Unifying Diquark and Molecular Models into a Universal Picture for the Exotics

Richard Lebed

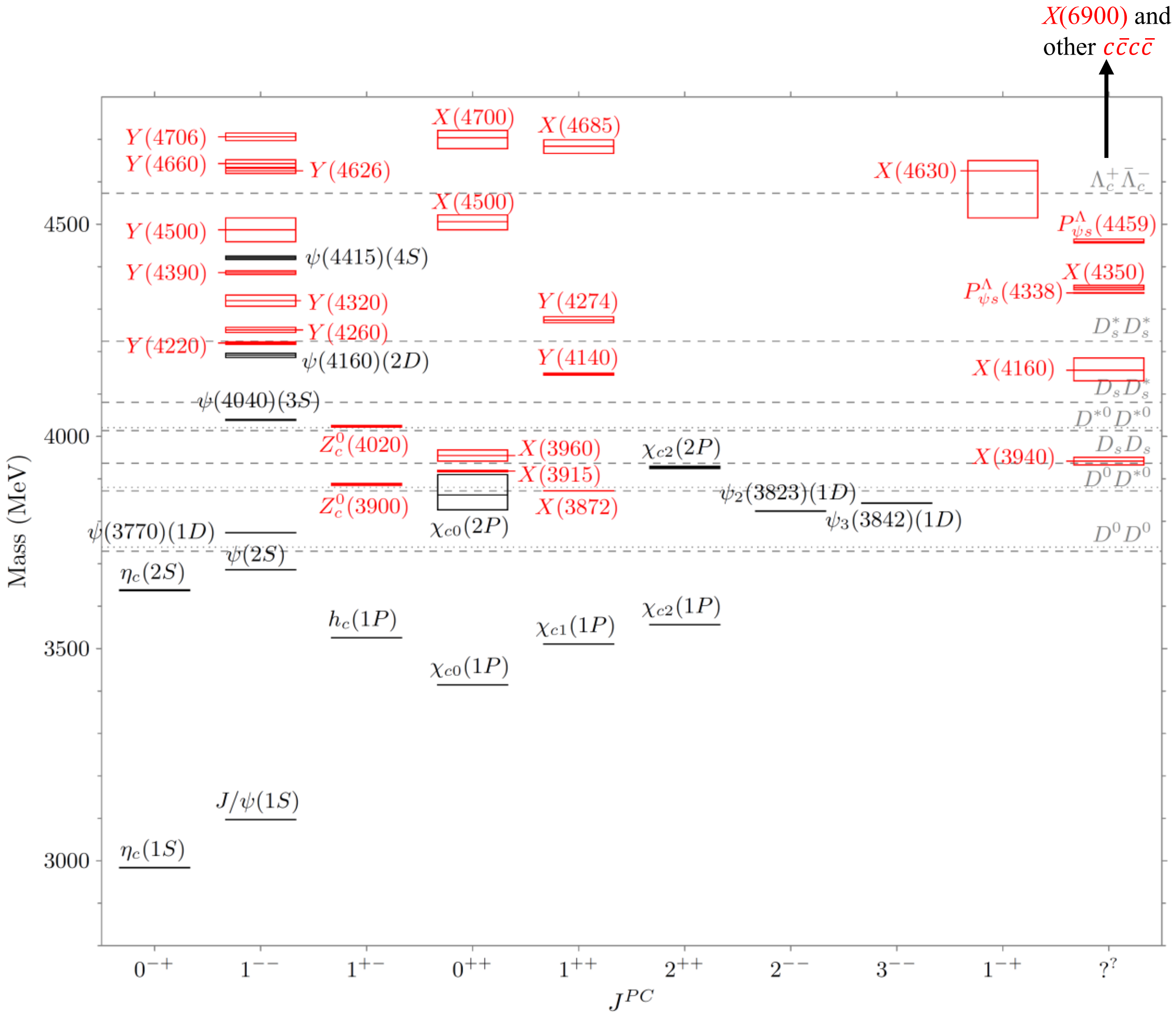
Arizona State University

Exotic Heavy Meson Spectroscopy and Structure with EIC

Center for Frontiers of Nuclear Science, Stony Brook University

August, 2022

Neutral hidden-charm system, August 2022



Several of the states are quite close to di-hadron thresholds

Most prominent example:

$m_{X(3872)} - m_{D^0} - m_{D^{*0}}$
$= -40 \pm 90$ keV

cf. the deuteron:

$m_d - m_p - m_n$
$= -2.2452(2)$ MeV

But many are *not* close to thresholds!

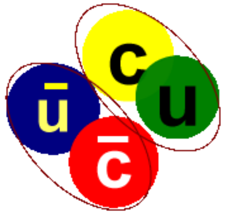
e.g., the 1^{--} Y states

Heavy-quark exotics census: August 2022

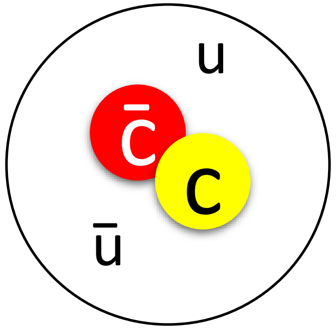
- **62** observed exotics, both tetraquarks and pentaquarks
  - 47 in the charmonium sector (neutral & charged, incl. open-strange)
  - 5 in the (much less explored) bottomonium sector
  - 4 with a single c quark (and an s, a u, and a d)
  - 1 with a single b quark (and an s, a u, and a d)
  - 4 with all c and $\bar{c}$ quarks
  - 1 with two c quarks
- A naïve count estimates well over 100 more exotics are waiting to be discovered

The internal structure of exotics is unresolved

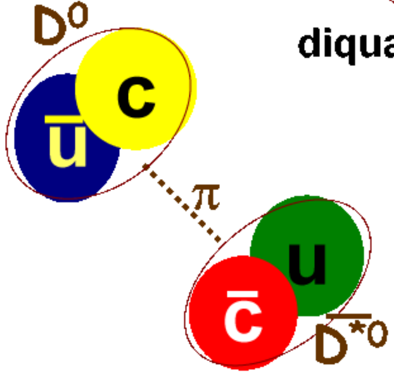
Mesons depicted here, but each model has a baryonic analogue



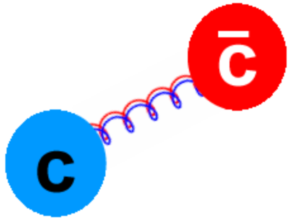
diquark-diantiquark



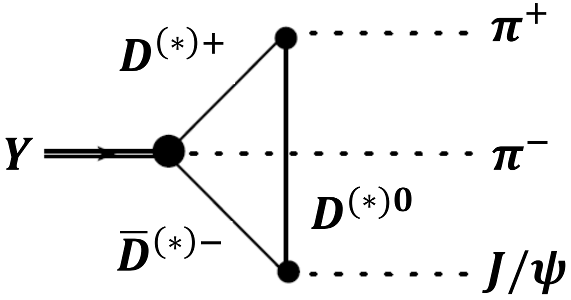
hadrocharmonium



D0 – D̄*0 “molecule”



qq̄-gluon“hybrid”



threshold/rescattering/cusp effect

Image adapted from Godfrey & Olsen,
Ann. Rev. Nucl. Part. Sci. **58** (2008) 51

“Each of the interpretations provides a natural explanation of parts of the data, but neither explains all of the data. It is quite possible that both kinds of structures appear in Nature. It may also be the case that certain states are superpositions of the compact and molecular configurations.”
—Karliner, Santopinto *et al.*, 2203.16583

The plan:

1) Develop a model that predicts a full spectrum for the expected exotics

2) Determine both the mass spectrum and decay patterns

3) Check whether yet-unobserved states are absent for some good reason

4) Seek to understand why some lie quite close to di-hadron thresholds

Why diquarks?

- Short-distance attraction of two color-$\mathbf{3}$ quarks into color-$\overline{\mathbf{3}}$ diquark is *fully half as strong* as combining $\mathbf{3}$ and $\overline{\mathbf{3}}$ into **color-neutral** singlet (*i.e.*, **diquark attraction** nearly as strong as the **confining attraction**)

- The $SU(2)$ analogue: Just as one computes a spin-spin coupling,

$$\vec{s}_1 \cdot \vec{s}_2 = \frac{1}{2}\left[(\vec{s}_1 + \vec{s}_2)^2 - \vec{s}_1{}^2 - \vec{s}_2{}^2\right],$$

  from two particles in representations 1 and 2 combined into representation 1+2:

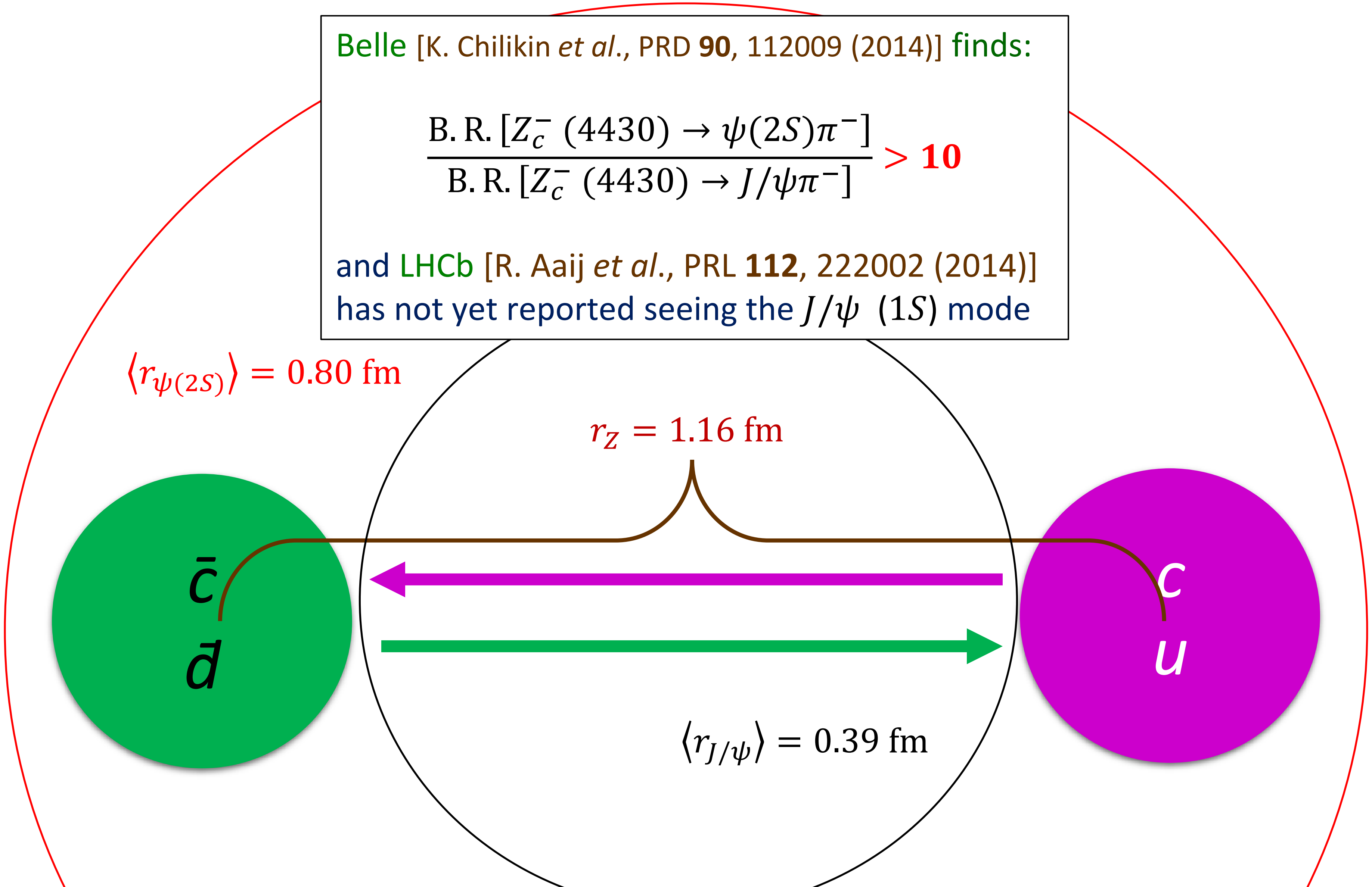
- If s_1, s_2 = spin $\frac{1}{2}$, and $\vec{s}_1 + \vec{s}_2$ = spin 0, get $-\frac{3}{4}$;
  if spin 1, get $+\frac{1}{4}$

Diquarks allow large, but still strongly bound, states

Belle [K. Chilikin *et al.*, PRD **90**, 112009 (2014)] finds:

$$\frac{\mathrm{B.R.}\,[Z_c^-\,(4430) \to \psi(2S)\pi^-]}{\mathrm{B.R.}\,[Z_c^-\,(4430) \to J/\psi\pi^-]} > \mathbf{10}$$

and LHCb [R. Aaij *et al.*, PRL **112**, 222002 (2014)] has not yet reported seeing the $J/\psi\ (1S)$ mode



The dynamical diquark *picture*:

Brodsky, Hwang, RFL [PRL **113**, 112001 (2014)]

- Heavy quarks provide nucleation points for diquark formation
- Separation of heavy quarks during production process (in heavy-hadron decays or high-energy collisions) leaves diquarks as identifiable constituent components of multiquark hadrons
- Diquark-antidiquark pair remain strongly connected by color flux tube → tetraquark $(Qq)_{\bar{\mathbf{3}}}(\bar{Q}\bar{q})_{\mathbf{3}}$
- Same color-triplet mechanism supports pentaquark formation, using a triquark: $[Q_{\mathbf{3}}(\bar{q}_1\bar{q}_2)_{\mathbf{3}}]_{\bar{\mathbf{3}}}(\bar{Q}\bar{q})_{\mathbf{3}}$ {RFL [PLB **749**, 454 (2015)]}

The dynamical diquark *model*:

RFL [PRD **96**, 116003 (2017)]

- Exotic eigenstate: the configuration once kinetic energy of the heavy di-(tri-)quarks converted into potential energy of the color flux tube
- Two heavy, slow sources connected by light degrees of freedom? That's the adiabatic approximation → ordinary Schrödinger equation
- In energy regions where only one potential-energy function important (*i.e.*, away from level crossings), have the single-channel approximation Together, these form the Born-Oppenheimer (BO) approximation
- BO potentials are same ones in lattice simulations of heavy-quark hybrids, labeled by axial quantum numbers such as in Σ_g^+, Π_u^-, *etc.*

Dynamical diquark model, first numerical results

Giron, RFL, Peterson [JHEP **05**, 061 (2019)]

- When the heavy sources coincide, BO potentials become degenerate (called *parity doubling* in atomic physics)
  → requires development of a coupled Schrödinger equation solver
- Our detailed simulations showed that all known exotics fit into the ground-state Σ_g^+ BO potential, in their $1S$, $1P$, $2S$, $2P$ orbitals
- Using the lattice-simulated BO potentials, the result for the $(cq)(\bar{c}\bar{q})$ $\Sigma_g^+(1S)$ multiplet-average mass naturally matches $m_{X(3872)}$, and those for $\Sigma_g^+(1P)$, $\Sigma_g^+(2S)$ beautifully match $m_{Y(4220)}$, $m_{Z_c(4430)}$, respectively
- But these are multiplet-average masses—Need to include fine structure

Dynamical diquark model, fine structure & isospin

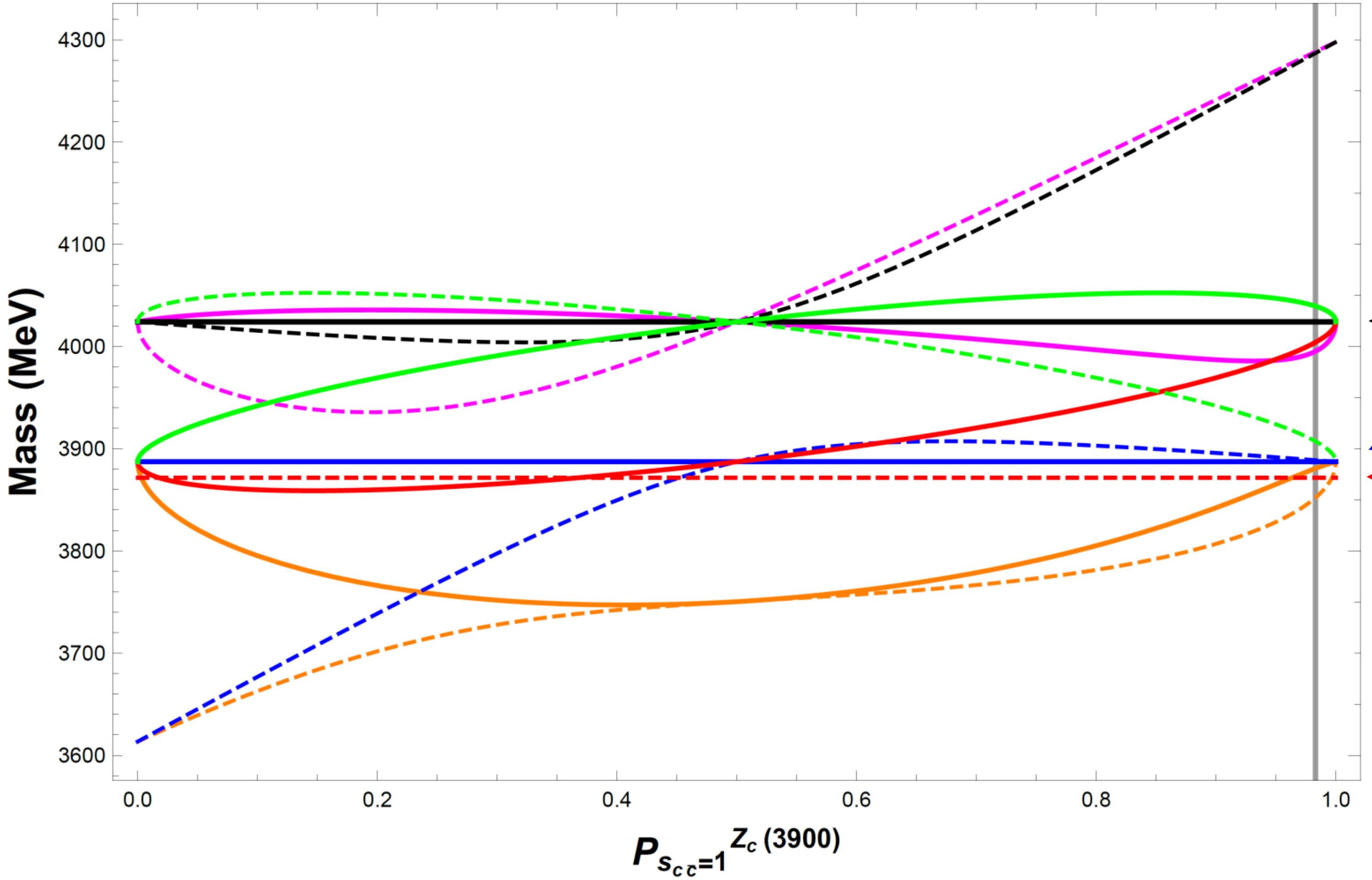
Giron, RFL, Peterson [JHEP **01**, 124 (2020)]

- With only a few known exotics in each multiplet, need to identify most physically important perturbation Hamiltonian operators
- *e.g.*, the multiplet $(cq)(\bar{c}\bar{q})\ \Sigma_g^+(1S)$ contains $6\ I=0$ and $6\ I=1$ states, and we know only $X(3872)$ [$I=0$], $Z_c(3900)$, $Z_c(4020)$ [$I=1$]
- Fixes 2 operators, taken to be:
  (1) quark spin-spin coupling *within* each diquark, and
  (2) isospin-spin exchange *between* diquarks (analogous to π exchange)
- Naturally predicts $X(3872)$ to be lightest narrow state in multiplet
- Naturally predicts $Z_c(3900)$ to decay preferentially to J/ψ ($s_{c\bar{c}}=1$) and $Z_c(4020)$ to h_c ($s_{c\bar{c}}=0$), as is observed

Dynamical diquark model, $(cq)(\bar{c}\bar{q})$ ground states

Figure from J. Giron, PhD dissertation (2021)

- The model also predicts masses for the other 9 states in $\Sigma_g^+(1S)$:



$Z_c(4020)$

$Z_c(3900)$

$X(3872)$

The orbitally excited $\Sigma_g^+(1P)$ multiplet

Giron, RFL [PRD **101**, 074032 (2020)]

- The lightest negative-parity states (like Y) live here:
  14 with $I = 0$, 14 with $I = 1$
- Multiplet $\Sigma_g^+(1P)$ contains precisely $4\ J^{PC} = 1^{--}, I = 0$ (Y) states
- Analysis requires more Hamiltonian operators: spin-orbit and tensor
- But which states are experimentally confirmed?
  BESIII data is rapidly improving, but still presents ambiguities
- Our analysis predicts full multiplet under several possible assignments:
  e.g., using $Y(4220)$, $Y(4320)$, $Y(4390)$ as inputs predicts
  the $Z_c(4240)$ [$J^{PC} = 0^{--}, I = 1$] seen by LHCb [PRL **122**, 22202 (2014)]

If the model is any good,
it must also apply to other flavor sectors

- Using the same Hamiltonian operators, apply to:

- the $(bq)(\bar{b}\bar{q})$ sector {Giron, RFL [PRD **102**, 014036 (2020)]}
  Here, just the masses of $Z_b(10610)$, $Z_b(10650)$, and their B.R.'s to $\Upsilon(nS)$, $h_b(nP)$ are enough to predict full $\Sigma_g^+(1S)$ multiplet
- the $(cs)(\bar{c}\bar{s})$ sector {Giron, RFL [PRD **102**, 014036 (2020)]}
  Here, $X(3915)$ (peculiar: no open-charm decay) is the lowest state, $X(4140)$ is analogue of $X(3872)$, and all other $\Sigma_g^+(1S)$ are predicted because the Hamiltonian has one less operator (zero isospin!)

If the model is any good, it must also apply to other flavor sectors

- Using the same Hamiltonian operators, apply to:

- the $(cq)(\bar{c}\bar{s})$ sector {Giron, Martinez, RFL [PRD **104**, 054001 (2021)]}
  Here, LHCb's recently observed $Z_{cs}(4000)$, $Z_{cs}(4220)$ [PRL **127**, 082001 (2021)] belong to $SU(3)_{\text{flavor}}$ multiplets of $J^{PC} = 1^{++}$ and 1^{+-}, but their strange members can mix, like K_{1A},K_{1B}
- the $(cu)(\bar{c}ud)$ and $(cs)(\bar{c}ud)$ pentaquarks
  {Giron, RFL [PRD **104**, 114028 (2021)]}
  Here, all the known nonstrange states:
  $P_c(4312)$, $P_c(4337)$, $P_c(4450)$, $P_c(4457)$, are easily accommodated

If the model is any good,
it must also apply to other flavor sectors

- Using the same Hamiltonian operators, apply to:
- the $(cc)(\bar{c}\bar{c})$ sector {Giron, RFL [PRD **102**, 074003 (2020)]}
  Here, identical particle constraints limit the number of allowed states
- Not easy to describe $(cc)(\bar{c}\bar{c})$ states using “conventional” di-hadron molecule picture
- Our calculations indicate $X(6900)$ [Sci. Bull. **65**, 1983 (2020)] observed by LHCb is almost certainly in an excited orbital, most likely $\Sigma_g^+(2S)$
- So where are the lower $(cc)(\bar{c}\bar{c})$ states?
  Preliminary results from CMS and ATLAS indicate seeing them!

But what about the closeness of some exotics to di-hadron thresholds?

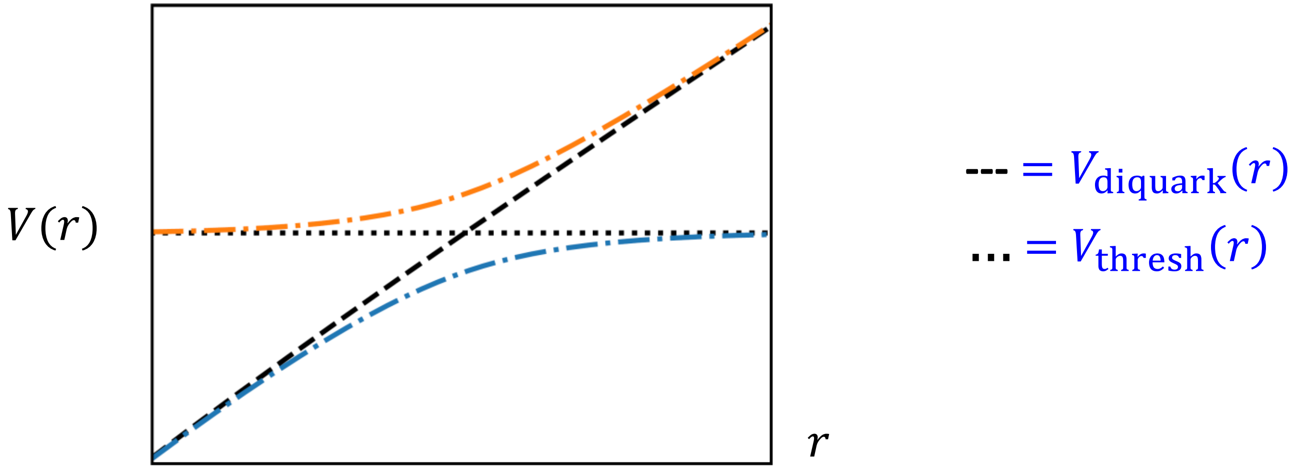
- Since the constituents are the same, *e.g.*, $(cq)(\bar{c}\bar{q})$ *vs.* $(c\bar{q})(\bar{c}q)$, some exotics will lie naturally close (~ 10 MeV) to thresholds

- But $m_{X(3872)} - m_{D^0} - m_{D^{*0}} = -40 \pm 90$ keV *cannot* be an accident!
- This binding energy much smaller than expected for a "conventional" hadron molecule—more likely a threshold rescattering effect [coupling to near-on-shell particle pair leads to enhanced amplitude]

- Much work done to explain some exotics as purely threshold effects, but not every threshold seems to have a prominent associated state

Diabatic corrections

- But what if both types of potentials are present (diquark-antidiquark and di-hadron threshold)?
- This is a well-known problem in atomic physics: One must perform coupled-channel calculation to find mixed-configuration eigenstates near level crossing, where adiabatic approximation fails
- Rigorous method to incorporate these effects: **diabatic approach**



Diabatic corrections

- Choose a separation $\boldsymbol{r}_0$ of heavy sources at which mixing is small
- Solve Schrödinger equation for eigenstates $|\xi_i(\boldsymbol{r}_0)\rangle$, where i labels unmixed diquark/di-hadron components
- Given interaction Hamiltonian for light degrees of freedom H_{light}, compute diabatic potential matrix $V_{ji}(\boldsymbol{r},\boldsymbol{r}_0) \equiv \langle\xi_j(\boldsymbol{r}_0)|H_{\text{light}}|\xi_i(\boldsymbol{r}_0)\rangle$
- The rest of the Hamiltonian is heavy-source kinetic-energy operator, $\mathrm{K} = \mathrm{diag}\{-\hbar^2\nabla^2/2\mu_i\}$
- Solve the coupled Schrödinger equation $[\mathrm{K}+\mathrm{V}(\boldsymbol{r})]\Psi(\boldsymbol{r}) = E\Psi(\boldsymbol{r})$ for eigenstates $|\Psi\rangle$, expressed as linear combinations of $|\xi_i\rangle$

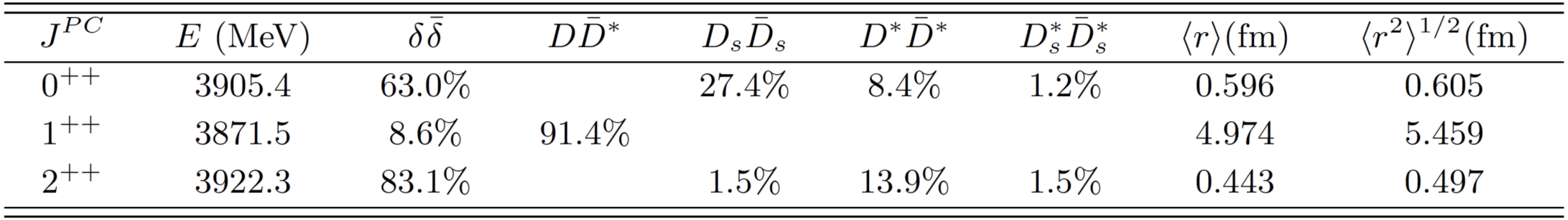
Diabatic corrections

- Only missing ingredient: What is the mixing potential of H_{light} (gives off-diagonal elements of diabatic potential matrix?
- Lattice simulations are able to calculate these (*e.g.*, string-breaking potential static energies), but in the meantime can model them as Gaussians that rapidly transition at the level crossing
- Diabatic approach first applied to mixing of hadron thresholds with conventional quarkonium: Bruschini & Gonzalez [PRD **102**, 074002 (2020)]
- We use the same techniques, but instead with diquark states
  The coupled-channel Schrödinger solver from prior work comes in very handy!
  (RFL, Martinez [2207.01101])

Diabatic framework first results

RFL, Martinez [2207.01101]

- It is not at all unnatural for a diquark state near a threshold to acquire a very large di-hadron component, while others do not:



J^{PC}	E (MeV)	$\delta\bar{\delta}$	$D\bar{D}^*$	$D_s\bar{D}_s$	$D^*\bar{D}^*$	$D_s^*\bar{D}_s^*$	$\langle r\rangle$(fm)	$\langle r^2\rangle^{1/2}$(fm)
0^{++}	3905.4	63.0%		27.4%	8.4%	1.2%	0.596	0.605
1^{++}	3871.5	8.6%	91.4%				4.974	5.459
2^{++}	3922.3	83.1%		1.5%	13.9%	1.5%	0.443	0.497

- Knowing explicitly the diquark (short-distance) as well as di-hadron (long-distance) components allows one to probe effects sensitive to short-distance structure, such as radiative decays: *e.g.*, B.R.[$X(3872) \longrightarrow \gamma\psi(2S)$]= $(4.5 \pm 2.0)\%$

Diabatic framework: The future

- Is this a true unification of diquark and molecular models? Here, the threshold potential is just treated as a free di-hadron state, but changing it to a binding potential would be trivial (*future work*)
- The calculations of 2207.01101 treat all $\Sigma_g^+(1S)$ states as degenerate, but fine structure is easy to incorporate into H_{light} (*future work*)
- Computing mass eigenvalues this way is rigorous only for states below or slightly above thresholds. States substantially above thresholds are broad resonances—Should be treated as poles in scattering amplitudes Here again, Bruschini & Gonzalez [PRD **104**, 074025 (2021)] provides a relevant diabatic framework (*future work*)

Summary & Conclusions

1) So many heavy-quark exotics have now been observed that
   a theory to predict their complete spectrum has become **imperative**
2) Molecules alone are not enough: Many exotics lie far from constituent thresholds
3) Models based upon diquarks hold promise:
   Fully predicted spectrum, whole state bound by strong QCD forces,
   many phenomenological successes (especially in the dynamical diquark model)
4) But many exotics *are* very close to thresholds⟶the adiabatic nature of the
   Born-Oppenheimer approximation can be generalized to the **diabatic approach**
   when di-hadron thresholds are nearby, unifying diquark & molecular pictures
5) First calculations of dynamical diquark model using diabatic framework complete,
   research in multiple future directions now underway

Backup Slides

For decades, hadronic spectroscopy was the core of high-energy physics

- 1947: Discovery of $\pi^{\pm}$, $K^{\pm}$, K^{0}
- 1950 ~ 1965: The hadron zoo; strangeness; the Eightfold Way;
  the quark model; color charge
- 1974: Charmonium; evidence for asymptotic freedom & QCD
- 1977: Bottomonium; 3rd generation of quarks needed for CP violation
- 1983: First full reconstruction of B meson decays
- 1983: W & Z bosons. Look for top quark! Look for Higgs!
  Look for BSM!!
- 1983– Hadron spectroscopy: Fill out the quark-model multiplets 🥱

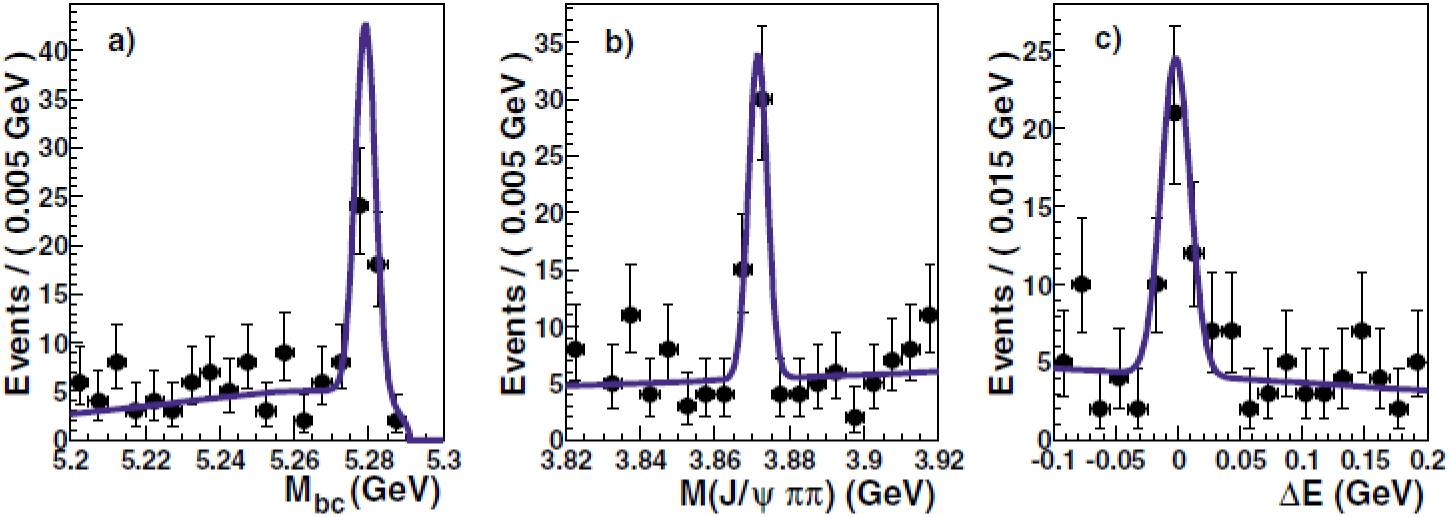
And then, in 2003…

The **Belle Collaboration** at KEK found evidence for a narrow new particle at 3872 MeV

In the broad mass range of charmonium, but behaves *very unlike* a pure $c\bar{c}$ state

Almost certainly a hadron of valence quark content $c\bar{c}q\bar{q}$

S.K. Choi *et al.*, Phys. Rev. Lett. **91** (2003) 262001

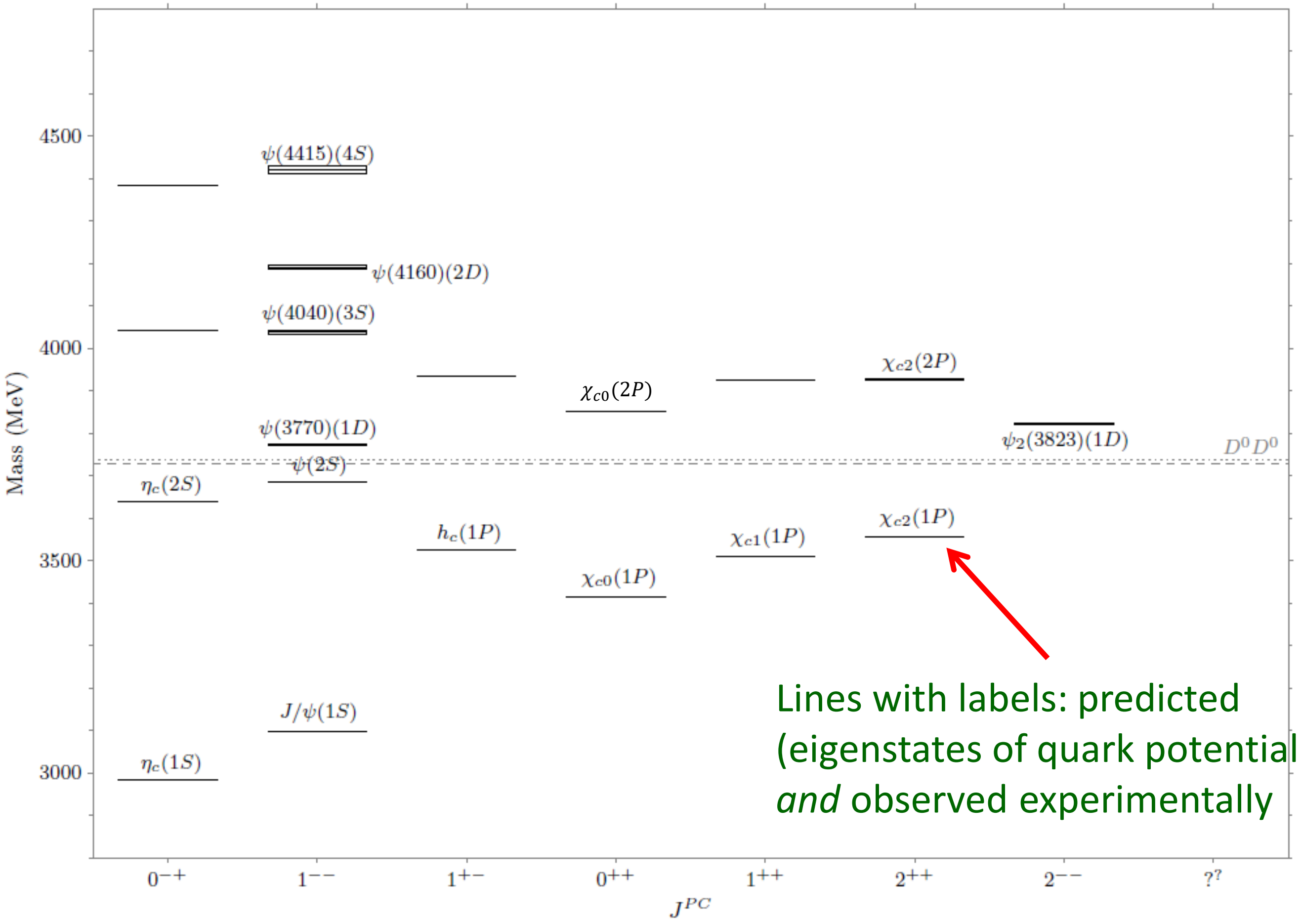


Reminder: The primary goal of Belle was the search for *CP* violation in the *B* system

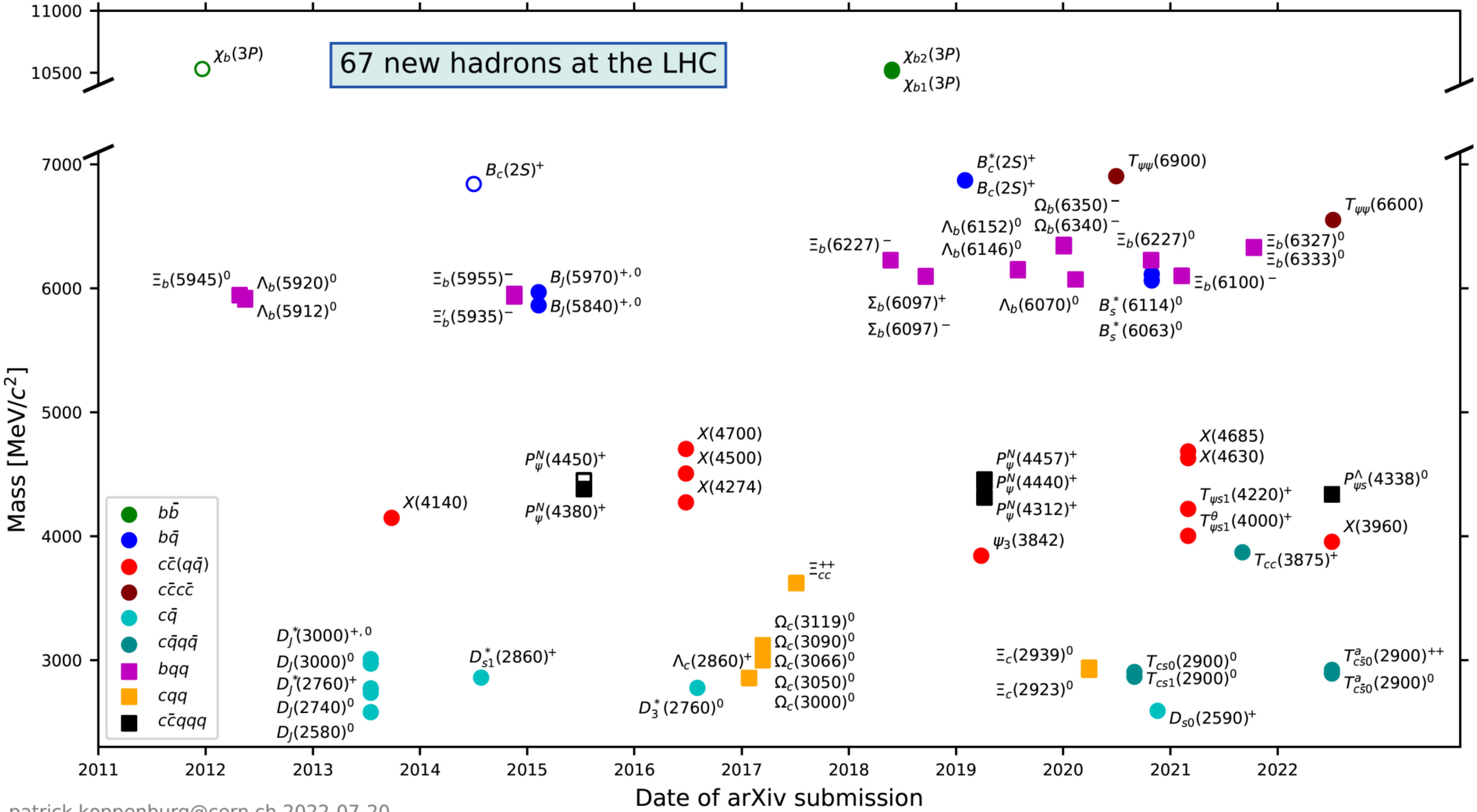
Do we really understand hadrons?

- Why was every hadron discovered in the quark model's first 50 years a $q\bar{q}$ meson or qqq baryon? Even Gell-Mann & Zweig saw other options:
  - $q\bar{q}q\bar{q}$, $q\bar{q}q\bar{q}q\bar{q}$, … (*tetraquark*, *hexaquark*, *…*)
  - $qqqq\bar{q}$, $qqqqqq\bar{q}$, … (*pentaquark*, *octoquark*, …)
- And with development of QCD and the discovery of gluons, other possibilities became available:
  - gg, ggg, … (*glueball*)
  - $q\bar{q}g$, $q\bar{q}gg$, … (*hybrid meson*)
- Are diquarks in their attractive color channel important hadrons subunits?
- Do molecules of hadrons (like deuterons) involving mesons form?
- **It is quite humbling that after 60 (50) years of the quark model (QCD), we still do not have undisputed answers to these fundamental questions!**
- Modern hadron spectroscopy aims at settling these issues
- Consequences could be far-ranging:
  - Neutron star models
  - Contributions of hadronic effects in rare B decays (BSM)
  - Phenomenology of *any* strongly coupled theory

What the charmonium system should look like



Lines with labels: predicted
(eigenstates of quark potential model)
and observed experimentally



patrick.koppenburg@cern.ch 2022-07-20

Even the Naming Scheme Has to Be Updated

LHCb Collaboration, 2206.15233

- "I found a hadron with valence-quark content c, s, u, d"

- "Which one? There are three kinds, not counting antiparticles:"
- $c\bar{s}u\bar{d}$: $T^{X}_{c\bar{s}\,J}(\mathrm{mass})^{++}$
- $c\bar{s}d\bar{u}$: $T^{X}_{c\bar{s}\,J}(\mathrm{mass})^{0}$
- $cs\bar{u}\bar{d}$: $T^{X}_{cs\,J}(\mathrm{mass})^{0}$
- Examples of all three of these types have already been observed!

J = total spin,
X = symbol for parity & isospin
(*e.g.*, a for $P = +$, $I = 1$)

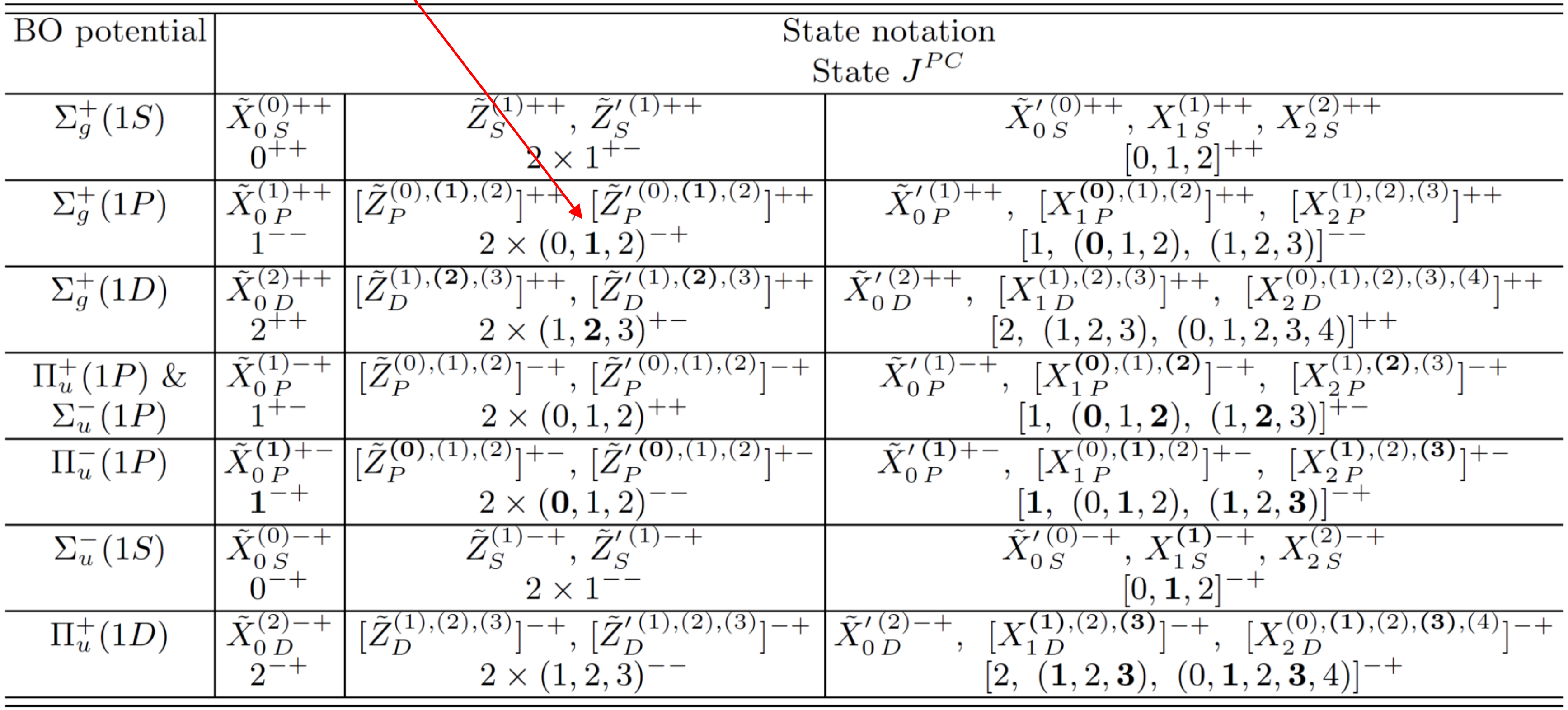
Not all exotic candidates have heavy quarks

- $\pi_1(1600)$ (discovered 1998) is believed to be a hybrid meson
  because its $J^{PC} = 1^{-+}$ is not accessible to $q\bar{q}$ states
- $f_0(1710)$ is believed to have a sizeable glueball component
  because the quark model predicts one fewer 0^{++} states than are seen, and of them
  $f_0(1710)$ shows up most prominently in J/ψ decays (a glue-rich environment)
- $\phi(2170)$ has a peculiar decay pattern and may be an $s\bar{s}g$ hybrid
  or the $s\bar{s}q\bar{q}$ tetraquark analogue to the $c\bar{c}q\bar{q}$ state $Y(4230)$

Exotics spectroscopy using BO potentials:

Tetraquarks

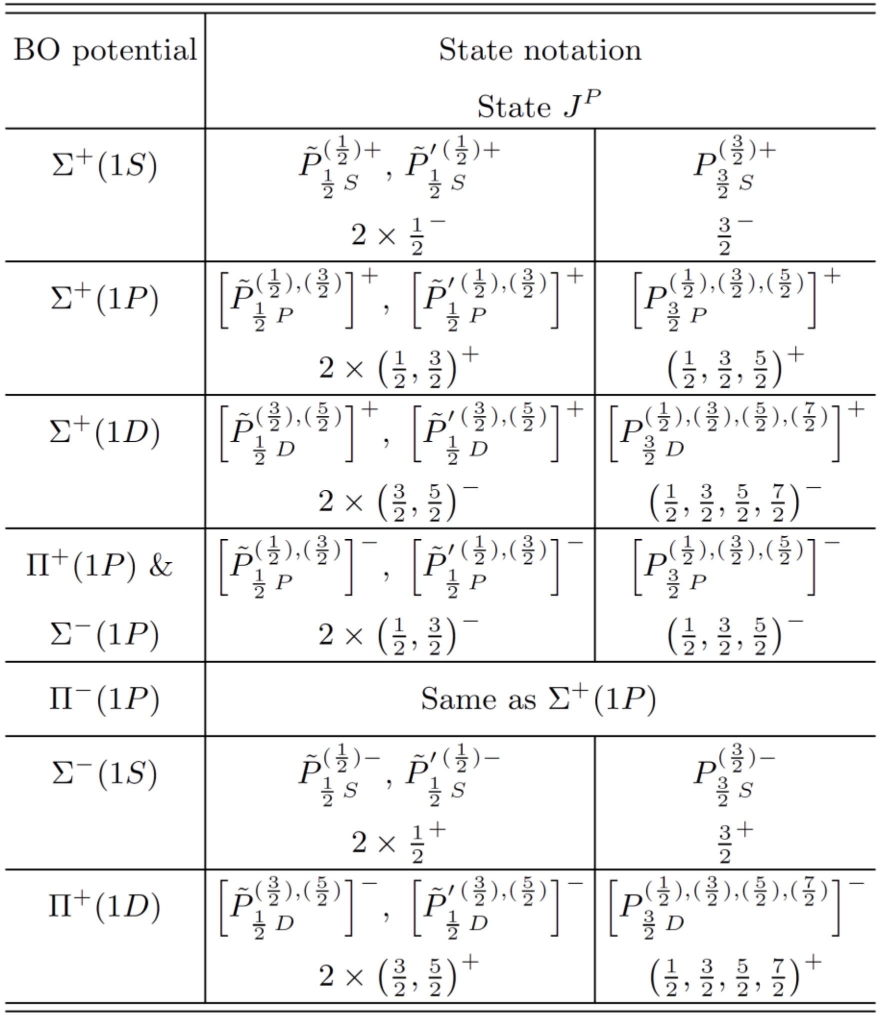
Boldface = exotic quantum numbers for $q\bar{q}$



BO potential	State notation State J^{PC}		
$\Sigma_g^+(1S)$	$\tilde{X}_{0\,S}^{(0)++}$ 0^{++}	$\tilde{Z}_{S}^{(1)++},\ \tilde{Z}'^{(1)++}_{S}$ $2\times 1^{+-}$	$\tilde{X}'^{(0)++}_{0\,S},\ X_{1\,S}^{(1)++},\ X_{2\,S}^{(2)++}$ $[0,1,2]^{++}$
$\Sigma_g^+(1P)$	$\tilde{X}_{0\,P}^{(1)++}$ 1^{--}	$[\tilde{Z}_{P}^{(0),\mathbf{(1)},(2)}]^{++},\ [\tilde{Z}'^{(0),\mathbf{(1)},(2)}_{P}]^{++}$ $2\times(0,\mathbf{1},2)^{-+}$	$\tilde{X}'^{(1)++}_{0\,P},\ [X_{1\,P}^{\mathbf{(0)},(1),(2)}]^{++},\ [X_{2\,P}^{(1),(2),(3)}]^{++}$ $[1,\ (\mathbf{0},1,2),\ (1,2,3)]^{--}$
$\Sigma_g^+(1D)$	$\tilde{X}_{0\,D}^{(2)++}$ 2^{++}	$[\tilde{Z}_{D}^{(1),\mathbf{(2)},(3)}]^{++},\ [\tilde{Z}'^{(1),\mathbf{(2)},(3)}_{D}]^{++}$ $2\times(1,\mathbf{2},3)^{+-}$	$\tilde{X}'^{(2)++}_{0\,D},\ [X_{1\,D}^{(1),(2),(3)}]^{++},\ [X_{2\,D}^{(0),(1),(2),(3),(4)}]^{++}$ $[2,\ (1,2,3),\ (0,1,2,3,4)]^{++}$
$\Pi_u^+(1P)$ & $\Sigma_u^-(1P)$	$\tilde{X}_{0\,P}^{(1)-+}$ 1^{+-}	$[\tilde{Z}_{P}^{(0),(1),(2)}]^{-+},\ [\tilde{Z}'^{(0),(1),(2)}_{P}]^{-+}$ $2\times(0,1,2)^{++}$	$\tilde{X}'^{(1)-+}_{0\,P},\ [X_{1\,P}^{\mathbf{(0)},(1),\mathbf{(2)}}]^{-+},\ [X_{2\,P}^{(1),\mathbf{(2)},(3)}]^{-+}$ $[1,\ (\mathbf{0},1,\mathbf{2}),\ (1,\mathbf{2},3)]^{+-}$
$\Pi_u^-(1P)$	$\tilde{X}_{0\,P}^{\mathbf{(1)}+-}$ $\mathbf{1}^{-+}$	$[\tilde{Z}_{P}^{\mathbf{(0)},(1),(2)}]^{+-},\ [\tilde{Z}'^{\mathbf{(0)},(1),(2)}_{P}]^{+-}$ $2\times(\mathbf{0},1,2)^{--}$	$\tilde{X}'^{\mathbf{(1)}+-}_{0\,P},\ [X_{1\,P}^{(0),\mathbf{(1)},(2)}]^{+-},\ [X_{2\,P}^{\mathbf{(1)},(2),\mathbf{(3)}}]^{+-}$ $[\mathbf{1},\ (0,\mathbf{1},2),\ (\mathbf{1},2,\mathbf{3})]^{-+}$
$\Sigma_u^-(1S)$	$\tilde{X}_{0\,S}^{(0)-+}$ 0^{-+}	$\tilde{Z}_{S}^{(1)-+},\ \tilde{Z}'^{(1)-+}_{S}$ $2\times 1^{--}$	$\tilde{X}'^{(0)-+}_{0\,S},\ X_{1\,S}^{\mathbf{(1)}-+},\ X_{2\,S}^{(2)-+}$ $[0,\mathbf{1},2]^{-+}$
$\Pi_u^+(1D)$	$\tilde{X}_{0\,D}^{(2)-+}$ 2^{-+}	$[\tilde{Z}_{D}^{(1),(2),(3)}]^{-+},\ [\tilde{Z}'^{(1),(2),(3)}_{D}]^{-+}$ $2\times(1,2,3)^{--}$	$\tilde{X}'^{(2)-+}_{0\,D},\ [X_{1\,D}^{\mathbf{(1)},(2),\mathbf{(3)}}]^{-+},\ [X_{2\,D}^{(0),\mathbf{(1)},(2),\mathbf{(3)},(4)}]^{-+}$ $[2,\ (\mathbf{1},2,\mathbf{3}),\ (0,\mathbf{1},2,\mathbf{3},4)]^{-+}$

Exotics spectroscopy using BO potentials:

Pentaquarks



BO potential	State notation State J^P	
$\Sigma^+(1S)$	$\tilde{P}^{(\frac{1}{2})+}_{\frac{1}{2}S}$, $\tilde{P}'^{(\frac{1}{2})+}_{\frac{1}{2}S}$ $2\times\frac{1}{2}^-$	$P^{(\frac{3}{2})+}_{\frac{3}{2}S}$ $\frac{3}{2}^-$
$\Sigma^+(1P)$	$\left[\tilde{P}^{(\frac{1}{2}),(\frac{3}{2})}_{\frac{1}{2}P}\right]^+$, $\left[\tilde{P}'^{(\frac{1}{2}),(\frac{3}{2})}_{\frac{1}{2}P}\right]^+$ $2\times\left(\frac{1}{2},\frac{3}{2}\right)^+$	$\left[P^{(\frac{1}{2}),(\frac{3}{2}),(\frac{5}{2})}_{\frac{3}{2}P}\right]^+$ $\left(\frac{1}{2},\frac{3}{2},\frac{5}{2}\right)^+$
$\Sigma^+(1D)$	$\left[\tilde{P}^{(\frac{3}{2}),(\frac{5}{2})}_{\frac{1}{2}D}\right]^+$, $\left[\tilde{P}'^{(\frac{3}{2}),(\frac{5}{2})}_{\frac{1}{2}D}\right]^+$ $2\times\left(\frac{3}{2},\frac{5}{2}\right)^-$	$\left[P^{(\frac{1}{2}),(\frac{3}{2}),(\frac{5}{2}),(\frac{7}{2})}_{\frac{3}{2}D}\right]^+$ $\left(\frac{1}{2},\frac{3}{2},\frac{5}{2},\frac{7}{2}\right)^-$
$\Pi^+(1P)$ & $\Sigma^-(1P)$	$\left[\tilde{P}^{(\frac{1}{2}),(\frac{3}{2})}_{\frac{1}{2}P}\right]^-$, $\left[\tilde{P}'^{(\frac{1}{2}),(\frac{3}{2})}_{\frac{1}{2}P}\right]^-$ $2\times\left(\frac{1}{2},\frac{3}{2}\right)^-$	$\left[P^{(\frac{1}{2}),(\frac{3}{2}),(\frac{5}{2})}_{\frac{3}{2}P}\right]^-$ $\left(\frac{1}{2},\frac{3}{2},\frac{5}{2}\right)^-$
$\Pi^-(1P)$	Same as $\Sigma^+(1P)$	
$\Sigma^-(1S)$	$\tilde{P}^{(\frac{1}{2})-}_{\frac{1}{2}S}$, $\tilde{P}'^{(\frac{1}{2})-}_{\frac{1}{2}S}$ $2\times\frac{1}{2}^+$	$P^{(\frac{3}{2})-}_{\frac{3}{2}S}$ $\frac{3}{2}^+$
$\Pi^+(1D)$	$\left[\tilde{P}^{(\frac{3}{2}),(\frac{5}{2})}_{\frac{1}{2}D}\right]^-$, $\left[\tilde{P}'^{(\frac{3}{2}),(\frac{5}{2})}_{\frac{1}{2}D}\right]^-$ $2\times\left(\frac{3}{2},\frac{5}{2}\right)^+$	$\left[P^{(\frac{1}{2}),(\frac{3}{2}),(\frac{5}{2}),(\frac{7}{2})}_{\frac{3}{2}D}\right]^-$ $\left(\frac{1}{2},\frac{3}{2},\frac{5}{2},\frac{7}{2}\right)^+$

e.g.,

$$\tilde{P}_{\frac{1}{2}} \equiv \left|\frac{1}{2}_{s_{qqq}}, 0_{s_{Q\bar{Q}}}\right\rangle_{S=\frac{1}{2}}$$

First numerical results of the model

[Giron, RFL, and Peterson, JHEP **05** (2019) 061]

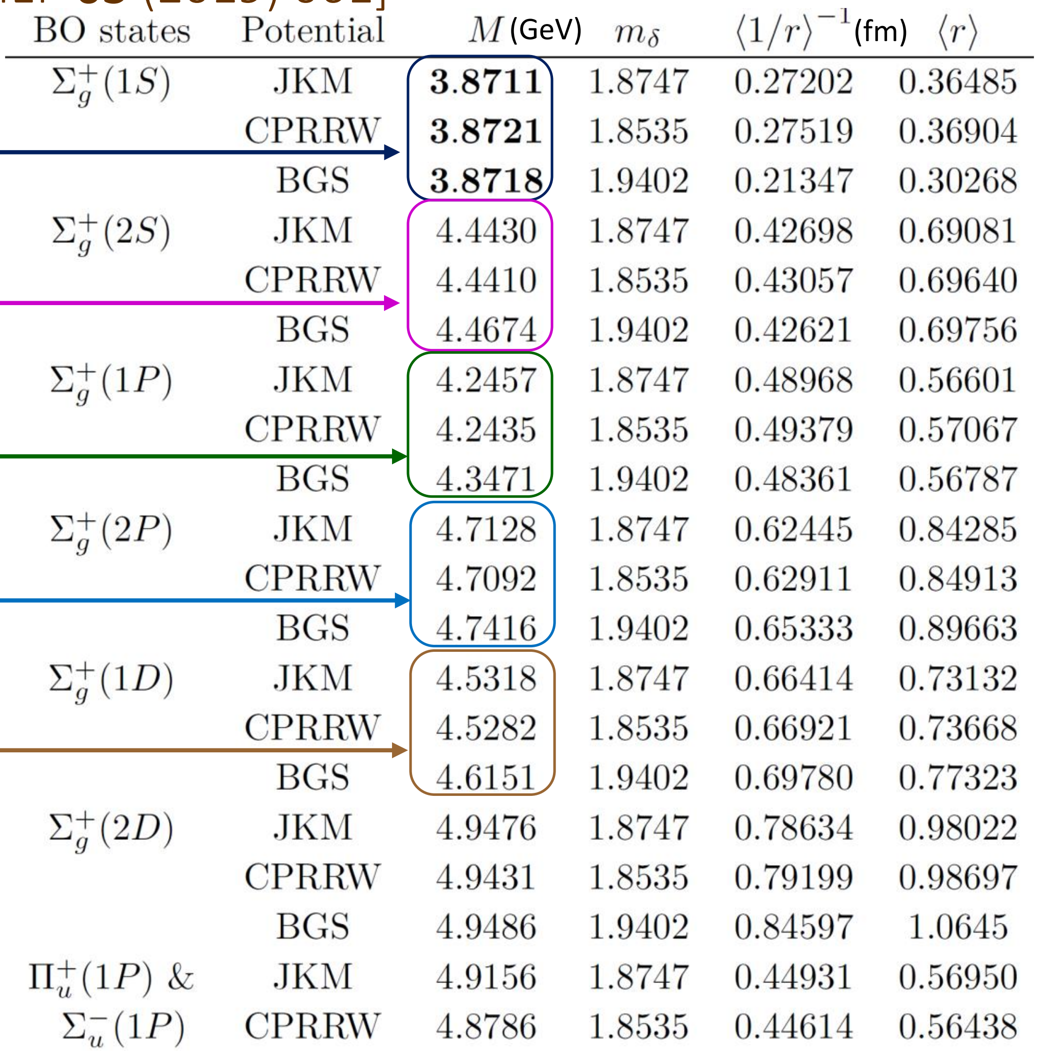
Fixed to $X(3872)$

Right atop $Z_c(4430)$

Right atop $Y(4220)$

Right atop $Y(4660)$

Right atop $X(4500)$



BO states	Potential	M (GeV)	m_δ	$\langle 1/r\rangle^{-1}$ (fm)	$\langle r\rangle$
$\Sigma_g^+(1S)$	JKM	**3.8711**	1.8747	0.27202	0.36485
	CPRRW	**3.8721**	1.8535	0.27519	0.36904
	BGS	**3.8718**	1.9402	0.21347	0.30268
$\Sigma_g^+(2S)$	JKM	4.4430	1.8747	0.42698	0.69081
	CPRRW	4.4410	1.8535	0.43057	0.69640
	BGS	4.4674	1.9402	0.42621	0.69756
$\Sigma_g^+(1P)$	JKM	4.2457	1.8747	0.48968	0.56601
	CPRRW	4.2435	1.8535	0.49379	0.57067
	BGS	4.3471	1.9402	0.48361	0.56787
$\Sigma_g^+(2P)$	JKM	4.7128	1.8747	0.62445	0.84285
	CPRRW	4.7092	1.8535	0.62911	0.84913
	BGS	4.7416	1.9402	0.65333	0.89663
$\Sigma_g^+(1D)$	JKM	4.5318	1.8747	0.66414	0.73132
	CPRRW	4.5282	1.8535	0.66921	0.73668
	BGS	4.6151	1.9402	0.69780	0.77323
$\Sigma_g^+(2D)$	JKM	4.9476	1.8747	0.78634	0.98022
	CPRRW	4.9431	1.8535	0.79199	0.98697
	BGS	4.9486	1.9402	0.84597	1.0645
$\Pi_u^+(1P)$ &	JKM	4.9156	1.8747	0.44931	0.56950
$\Sigma_u^-(1P)$	CPRRW	4.8786	1.8535	0.44614	0.56438

Fine structure of the multiplets: The model

- All that is known about states $\Sigma_g^+(1S)$ multiplet can be incorporated using a 3-parameter Hamiltonian:

$$H = M_0 + 2\kappa_{qQ}(\mathbf{s}_q \cdot \mathbf{s}_Q + \mathbf{s}_{\bar{q}} \cdot \mathbf{s}_{\bar{Q}}) + V_0\, \boldsymbol{\tau}_q \cdot \boldsymbol{\tau}_{\bar{q}}\, \boldsymbol{\sigma}_q \cdot \boldsymbol{\sigma}_{\bar{q}}$$

Common multiplet mass (as computed above)

Internal diquark spin-spin coupling

Isospin-spin coupling of light quarks (same form as in πNN exchange)

- In addition, the pairs of states with degenerate J^{PC} can mix:

0^{++} $$\begin{pmatrix} \bar{X}_0 \\ \bar{X}_0' \end{pmatrix} = \begin{pmatrix} \cos\theta_X & \sin\theta_X \\ -\sin\theta_X & \cos\theta_X \end{pmatrix} \begin{pmatrix} X_0 \\ X_0' \end{pmatrix}$$

1^{+-} $$\begin{pmatrix} \bar{Z} \\ \bar{Z}' \end{pmatrix} = \begin{pmatrix} \cos\theta_Z & \sin\theta_Z \\ -\sin\theta_Z & \cos\theta_Z \end{pmatrix} \begin{pmatrix} Z \\ Z' \end{pmatrix}$$

Fine structure in the $P = -$ states

[Giron & RFL, PRD **101** (2020) 074032]

- Hamiltonian for $L > 0$ states like $\Sigma_g^+(1P)$ requires 2 additional operators, spin-orbit and tensor:

$$\Delta H_{LS} = V_{LS}\,\mathbf{L}\cdot\mathbf{S} \quad , \qquad \Delta H_T = V_T\,\boldsymbol{\tau}_q\cdot\boldsymbol{\tau}_{\bar{q}}\;S_{12}^{(q\bar{q})},$$

$$S_{12} \equiv 3\,\boldsymbol{\sigma}_1\cdot\boldsymbol{r}\,\boldsymbol{\sigma}_2\cdot\boldsymbol{r}/r^2 - \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$$

- Since only 5 Hamiltonian parameters for 28 states (as well as mixing angles, *e.g.*, 3 for the 4 Y states), the system is already almost completely constrained by data
- Example: The fits with lowest χ^2 predict the sole $\Sigma_g^+(1P)$ 0^{--}, $I = 1$ state to match the candidate $Z_c(4240)$ [seen in LHCb's $Z_c(4430)$ paper PRL **112** (2014) 222002]

Fine structure in $c\bar{c}s\bar{s}$ and $b\bar{b}q\bar{q}$ states

[Giron & RFL, PRD **102** (2020) 014036]

- Several exotic candidates [$Y(4140), Y(4274), X(4350), X(4500), Y(4626), X(4700)$] so far seen only to decay to $J/\psi\ \phi$ or to $D_s\bar{D}_{sJ}$,
  hence are natural $c\bar{c}s\bar{s}$ candidates
- Furthermore, $X(3915)$ (likely 0^{++}) is a weird state:
  Does not fit well as $c\bar{c}$ or $c\bar{c}q\bar{q}$ (no open-charm decays)
  Proposed lowest $c\bar{c}s\bar{s}$ state [RFL & Polosa, PRD **93** (2016) 094024]
- 0 isospin$\Longrightarrow \Sigma_g^+(1S)$: only 2 Hamiltonian parameters, 6 states
- $X(3915), Y(4140), X(4350)$ fit well in $c\bar{c}s\bar{s}\ \Sigma_g^+(1S)$ multiplet
- But $Y(4274)$ does not! Fits well as conventional $\chi_{c1}(3P)$

Fine structure in $c\bar{c}s\bar{s}$ and $b\bar{b}q\bar{q}$ states

[Giron & RFL, PRD **102** (2020) 014036]

- For $b\bar{b}q\bar{q}$, only known $\Sigma_g^+(1S)$ ($P = +$) candidates are $Z_b(10610)$ & $Z_b(10650)$, both $J^{PC} = 1^{+-}$, $I = 1$
- But here one has an important extra piece of information: Both Z_b's decay to $\Upsilon(1S)$, $\Upsilon(2S)$, $\Upsilon(3S)$ and to $h_b(1P)$, $h_b(2P)$
- And $Z_b(10610)$ prefers Υ ($s_{b\bar{b}} = 1$),
while $Z_b(10650)$ prefers h_b ($s_{b\bar{b}} = 0$), by roughly a $3{:}\,1$ ratio
⇒ 2 masses and $P_{s_{b\bar{b}}=1}\big(Z_b(10610)\big)$ are enough to fix
all 3 Hamiltonian parameters and all 12 $\Sigma_g^+(1S)$ masses:

J^{PC}	$I = 0$		$I = 1$	
0^{++}	10569.7	10695.7	10575.4	10644.9
1^{++}	10598.9		10621.4	
1^{+-}	10613.2	10691.1	**10607.2**	**10652.2**
2^{++}	10637.9		10660.4	

(MeV)

The $c\bar{c}c\bar{c}$ states

[Giron & RFL, 2008.01631]

- LHCb's observation of a state $X(6900)$ decaying to $J/\psi\, J/\psi$ [2006.16957] is BIG NEWS for hadron spectroscopy
- Sits $\sim 700\ \mathrm{MeV}$ above threshold
  No likely molecular binding mechanism
- LHCb also offers evidence for a 2nd $J/\psi\, J/\psi$ resonance:
  Either $X(6500)$ or $X(6740)$
- What state is $X(6900)$ in the dynamical diquark model?
  Identical cc quarks⟹Only 3 $\Sigma_g^+(nS)$ & 7 $\Sigma_g^+(nP)$ states
- We calculate that only one LHCb scenario fits the model, with $X(6900)$ being in $\Sigma_g^+(2S)$ and $X(6740)$ in $\Sigma_g^+(1P)$

Phenomenological modeling

[2203.16583]

- The internal structure of heavy-quark exotics is not yet resolved, so multiple approaches must continue to be developed
- Heavy-quark ($m_Q \gg \Lambda_{\mathrm{QCD}}$) hadrons (especially multiquark exotics) admit features not available for light-quark ones:
- Usually fewer decay modes (hence narrower); anomalous decay modes (*e.g.*, $X(3872) \to J/\psi\,\rho$); small KE for m_Q hence heavy-quarks nucleates quark clusters (Hadronic molecules? Diquark compounds?)
- Large m_Q allows for scale separation from lighter d.o.f.: effective field theory, Born-Oppenheimer approximation

Phenomenological modeling

[2203.16583]

- Many candidates lie near di-hadron thresholds (*e.g.*, $X(3872)$ to $D^0\overline{D}^{*0}$)
  Hadron molecules? Threshold effects? Configuration mixing?
- Do b and c systems have analogous states?
  Do they form full isospin and SU(3)-flavor multiplets?
- No single picture simultaneously explains all exotic candidates
  Multiple perspectives needed to develop comprehensive understanding