcal{O}(Y^{-1})$$

- Back-reaction of the quantum evolution on $\rho_s(Y)$ induces corrections of order $\sqrt{\bar{\alpha}_s}\ln(Y)$ (larger than single log).

Results for fixed coupling

- For fixed coupling, we find the pre-asymptotic behaviour

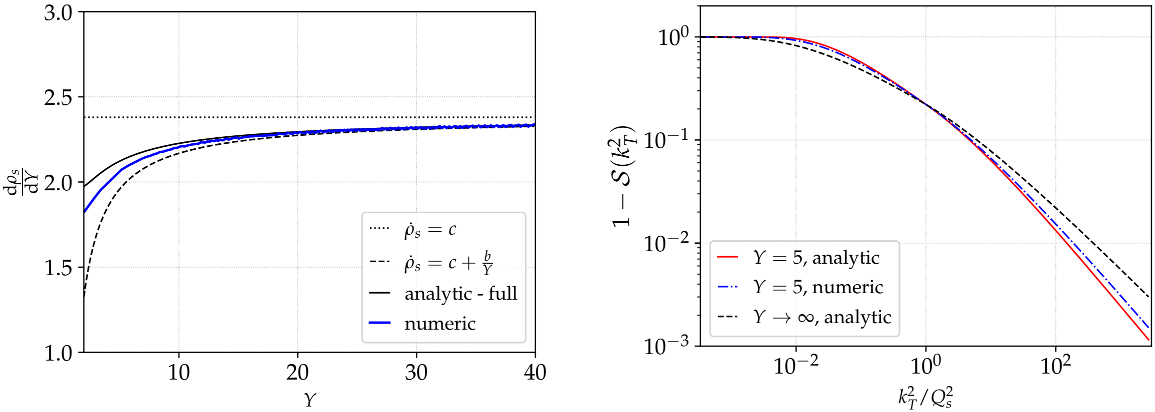
$$\frac{\hat{q}(Y,x)L}{Q_s^2(L)} = \begin{cases} \exp\left(\beta x - \frac{\beta x^2}{4cY}\right)\left[1+\beta x - \frac{3x}{c(1+c)Y}\left(1+\frac{\beta(c+4)x}{6}\right)+\mathcal{O}\left(\frac{1}{Y^2}\right)\right] & \text{if } x \geq 0 \\ \exp\left(2\beta x - \frac{3}{c(1+c)}\frac{x}{Y}+\mathcal{O}\left(\frac{1}{Y^2}\right)\right) & \text{if } x < 0\,. \end{cases} \tag{1}$$

with

$$\rho_s(Y) = cY - \frac{3c}{(1+c)}\ln(Y) - \frac{6c\sqrt{2\pi(c-1)}}{(1+c)^2}\frac{1}{\sqrt{Y}} + \mathcal{O}(Y^{-1})$$

PC, Mehtar-Tani 2109.12041

Some plots



- Sub-asymptotic corrections enable one to have a good agreement with the numeric.
- Analytic results can be systematically improved.

Running α_s: proof of the Iancu & Triantafyllopoulos conjecture

- The large Y development of $\rho_s(Y)$ for the rc-evolution has been found numerically in Iancu, Triantafyllopoulos 1405.3525

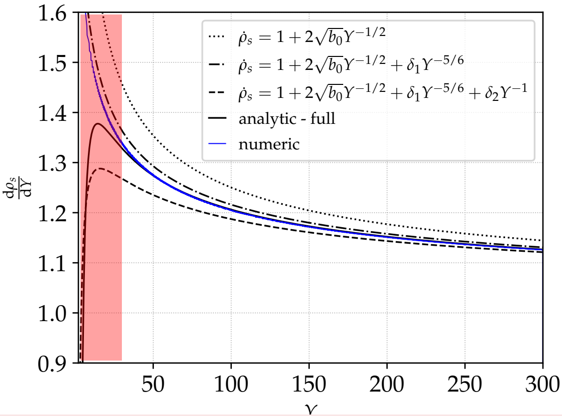
- With the TW technique, we can prove and improve their results

$$\rho_s(Y) = Y + 2\sqrt{4b_0 Y} + 3\xi_1 (4b_0 Y)^{1/6} + \left(\frac{1}{4} - 2b_0\right)\ln(Y) + \kappa + \frac{7\xi_1^2}{180}\frac{1}{(4b_0 Y)^{1/6}} + \xi_1\left(\frac{5}{108} + 18b_0\right)\frac{1}{(4b_0 Y)^{1/3}} + b_0\,(1 - 8b_0)\,\frac{\ln(Y)}{\sqrt{4b_0 Y}} + O\left(Y^{-1/2}\right)$$

- However, the development is divergent at moderate values of $Y \Longrightarrow$ divergent series.

PC, Mehtar-Tani, to appear

Front velocity of the rc equation

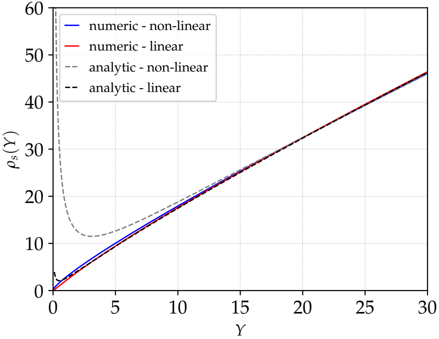


- Excellent agreement at large Y.
- At small Y, the asymptotic development fails to converge.

Phenomenology

Analytic formulas at small Y

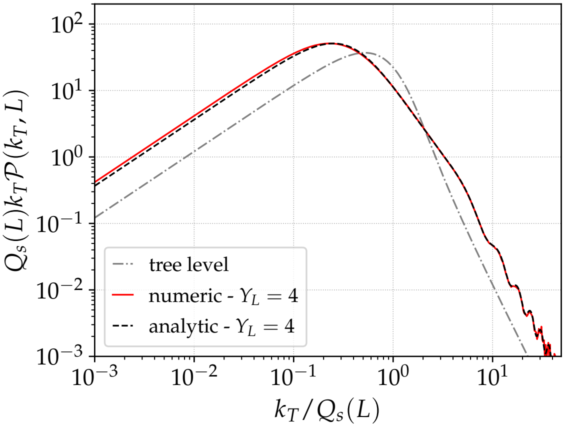
- Solution: drop the terms which make the series divergent.



- Use this analytic form for $\rho_s(Y)$ to compute $\hat{q}(Y,\rho)$ as a Taylor series around ρ_s:

$$\hat{q}(Y,x) = \hat{q}_0\, e^{\rho_s(Y)-Y}\left[1 + \frac{\dot{\rho}_s - 1}{\dot{\rho}_s}x + \frac{1}{2}\left(\left(\frac{\dot{\rho}_s - 1}{\dot{\rho}_s}\right)^2 + \frac{\ddot{\rho}_s}{\dot{\rho}_s^3} - \frac{\bar{\alpha}_s(\rho_s)}{\dot{\rho}_s}\right)x^2 + \mathcal{O}(x^2)\right]$$

Final result

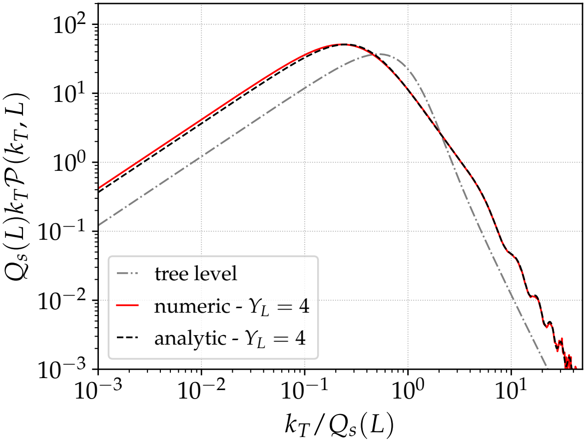


- Very good agreement over several orders of magnitude.
- Shape driven by the universality, and not by the initial condition.

Initial conditions for BK evolution

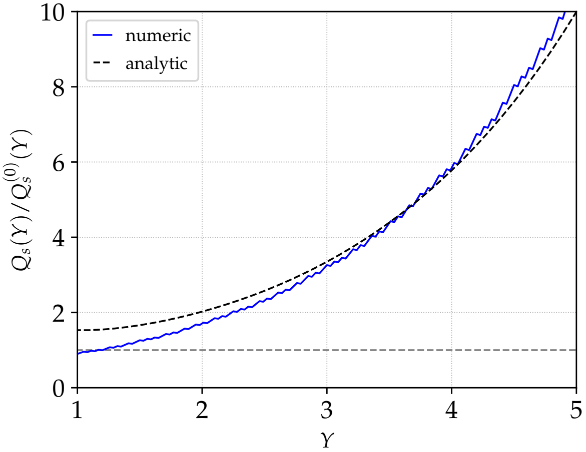
- Analytic formulas for the TMB or the dipole S-matrix $\mathcal{S}$ that resum to all orders gluon fluctuations enhanced by $\alpha_s \ln^2(A^{1/3})$.
- Down to $Y \sim 2$, the shape of $\mathcal{S}$ is driven by the universal behaviour of the traveling front.
- Physically motivated new initial condition for the BK-JIMWLK evolution equations.

see also Dumitru, Mantysaari, Paatelainen, 2103.11682



p_T-broadening in heavy-ion collisions

- $\langle k_\perp^2 \rangle$ responsible for the dijet azimuthal decorrelation related to the "renormalized" value of ρ_s.
- For realistic values of Y, enhancement factor of order $2-6$ compared to a tree-level estimation!
- Need to include single log corrections to reach greater precision.



Summary

- Study of the effect of radiative corrections on transverse momentum broadening in a dense QCD medium for large system sizes.
- TMB satisfies extended geometric scaling.
- Radiative corrections yield super-diffusive behaviour in momentum space, and a heavy tail with power index smaller than the typical Rutherford behaviour.
- The DLA non-linear evolution equations share similar mathematical properties as equations for wave front propagation into unstable states.
- Enable to compute the universal behaviour of the TMB distribution, valid down to realistic values of the system size.

THANK YOU!