Bound states in the three-body scattering formalism

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Three-body processes and hadronic spectrum

- Interesting resonances decay to three-particle final states
  - $X(3872)$, $\pi_1(1600)$, $N^*(1440)$, $a_1(1420)$, $a_1(1260)$, ...

- Interpretations
  - molecules
  - diquark-antidiquark
  - hybrids
  - kinematical effects
Three-body processes and hadronic spectrum

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  - $X(3872), \pi_1(1600), N^*(1440), a_1(1420), a_1(1260), \ldots$

- Interpretations
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Intensity of the $1^{++} 0^+ f_0 \pi$ $P$ wave

COMPASS 2008

$0.1 < t'/(GeV/c)^2 < 0.113$

Number of events/(20 MeV/c^2)

On the nature of $a_1(1420)$
Three-body processes and hadronic spectrum

- Interesting resonances decay to three-particle final states
  \* $X(3872)$, $\pi_1(1600)$, $N^*(1440)$, $\sigma_1(1420)$, $\sigma_1(1260)$, ...

- Interpretations
  - molecules
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  - hybrids
  - kinematical effects

Additional energy level, which implies that $N\pi$ elastic scattering alone does not render a low-lying Roper resonance. The current status indicates that the $N^*(1440)$ might arise as dynamically generated resonance from coupling to other channels, most notably the $N\pi\pi$. 

On the nature of $\sigma_1(1420)$
Three-body processes and hadronic spectrum

Interesting resonances decay to three-particle final states

- \( X(3872), \pi_1(1600), N^*(1440), \sigma_1(1420), \sigma_1(1260), \ldots \)

Interpretations

GOAL: three-body scattering formalism

- properties of hadrons from the (lattice) QCD
- convenient three-body framework for phenomenology

additional energy level, which implies that \( N\pi \) elastic scattering alone does not render a low-lying Roper resonance. The current status indicates that the \( N^*(1440) \) might arise as dynamically generated resonance from coupling to other channels, most notably the \( N\pi\pi \).
The B-matrix approach

- Physical degrees of freedom (domain of integration)
- Simple parametrization with clear interpretation

Three-body amplitude

\[ A_{\ell' m_{\ell'}; \ell m_{\ell}}(p', s, p) \]

- pair-spectator
- partial waves
- symmetrization

\[ A = M_2 B M_2 + M_2 \int B \tau A \]

Three-body scattering amplitudes in the B-matrix formalism
The B-matrix approach

- Physical degrees of freedom (domain of integration)
- Simple parametrization with clear interpretation

\[ \tilde{A} = B + \int B M_2 \tau \tilde{A} \]

Three-body amplitude

\[ A_{\ell' m_{\ell'} ; \ell m_{\ell}} (p', s, p) \]

- pair-spectator
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Path to three-body physics from the lattice QCD

Lattice QCD  FV Spectrum  Amplitudes  Particle properties
Path to three-body physics from the lattice QCD

Lattice QCD  FV Spectrum  Amplitudes  Particle properties

(A) Finite volume spectrum  (B) Quantization Condition  (C) Three-body K-matrix

Relativistic, model-independent, three-particle quantization condition
Hansen, Sharpe, Phys. Rev. D 90 (2014) 11, 116003

Three-body unitarity in finite volume
Three-body spectra — status

Currently three $\pi$ and three $K$ at $l=3$, $S$ and $D$ waves

Hansen et al. (HadSpec), Phys. Rev. Lett. 126 (2021), 012001

Blanton et al. JHEP 10 (2021) 023
Path to three-body physics from the lattice QCD

(A) Finite volume spectrum  (B) Quantization Condition  (C) Three-body K-matrix
Path to three-body physics from the lattice QCD

Lattice QCD \quad \text{FV Spectrum} \quad \text{Amplitudes} \quad \text{Particle properties}

(a) K-matrix + two-body subprocesses \quad (b) \text{Integral equations} \quad (c) Three-body amplitudes

(d) Amplitudes \textit{analytically continued} to the unphysical Riemann sheets
Solving the REFT three-body ladder equation

- Ladder approximation, $B = G + (R=0)$

- Numerical solution of the three-body EFT equations

- Similar studies

- weakly interacting system in $\pi^+\pi^+$ and $\pi^+\pi^+\pi^+$
  

- decay $a_1(1260) \rightarrow \rho^0\pi^- \rightarrow \pi^-\pi^+\pi^-$
  
  
  * Sadasivan et al., Phys. Rev. D 105 (2022) 5, 054020
Solving the REFT three-body ladder equation

- Ladder approximation, \( B = G + (R=0) \)

\[
\begin{align*}
B &= G + (R=0) \\
&= \sum_{\text{terms}}
\end{align*}
\]

\( \mathcal{M}_2^{-1} \sim \frac{1}{a} - i\rho_2 \)

\[
\lim_{\sigma,\sigma' \to \sigma_b} A_3 = \frac{g}{\sigma' - \sigma_b} A_2 \frac{g}{\sigma - \sigma_b}
\]

Sadasivan et al., Phys. Rev. D 105 (2022) 5, 054020
Solving the REFT three-body ladder equation

Three-body scattering amplitudes in the B-matrix formalism
Solving the REFT three-body ladder equation
Solving the REFT three-body ladder equation

Bound-state—spectator amplitude

Break-up amplitude
Numerical procedure

- Discretization of the integral equation → $N$ linear equations (Matrix equation)
- Regulation of the bound-state pole via the $i\epsilon$ prescription

$$D(\sigma_p, s, \sigma_k) = -\mathcal{M}_2(\sigma_p)G(\sigma_p, s, \sigma_k)\mathcal{M}_2(\sigma_k)\mathcal{M}_2(\sigma_p)$$

$$D(s) = \lim_{\epsilon \to 0^+} \lim_{N \to \infty} d(s; N, \epsilon)$$
Numerical procedure

- Discretization of the integral equation → N linear equations (Matrix equation)
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$$D(\sigma_p, s, \sigma_k) = -\mathcal{M}_2(\sigma_p)G(\sigma_p, s, \sigma_k)\mathcal{M}_2(\sigma_k) - \mathcal{M}_2(\sigma_p) \int_0^{(\sqrt{s}-m)^2} \frac{d\sigma_q}{2\pi} G(\sigma_p, s, \sigma_q) \tau(\sigma_q, s) D(\sigma_q, s, \sigma_k)$$

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Numerical procedure

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\[
D(\sigma_p, s, \sigma_k) = \mathcal{M}_2(\sigma_p) \mathcal{G}(\sigma_p, s, \sigma_k) \mathcal{M}_2(\sigma_k) - \mathcal{M}_2(\sigma_p) \int_0^{(\sqrt{s} - m)^2} \frac{d\sigma_q}{2\pi} \mathcal{G}(\sigma_p, s, \sigma_q) \tau(\sigma_q, s) D(\sigma_q, s, \sigma_k)
\]

Amputation

\[
D(\sigma_p, s, \sigma_k) = \mathcal{M}_2(\sigma_p) d(\sigma_p, s, \sigma_k) \mathcal{M}_2(\sigma_k)
\]

\[
d(s) = \lim_{\epsilon \to 0^+} \lim_{N \to \infty} d(s; N, \epsilon)
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**Numerical procedure**

- Discretization of the integral equation → **N linear equations (Matrix equation)**
- Regulation of the bound-state pole via the \( i\epsilon \) prescription

\[
d(s) = \lim_{\epsilon \to 0^+} \lim_{N \to \infty} d(s; N, \epsilon)
\]

\[
d(N, \epsilon) = -G(\epsilon) - P \cdot G(\epsilon) \cdot \mathcal{M} \cdot d(N, \epsilon)
\]

\[
D(\sigma_p, s, \sigma_k) = \mathcal{M}_2(\sigma_p) d(\sigma_p, s, \sigma_k) \mathcal{M}_2(\sigma_k)
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\[
d(N, \epsilon) = -G(\epsilon) - P \cdot G(\epsilon) \cdot \mathcal{M} \cdot d(N, \epsilon)
\]

\[
d(N, \epsilon) = -[1 + P \cdot G(\epsilon) \cdot \mathcal{M}]^{-1} \cdot G(\epsilon)
\]

- Systematics: unitarity test, convergence test
- Different methods: "Brute force", explicit pole removal, spline-based quadratures
Example results, $M^2 = 3m^2$ (ma=2)

Three-body scattering amplitudes in the B-matrix formalism
Example result, three-body scattering length

This work, $\eta = 15$

Romero-López et al.  
NREFT $\Lambda = 0.75m$

Romero-Lopez et al., JHEP 10 (2019) 007

Example result, 2→3 amplitude

We are not limited to energies below the three-body threshold
Analytic continuation to complex energies
Three-body scattering amplitudes in the B-matrix formalism

Analytic continuation to complex energies

Amplitude below the physical threshold

\[ \varphi b \quad 3\varphi \quad \sqrt{s} \]
Analytic continuation to complex energies

Amplitude below the physical threshold

Three-body bound state poles
Integral equation below the threshold

- Avoid crossing the singularities in the integration

\[ d(p', s, p) = -G(p', s, p) - \int_{0}^{q_{\text{max}}} \frac{dq q^2}{(2\pi)^2 \omega_q} G(p', s, q) \mathcal{M}_2(q, s) d(q, s, p) \]
Integral equation below the threshold

Avoid crossing the singularities in the integration

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Integral equation below the threshold

- Avoid crossing the singularities in the integration

\[ d(p', s, p) = -G(p', s, p) - \int_0^{q_{\text{max}}} \left( \begin{array}{c} \text{Inhomogeneous term} \\ \text{Homogeneous term} \end{array} \right) \]
Contour deformation in momentum variable

\[
\text{Im}(s) < 0 \quad \text{Im}(s) > 0
\]
Homogeneous term also contributes singularities to the solution

\[ d(p', s, p) = -G(p', s, p) - \int_0^{q_{\text{max}}} \frac{dq}{(2\pi)^2} \frac{q^2}{\omega_q} G(p', s, q) M_2(q, s) d(q, s, p) \]
Integral equation below the threshold

Homogeneous term also contributes singularities to the solution

\[ d(p', s, p) = -G(p', s, p) - \int_0^{q_{\text{max}}} \frac{dq}{(2\pi)^2} \omega_q G(p', s, q) M_2(q, s) d(q, s, p) \]
Integral equation below the threshold

- Homogeneous term also contributes singularities to the solution

\[
d(p', s, p) = -G(p', s, p) - \int_{0}^{q_{\text{max}}} \frac{dq \, q^2}{(2\pi)^2 \omega_q} G(p', s, q) \mathcal{M}_2(q, s) \, d(q, s, p)
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\]
Three-body scattering amplitudes in the B-matrix formalism

Integral equation below the threshold

Analytic continuation of the three-body scattering equation in the presence of bound state
Dawid, Briceño, Jackura, Islam, in preparation

Inhomogeneous term

Homogeneous term also contributes singularities to the solution

\[ \frac{\omega_q^2}{\omega_q} G(p', s, q) M_2(q, s) d(q, s, p) \]

Self-consistency of the contour

Original OPE cuts

"inherits" the OPE cuts

Solution
Results — Amplitudes

ma=2, pole at $s/m^2 = 7.25$

ma=16, pole at $s/m^2 = 8.78$

second pole at $s/m^2 = 8.98$
Efimov physics

Binding energies of the three-body bound states

- $n=1$
- $n=2$
- $n=3$
Efimov physics

Binding energies of the three-body bound states

$n=1$

Ratios of binding energies approach Efimov’s scaling constant

\[
R_1 = \frac{E_{b2}}{E_{b1}} \quad R_2 = \frac{E_{b3}}{E_{b2}}
\]
Summary

- Three-body scattering
  - relevant for some of the most intriguing states
  - phenomenology & Lattice QCD
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- Resonance properties from LQCD
  - three-body spectra
  - quantization condition
  - integral equations
  - analytic continuation
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  - two-body bound state
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- **Analytic continuation**
  - circular cut
  - Efimov states
Thank You