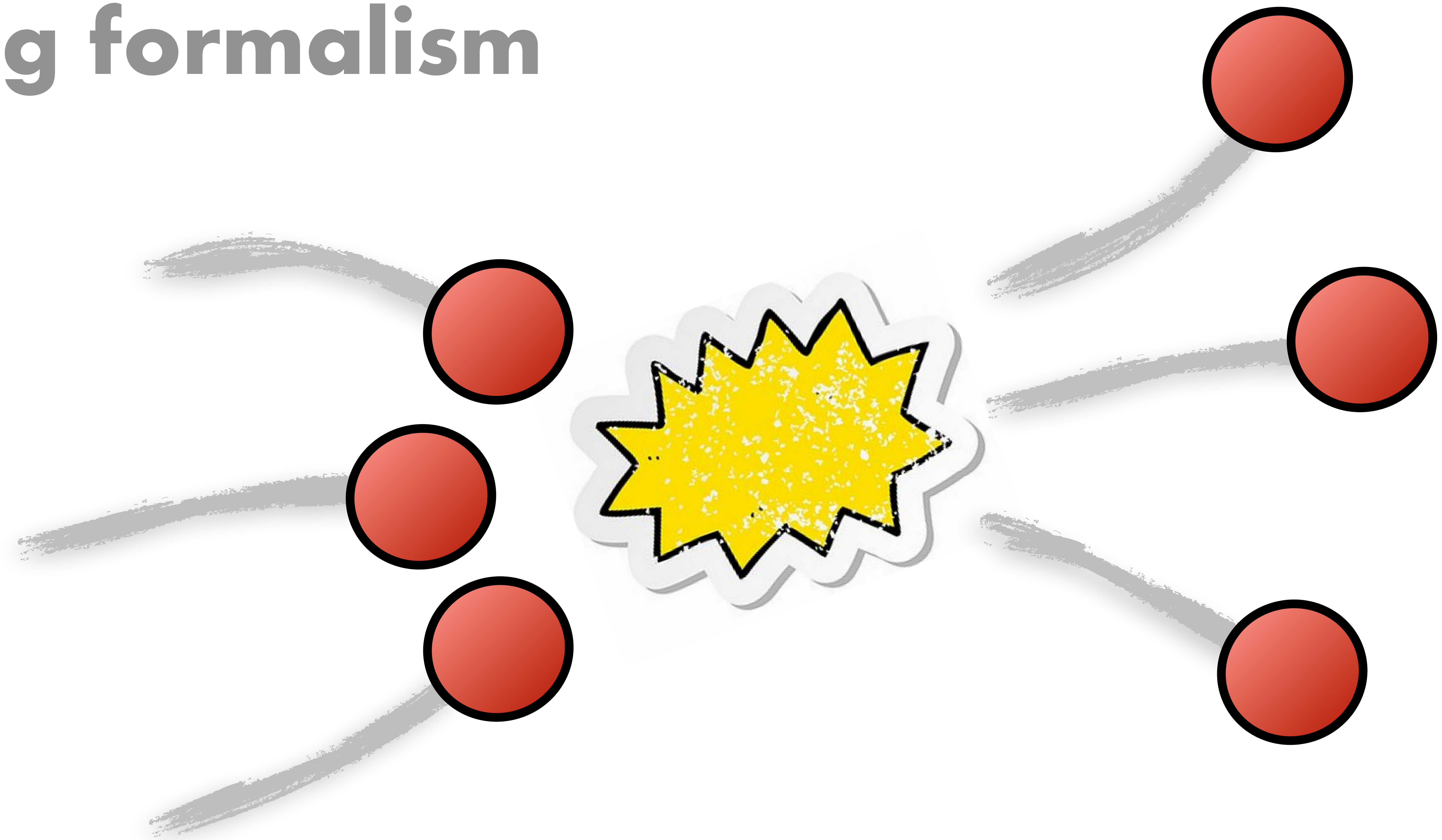
Bound states in the three-body scattering formalism



Sebastian M. Dawid

Indiana University Bloomington

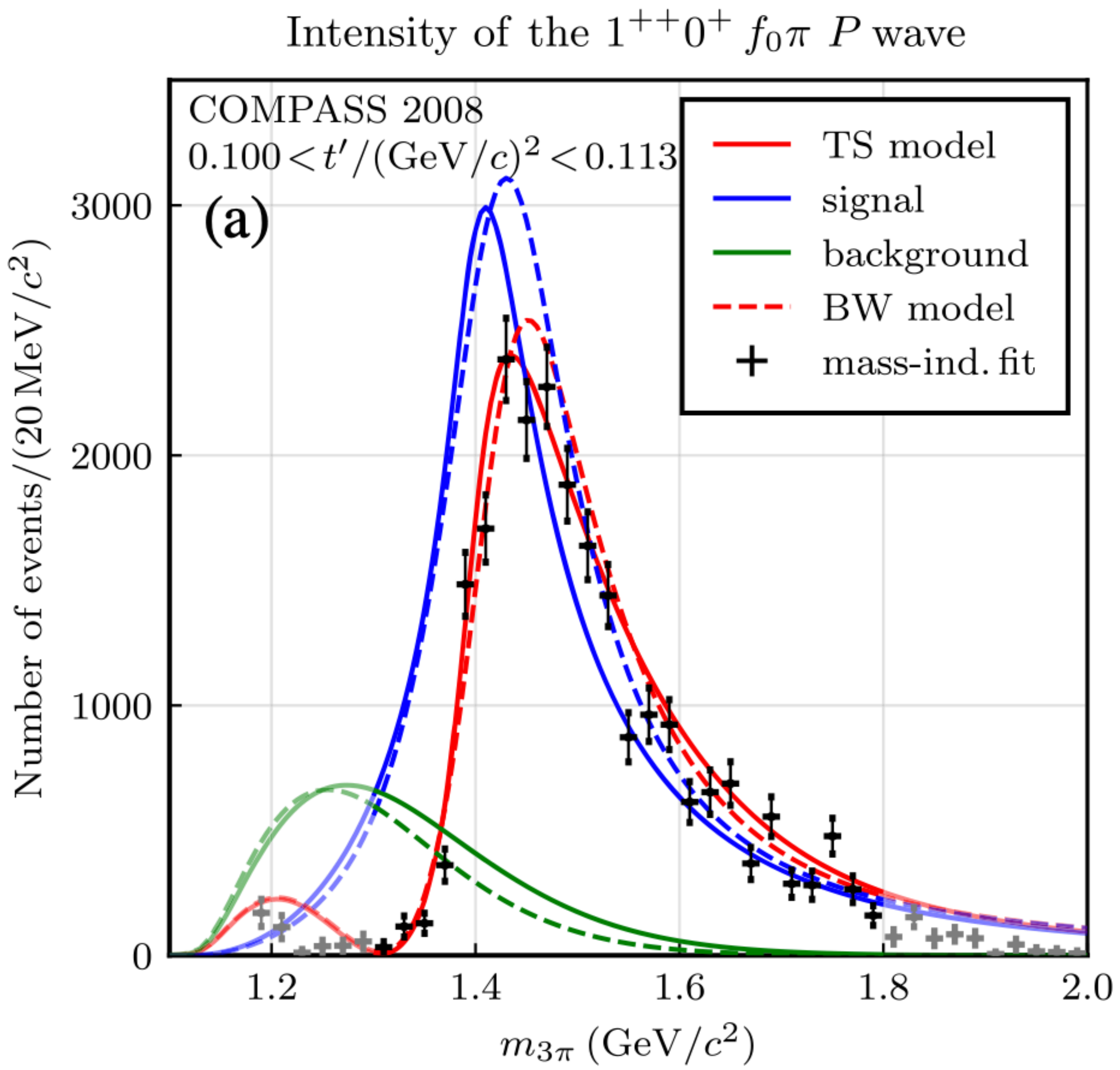
EIC workshop, August 17th, 2022

Three-body processes and hadronic spectrum

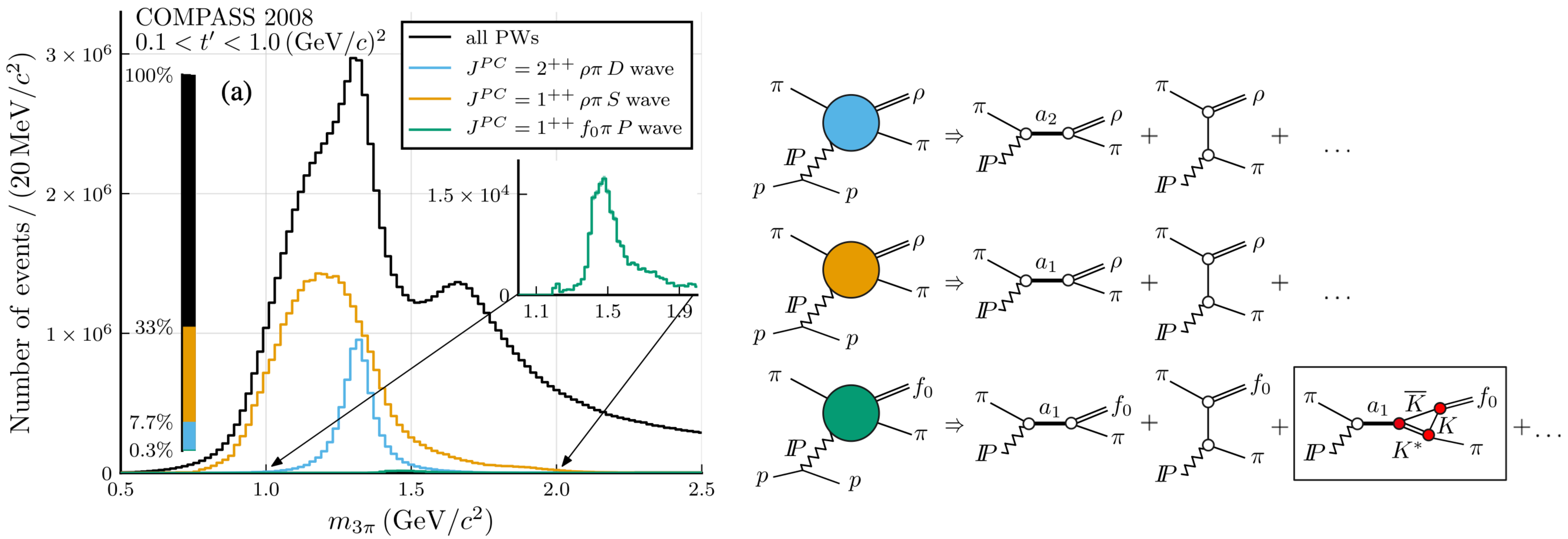
- Interesting resonances decay to three-particle final states
  - $X(3872)$, $\pi_1(1600)$, $N^*(1440)$, $a_1(1420)$, $a_1(1260)$, ...
- Interpretations
  - molecules
  - hybrids
  - diquark-antidiquark
  - kinematical effects

Three-body processes and hadronic spectrum

- Interesting resonances decay to three-particle final states
  - $X(3872)$, $\pi_1(1600)$, $N^*(1440)$, $a_1(1420)$, $a_1(1260)$, …
- Interpretations
  - molecules
  - diquark-antidiquark
  - hybrids
  - kinematical effects



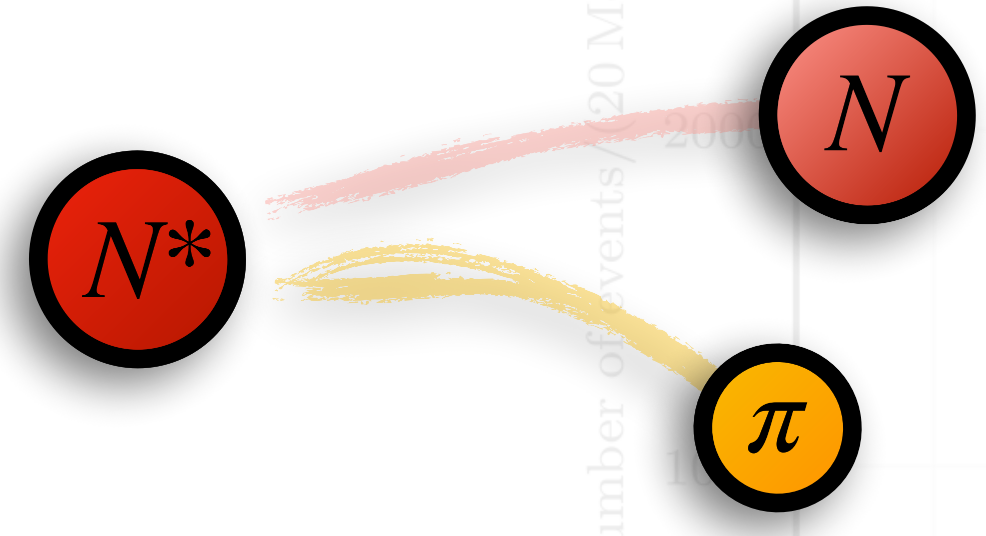
On the nature of $a_1(1420)$
Mikhasenko, Ketzer, Sarantsev, Phys. Rev. D 91, 094015 (2015)



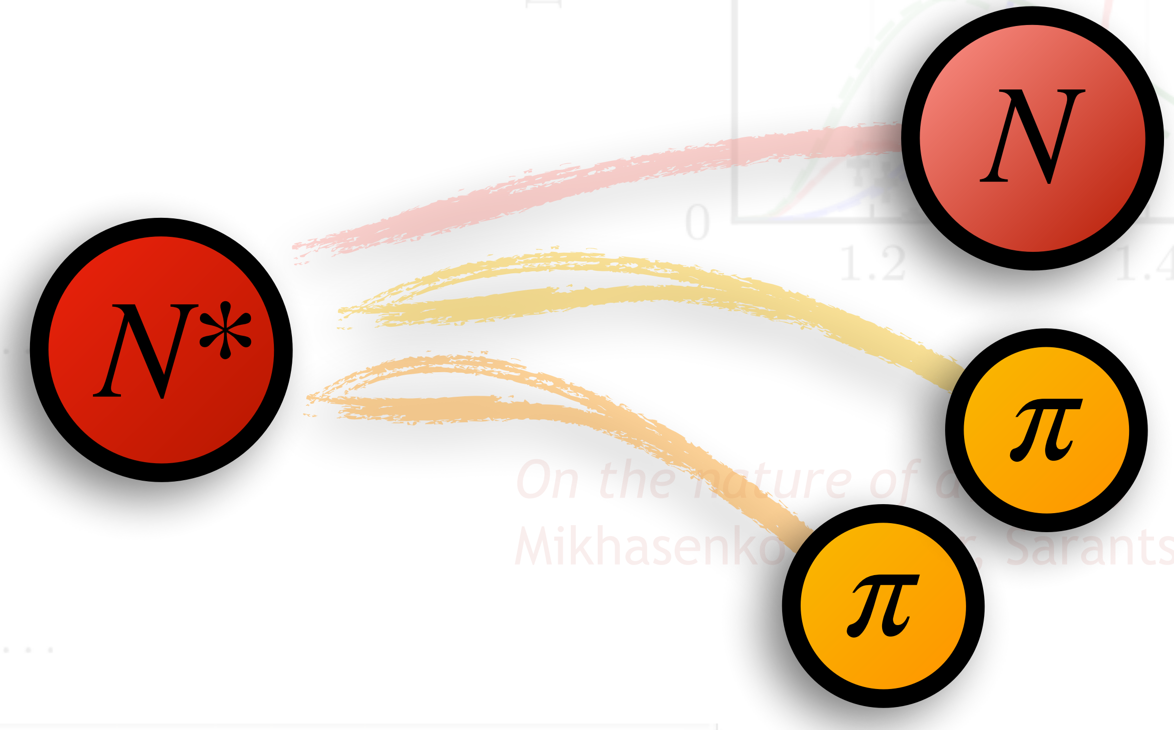
Triangle singularity as the origin of the $a_1(1420)$
COMPASS, Phys. Rev. Lett. 127 (2021) 8, 082501

Three-body processes and hadronic spectrum

- Interesting resonances decay to three-particle final states
  - $X(3872)$, $\pi_1(1600)$, $N^*(1440)$, $a_1(1420)$, $a_1(1260)$, …
- Interpretations
  - molecules
  - diquark-antidiquark
  - hybrids
  - kinematical effects



55-75%



30-40%

Pion-nucleon scattering in the Roper channel from lattice QCD
Lang, Leskovec, Padmanath, Prelovsek, Phys. Rev. D 95 (2017) 1, 014510

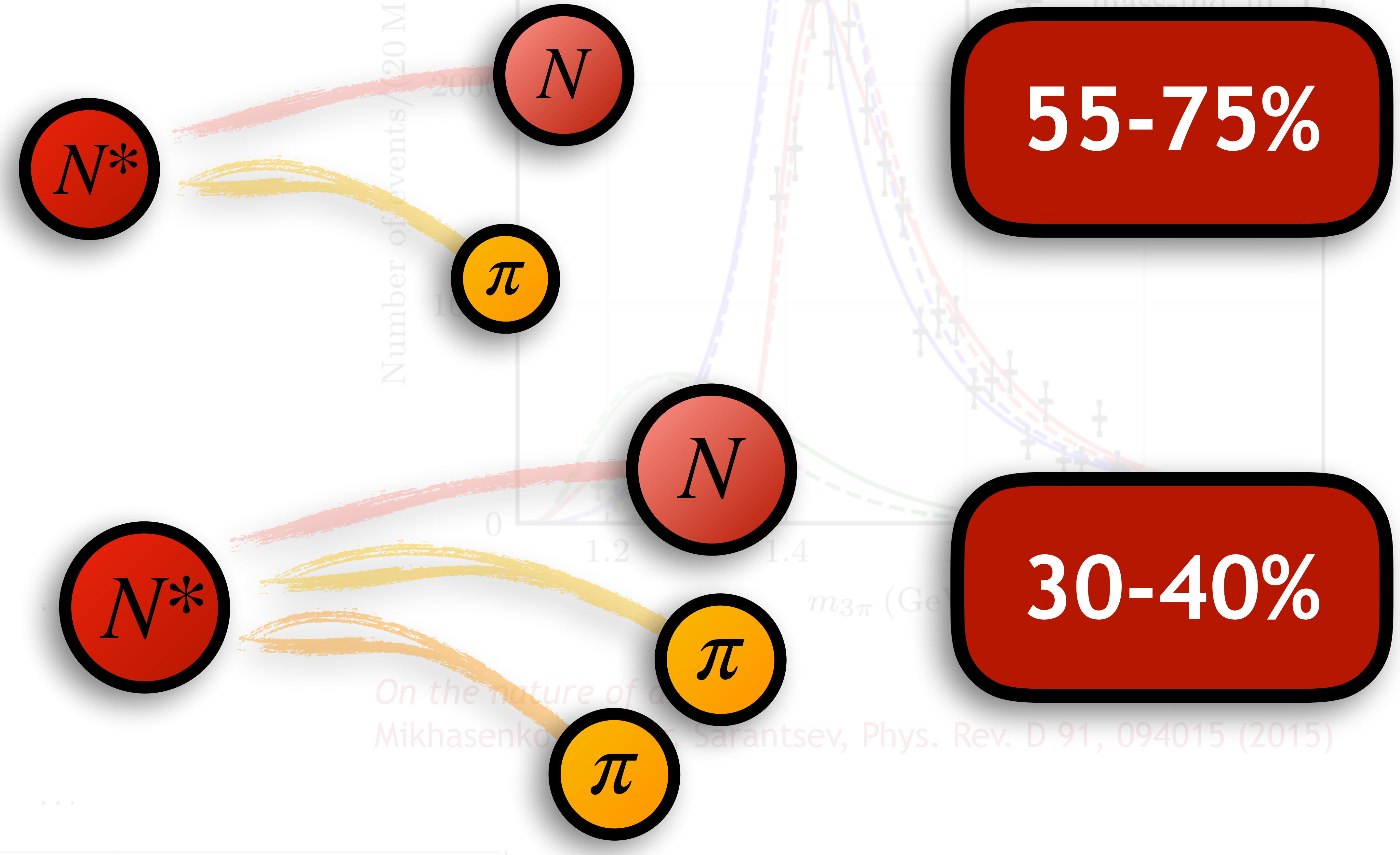
> additional energy level, which implies that $N\pi$ elastic scattering alone does not render a low-lying Roper resonance. The current status indicates that the $N^*(1440)$ might arise as dynamically generated resonance from coupling to other channels, most notably the $N\pi\pi$.

Three-body processes and hadronic spectrum

- Interesting resonances decay to three-particle final states
  - ✱ $X(3872)$, $\pi_1(1600)$, $N^*(1440)$, $a_1(1420)$, $a_1(1260)$, …
- Interpretations

GOAL: three-body scattering formalism

- ✱ properties of hadrons from the (lattice) QCD
- ✱ convenient three-body framework for phenomenology

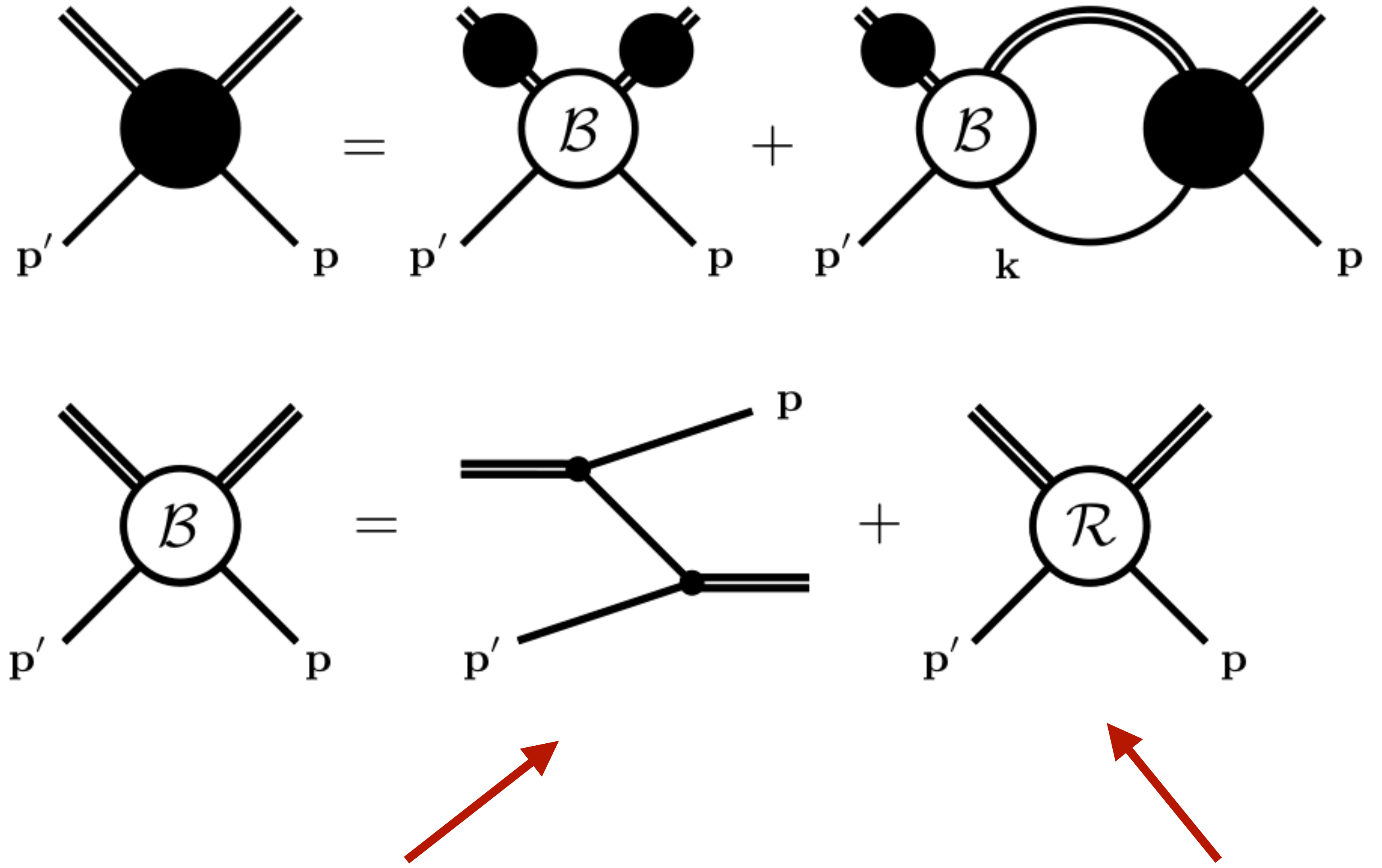


Pion-nucleon scattering in the Roper channel from lattice QCD
Lang, Leskovec, Padmanath, Prelovsek, Phys. Rev. D 95 (2017) 1, 014510

additional energy level, which implies that $N\pi$ elastic scattering alone does not render a low-lying Roper resonance. The current status indicates that the $N^*(1440)$ might arise as dynamically generated resonance from coupling to other channels, most notably the $N\pi\pi$.

The B-matrix approach

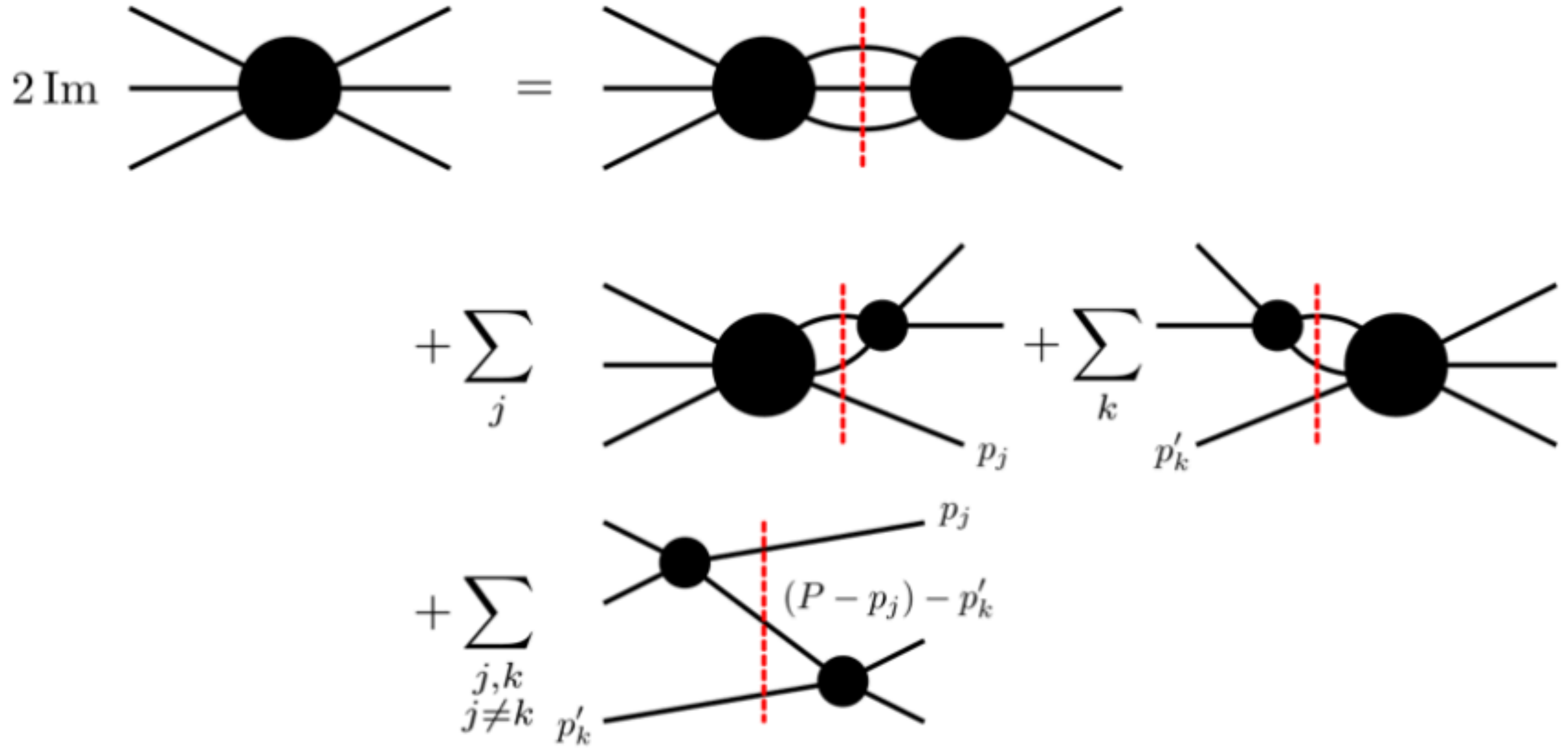
- Physical degrees of freedom **(domain of integration)**
- Simple parametrization with clear interpretation



One Particle Exchange

Short Range Interactions

$$\mathcal{A} = \mathcal{M}_2\,\mathcal{B}\,\mathcal{M}_2 + \mathcal{M}_2 \int \mathcal{B}\,\tau\,\mathcal{A}$$



Three-body amplitude

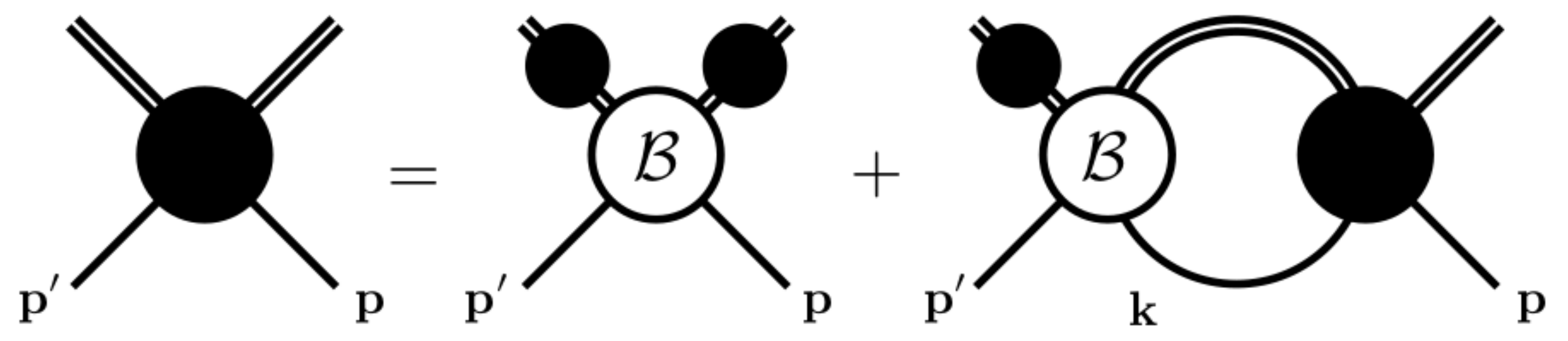
$$\mathcal{A}_{\ell' m_{\ell'};\ell m_\ell}(p', s, p)$$

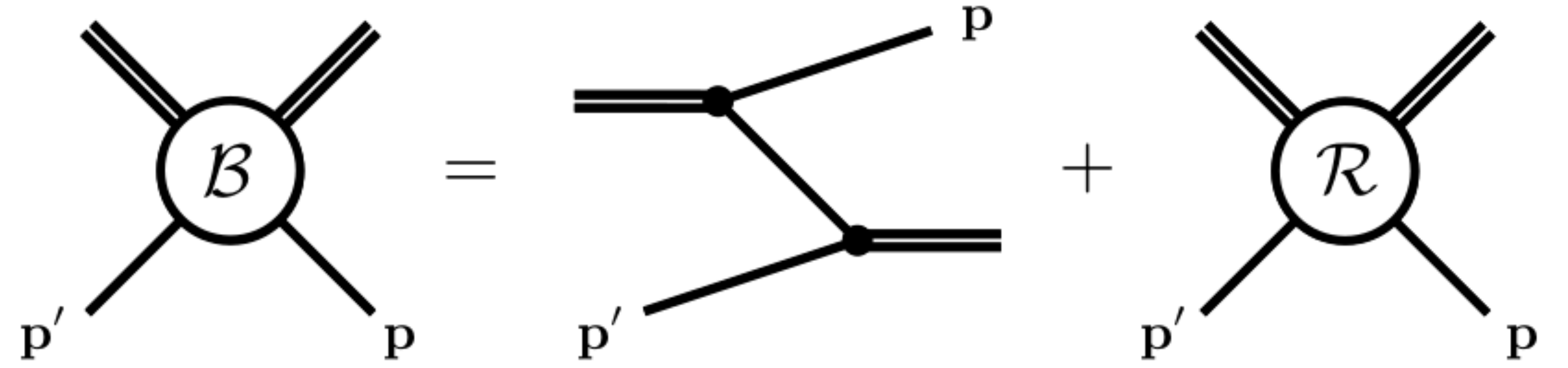
- pair-spectator
- partial waves
- symmetrization

Phenomenology of relativistic 3 → 3 reaction amplitudes within the isobar approximation
Jackura et al. (JPAC), Eur. Phys. J. C (2019) 1, 56

The B-matrix approach

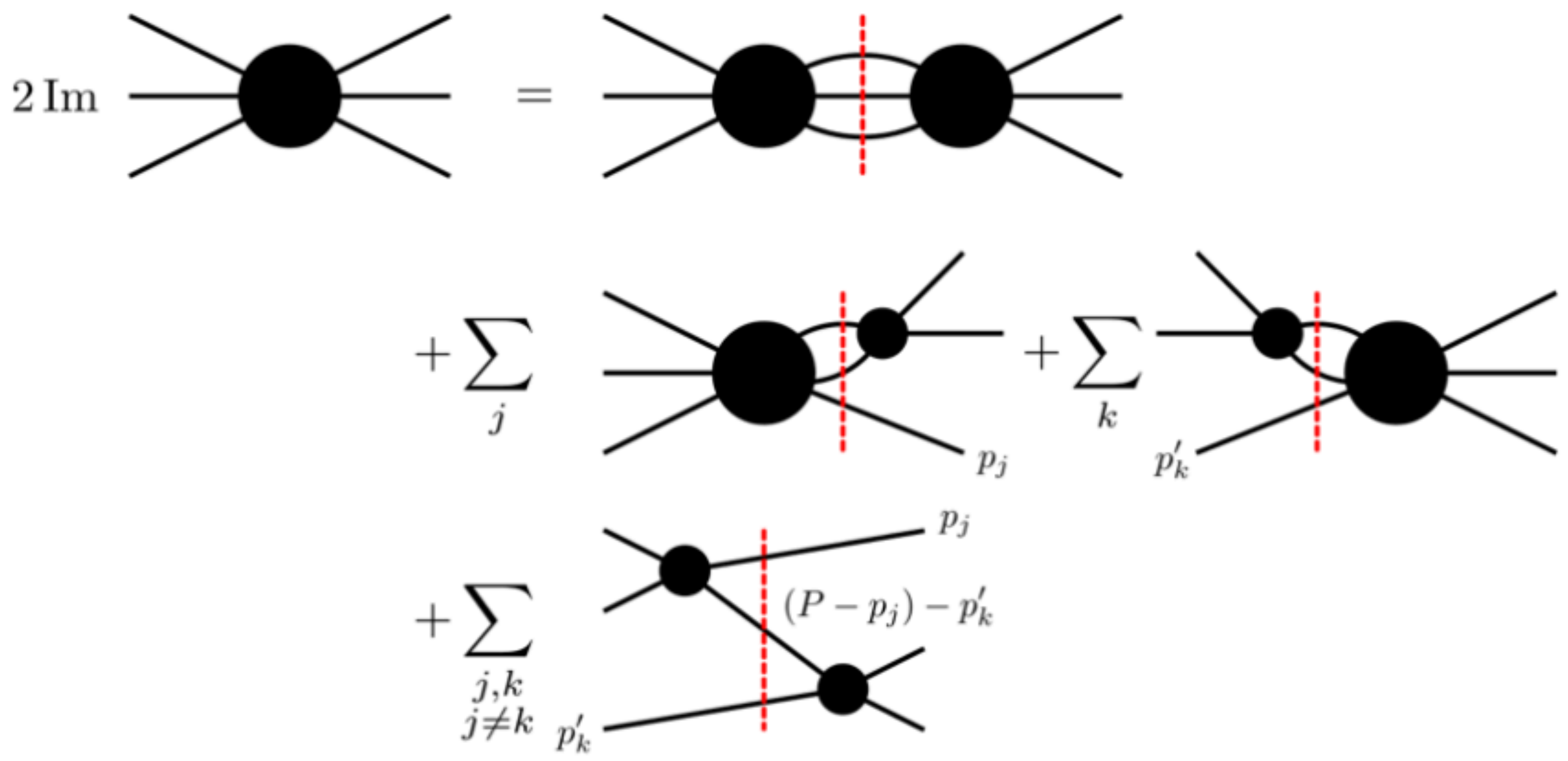
- Physical degrees of freedom **(domain of integration)**
- Simple parametrization with clear interpretation





One Particle Exchange

Short Range Interactions



Three-body amplitude

$$\mathcal{A}_{\ell' m_{\ell'};\ell m_\ell}(p', s, p)$$

- pair-spectator
- partial waves
- symmetrization

$$\tilde{A} = \mathcal{B} + \int \mathcal{B}\,\mathcal{M}_2\,\tau\,\tilde{A}$$

$$\mathcal{M}_2 = \mathcal{K} + \mathcal{K}\,i\rho\,\mathcal{M}_2$$

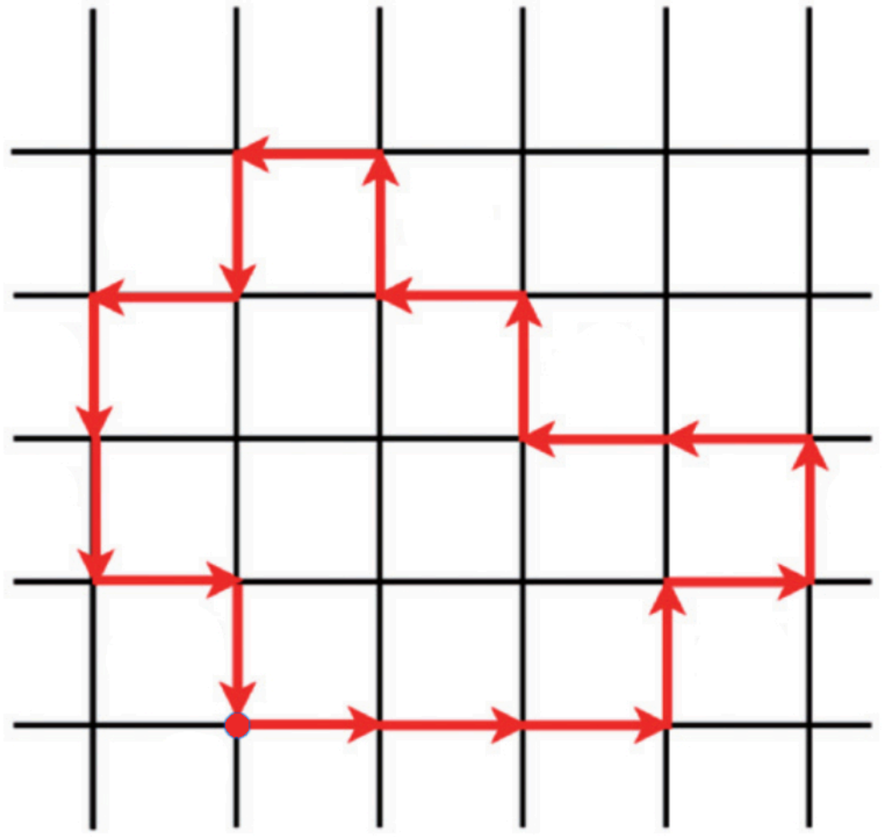
approximation

Jackura et al. (JPAC), Eur. Phys. J. C (2019) 1, 56

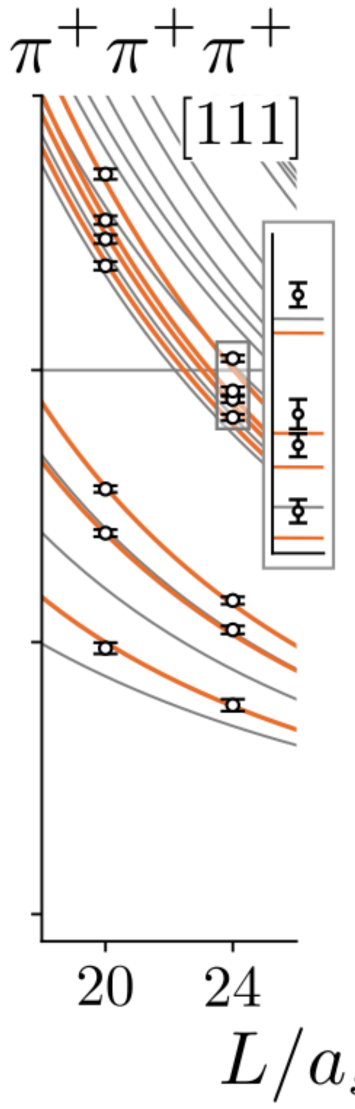
Relativistic, model-independent, three-particle quantization condition
Hansen, Sharpe, Phys. Rev. D 90 (2014) 11, 116003

Three-body unitarity in finite volume
Mai, Döring, Eur. Phys. J. A 53 (2017) 12, 240

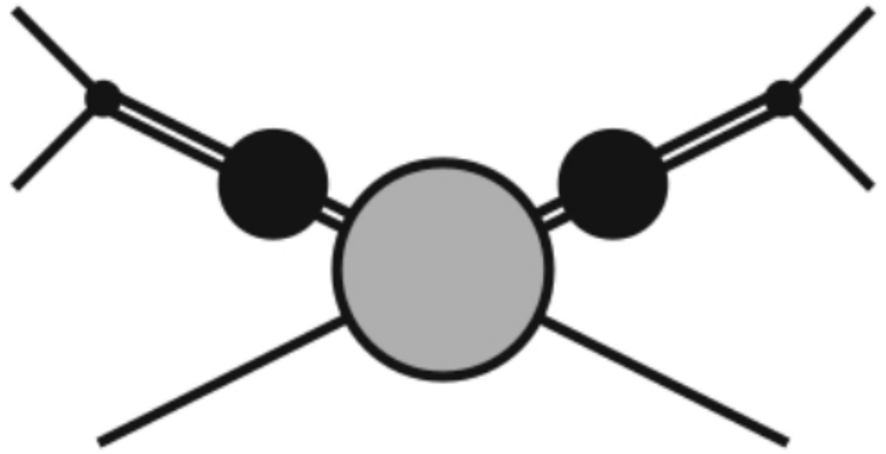
Path to three-body physics from the lattice QCD



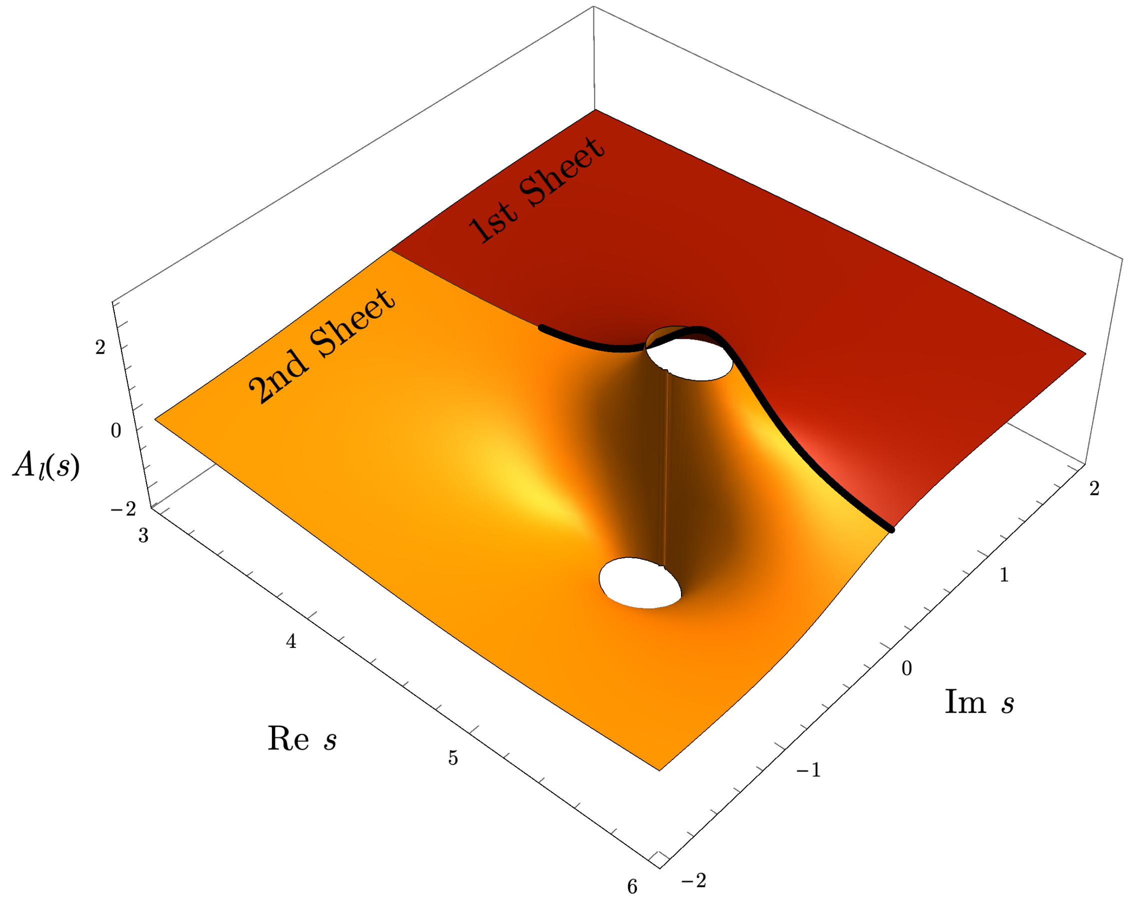
Lattice QCD



FV Spectrum



Amplitudes

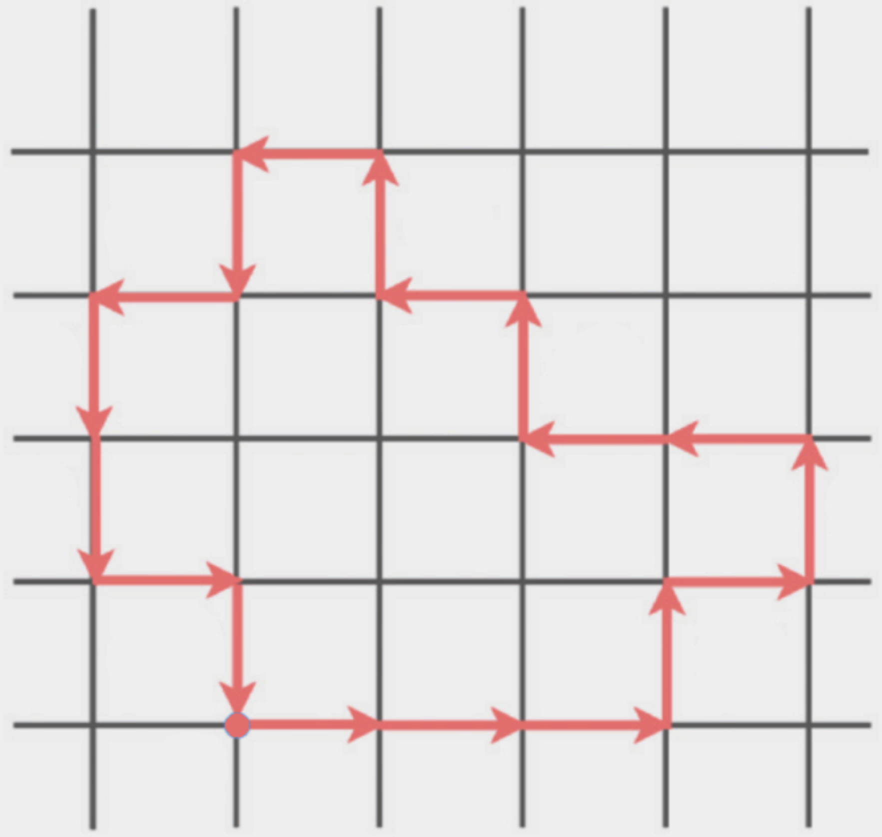


Particle properties

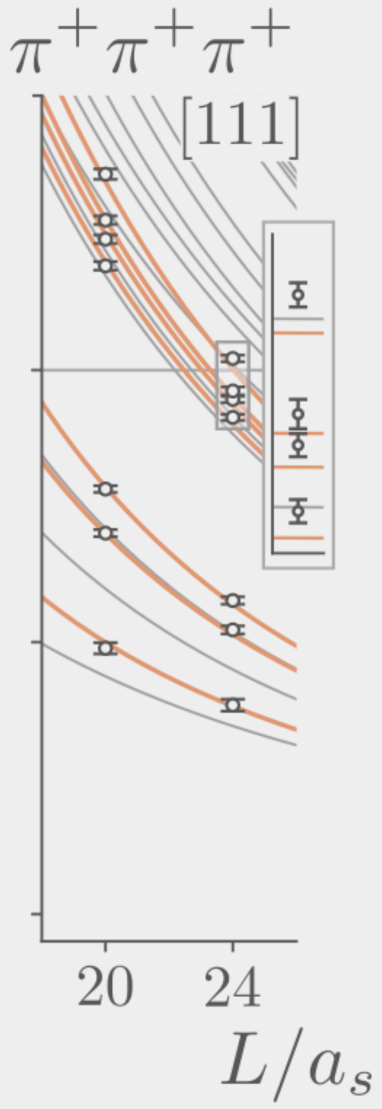
Relativistic, model-independent, three-particle quantization condition
Hansen, Sharpe, Phys. Rev. D 90 (2014) 11, 116003

Three-body unitarity in finite volume
Mai, Döring, Eur. Phys. J. A 53 (2017) 12, 240

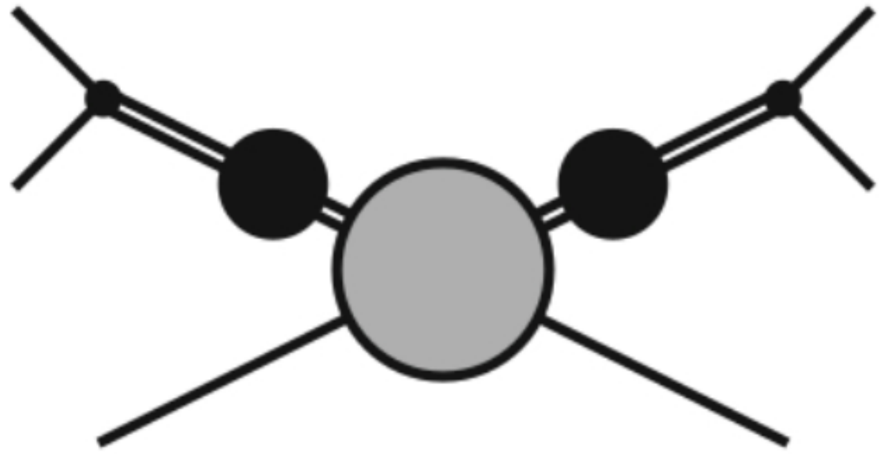
Path to three-body physics from the lattice QCD



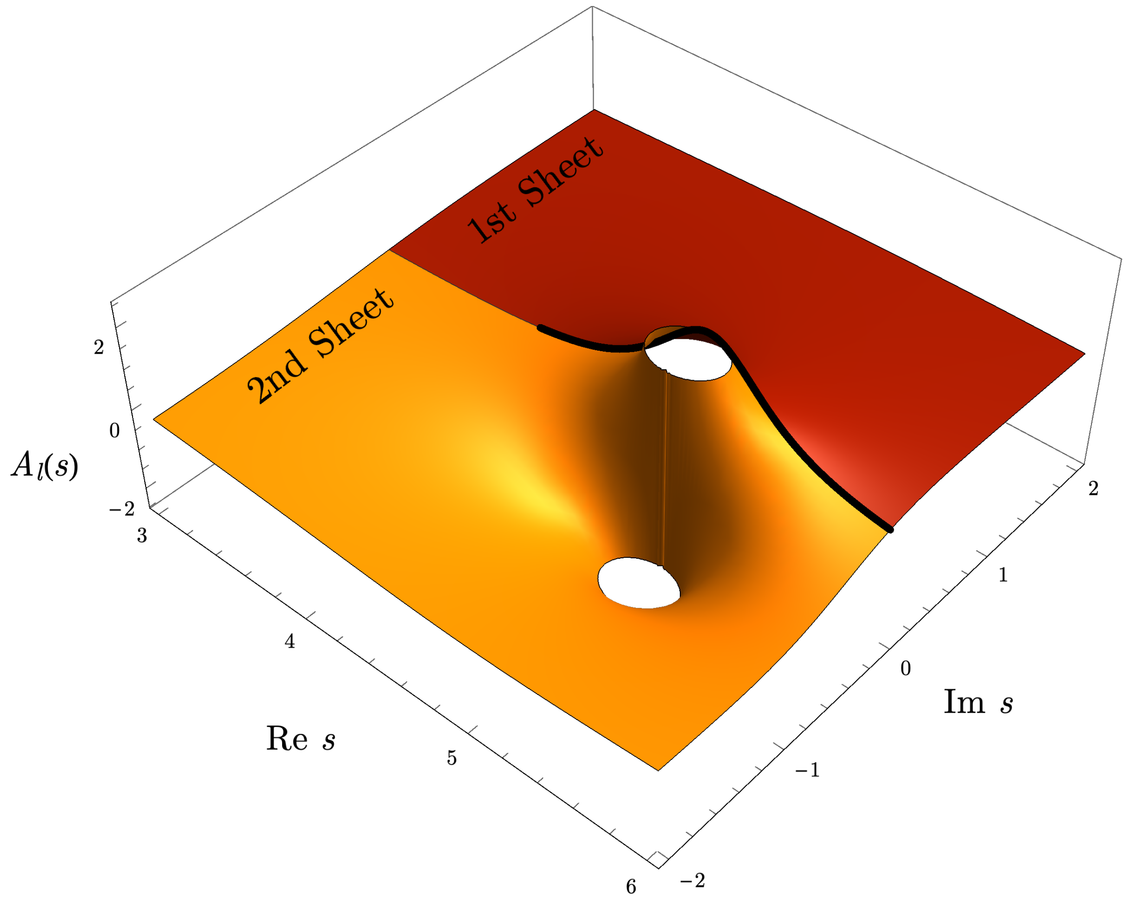
Lattice QCD



FV Spectrum



Amplitudes



Particle properties

FV

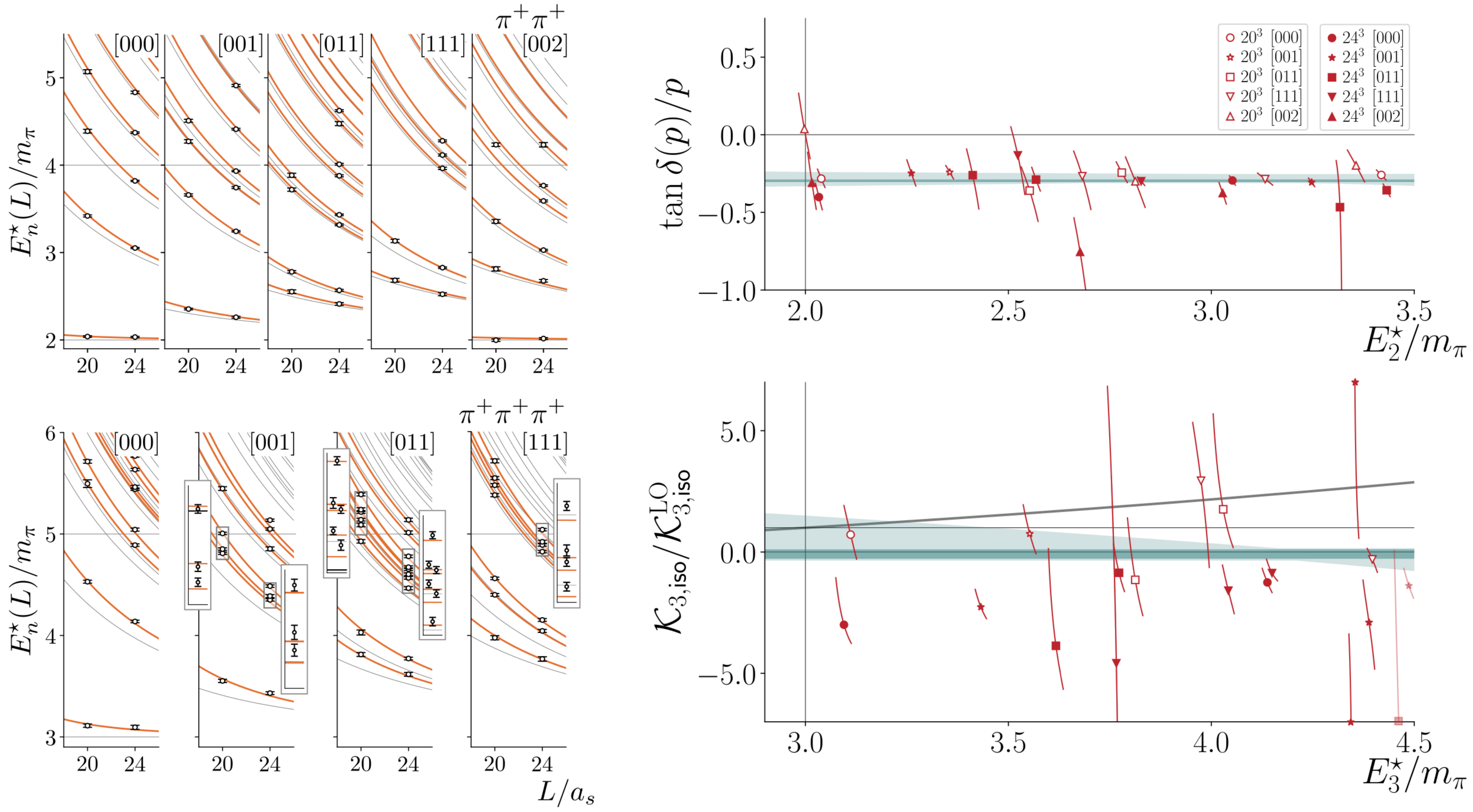
(A) Finite volume spectrum

(B) Quantization Condition

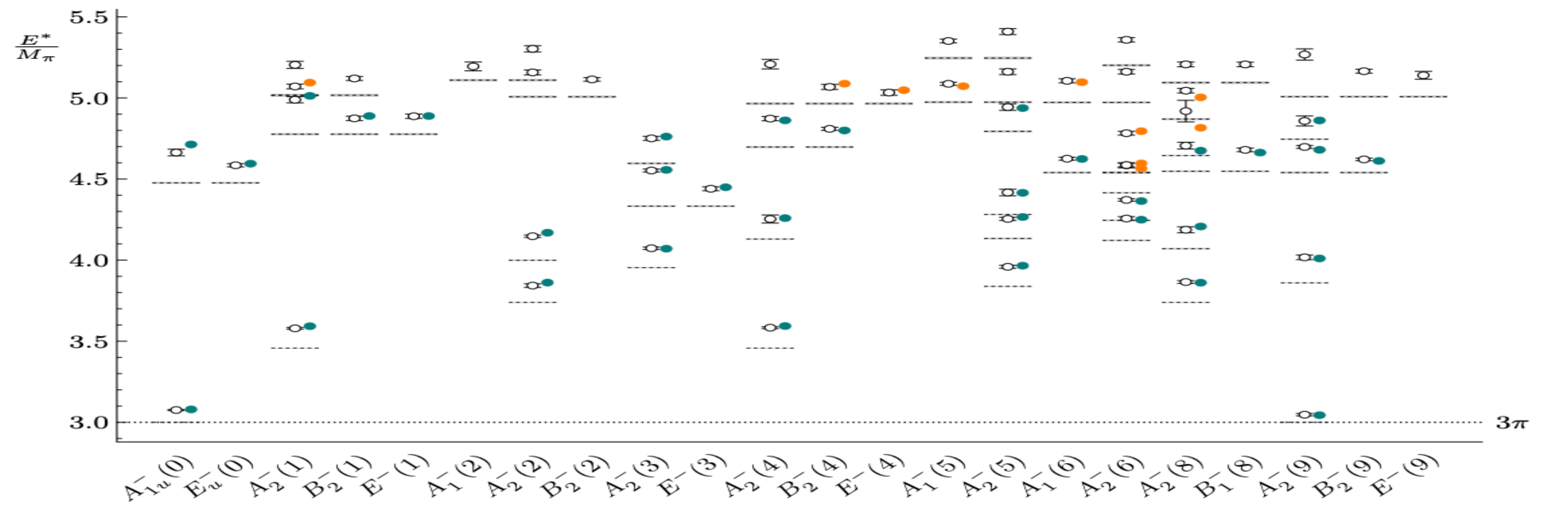
(C) Three-body K-matrix

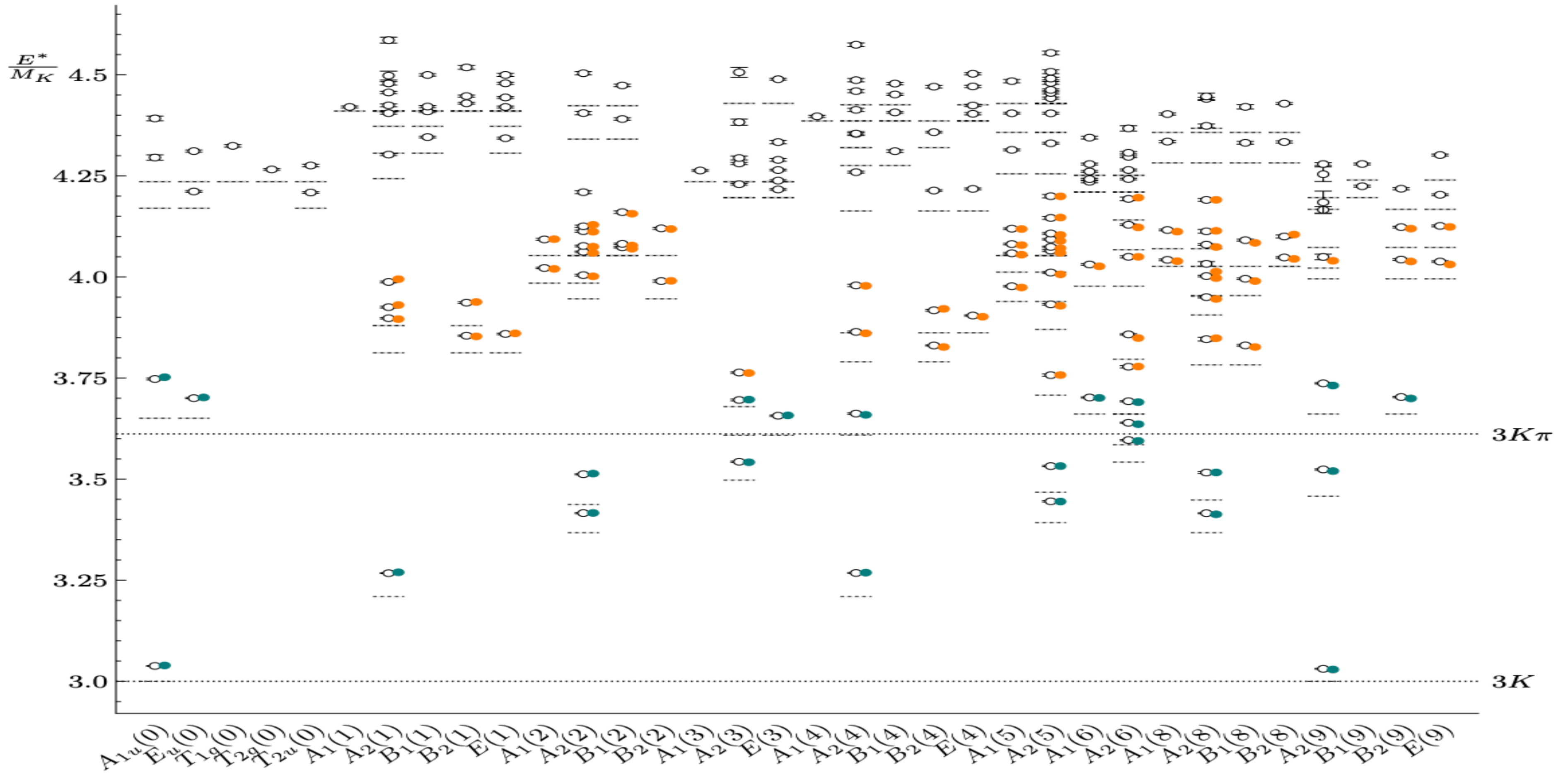
Three-body spectra — status

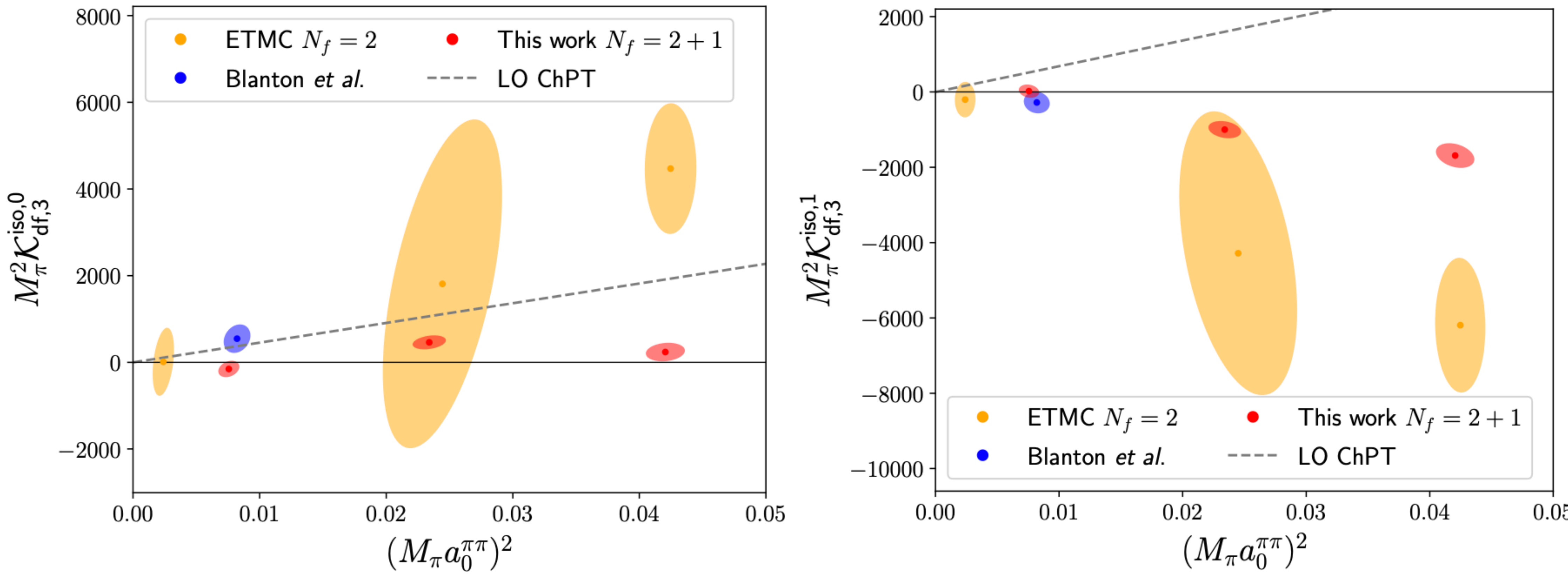
- Currently three π and three K at I=3, S and D waves



Hansen et al. (HadSpec), Phys. Rev. Lett. 126 (2021), 012001



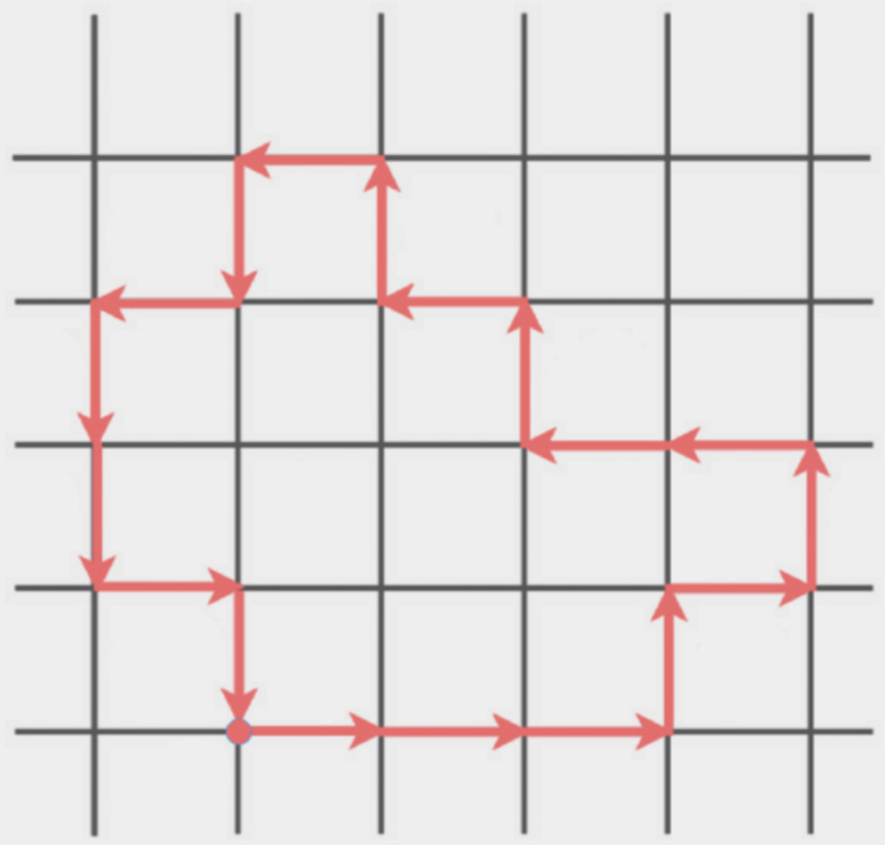




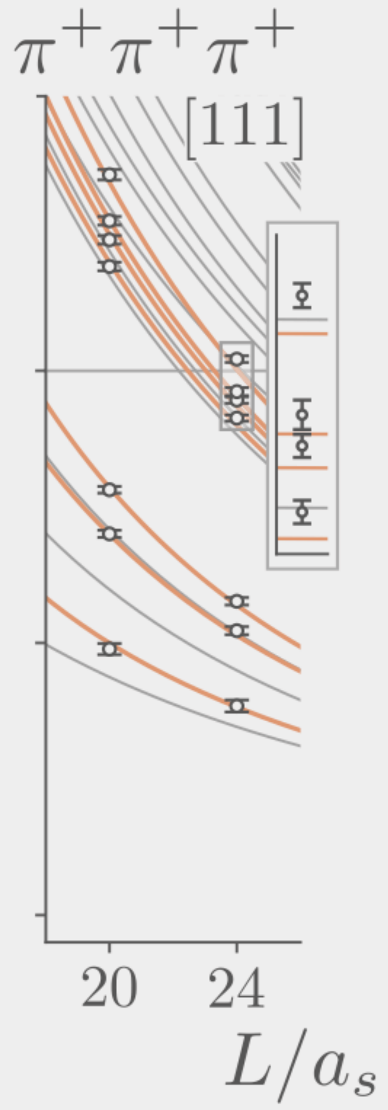
Blanton et al. *JHEP* 10 (2021) 023

(a) $\mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},0}$ (b) $\mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},1}$

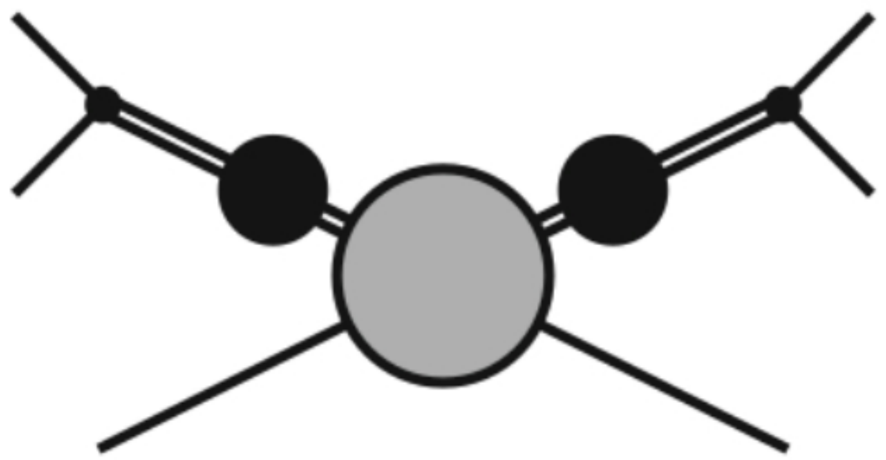
Path to three-body physics from the lattice QCD



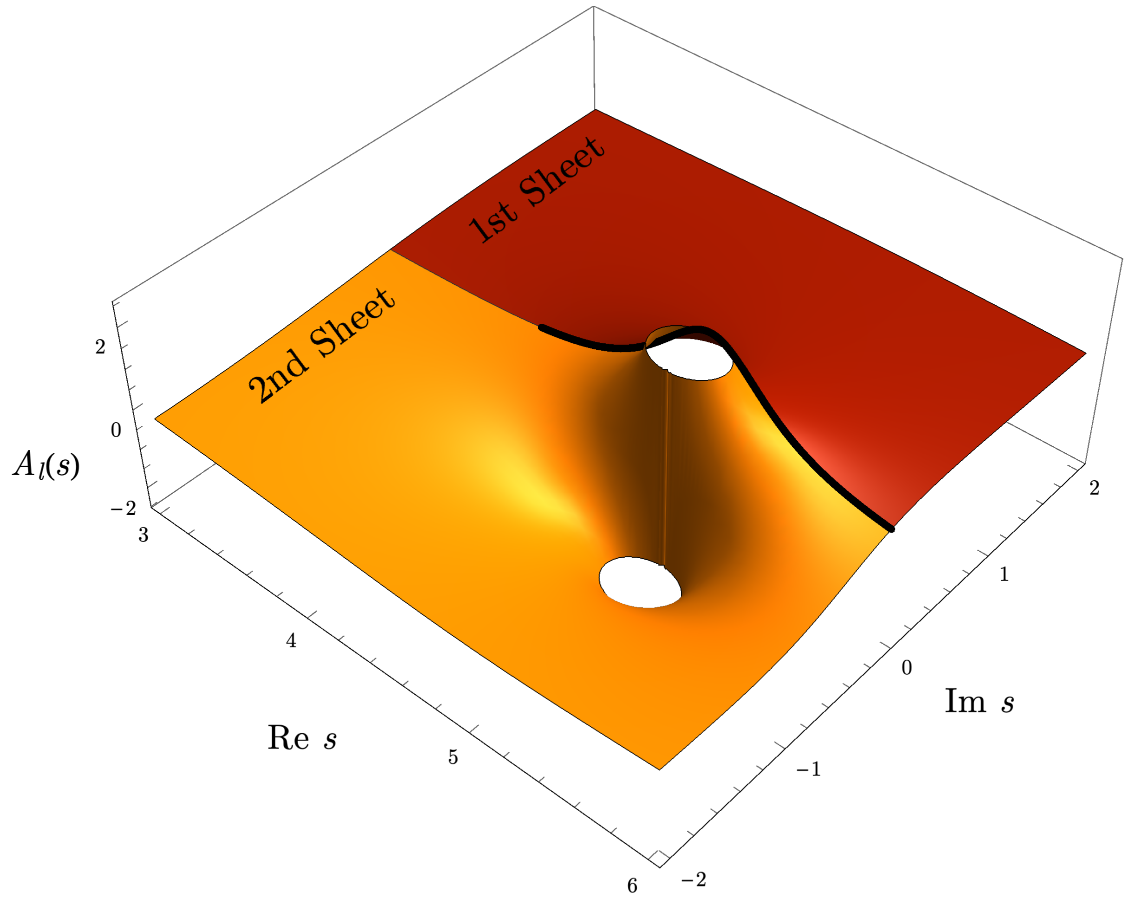
Lattice QCD



FV Spectrum



Amplitudes

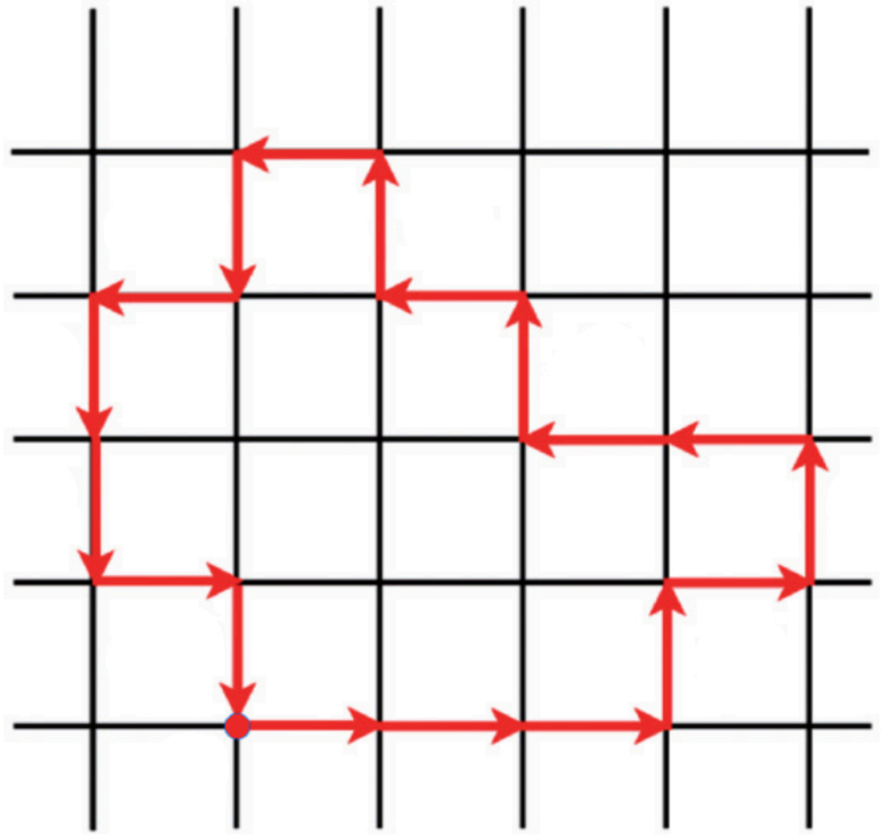


Particle properties

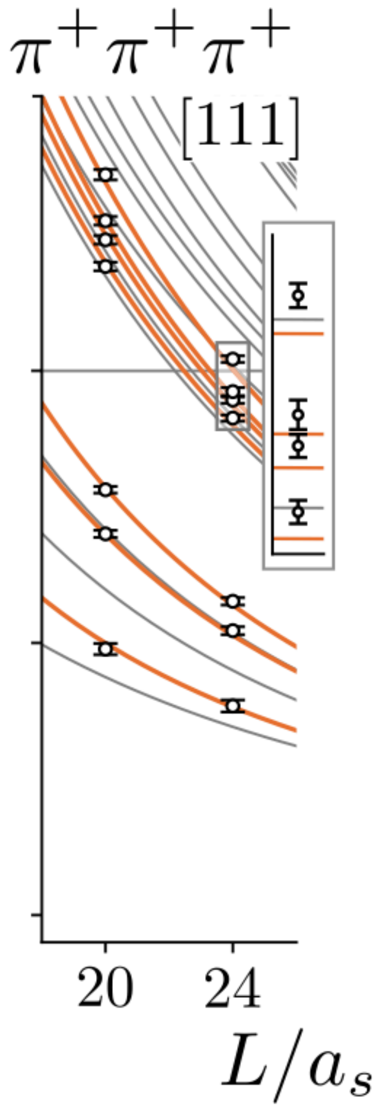
FV

(A) Finite volume spectrum (B) Quantization Condition (C) Three-body K-matrix

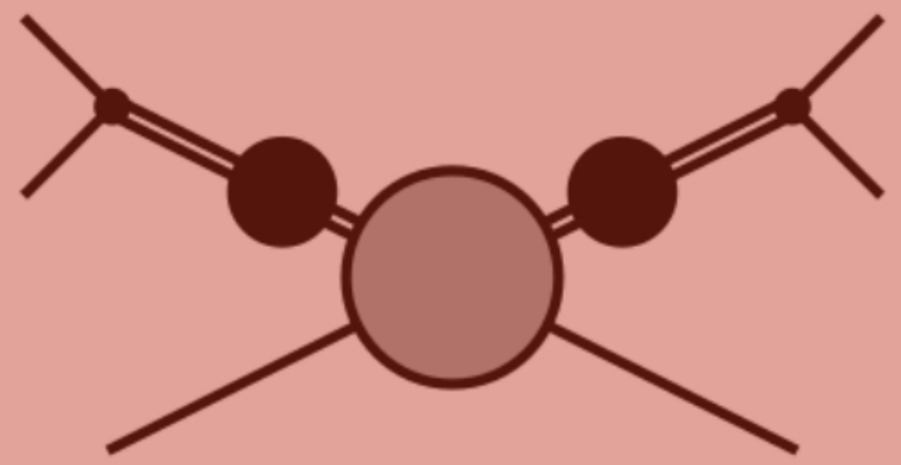
Path to three-body physics from the lattice QCD



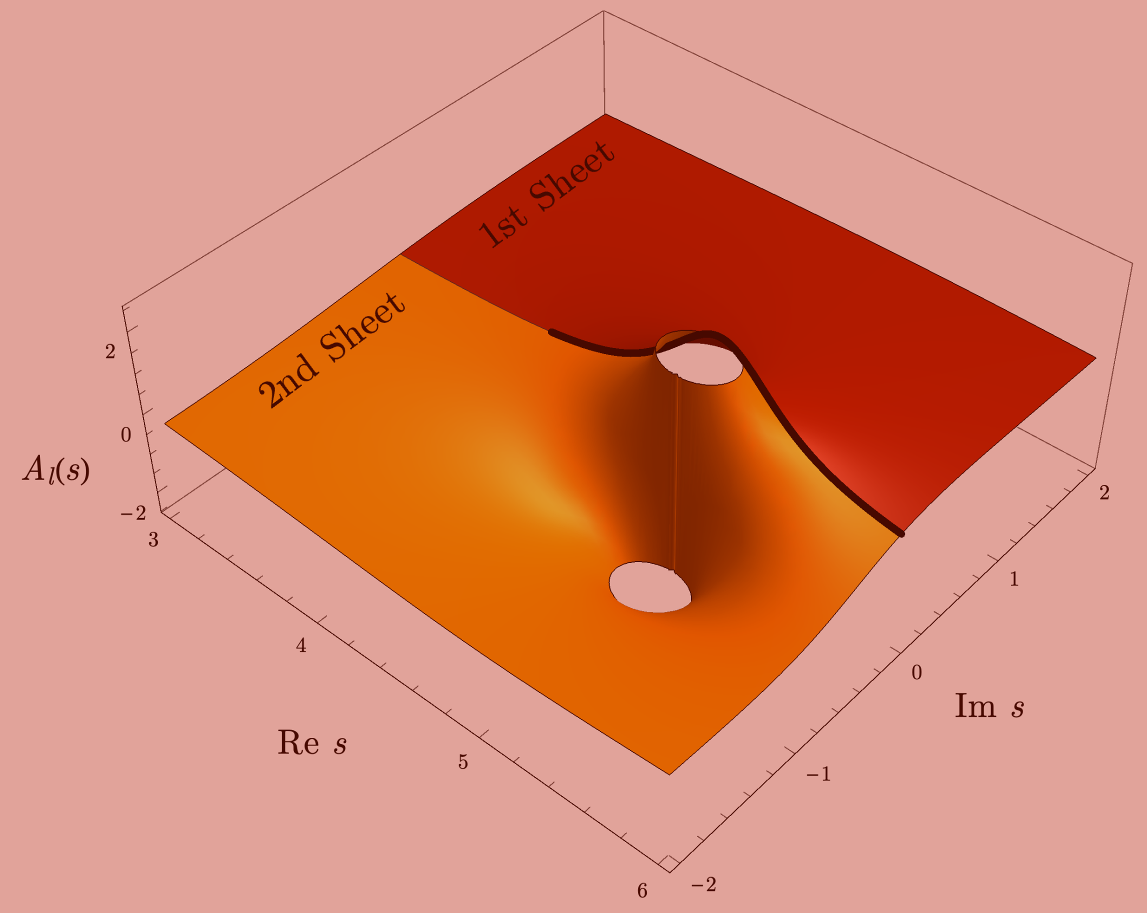
Lattice QCD



FV Spectrum



Amplitudes



Particle properties

IV

(a) K-matrix + two-body subprocesses

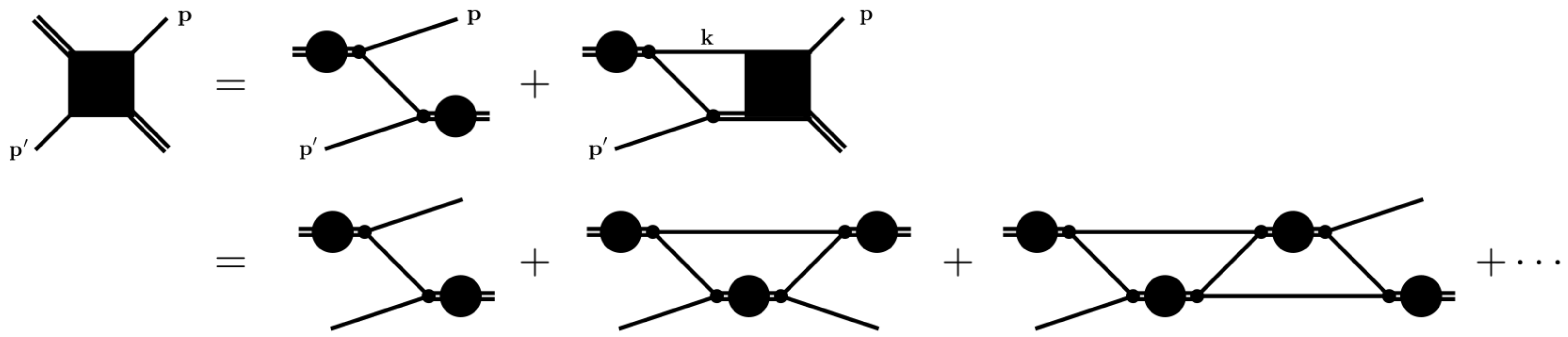
(b) Integral equations

(c) Three-body amplitudes

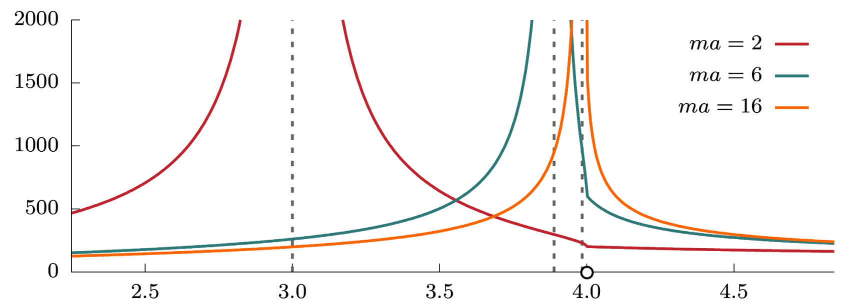
(d) Amplitudes analytically continued to the unphysical Riemann sheets

Solving the REFT three-body ladder equation

- Ladder approximation, *B* = *G* + ***(R=0)***

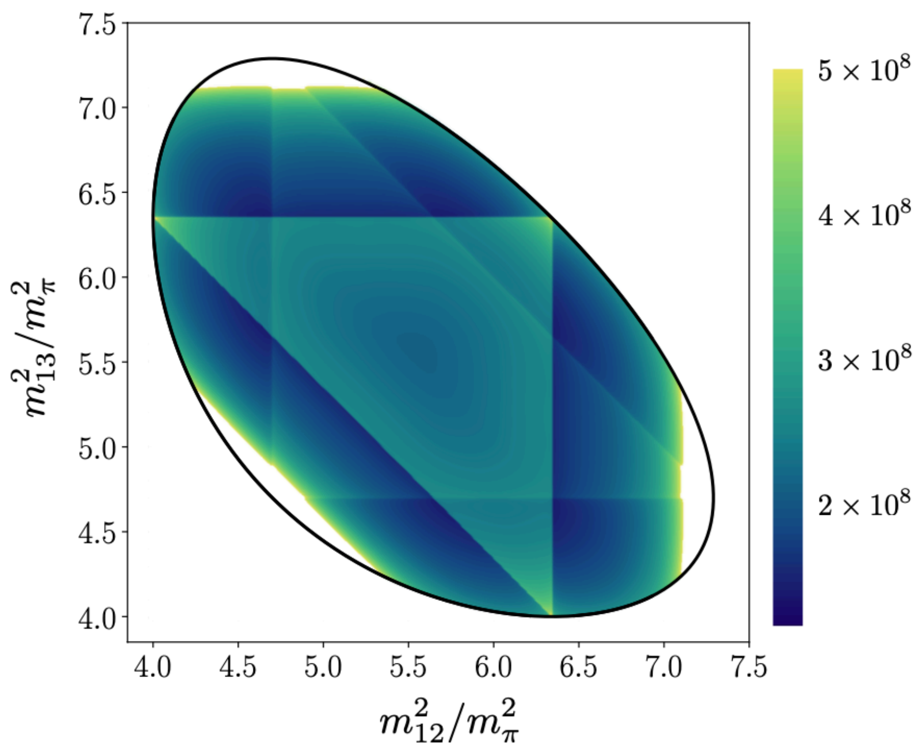


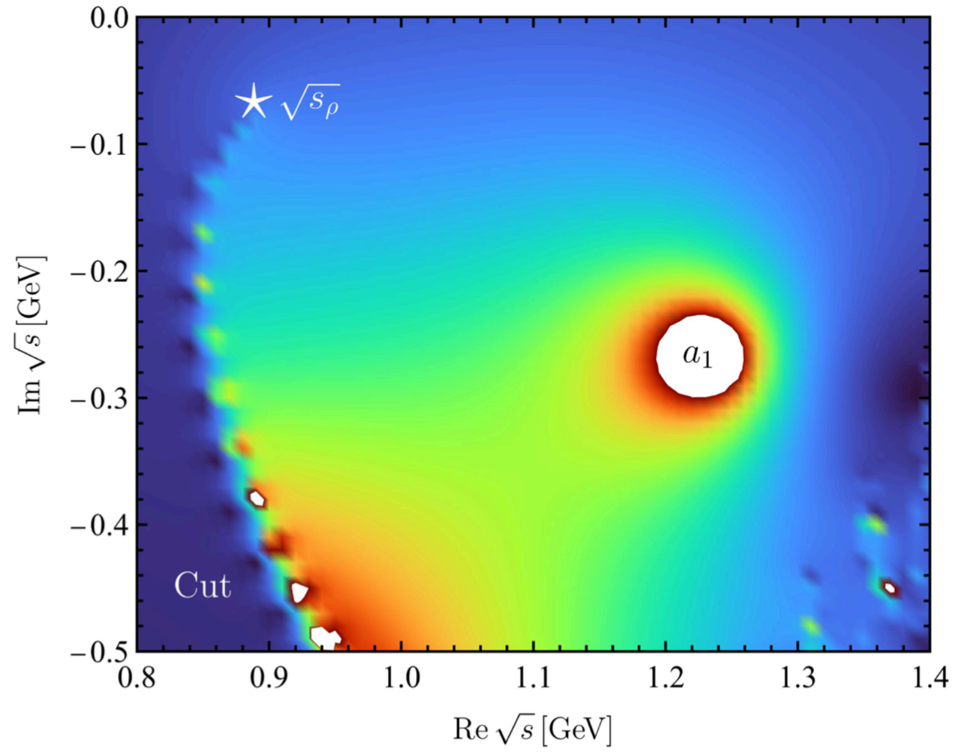
- Numerical solution of the three-body EFT equations
- Similar studies



$$\mathcal{M}_2^{-1} \sim -\frac{1}{a} - i\rho_2$$

$$\lim_{\sigma,\sigma'\to\sigma_b} \mathcal{A}_3 = \frac{g}{\sigma' - \sigma_b}\mathcal{A}_2\frac{g}{\sigma - \sigma_b}$$

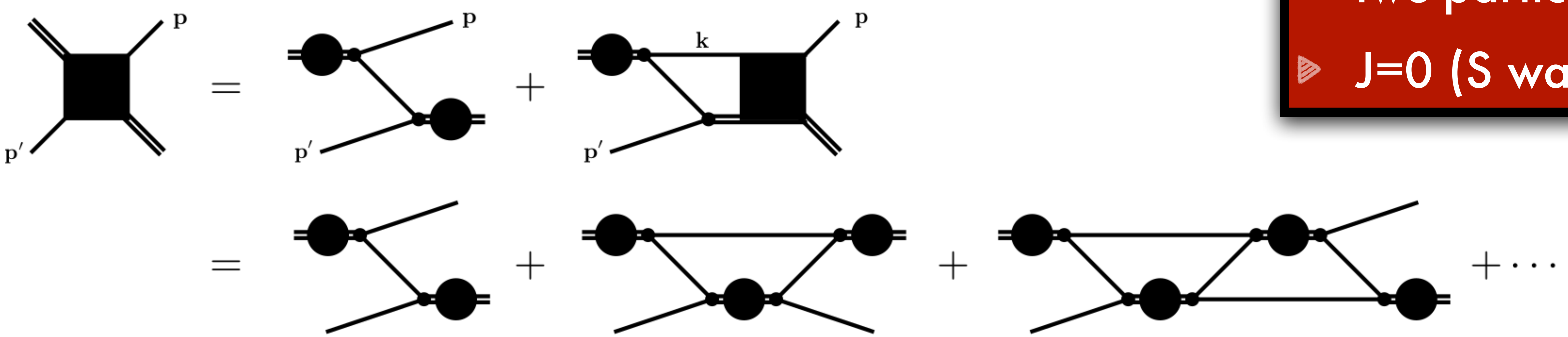




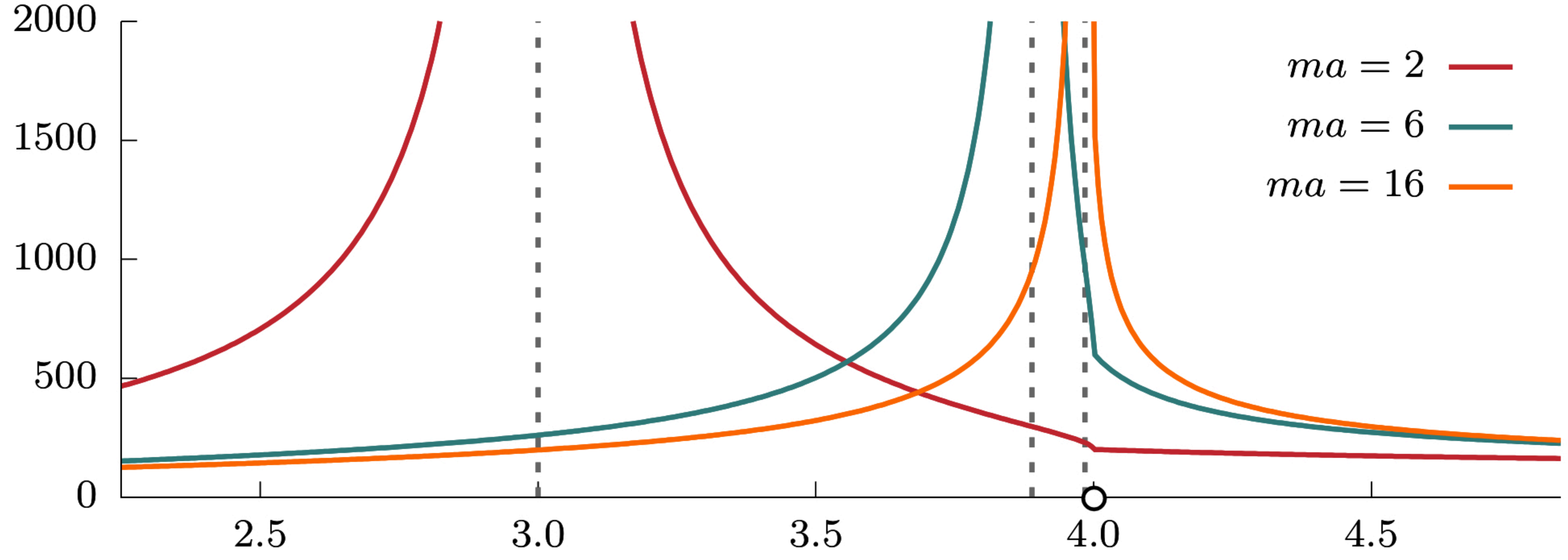
* weakly interacting system in $\pi^+\pi^+$ and $\pi^+\pi^+\pi^+$
  Hansen et al., Phys. Rev. Lett. 126 (2021), 012001
* decay $a_1(1260) \to \rho^0\pi^- \to \pi^-\pi^+\pi^-$
  Sadasivan et al., Phys. Rev. D 101 (2020) 9, 094018
  Sadasivan et al., Phys. Rev. D 105 (2022) 5, 054020

Solving the REFT three-body ladder equation

- Ladder approximation, *B* = *G* + ***(R=0)***



- two-particle bound state + particle
- J=0 (S wave)



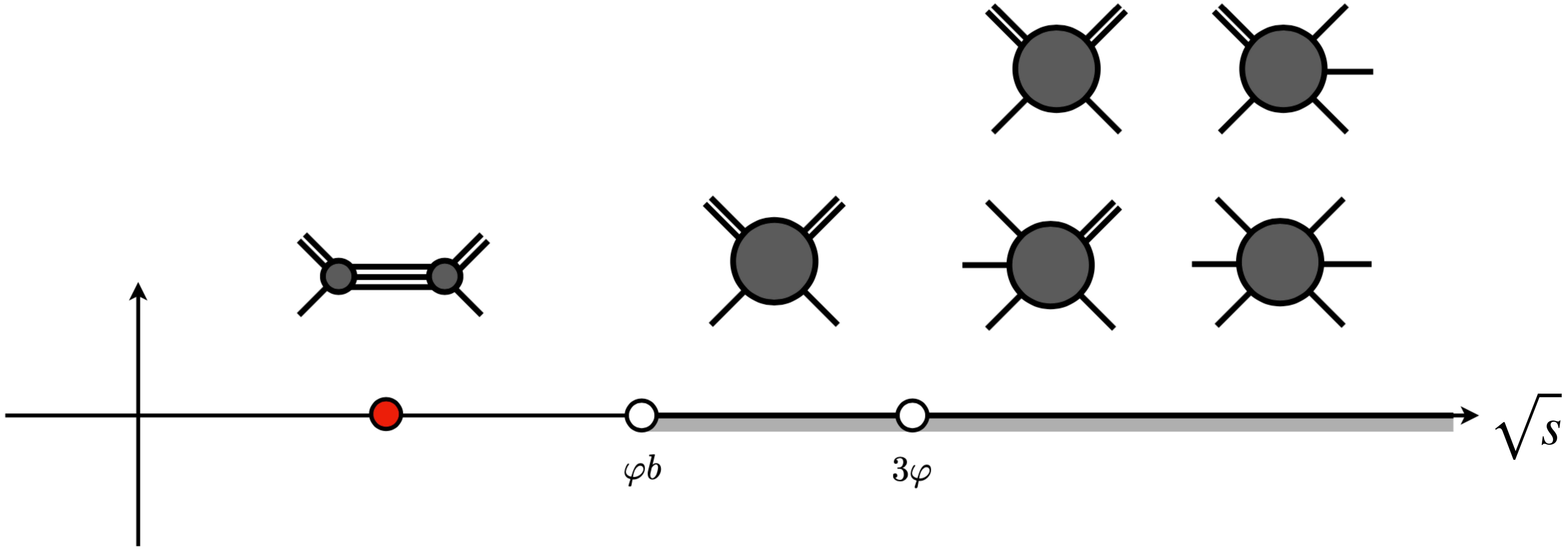
$$\mathcal{M}_2^{-1} \sim -\frac{1}{a} - i\rho_2$$

$$\lim_{\sigma,\sigma' \to \sigma_b} \mathcal{A}_3 = \frac{g}{\sigma' - \sigma_b} \mathcal{A}_2 \frac{g}{\sigma - \sigma_b}$$

Sadasivan et al., Phys. Rev. D 105 (2022) 5, 054020

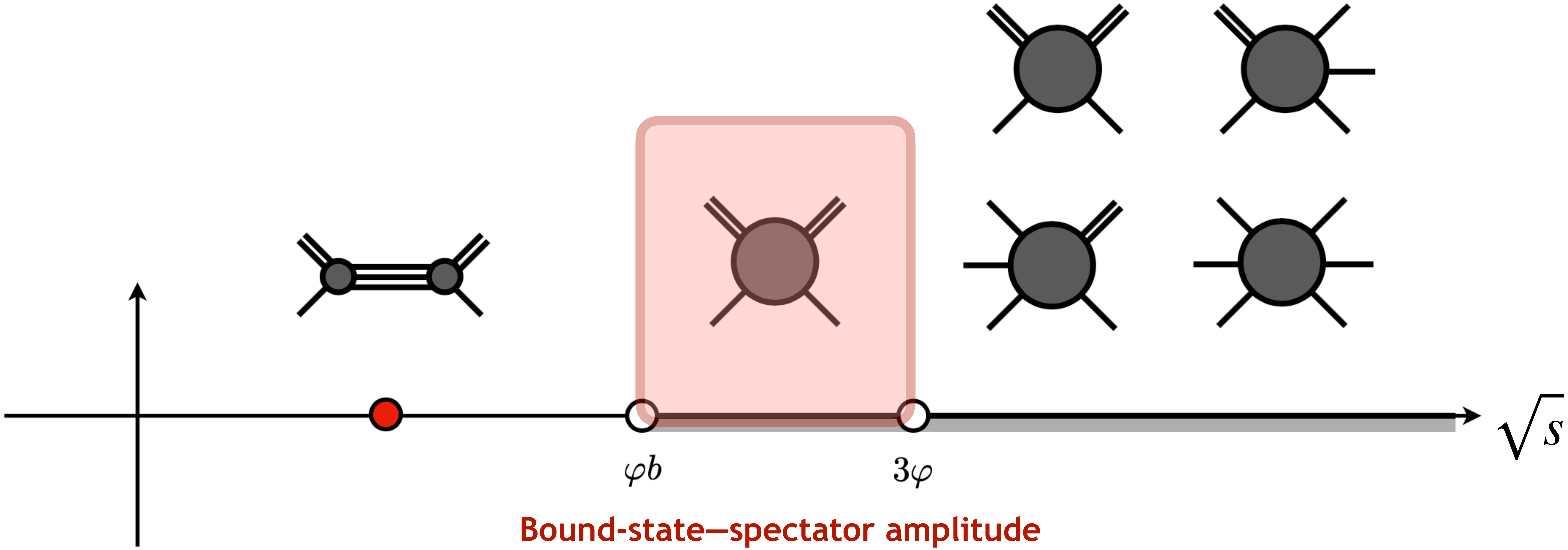
Solving relativistic three-body integral equations in the presence of bound states
Jackura, Briceño, Dawid, Islam, McCarty, Phys.Rev.D 104 (2021) 1, 014507

Solving the REFT three-body ladder equation



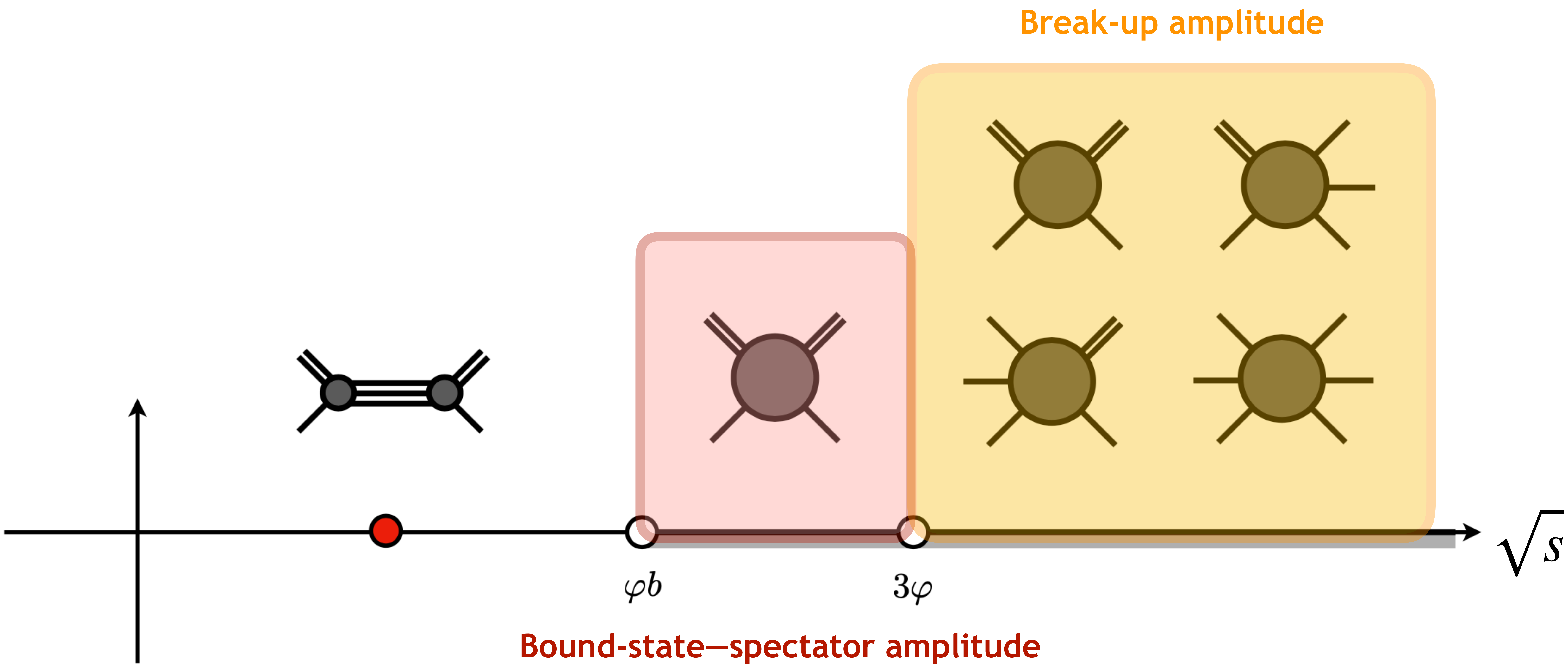
Solving relativistic three-body integral equations in the presence of bound states
Jackura, Briceño, Dawid, Islam, McCarty, Phys.Rev.D 104 (2021) 1, 014507

Solving the REFT three-body ladder equation



Bound-state—spectator amplitude

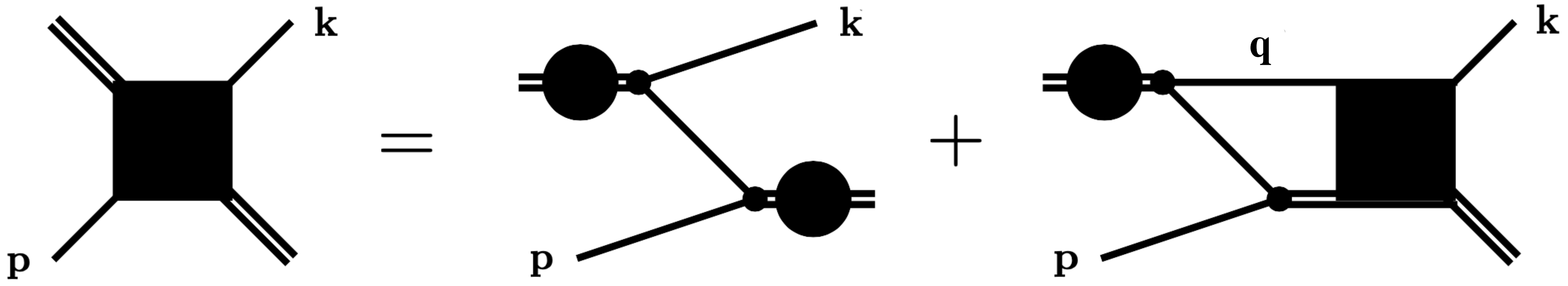
Solving the REFT three-body ladder equation



Numerical procedure

- Discretization of the integral equation $\longrightarrow$ **N linear equations** **(Matrix equation)**
- Regulation of the bound-state pole via the $i\epsilon$ **prescription**

$$d(s) = \lim_{\epsilon\to 0^+} \lim_{N\to\infty} d(s; N, \epsilon)$$

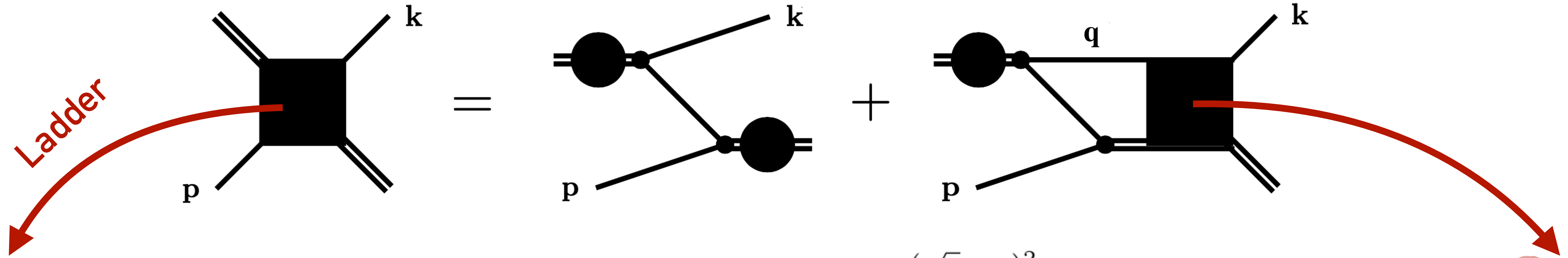


$$D(\sigma_p, s, \sigma_k) = -\mathcal{M}_2(\sigma_p) G(\sigma_p, s, \sigma_k) \mathcal{M}_2(\sigma_k) - \mathcal{M}_2(\sigma_p) \int_0^{(\sqrt{s}-m)^2} \frac{d\sigma_q}{2\pi} G(\sigma_p, s, \sigma_q)\, \tau(\sigma_q, s)\, D(\sigma_q, s, \sigma_k)$$

Numerical procedure

- Discretization of the integral equation $\longrightarrow$ **N linear equations** (Matrix equation)
- Regulation of the bound-state pole via the $i\epsilon$ prescription

$$d(s) = \lim_{\epsilon \to 0^+} \lim_{N \to \infty} d(s; N, \epsilon)$$

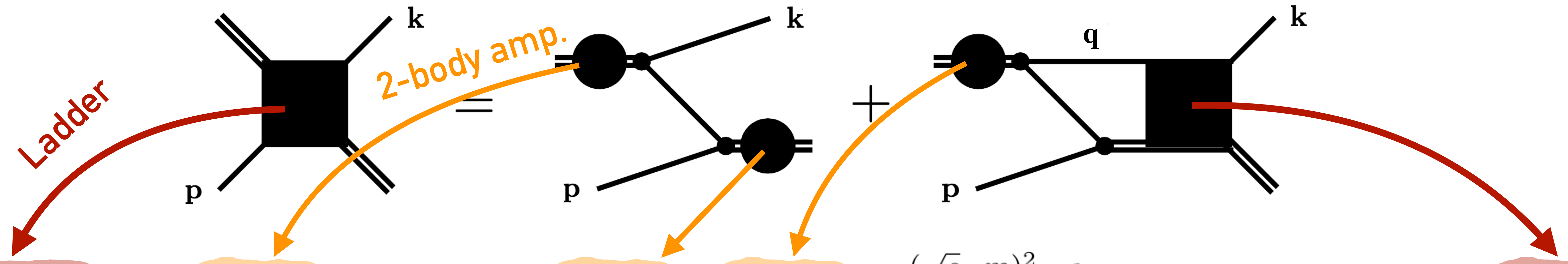


$$D(\sigma_p, s, \sigma_k) = -\mathcal{M}_2(\sigma_p) G(\sigma_p, s, \sigma_k) \mathcal{M}_2(\sigma_k) - \mathcal{M}_2(\sigma_p) \int_0^{(\sqrt{s}-m)^2} \frac{d\sigma_q}{2\pi} G(\sigma_p, s, \sigma_q)\, \tau(\sigma_q, s)\, D(\sigma_q, s, \sigma_k)$$

Numerical procedure

- Discretization of the integral equation $\longrightarrow$ **N linear equations (Matrix equation)**
- Regulation of the bound-state pole via the $i\epsilon$ prescription

$$d(s) = \lim_{\epsilon\to 0^+} \lim_{N\to\infty} d(s; N, \epsilon)$$

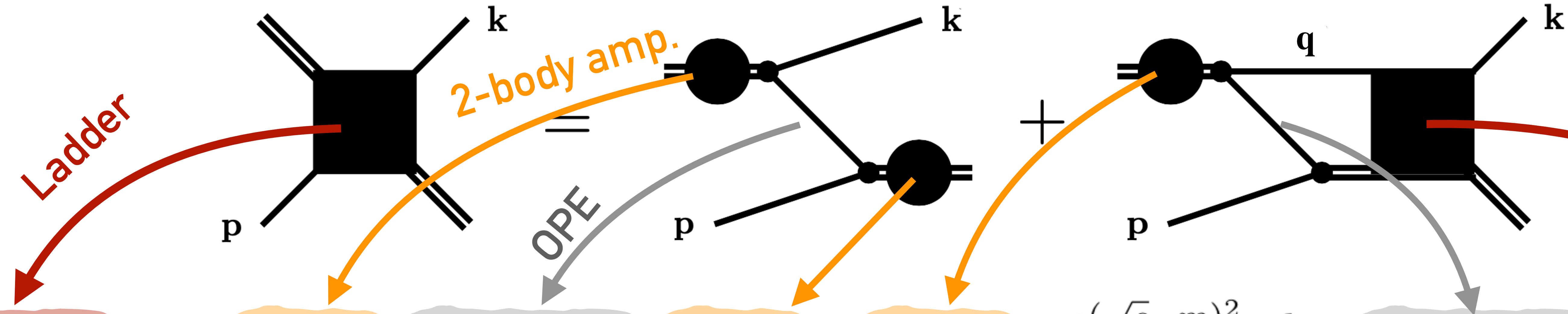


$$D(\sigma_p, s, \sigma_k) = -\mathcal{M}_2(\sigma_p) G(\sigma_p, s, \sigma_k) \mathcal{M}_2(\sigma_k) - \mathcal{M}_2(\sigma_p) \int_0^{(\sqrt{s}-m)^2} \frac{d\sigma_q}{2\pi} G(\sigma_p, s, \sigma_q)\, \tau(\sigma_q, s)\, D(\sigma_q, s, \sigma_k)$$

Numerical procedure

- Discretization of the integral equation $\longrightarrow$ **N linear equations** **(Matrix equation)**
- Regulation of the bound-state pole via the $i\epsilon$ prescription

$$d(s) = \lim_{\epsilon \to 0^+} \lim_{N \to \infty} d(s; N, \epsilon)$$



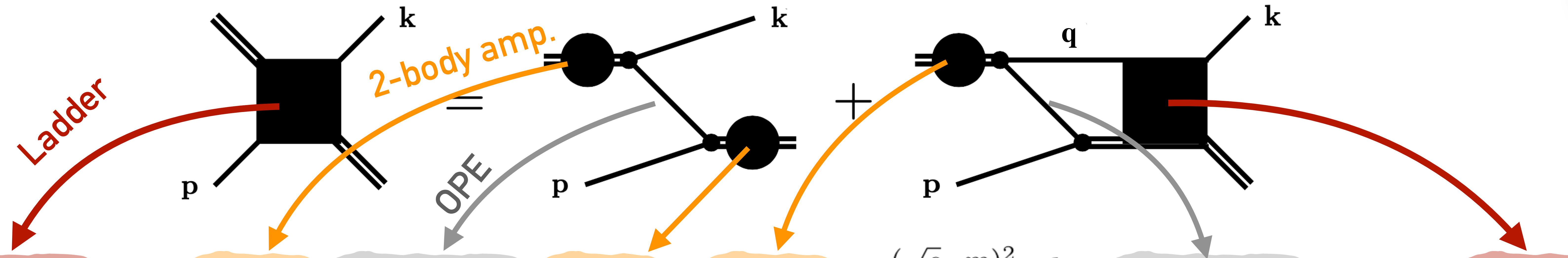
$$D(\sigma_p, s, \sigma_k) = -\mathcal{M}_2(\sigma_p) G(\sigma_p, s, \sigma_k) \mathcal{M}_2(\sigma_k) - \mathcal{M}_2(\sigma_p) \int_0^{(\sqrt{s}-m)^2} \frac{d\sigma_q}{2\pi} G(\sigma_p, s, \sigma_q) \tau(\sigma_q, s) D(\sigma_q, s, \sigma_k)$$

Solving relativistic three-body integral equations in the presence of bound states
Jackura, Briceño, Dawid, Islam, McCarty, Phys.Rev.D 104 (2021) 1, 014507

Numerical procedure

- Discretization of the integral equation $\longrightarrow$ **N linear equations** **(Matrix equation)**
- Regulation of the bound-state pole via the $i\epsilon$ **prescription**

$$d(s) = \lim_{\epsilon \to 0^+} \lim_{N \to \infty} d(s; N, \epsilon)$$



$$D(\sigma_p, s, \sigma_k) = -\mathcal{M}_2(\sigma_p) G(\sigma_p, s, \sigma_k) \mathcal{M}_2(\sigma_k) - \mathcal{M}_2(\sigma_p) \int_0^{(\sqrt{s}-m)^2} \frac{d\sigma_q}{2\pi} G(\sigma_p, s, \sigma_q) \tau(\sigma_q, s) D(\sigma_q, s, \sigma_k)$$

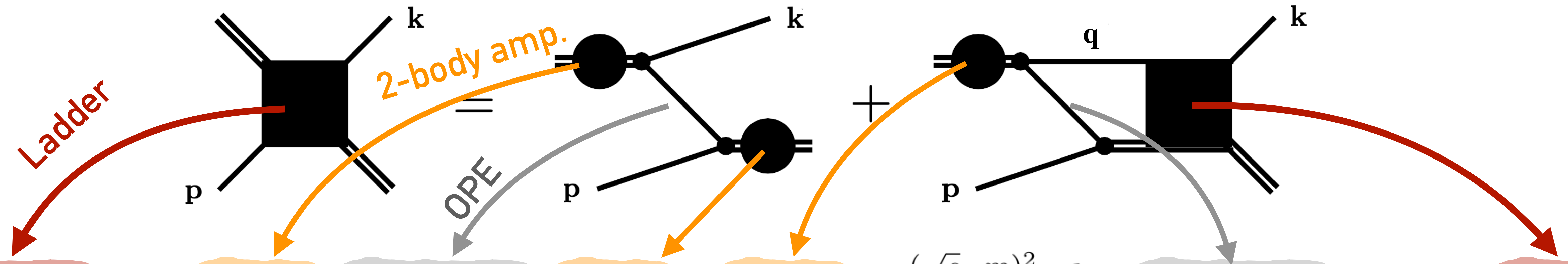
Amputation

$$D(\sigma_p, s, \sigma_k) = \mathcal{M}_2(\sigma_p) d(\sigma_p, s, \sigma_k) \mathcal{M}_2(\sigma_k)$$

Numerical procedure

- Discretization of the integral equation $\longrightarrow$ **N linear equations** **(Matrix equation)**
- Regulation of the bound-state pole via the $i\epsilon$ **prescription**

$$d(s) = \lim_{\epsilon \to 0^+} \lim_{N \to \infty} d(s; N, \epsilon)$$



$$D(\sigma_p, s, \sigma_k) = -\mathcal{M}_2(\sigma_p) G(\sigma_p, s, \sigma_k) \mathcal{M}_2(\sigma_k) - \mathcal{M}_2(\sigma_p) \int_0^{(\sqrt{s}-m)^2} \frac{d\sigma_q}{2\pi} G(\sigma_p, s, \sigma_q) \tau(\sigma_q, s) D(\sigma_q, s, \sigma_k)$$

Discretization

Amputation

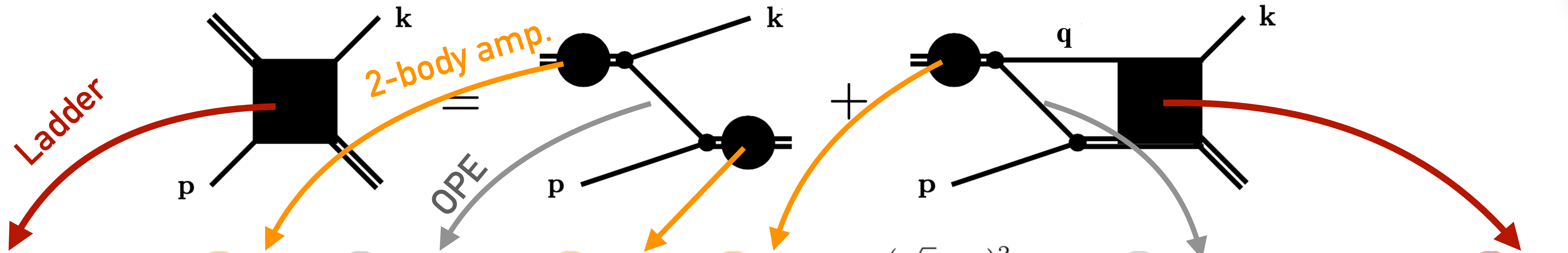
$$D(\sigma_p, s, \sigma_k) = \mathcal{M}_2(\sigma_p) d(\sigma_p, s, \sigma_k) \mathcal{M}_2(\sigma_k)$$

$$\boldsymbol{d}(N, \epsilon) = -\boldsymbol{G}(\epsilon) - \boldsymbol{P} \cdot \boldsymbol{G}(\epsilon) \cdot \boldsymbol{\mathcal{M}} \cdot \boldsymbol{d}(N, \epsilon)$$

Numerical procedure

- Discretization of the integral equation $\longrightarrow$ **N linear equations (Matrix equation)**
- Regulation of the bound-state pole via the $i\epsilon$ prescription

$$d(s) = \lim_{\epsilon\to 0^+} \lim_{N\to\infty} d(s; N, \epsilon)$$



$$D(\sigma_p, s, \sigma_k) = -\mathcal{M}_2(\sigma_p) G(\sigma_p, s, \sigma_k) \mathcal{M}_2(\sigma_k) - \mathcal{M}_2(\sigma_p) \int_0^{(\sqrt{s}-m)^2} \frac{d\sigma_q}{2\pi} G(\sigma_p, s, \sigma_q) \tau(\sigma_q, s) D(\sigma_q, s, \sigma_k)$$

Discretization

Amputation

$$D(\sigma_p, s, \sigma_k) = \mathcal{M}_2(\sigma_p) d(\sigma_p, s, \sigma_k) \mathcal{M}_2(\sigma_k)$$

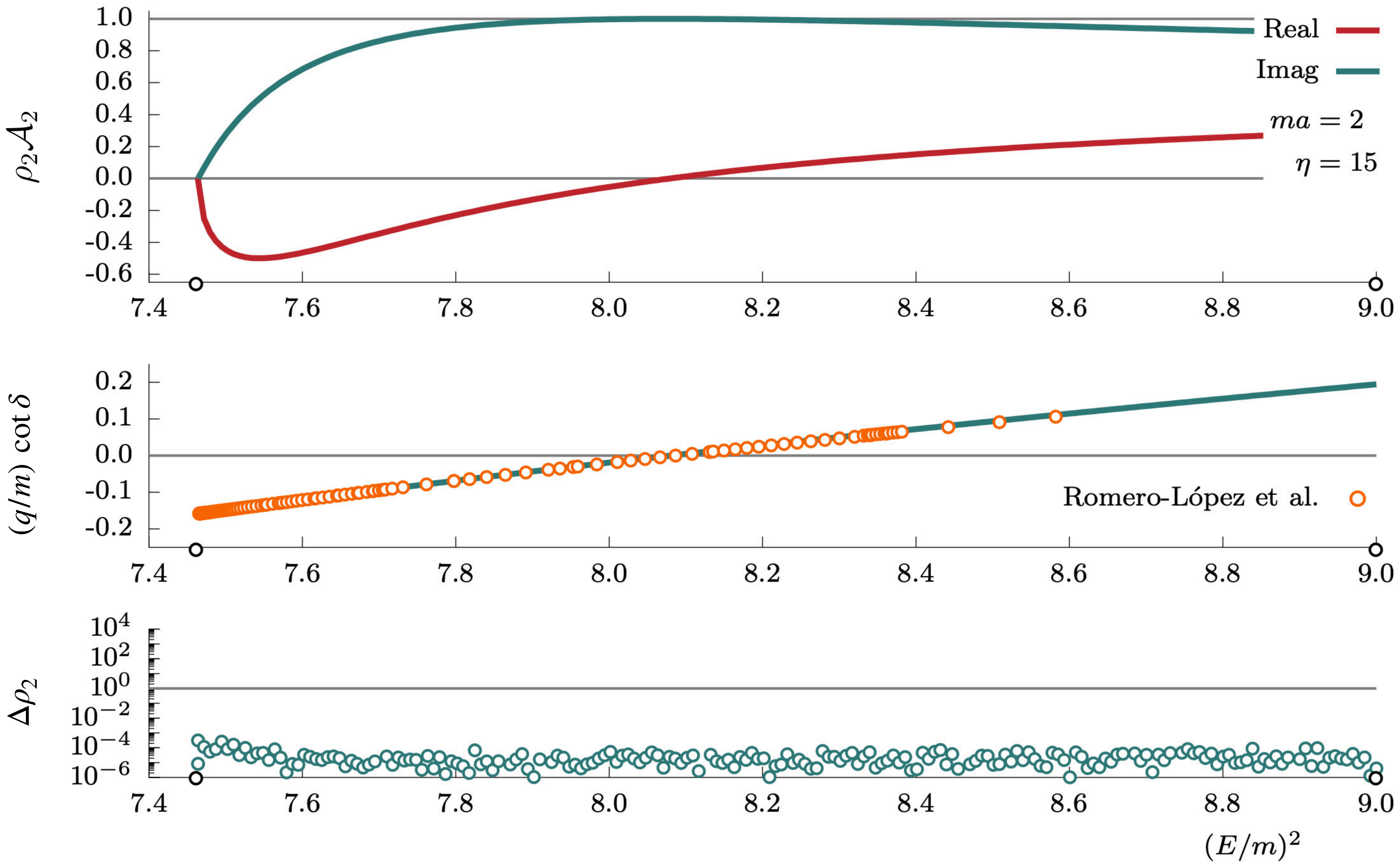
$$\boldsymbol{d}(N,\epsilon) = -\boldsymbol{G}(\epsilon) - \boldsymbol{P} \cdot \boldsymbol{G}(\epsilon) \cdot \boldsymbol{\mathcal{M}} \cdot \boldsymbol{d}(N,\epsilon)$$

Matrix inversion

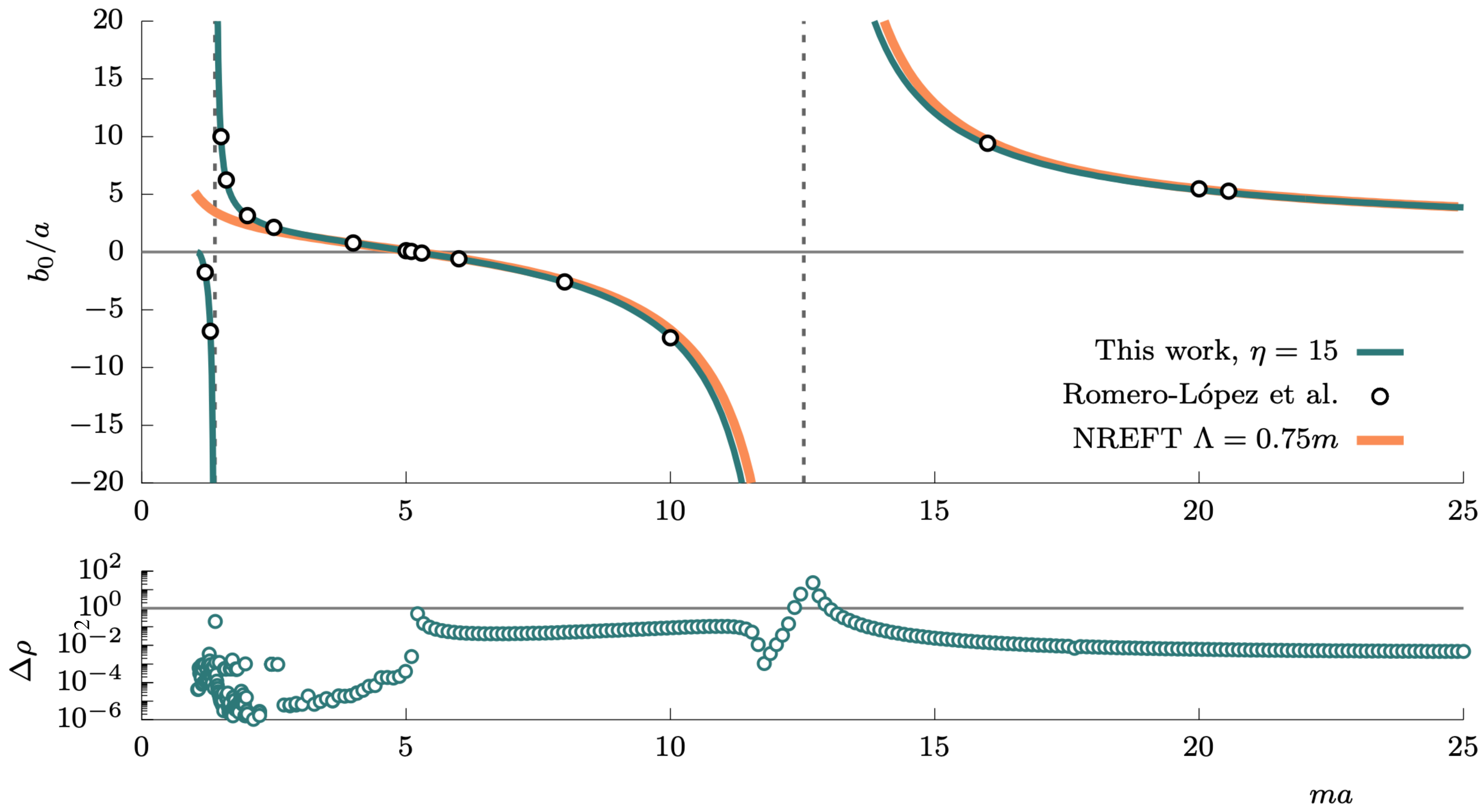
$$\boldsymbol{d}(N,\epsilon) = -\left[\mathbb{1} + \boldsymbol{P} \cdot \boldsymbol{G}(\epsilon) \cdot \boldsymbol{\mathcal{M}}\right]^{-1} \cdot \boldsymbol{G}(\epsilon)$$

- Systematics: unitarity test, convergence test
- Different methods: "Brute force", explicit pole removal, spline-based quadratures

Example results, $M^2 = 3m^2$ (ma=2)



Example result, three-body scattering length

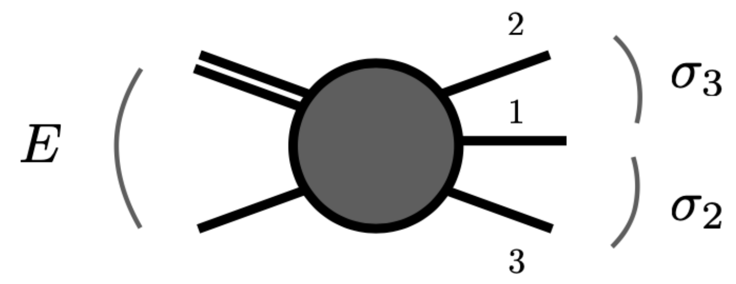


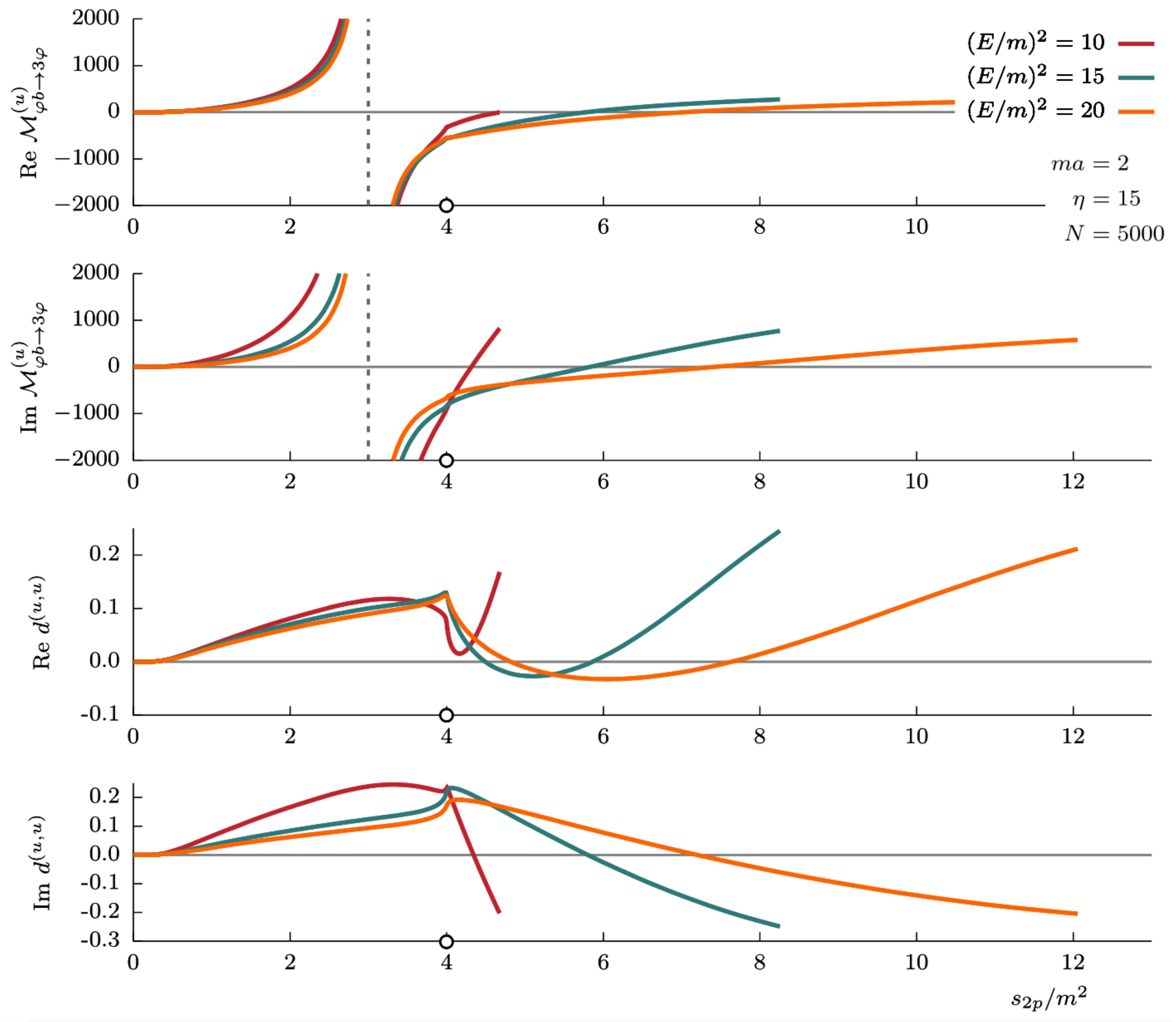
Romero-Lopez et al., JHEP 10 (2019) 007

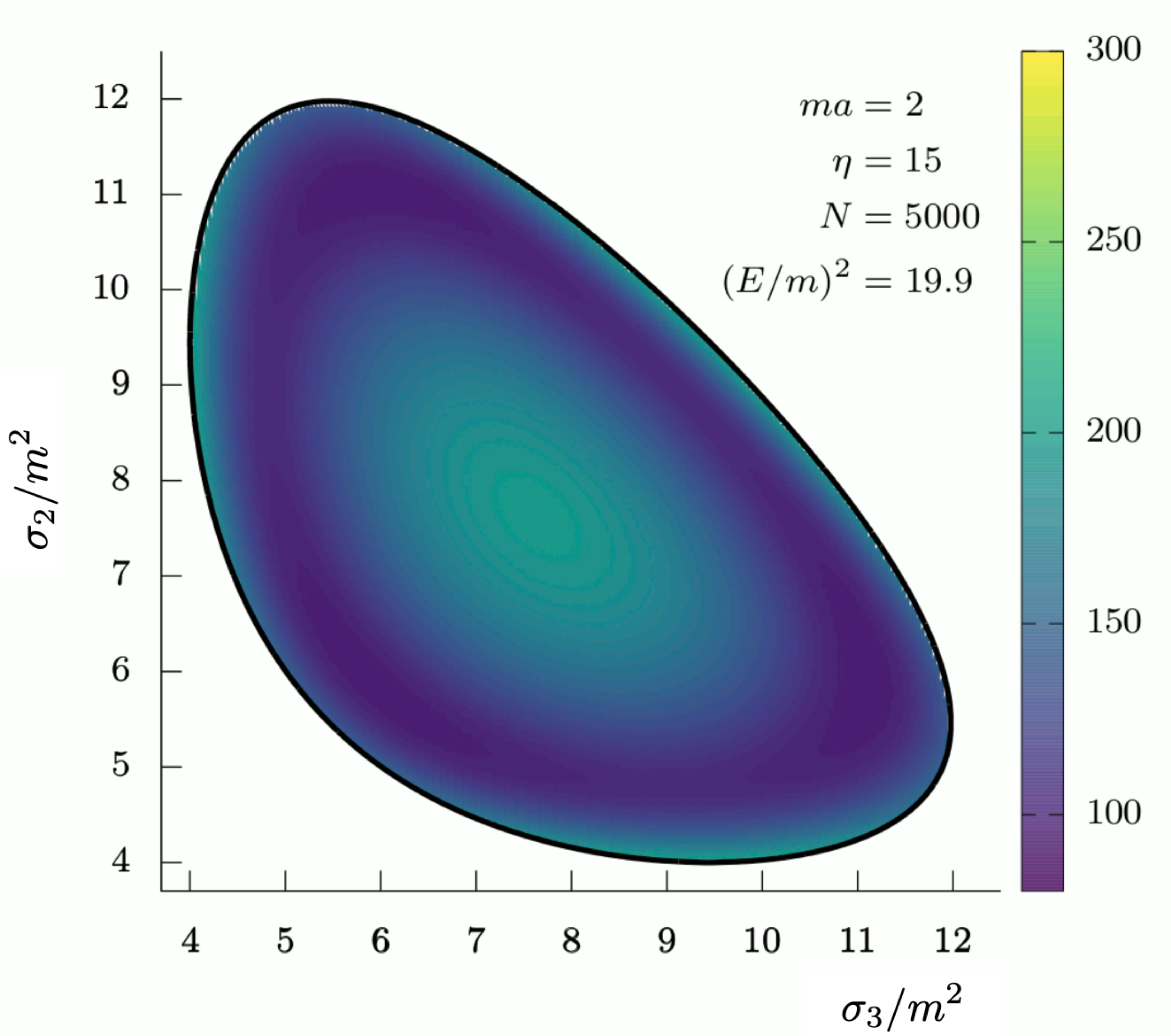
Bedaque et al., Nucl. Phys. A 646 (1999) 444

Example result, 2→3 amplitude

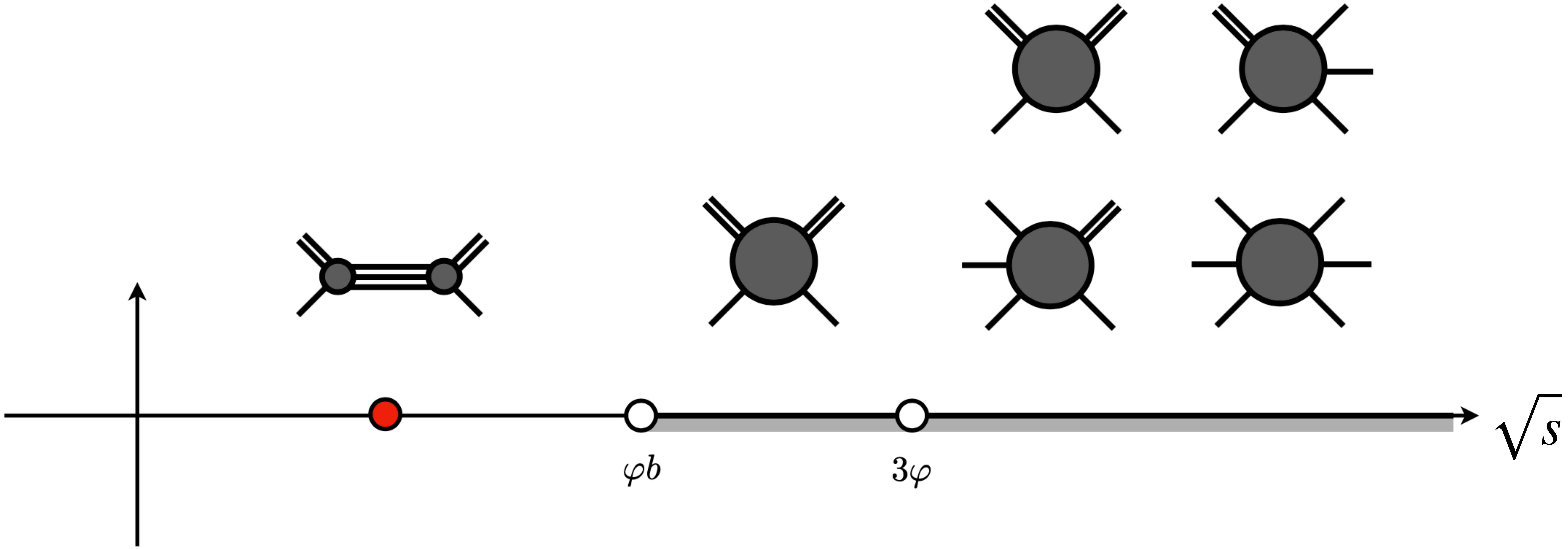
- We are not limited to energies below the three-body threshold



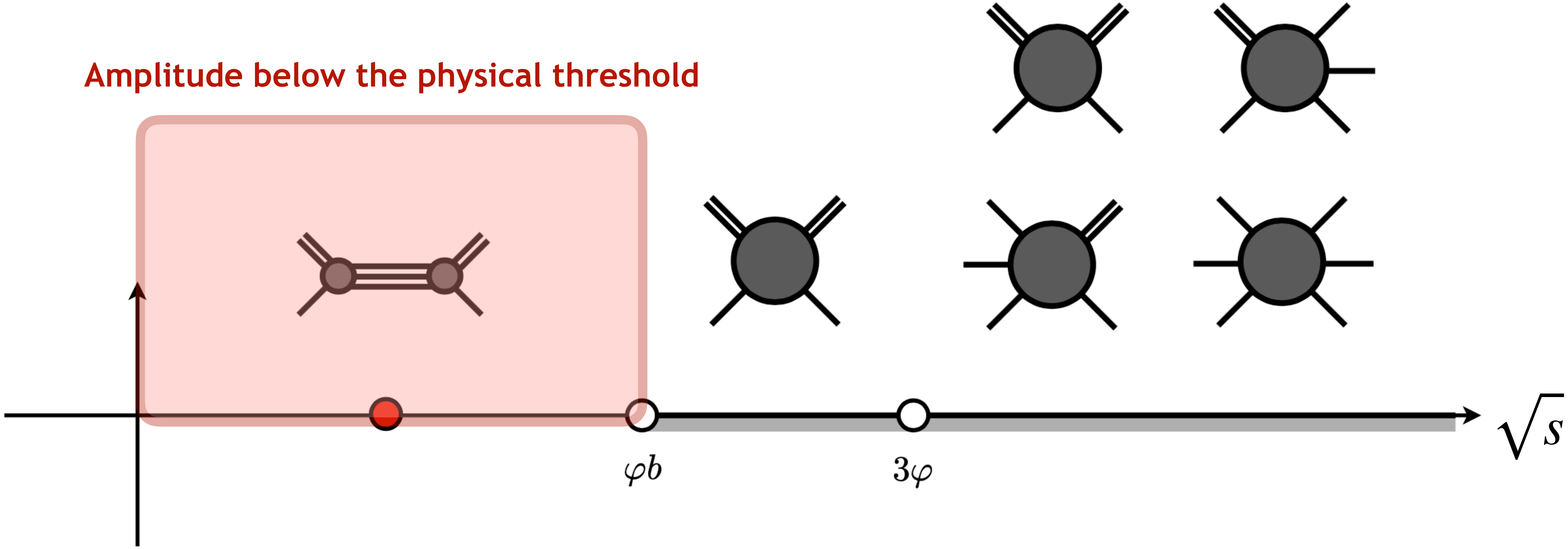




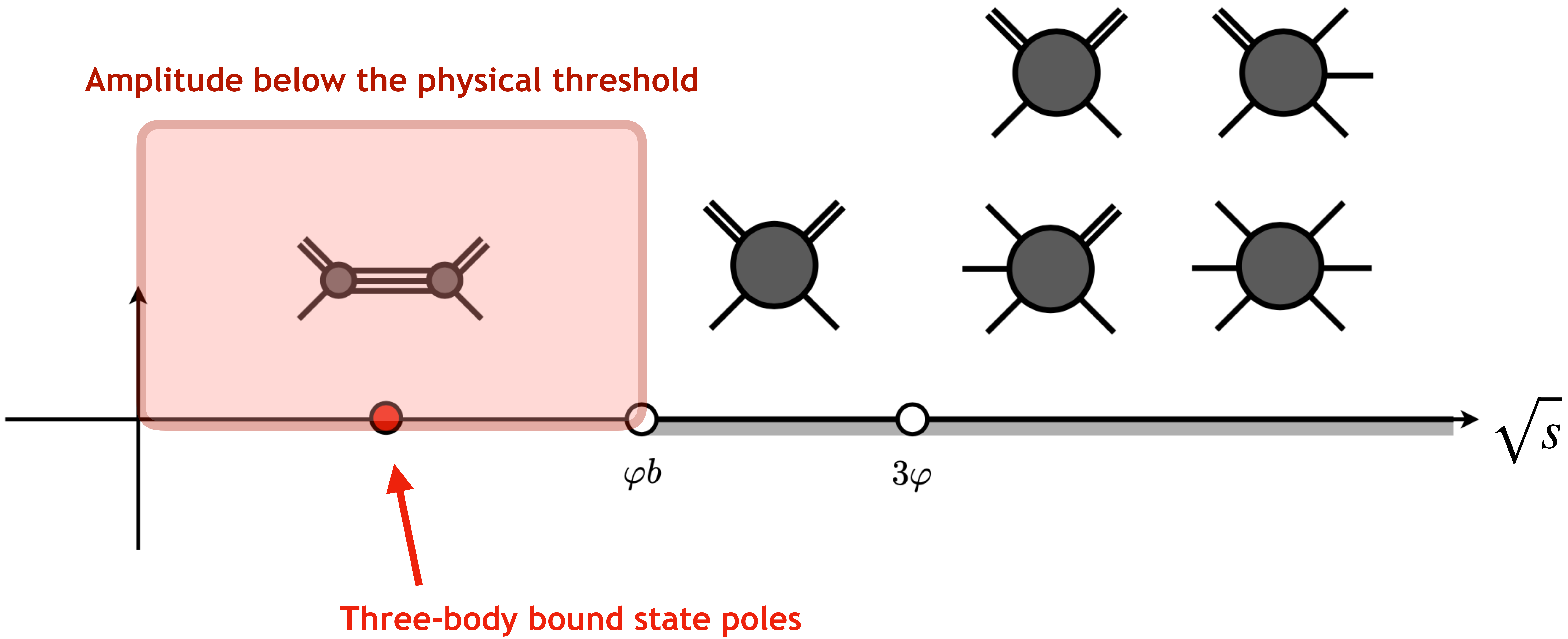
Analytic continuation to complex energies



Analytic continuation to complex energies



Analytic continuation to complex energies



Integral equation below the threshold

- Avoid crossing the singularities in the integration

$$d(p',s,p) = -G(p',s,p) - \int_0^{q_{\max}} \frac{dq\, q^2}{(2\pi)^2 \omega_q} G(p',s,q)\mathcal{M}_2(q,s)\, d(q,s,p)$$

Integral equation below the threshold

- Avoid crossing the singularities in the integration

$$d(p', s, p) = -G(p', s, p) - \int_0^{q_{\max}} \frac{dq\, q^2}{(2\pi)^2 \omega_q} G(p', s, q) \mathcal{M}_2(q, s)\, d(q, s, p)$$

Integration kernel — $\frac{dq\, q^2}{(2\pi)^2 \omega_q} G(p', s, q) \mathcal{M}_2(q, s)$

Solution — $d(q, s, p)$

Integral equation below the threshold

- Avoid crossing the singularities in the integration

$$d(p',s,p) = -G(p',s,p) - \int_0^{q_{\rm max}} \frac{dq\, q^2}{(2\pi)^2 \omega_q} G(p',s,q)\mathcal{M}_2(q,s)\, d(q,s,p)$$

Integral equation below the threshold

- Avoid crossing the singularities in the integration

Inhomogeneous term

$$d(p', s, p) = -G(p', s, p) - \int_0^{q_{\max}} \frac{dq\, q^2}{(2\pi)^2 \omega_q} G(p', s, q) \mathcal{M}_2(q, s)\, d(q, s, p)$$

Integral equation below the threshold

- Avoid crossing the singularities in the integration

Homogeneous term

Inhomogeneous term

$$d(p', s, p) = -G(p', s, p) - \int_0^{q_{\max}} \frac{dq\, q^2}{(2\pi)^2 \omega_q} G(p', s, q) \mathcal{M}_2(q, s)\, d(q, s, p)$$

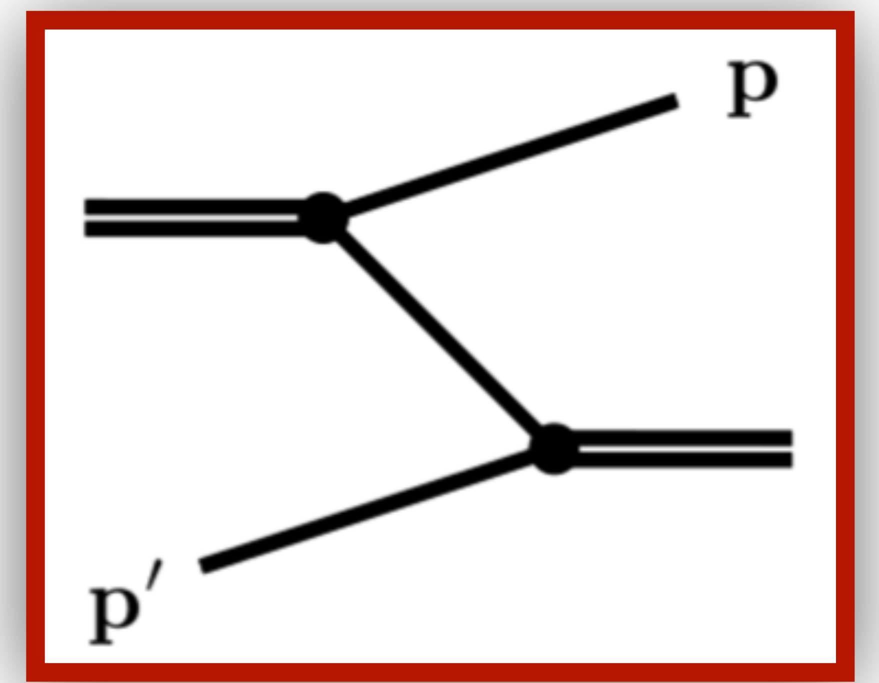
Integral equation below the threshold

- Avoid crossing the singularities in the integration

Homogeneous term

Inhomogeneous term

$$d(p', s, p) = -G(p', s, p) - \int_0^{q_{\max}} \frac{dq\, q^2}{(2\pi)^2 \omega_q} G(p', s, q) \mathcal{M}_2(q, s)\, d(q, s, p)$$

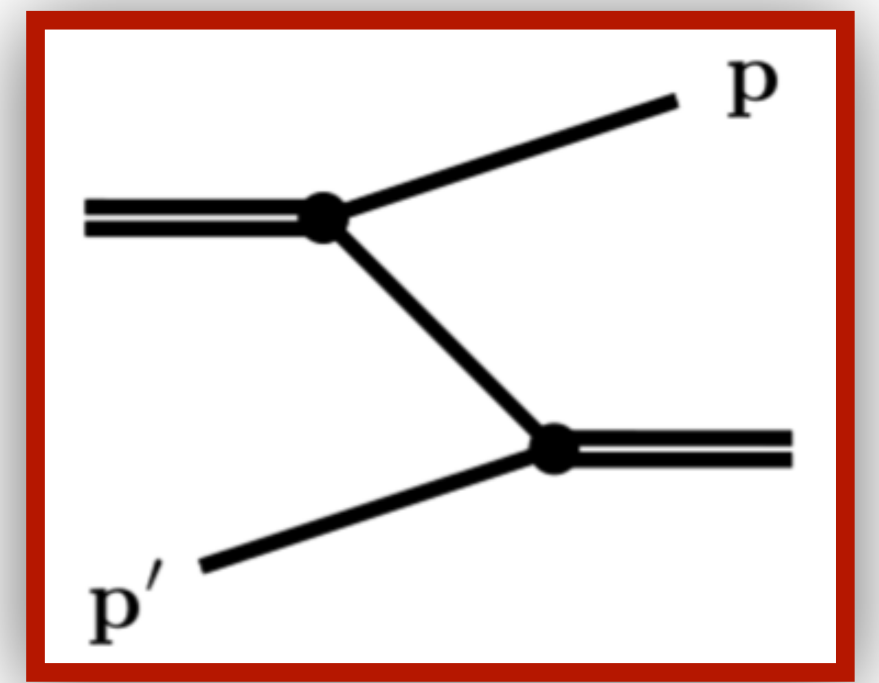


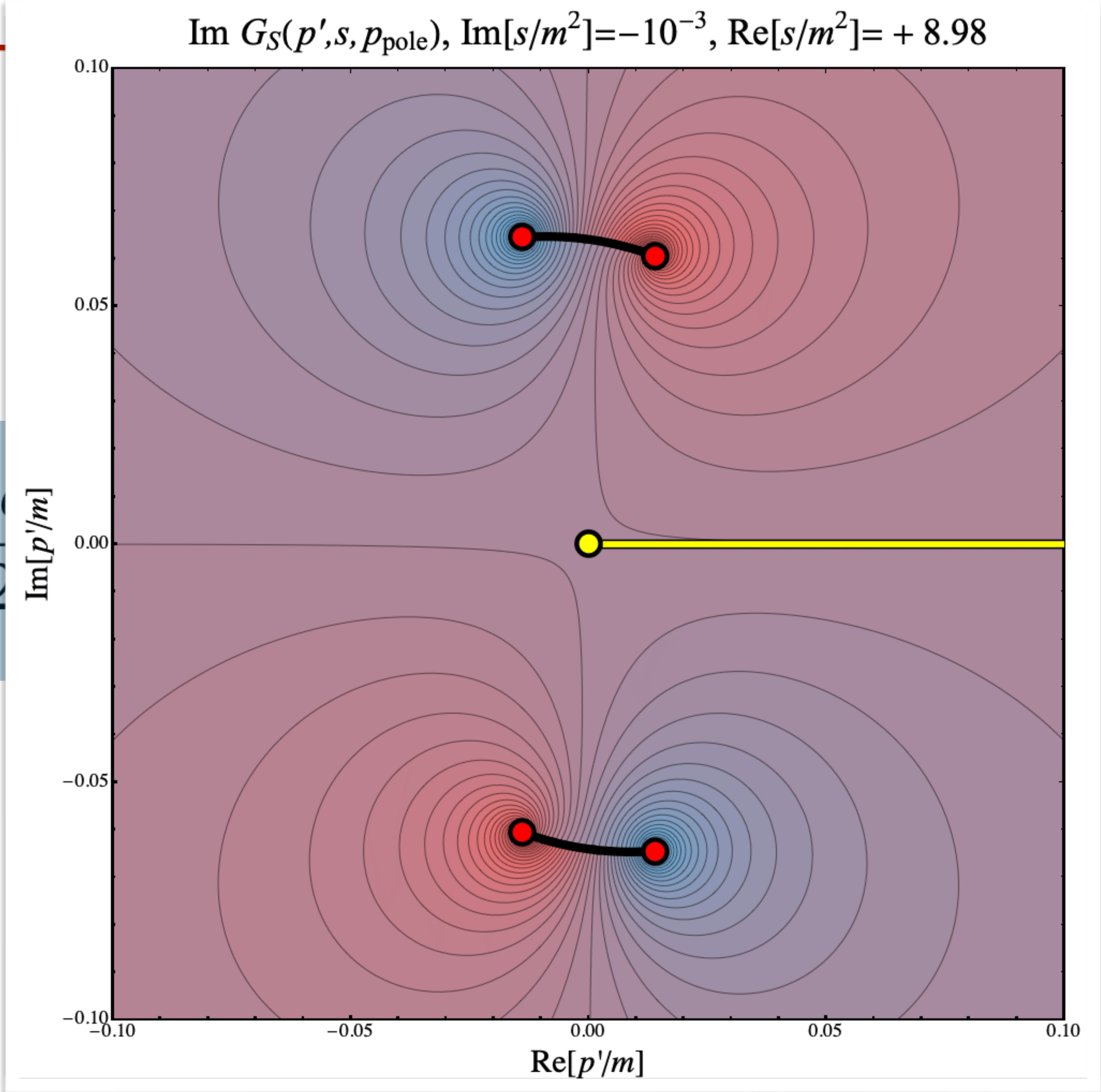
Integral equation below the threshold

- Avoid crossing the singularities in the

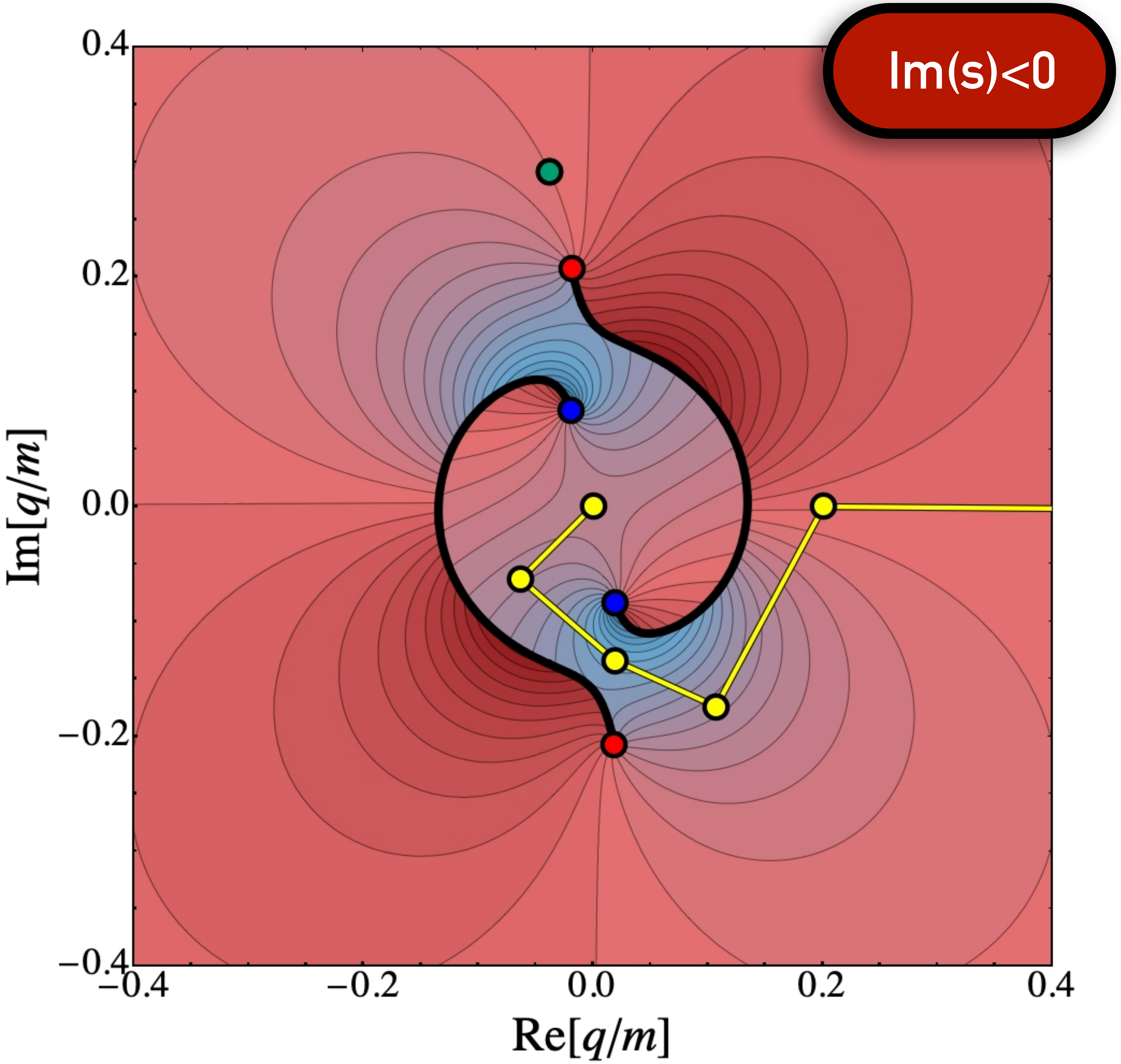
Inhomogeneous term

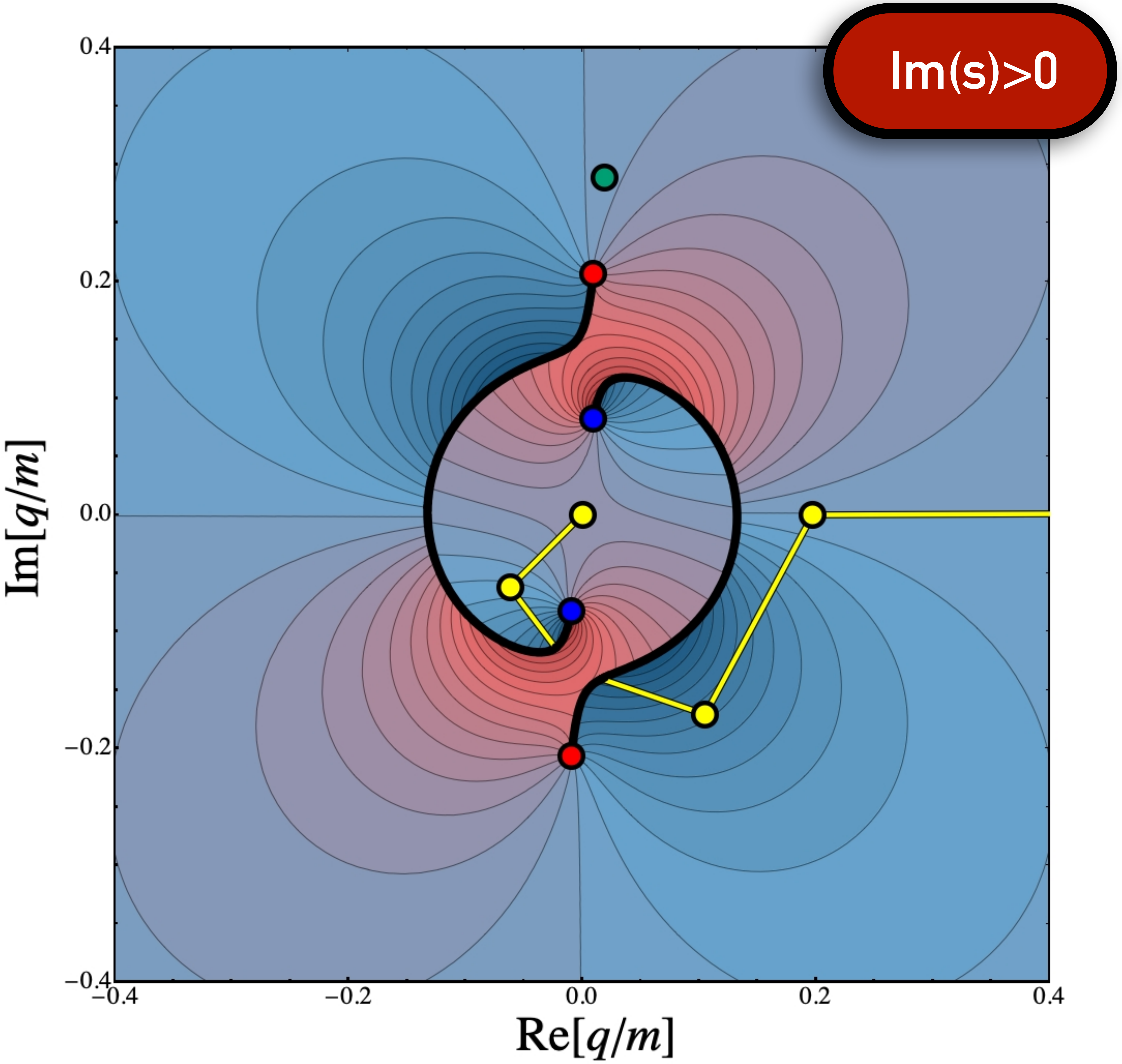
$$d(p', s, p) = -G(p', s, p) - \int_0^{q_{\max}} \frac{\ \ }{(2 \ \ }$$





Contour deformation in momentum variable





Integral equation below the threshold

- Homogeneous term also contributes singularities to the solution

$$d(p', s, p) = -G(p', s, p) - \int_0^{q_{\max}} \frac{dq\, q^2}{(2\pi)^2 \omega_q} G(p', s, q) \mathcal{M}_2(q, s)\, d(q, s, p)$$

Integral equation below the threshold

- Homogeneous term also contributes singularities to the solution

Inhomogeneous term

$$d(p', s, p) = \boxed{-G(p', s, p)} - \int_0^{q_{\max}} \frac{dq\, q^2}{(2\pi)^2 \omega_q} G(p', s, q) \mathcal{M}_2(q, s)\, d(q, s, p)$$

Integral equation below the threshold

- Homogeneous term also contributes singularities to the solution

Inhomogeneous term

$$d(p', s, p) = -G(p', s, p) - \int_0^{q_{\max}} \frac{dq\, q^2}{(2\pi)^2 \omega_q} G(p', s, q) \mathcal{M}_2(q, s)\, d(q, s, p)$$

Solution "inherits" the OPE cuts

Integral equation below the threshold

- Homogeneous term also contributes singularities to the solution

Inhomogeneous term

Homogeneous term

$$d(p', s, p) = -G(p', s, p) - \int_0^{q_{\max}} \frac{dq\, q^2}{(2\pi)^2 \omega_q} G(p', s, q) \mathcal{M}_2(q, s)\, d(q, s, p)$$

Solution "inherits" the OPE cuts

Integral equation below the threshold

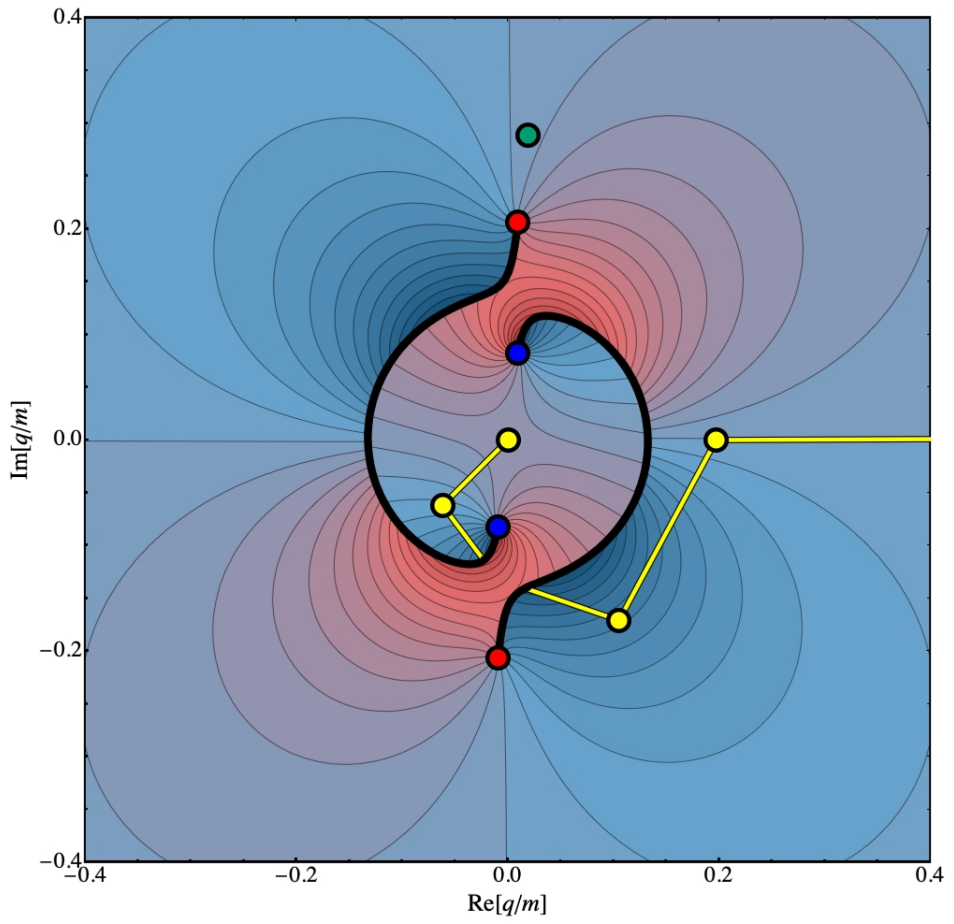
- Homogeneous term also contributes singularities to the solution

Homogeneous term

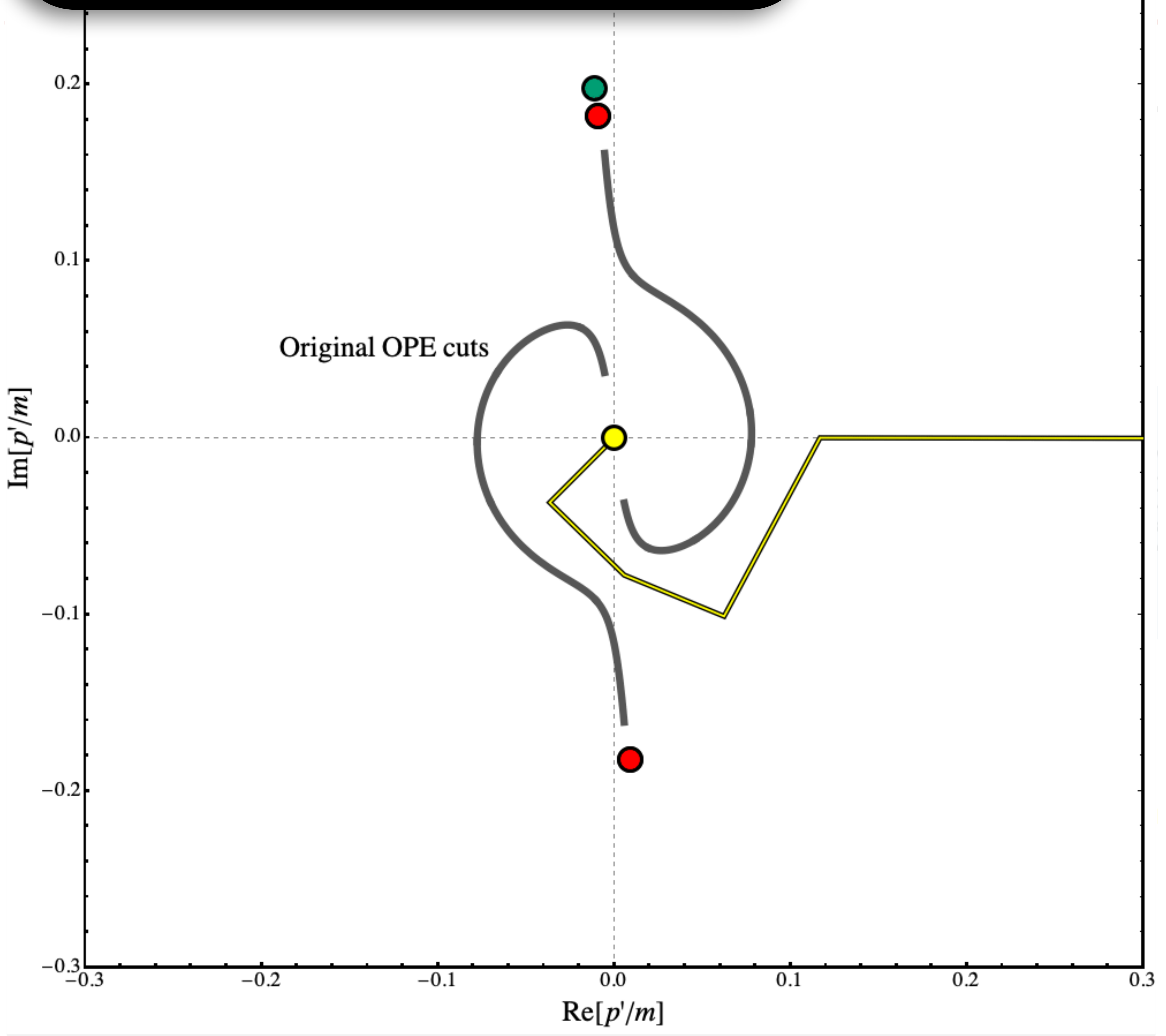
Inhomogeneous term

$$d(p', s, p) = -G(p', s, p) - \int_0^{q_{\max}} \frac{dq\, q^2}{(2\pi)^2 \omega_q} G(p', s, q) \mathcal{M}_2(q, s)\, d(q, s, p)$$

Solution "inherits" the OPE cuts



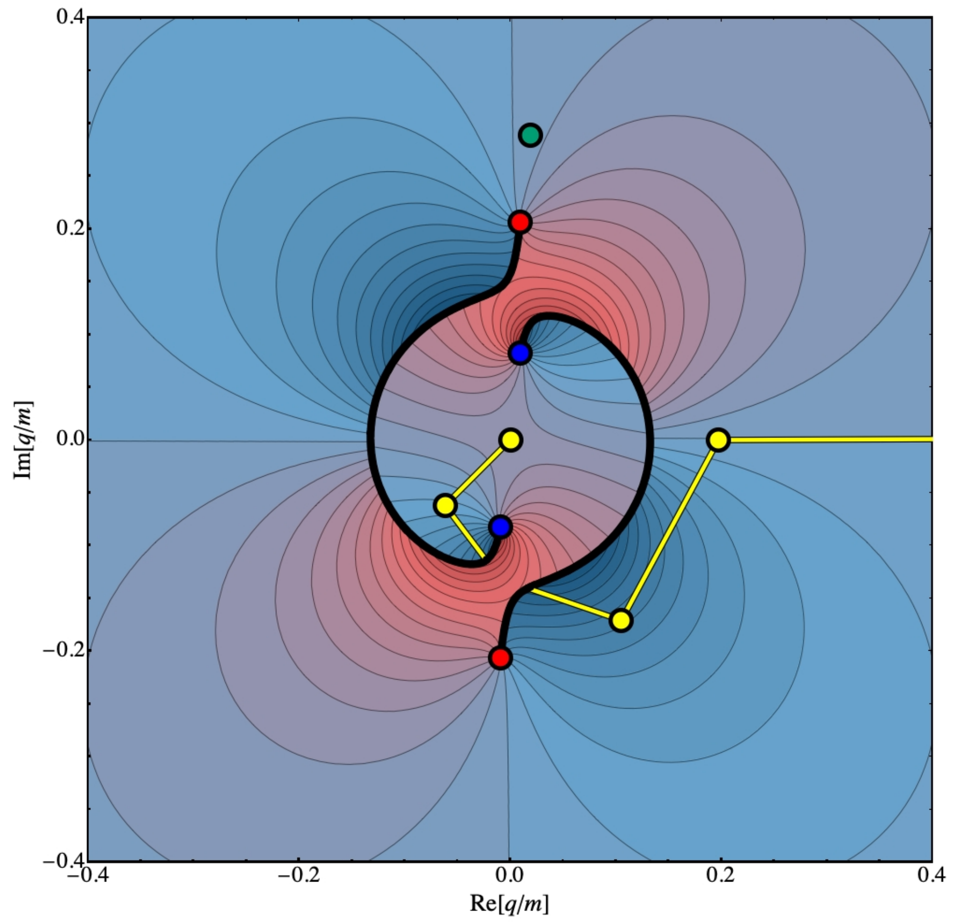
Self-consisentcy of the contour



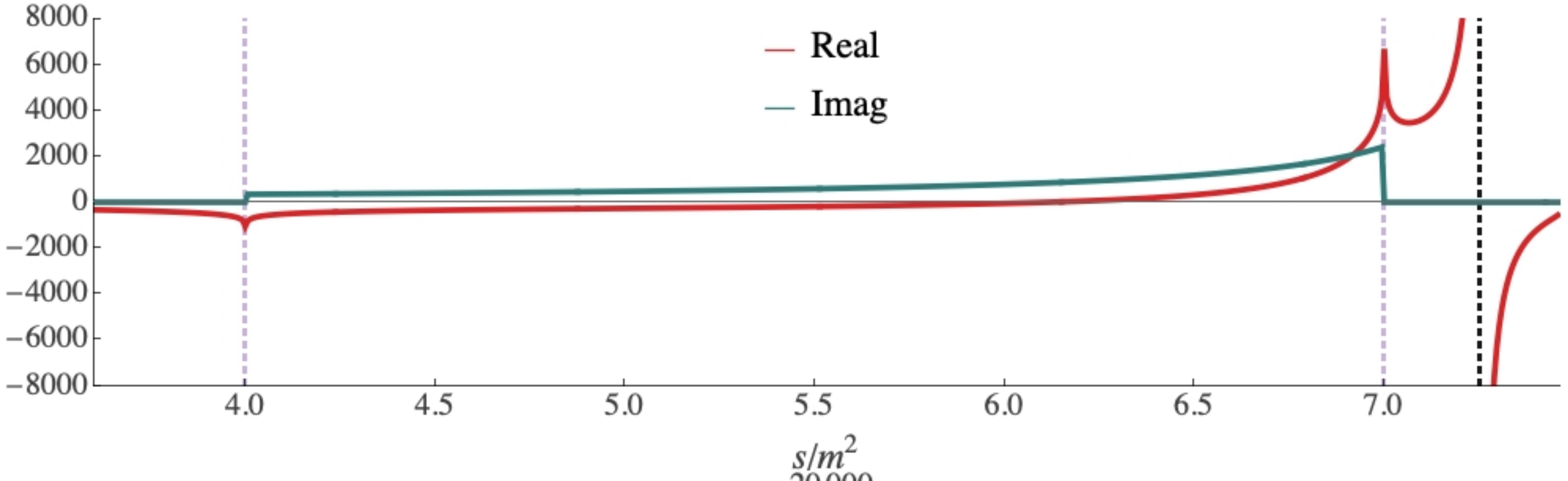
larities to the solution

Homogeneous term

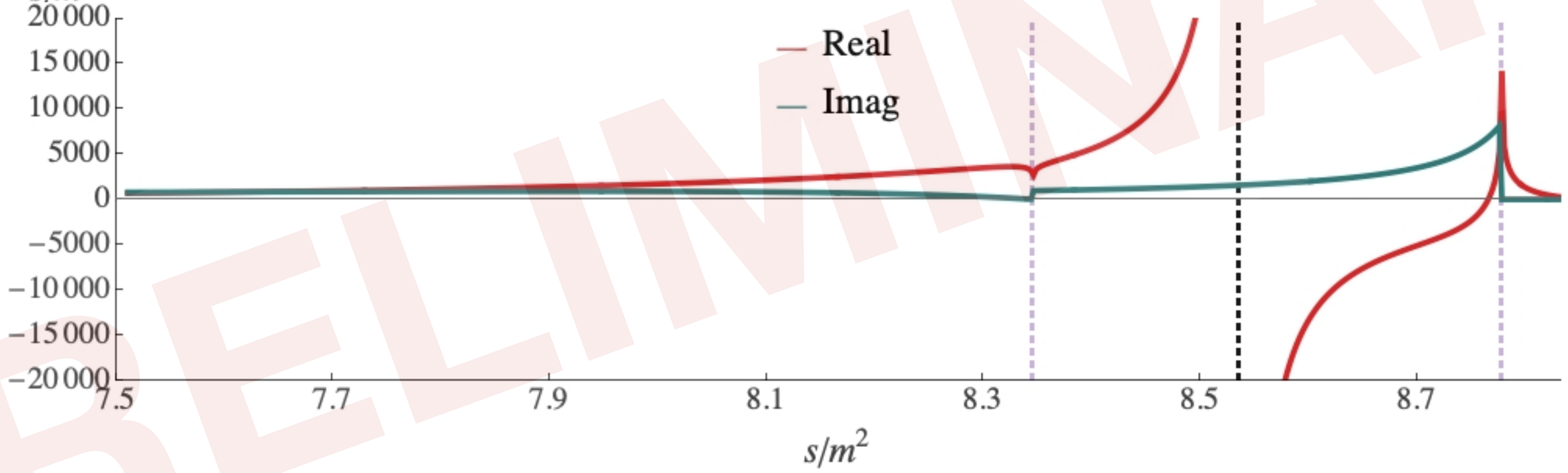
$$\frac{\;^2}{\omega_q} G(p', s, q)\mathcal{M}_2(q, s)\, d(q, s, p)$$



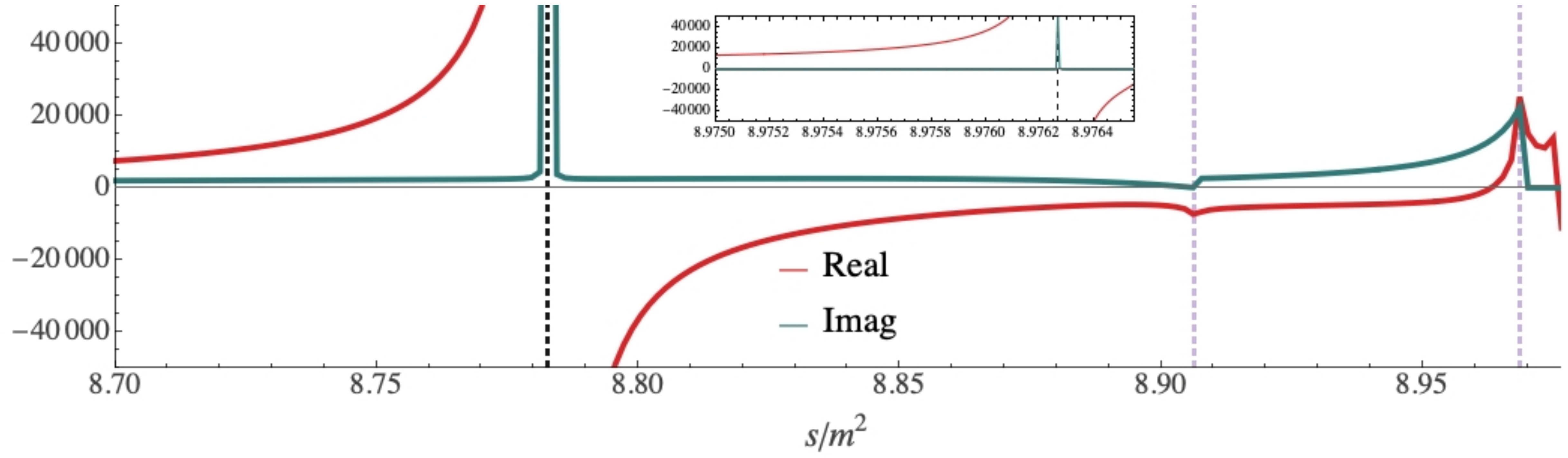
Results – Amplitudes



ma=2, pole at $s/m^2 = 7.25$

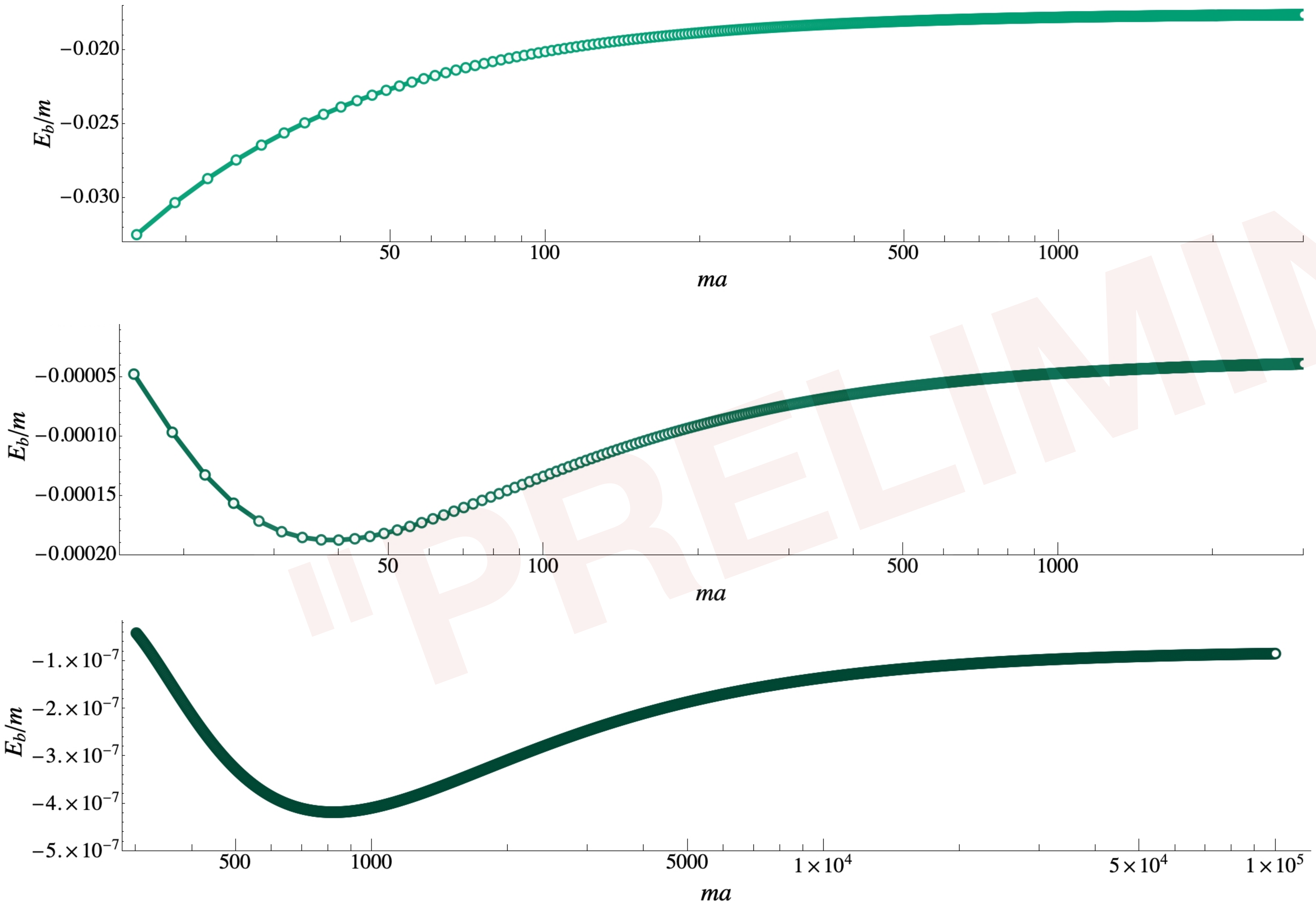


ma=16, pole at $s/m^2 = 8.78$
second pole at $s/m^2 = 8.98$



Efimov physics

- **Binding energies of the three-body bound states**



n=1

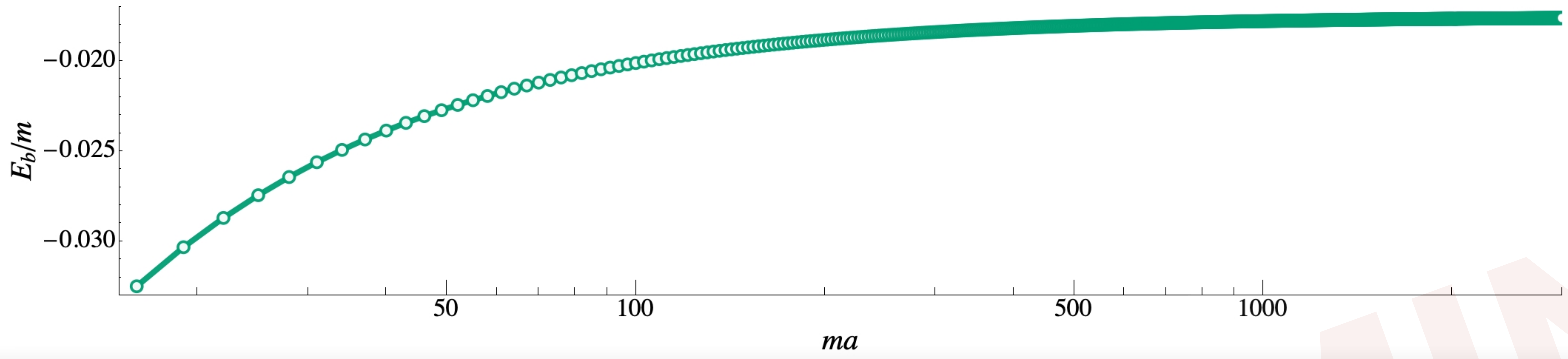
n=2

n=3

Analytic continuation of the three-body scattering equation in the presence of bound state
Dawid, Briceño, Jackura, Islam, in preparation

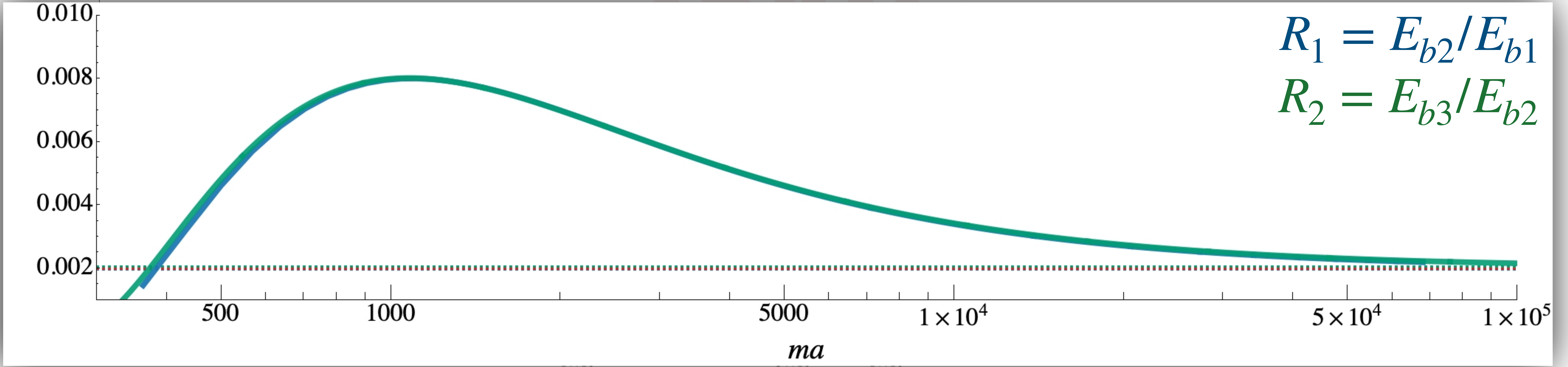
Efimov physics

- Binding energies of the three-body bound states



n=1

Ratios of binding energies approach Efimov's scaling constant



$R_1 = E_{b2}/E_{b1}$

$R_2 = E_{b3}/E_{b2}$

ma

Summary

- **Three-body scattering**
  - relevant for some of the most intriguing states
  - phenomenology & Lattice QCD

Summary

☑ Three-body scattering
- relevant for some of the most intriguing states
- phenomenology & Lattice QCD

☑ Resonance properties from LQCD
- three-body spectra
- quantization condition
- integral equations
- analytic continuation

Summary

☑ **Three-body scattering**
- relevant for some of the most intriguing states
- phenomenology & Lattice QCD

☑ **Resonance properties from LQCD**
- three-body spectra
- quantization condition
- integral equations
- analytic continuation

☑ **Solution of the ladder equation**
- two-body bound state
- systematic numerical procedure

Summary

- **Three-body scattering**
  - relevant for some of the most intriguing states
  - phenomenology & Lattice QCD

- **Resonance properties from LQCD**
  - three-body spectra
  - quantization condition
  - integral equations
  - analytic continuation

- **Solution of the ladder equation**
  - two-body bound state
  - systematic numerical procedure

- **Analytic continuation**
  - circular cut
  - Efimov states

THANK YOU