Isospin Breaking corrections to leptonic meson decays

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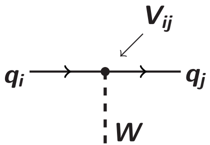
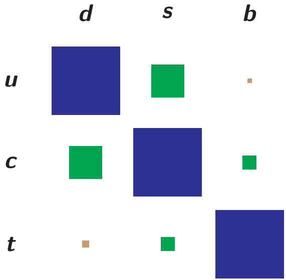
Introduction

u up	c charm	t top
d down	s strange	b bottom

- charged weak interaction ($W^{\pm}$): changes "up"-type quarks into a "down"-type quarks
- mixes different generation of quarks
- quark-mixing Cabibbo–Kobayashi–Maskawa (CKM) matrix

$$V_{\mathrm{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

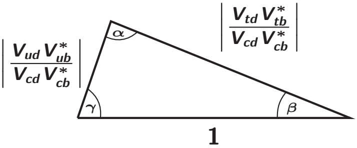


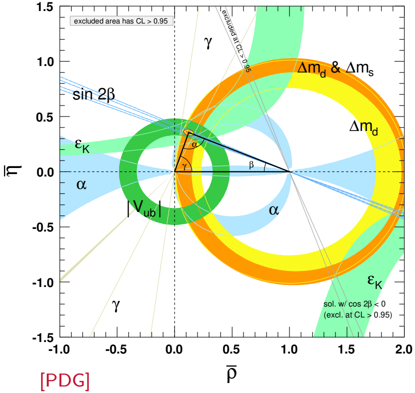


Unitarity of the CKM matrix

- within the Standard Model CKM matrix is unitary $\boldsymbol{V}_{\mathrm{CKM}} \boldsymbol{V}_{\mathrm{CKM}}^{\dagger} = \mathbb{1}$
  $\rightarrow$ test the Standard Model by finding deviations from Unitarity
- example

$$\boldsymbol{V_{ud} V_{ub}^* + V_{cd} V_{cb}^* + V_{td} V_{tb}^* = 0}$$

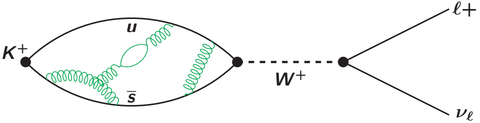




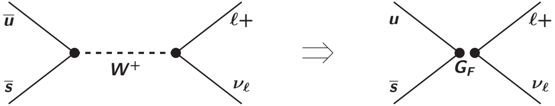
[PDG]

V_{us} from leptonic Kaon decays

- leptonic Kaon decay $K^+ \to \ell^+ \nu_\ell$



- effective weak Hamiltonian

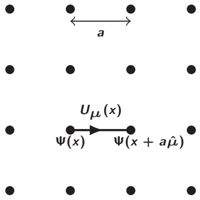


- decay rate (can be measured experimentally)

$$\Gamma(K^+ \to \ell^+ \nu_\ell) = \frac{G_F^2 \, |V_{us}|^2 \, f_K^2}{8\pi} \, M_K \, m_\ell^2 \left(1 - \frac{m_\ell^2}{M_K^2}\right)^2$$

- known factors (Fermi constant G_F, masses m)
- kaon decay constant f_K, can be calculated on the lattice
- CKM matrix element V_{us}

QCD on the lattice



- Wick rotation $(t \to -ix_0)$ to Euclidean space-time
- Discretize space-time by a hypercubic lattice Λ
- Quantize QCD using Euclidean path integrals

$$\langle A \rangle = \frac{1}{Z} \int \mathcal{D}[\Psi, \overline{\Psi}]\mathcal{D}[U]\, e^{-S_E[\Psi,\overline{\Psi},U]}\, A(U, \Psi, \bar{\Psi})$$

$\longrightarrow$ can be split into fermionic and gluonic part

- Calculate gluonic expectation values using Monte Carlo techniques:

$$\langle\langle A \rangle_F \rangle_G = \int \mathcal{D}[U]\, \langle A \rangle_F\, P(U) \approx \frac{1}{N_{\text{cfg}}} \sum_{n=1}^{N_{\text{cfg}}} \langle A \rangle_F$$

average over gluonic gauge configurations U distributed according to

$$P(U) = \frac{1}{Z} (\det D)^{N_f}\, e^{-S_G[U]}$$

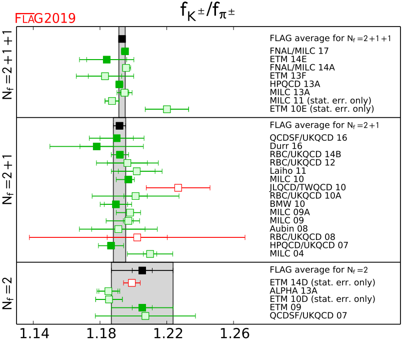
- extrapolate to the continuum $(a \to 0)$ and infinite volume $(V \to \infty)$

f_K/f_π from the lattice

- pseudoscalar meson decay constant from the lattice
- axial-vector matrix element

$$\mathcal{A}_K = \langle 0 |\, \overline{u}\gamma_0\gamma_5 s\, | K \rangle = M_K f_K$$

- overview Kaon/Pion decay constants



- results with precision $< \mathbf{1\%}$

Isospin Breaking Corrections

- lattice calculations usually done in the isospin symmetric limit
- two sources of isospin breaking effects
  - different masses for up- and down quark (of $\mathcal{O}((m_d - m_u)/\Lambda_{\mathrm{QCD}})$)
  - Quarks have electrical charge (of $\mathcal{O}(\alpha)$)
- lattice calculation aiming at **1%** precision requires to include isospin breaking

- separation of strong IB and QED effects requires renormalization scheme
- definition of “physical point” in a “QCD only world” also scheme dependent

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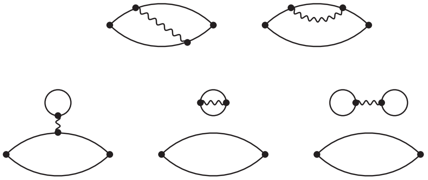
- Euclidean path integral including QED

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}[\Psi, \overline{\Psi}] \mathcal{D}[U] \mathcal{D}[A] \; O \; e^{-S_F[\Psi, \overline{\Psi}, U, A]} \, e^{-S_G[U]} \, e^{-S_\gamma[A]}$$

Expansion around IB symmetric (eg IB corrections to meson masses)

- perturbative expansion in α [RM123 Collaboration, Phys.Rev. **D87**, 114505 (2013)]

$$\langle O \rangle = \langle O \rangle_{e=0} + \frac{1}{2} e^2 \, \frac{\partial^2}{\partial e^2} \langle O \rangle \Big|_{e=0} + \mathcal{O}(\alpha^2)$$



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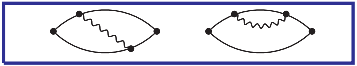
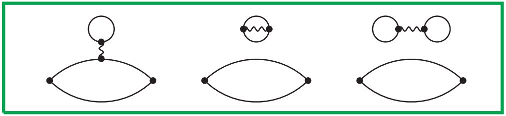


electro-quenched approximation



sea-quark effects

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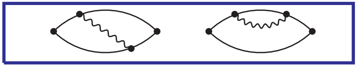

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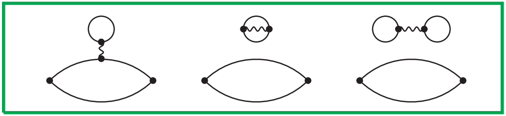
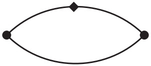


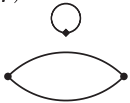
sea-quark effects

- perturbative expansion in $\Delta m_f = (m_f^0 - m_f)$ [G.M. de Divitiis *et al*, JHEP 1204 (2012) 124]

$$\langle O \rangle_{m_f} = \langle O \rangle_{m_f^0} + \Delta m_f \frac{\partial}{\partial m_f} \langle O \rangle \Big|_{m_f^0} + \mathcal{O}\left(\Delta m_f^2\right)$$



sea quark effects:
quark-disconnected diagrams



Tuning to physical point in a QCD+QED, $m_u \neq m_d$

- bare quark masses are free parameters in QCD

  $\rightarrow$ choose input quark masses such that a set of hadron masses receive their experimentally measured value including QED, e.g.

$$\left(m_u,\, m_d,\, m_s,\, \ldots\right) \qquad \longrightarrow \qquad \left(M_{\pi^+},\, M_{K^+},\, M_{K^0},\, \ldots\right)$$

$$m_{\pi^+}^{\text{exp}} = \left[m_\pi^0 + \alpha m_{\pi^+}^{\text{QED}} + \Delta m_d\; m_{\pi^+}^{\Delta m_d} + \Delta m_u\; m_{\pi^+}^{\Delta m_u}\right]$$
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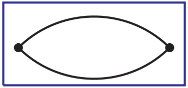

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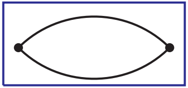

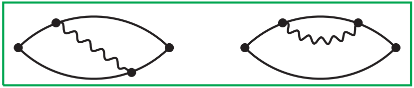

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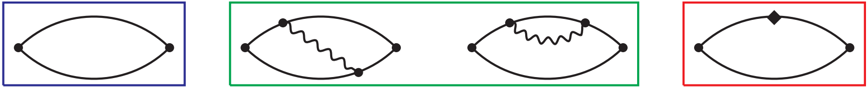

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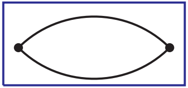

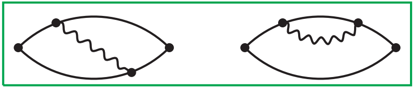

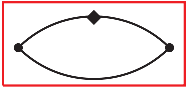

(plus another mass to determine the scale, e.g. the Ω-Baryon)

Renormalisation Scheme

- How much of IB comes from QED and how much from $m_u \neq m_d$?

$$X^{\phi} = \overline{X} + X^{SU(2)} + X^{\gamma} \qquad\qquad X^{\mathrm{QCD}} \equiv \overline{X} + X^{SU(2)}$$

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$m_u = m_d,\ \alpha = 0$ $\qquad$ $SU(2)$-Breaking correction

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$m_u = m_d,\ \alpha = 0$ $\qquad$ $SU(2)$-Breaking correction $\qquad$ QED correction

- Answer depends on the renormalisation scheme!

- define arbitrary unphysical values $\overline{\Pi}$ and Π^{QCD} for a set of quantities Π

$\alpha = 0,\ m_u = m_d$ $\qquad$ $\alpha = 0,\ m_u \neq m_d$

- ambiguities in choosing different schemes
- intermediate quantities can only be compared when using schemes, where ambiguities are of higher order
  $\longrightarrow$ efforts in this direction:
  [S. Aoki *et al*, Eur.Phys.J.C 80 (2020), http://flag.unibe.ch/2019/]
  [Whitepaper $g-2$ Theory Initiative, arXiv:2006.04822]

Renormalisation Scheme - examples

- e.g. GRS scheme [J. Gasser *et al*, Eur. Phys. J. C32, 97]
  0. use quark masses in some massless renormalisation scheme
  $$\Pi = (m_{ud} = 1/2(m_u + m_d), \delta m = (m_u - m_d), m_s)$$
  1. determine physical quark masses $\Pi^{\phi} = (m_{ud}^{\phi}, \delta m^{\phi}, m_s^{\phi})$
  2. define quark masses to be equal in QCD and QCD+QED: $\Pi^{\mathrm{QCD}} = \Pi^{\phi}$
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- GRS/χPT-inspired mesonic scheme [S. Borsanyi *et al*, Phys. Rev. Lett. 111 (25), 252001]
  use the following set of quantities $\boldsymbol{\Pi = (M_{ud}^2, \Delta M^2, 2M_{K\chi}^2)}$

$$\boldsymbol{M_{ud}^2 = \frac{1}{2}\left(M_{uu}^2 + M_{dd}^2\right) \qquad \Delta M^2 = M_{uu}^2 - M_{dd}^2 \qquad 2M_{K\chi}^2 = M_{K^+}^2 + M_{K^0}^2 - M_{\pi^+}^2}$$

in χPT [J. Bijnens, and N. Danielsson, Phys. Rev. D75, 014505] $\boldsymbol{\Pi = 2B(m_{ud}, \delta m, m_s) + \cdots}$

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- GRS/χPT-inspired mesonic scheme [S. Borsanyi *et al*, Phys. Rev. Lett. 111 (25), 252001]
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$$\boldsymbol{M_{ud}^2 = \frac{1}{2}\left(M_{uu}^2 + M_{dd}^2\right) \qquad \Delta M^2 = M_{uu}^2 - M_{dd}^2 \qquad 2M_{K\chi}^2 = M_{K^+}^2 + M_{K^0}^2 - M_{\pi^+}^2}$$

  in χPT [J. Bijnens, and N. Danielsson, Phys. Rev. D75, 014505] $\boldsymbol{\Pi = 2B(m_{ud}, \delta m, m_s) + \cdots}$

- other schemes possible, e.g. [M. Di Carlo *et al*, Phys. Rev. D100 (3), 034514]

Decay rate leptonic meson decays

- P^+ decay rate in rest frame ($P = \{\pi, K\}$)

$$\Gamma(P^+ \to \ell^+\nu_\ell) = K \sum_{r,s} |\mathcal{M}^{r,s}|^2$$

summed over spins r, s of final state ($\ell\nu_\ell$)

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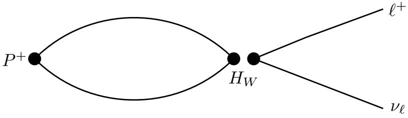
- matrix element

$$\mathcal{M}^{r,s} = \langle \ell^+, r; \nu_\ell, s | H_W | P^+ \rangle$$



- weak Hamiltonian H_W
- tree-level matrix element (hadronic and leptonic part factorisable)

$$\mathcal{M}_0^{r,s} = f_P \, M_P \, \left(\overline{u}^r_{\nu_\ell} \, \gamma^\mu_L \, v^s_\ell \right) \qquad \gamma^\mu_L = \gamma^\mu (1 - \gamma_5)$$

Decay rate leptonic meson decays

- tree-level decay rate

$$\Gamma^0(P^+ \to \ell^+\nu_\ell) = \frac{G_F^2\,|V_{ij}|^2\,f_P^2}{8\pi}\;M_P\,m_\ell^2\left(1-\frac{m_\ell^2}{M_P^2}\right)^2$$

- full QCD+QED decay rate

$$\Gamma = \Gamma^0 + \delta\Gamma = \Gamma^0(1+\delta R) \qquad\qquad \delta R = \delta\Gamma/\Gamma_0$$

Decay rate leptonic meson decays

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$$\delta\Gamma = \delta K \sum_{r,s} \left|\mathcal{M}_0^{r,s}\right|^2 + 2K_0 \sum_{r,s} \Re(\mathcal{M}_0^{r,s} \; \delta\mathcal{M}^{r,s,*})$$

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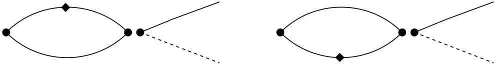
with

$$\mathcal{M}_0^{r,s} = f_P \, M_P \, \left(\overline{u}_{\nu_\ell}^r \, \gamma_L^\mu \, v_\ell^s\right) \qquad\qquad \delta\mathcal{M}^{r,s} = \overline{u}_{\nu_\ell}^r \, \delta\widetilde{\mathcal{M}} \, v_\ell^s$$

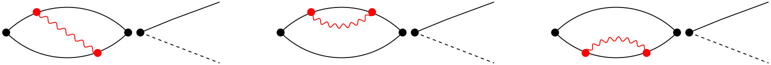
$$\Rightarrow \qquad \sum_{r,s} \Re(\mathcal{M}_0^{r,s} \; \delta\mathcal{M}^{r,s,*}) = f_P \, M_P \, \mathrm{Tr}[\not{p}_\nu \delta\widetilde{\mathcal{M}}(-\not{p}_\ell + i m_\ell)\gamma_L^\mu]$$

perturbative expansion - leptonic meson decay

- mass-shift corrections $\mathcal{O}(\Delta m)$



- quark QED corrections $\mathcal{O}(e_q^2)$



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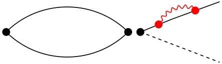
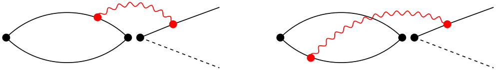


- quark-lepton QED correction $\mathcal{O}(e_\ell e_q)$



perturbative expansion - leptonic meson decay

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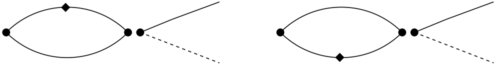

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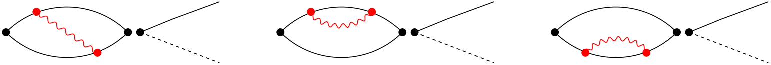
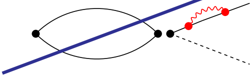


- ~~lepton QED corrections $\mathcal{O}(e_\ell^2)$~~



$\rightarrow$ absorbed in renormalisation of lepton

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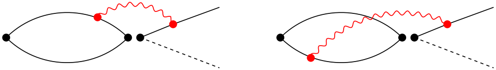

perturbative expansion - leptonic meson decay

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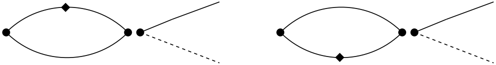

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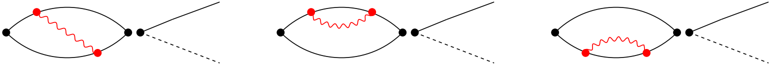

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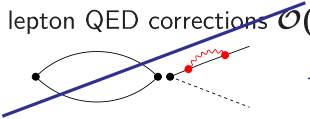

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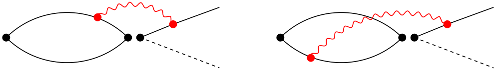

factorisable

perturbative expansion - leptonic meson decay

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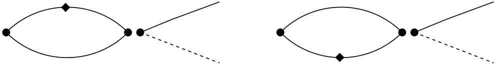

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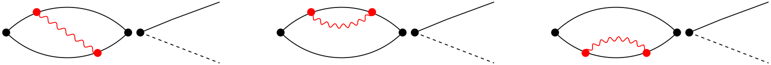

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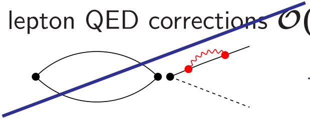

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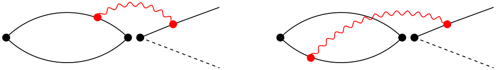
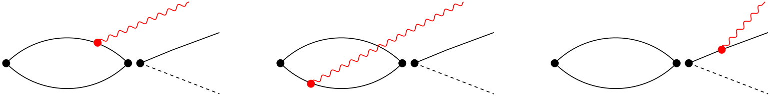


factorisable non-factorisable

IB corrections to leptonic meson decay

- Infrared divergencies cancled by diagrams with one final state photon

$$\Gamma(K^+ \to \ell^+ \nu_\ell, \alpha) + \Gamma(K^+ \to \ell^+ \nu_\ell \gamma)$$



- pioneering work to calculate IB correction to decay rate by RM123-Soton
  - formalism developed in [N. Carrasco *et al*, Phys.Rev. **D91**, 074506 (2015)]
  - finite volume effects [V. Lubicz *et al*, Phys. Rev. **D95**, 034504 (2017)]
  - first lattice results [M. Di Carlo *et al*, arXiv:1904.08731], [D. Giusti *et al*, Phys. Rev. Lett. **120**, 072001 (2018)]
  - final state photon radiation [A. Desiderio *et al*, arXiv:2006.05358]

- this work: calculation directly at the physical point

Lattice Setup

- $N_f = 2 + 1$ Möbius Domain Wall Fermions
- near physical quark masses
- inverse lattice spacing $a^{-1} = 1.730(4)$ GeV
- $48^3 \times 96$ with $L_s = 24$

- Feynman gauge and QED$_L$ for photon propagators

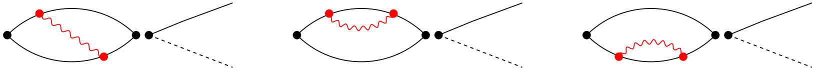
$$\Delta_{\mu\nu}(x-y) = \langle A_\mu(x) A_\nu(y) \rangle = \delta_{\mu\nu} \frac{1}{N} \sum_{k,\vec{k}\neq 0} \frac{e^{ik\cdot(x-y)}}{\hat{k}^2}$$

- use stochastic photon fields $A_\mu(x)$ to estimate $\Delta_{\mu\nu}(x-y)$
[D. Giusti et al. Phys.Rev. D95 (2017) 114504]
- em vertices using local vector currents $\gamma_\mu A_\mu = \not{A}$

- all results shown in this talk are preliminary

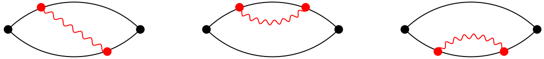
factorisable QED diagrams

- factorisable diagrams QED correction



$\rightarrow$ hadronic and leptonic part can be factorised (as in tree-level)

- only need to calculate



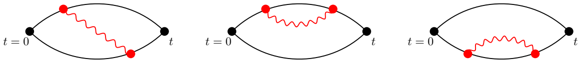
- IB correction from factorisable diagrams

$$\delta^{qq}\mathcal{M}^{rs} = \left(\overline{u}^{r}_{\nu_\ell}\,\gamma^{\mu}_{L}\,v^{s}_{\ell}\right)\delta\mathcal{A}$$

with

$$\delta\mathcal{A} = \delta\langle 0|\overline{q}_1\gamma_0\gamma_5 q_2|P^+\rangle$$

IB correction from factorisable diagrams

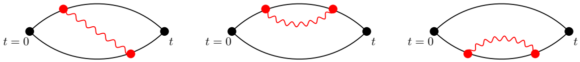


- correlators w/o QED (example: Kaon)

$$C^0_{PP}(t) = \langle 0|(\overline{s}\gamma_5 u)(\overline{u}\gamma_5 s)|0\rangle = A_0\, e^{-m_0 t} \qquad A_0 = \frac{\phi_0^2}{2m_0}$$

$$C^0_{AP}(t) = \langle 0|(\overline{s}\gamma_0\gamma_5 u)(\overline{u}\gamma_5 s)|0\rangle = B_0\, e^{-m_0 t} \qquad B_0 = \frac{\phi_0 \mathcal{A}_0}{2m_0}$$

IB correction from factorisable diagrams



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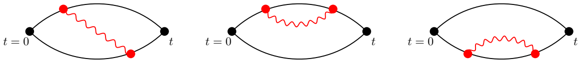
- full QCD+QED correlator $C(t) = C^0 + \delta C(t) = (A_0 + \delta A)e^{-t(m_0+\delta m)}$

IB correction from factorisable diagrams



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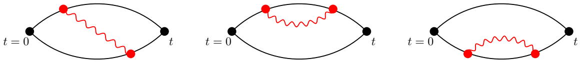
- $\mathcal{O}(\alpha)$ QED corrections

$$\frac{\delta C_{PP}(t)}{C^0_{PP}(t)} = \frac{\delta A}{A_0} - \delta m\, t = 2\frac{\delta\phi}{\phi_0} - \frac{\delta m}{m_0} - \delta m\, t$$

$$\frac{\delta C_{AP}(t)}{C^0_{AP}(t)} = \frac{\delta B}{B_0} - \delta m\, t = \frac{\delta\phi}{\phi_0} + \frac{\delta \mathcal{A}}{\mathcal{A}_0} - \frac{\delta m}{m_0} - \delta m\, t$$

IB correction from factorisable diagrams



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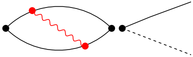
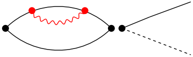
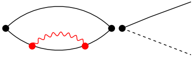
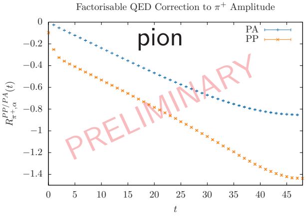
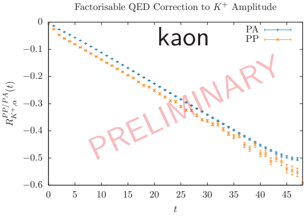
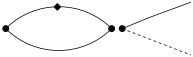
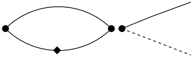
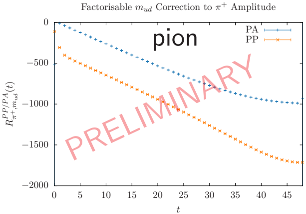
Results - factorisable contributions [Plots by A. Yong]

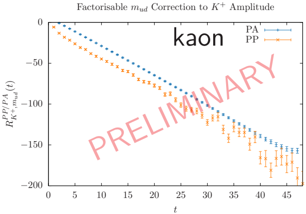








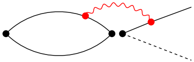



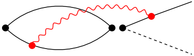




non-factorisable diagrams

- non-factorisable diagrams
- include lepton in lattice calculation
- neutrino can be done anlytically





- amputated weak Hamiltonian and matrix element

$$\overline{H}_W^{\alpha} = (\gamma_\mu^L \ell)^\alpha \, (\overline{q}_1 \gamma_\mu^L q_2) \qquad \overline{\mathcal{M}}^{r,\alpha} = \langle \ell^+, r | \overline{H}_W^{\alpha} | P^+ \rangle = (\widetilde{\mathcal{M}} v_\ell^r)^\alpha$$

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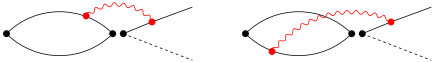

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- Euclidean three-point function

$$C^{\alpha\beta}(t_\ell, t_H, t_P) = \left\langle \overline{\ell}^{\alpha}(t_\ell) \, \overline{H}_W^{\beta}(t_H) \, \phi_P^\dagger(t_P) \right\rangle$$
$$= C_0^{\alpha\beta}(t_\ell, t_H, t_P) + \delta C^{\alpha\beta}(t_\ell, t_H, t_P) + \mathcal{O}(\alpha^2)$$

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- recap: QED correction to decay rate

$$\delta\Gamma \sim \sum_{r,s} \Re(\mathcal{M}_0^{r,s} \; \delta\mathcal{M}^{r,s,*})$$

$$\sum_{r,s} \Re(\mathcal{M}_0^{r,s} \; \delta\mathcal{M}^{r,s,*}) = f_P \, M_P \, \mathrm{Tr}[\not{p}_\nu \delta\widetilde{\mathcal{M}}(-\not{p}_\ell + i m_\ell)\gamma_L^{\mu}]$$

extract correction to decay rate

- spectral representation

$$\delta^{\ell q} C^{\alpha\beta}(t_\ell, t_H, t_P) = \frac{\phi_0 \left[\delta^{\ell q} \widetilde{\mathcal{M}} \left(-\not{p}_\ell + i m_\ell\right)\right]_{\alpha\beta}}{4 E_\ell M_P} e^{-(t_H - t_P) M_P} e^{-(t_\ell - t_H) E_\ell}$$

extract correction to decay rate

part of correction to decay rate

- spectral representation

$$\delta^{\ell q} C^{\alpha\beta}(t_\ell, t_H, t_P) = \frac{\phi_0 \boxed{\left[\delta^{\ell q} \widetilde{\mathcal{M}} \left(-\not{p}_\ell + i m_\ell\right)\right]_{\alpha\beta}}}{4 E_\ell M_P} e^{-(t_H - t_P) M_P} e^{-(t_\ell - t_H) E_\ell}$$

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- include the lepton trace

$$\mathrm{Tr}\left[\not{p}_\nu \delta^{\ell q} C(t_H) \gamma_L^\mu\right] \propto \mathrm{Tr}\left[\not{p}_\nu \delta^{\ell q} \widetilde{\mathcal{M}}\left(-\not{p}_\ell + i m_\ell\right) \gamma_L^\mu\right]$$

extract correction to decay rate

part of correction to decay rate

- spectral representation

$$\delta^{\ell q} C^{\alpha\beta}(t_\ell, t_H, t_P) = \frac{\phi_0 \boxed{\left[\delta^{\ell q}\widetilde{\mathcal{M}}\left(-\not{p}_\ell + i m_\ell\right)\right]_{\alpha\beta}}}{4E_\ell M_P} e^{-(t_H - t_P)M_P} e^{-(t_\ell - t_H)E_\ell}$$

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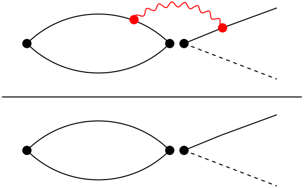
- $\delta^{\ell q}\Gamma/\Gamma_0$ can be obtained from long distance behaviour of

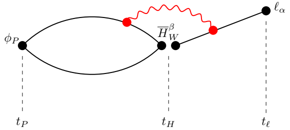
$$R(t_\ell, t_H, t_P) = \frac{\mathrm{Tr}\left[\not{p}_\nu \delta^{\ell q} C(t_\ell, t_H, t_P)\gamma_L^\mu\right]}{\mathrm{Tr}\left[\not{p}_\nu C^0(t_\ell, t_H, t_P)\gamma_L^\mu\right]}$$



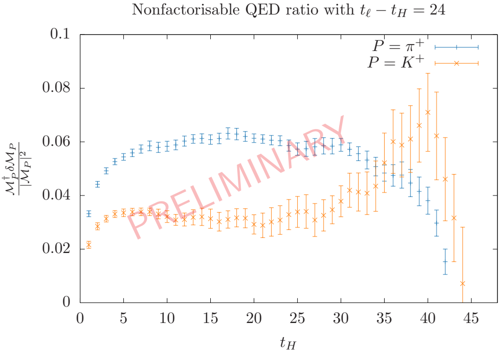
$$\longrightarrow \quad \delta^{\ell q}\Gamma/\Gamma_0$$

Results - non-factorisable contributions

- lattice calculation



- lepton: free Domain Wall Fermion with muon mass as pole mass
- twisted boundary conditions for muon for energy/momentum conservation



[Plot by A. Yong]

$$R(t_H) = \frac{\mathrm{Tr}\left[\not{p}_\nu \delta^{\ell q} C(t_H) \gamma_0^L\right]}{\mathrm{Tr}\left[\not{p}_\nu C^0(t_H) \gamma_0^L\right]}$$

$$\longrightarrow \quad \delta^{\ell q}\Gamma/\Gamma_0$$

Next step (putting everything together)

- Decay rate

$$\Gamma(P^+ \to \ell^+ \nu_\ell \,[\gamma]) = \Gamma^0(P^+ \to \ell^+ \nu_\ell)\,(1 + \delta R_P)$$

with

$$\Gamma^0(P^+ \to \ell^+ \nu_\ell) = \frac{G_F^2 \, |V_{ij}|^2 \, f_P^2}{8\pi} \; M_P \, m_\ell^2 \left(1 - \frac{m_\ell^2}{M_P^2}\right)^2$$

- ratio of pion and kaon decay rates

$$\frac{\Gamma(K^+ \to \mu^+ \nu_\mu \,[\gamma])}{\Gamma(\pi^+ \to \mu^+ \nu_\mu \,[\gamma])} = \left|\frac{V_{us}}{V_{ud}}\right|^2 \left|\frac{f_K}{f_\pi}\right|^2 \frac{M_\pi^3}{M_K^3} \left(\frac{M_K^2 - m_\mu^2}{M_\pi^2 - m_\mu^2}\right)^2 (1 + \delta R_K - \delta R_\pi)$$

Next step (putting everything together)

- Decay rate

$$\Gamma(P^+ \to \ell^+ \nu_\ell\,[\gamma]) = \Gamma^0(P^+ \to \ell^+ \nu_\ell)\,(1 + \delta R_P)$$

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experimental input

Next step (putting everything together)

- Decay rate

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experimental input   calculation

Next step (putting everything together)

- Decay rate

$$\Gamma(P^+ \to \ell^+\nu_\ell\,[\gamma]) = \Gamma^0(P^+ \to \ell^+\nu_\ell)\,(1+\delta R_P)$$

with

$$\Gamma^0(P^+ \to \ell^+\nu_\ell) = \frac{G_F^2\,|V_{ij}|^2\,f_P^2}{8\pi}\;M_P\,m_\ell^2\left(1-\frac{m_\ell^2}{M_P^2}\right)^2$$

- ratio of pion and kaon decay rates

$$\boxed{\frac{\Gamma(K^+ \to \mu^+\nu_\mu\,[\gamma])}{\Gamma(\pi^+ \to \mu^+\nu_\mu\,[\gamma])}} = \left|\frac{V_{us}}{V_{ud}}\right|^2 \boxed{\left|\frac{f_K}{f_\pi}\right|^2} \boxed{\frac{M_\pi^3}{M_K^3}\left(\frac{M_K^2-m_\mu^2}{M_\pi^2-m_\mu^2}\right)^2} \boxed{(1+\delta R_K-\delta R_\pi)}$$

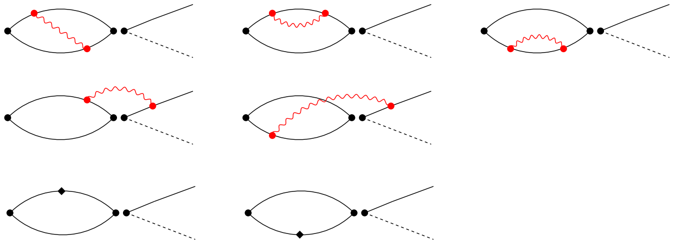
experimental input calculation

$\rightarrow$ determine V_{us}/V_{ud}

Summary

- lattice determinations of f_K, f_π have reached precision of $\lesssim$ **1%**
  $\rightarrow$ isospin breaking correction become important
  $\rightarrow$ necessary to improve determination of CKM matrix elements

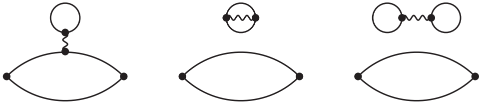
- preliminary results for isospin corrections to leptonic meson decays
  - physical point ensemble
  - factorisable and non-factorisable QED corrections to decay rate



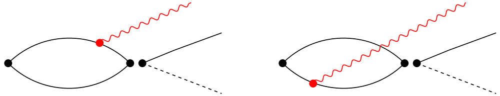
  - currently finalising the result for V_{us}/V_{ud}

Outlook

- disconnected diagrams & sea-quark effects, e.g.



- renormalisation of the weak Hamiltonian including QED
  $\rightarrow$ V_{us} and V_{ud} separately
- diagrams with final state photon



- semi-leptonic meson decays $K \rightarrow \pi\ell\nu$

$K_{\ell 3}^{\pm}$: $K^{\pm} \rightarrow \pi^0 \ell^{\pm} \nu_\ell$ $\qquad$ $K_{\ell 3}^{0}$: $K^{0} \rightarrow \pi^{\pm} \ell^{\mp} \nu_\ell$

$\rightarrow$ determination of V_{us}

Thank you