From the Kondo Problem to Wilson Lines — Defects in Many-Body Systems

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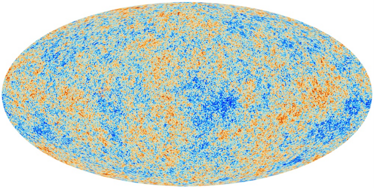
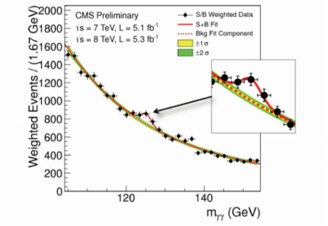
Quantum Field Theory is a universal mathematical structure that follows from two central pillars of modern physics

- Quantum Mechanics
- Special Relativity

It is the main framework in Elementary Particles, Statistical Mechanics, Condensed Matter, Stochastic Processes, and Cosmology.





There are many possible models in QFT. A fundamental question is to understand how they are related.

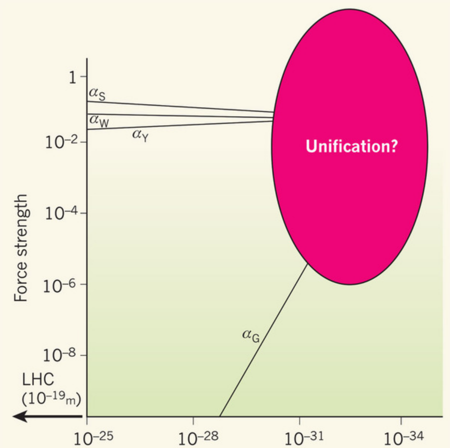
For example, we can start from some given model and deform the Feynman integral (or the Hamiltonian) by

$$\left\langle e^{\int d^d x \sum_i \lambda^i \mathcal{O}_i} \right\rangle$$

where the $\mathcal{O}_i$ are various functions of the degrees of freedom.

The most fundamental and confusing observation is that the λ^i are not really well defined numbers [Landau et al, Gell-Mann Low, Kadanoff, Wilson...]!!

Their actual numerical value depends on the resolution of the experiment.



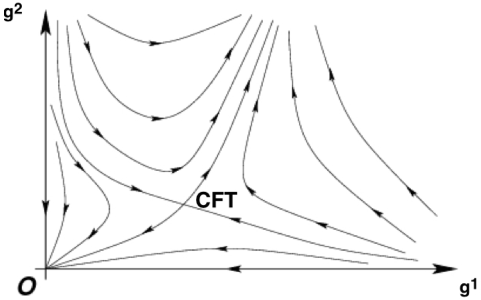
As we decrease the resolution (i.e. we coarse grain) the various options are

- Irrelevant: the coupling λ decreases as we go to long distances.
- Relevant: the coupling λ increases as we go to long distances.
- Exactly Marginal: the coupling does not change: it is a genuine number.

The last option is pretty rare but it appears in supersymmetric theories and in 2d models such as the Ashkin-Teller model.
Note: the irrelevant (relevant) couplings could decrease (increase) as a power law or logarithmically in the resolution.

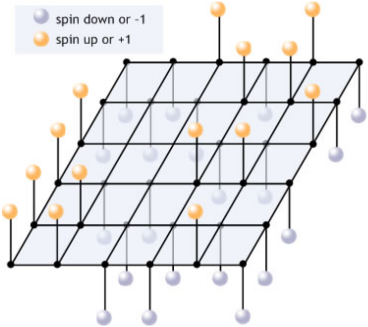
Typically: Only finitely many relevant couplings!

By removing all the relevant couplings by hand (turning knobs/fine tuning) we find a theory that at sufficiently long distances no longer depends on the resolution.



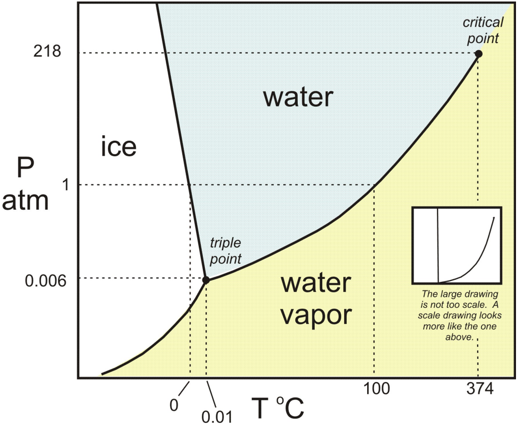
We call such points that are invariant under coarse graining *fixed points*. A more correct way to think about such flows and fixed points is as a gradient flow

Example 1: A Ferromagnet.

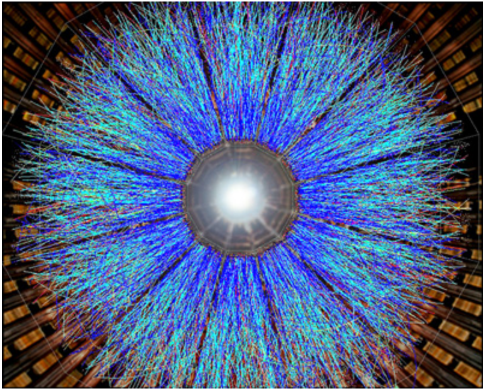


L^d spins with nearest-neighbor interaction energy $J > 0$ if they are misaligned.

Example 2: The water-vapor transition.



Example 3: The confinement-deconfinement transition in $SU(2)$ Yang-Mills theory.



Hadrons turn into a plasma of gluons.

All of these examples have at some T_c the **same** fixed point, which is the 3d Ising model! Long range correlations develop and micro structure becomes irrelevant.

Ginzburg-Landau theory:

$$H = \int d^d x \left(r(\nabla M)^2 + cM^2 + \lambda M^4 + ... \right)$$

The partition function

$$Z = \int [dM] e^{-H}$$

encodes all the thermodynamics properties at the phase transition.

Some experimentally interesting quantities are the exponents $\alpha, \beta, \gamma, \delta, \eta, \nu$

$$C \sim (T - T_c)^{-\alpha} , \qquad M \sim (T_c - T)^{\beta} , \qquad \chi \sim (T - T_c)^{-\gamma} ,$$

$$M \sim h^{1/\delta} , \qquad \langle M(\vec{n}) M(0) \rangle \sim \frac{1}{|\vec{n}|^{d-2+\eta}} , \qquad \xi \sim (T - T_c)^{-\nu} .$$

Amazingly, one discovers four relations between these 6 quantities:

$$\alpha + 2\beta + \gamma = 2 \ ,$$

$$\gamma = \beta(\delta - 1) \ ,$$

$$\gamma = \nu(2 - \eta) \ ,$$

$$\nu d = 2 - \alpha \ .$$

The explanation of this miracle is that at T_c the symmetry of the system is enhanced:

$$SO(3) \times \mathbb{R}^3 \to SO(3) \times \mathbb{R}^3 \times \mathbb{R}_+ \ ,$$

with

$$\mathbb{R}_+ : \ x \to \lambda x$$

and $\lambda \in \mathbb{R}_+$.

The dilation charge associated to $\mathbb{R}_+$ is called Δ. It can be diagonalized. If we have a local operator $\mathcal{O}$ in the theory, $\Delta(\mathcal{O})$ would uniquely determine its two-point correlator

$$\langle \mathcal{O}(n)\mathcal{O}(0)\rangle \sim \frac{1}{n^{2\Delta(\mathcal{O})}} \ .$$

Local operators could also have spin s, but we suppress it in the meantime.

We can make contact with our previous terminology:

- Relevant: $\Delta \leq 3$
- Irrelevant: $\Delta \geq 3$

(For $\Delta = 3$ a separate analysis of $\langle \mathcal{O}(n)\mathcal{O}(n')\mathcal{O}(n'') \rangle$ is necessary.)

The relevant operators appear in phase diagrams and the irrelevant ones disappear at long distances.

(irrelevant operators can be dangerously irrelevant and affect the phase diagram, but not the fixed point – e.g. perovskite materials)

In the 3d Ising model, there are two relevant operators: $M(x)$ and $\epsilon(x)$.

$$\Delta_M = 0.518... \,, \qquad \Delta_\epsilon = 1.413...$$

- Ferromagnet: magnetic field and temperature.
- water-vapor: pressure and temperature.
- SU(2) Yang-Mills: fundamental quark mass and temperature.

These two numbers are "fundamental constants."

The four miraculous relations among $\alpha, \beta, \gamma, \delta, \eta, \nu$ can be simply understood from scale invariance:

$$\alpha = \frac{d - 2\Delta_\epsilon}{d - \Delta_\epsilon} \ ,$$

$$\beta = \frac{\Delta_M}{d - \Delta_\epsilon} \ ,$$

$$\gamma = \frac{d - 2\Delta_M}{d - \Delta_\epsilon} \ ,$$

$$\delta = \frac{d - \Delta_M}{\Delta_M} \ ,$$

$$\eta = 2 - d + 2\Delta_M \ ,$$

$$\nu = \frac{1}{d - \Delta_\epsilon} \ .$$

This is how these relations were originally explained.

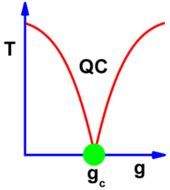
There has been a lot of recent progress based on the observation that the symmetry is actually **bigger!!**

$$SO(3) \times \mathbb{R}^3 \to SO(3) \times \mathbb{R}^3 \times \mathbb{R}_+ \to SO(4,1)$$

Theories with this big symmetry are called (3d) **Conformal Field Theories** (CFTs). There are many such theories. The 3d Ising model is perhaps the simplest nontrivial example. [Polyakov?!]

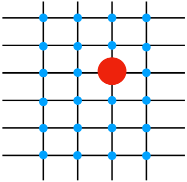
Applications of Conformal Field Theories:

- Quantum phase transitions



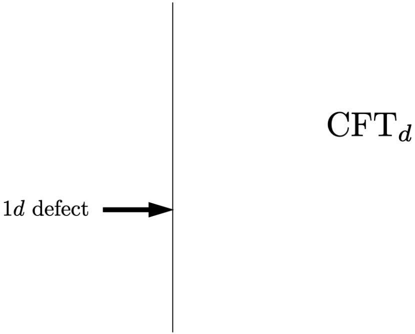
Many fixed points of the renormalization group can be constructed starting from lattice quantum systems and tuning to second order phase transitions at zero temperature. This can be done in 1+1, 2+1, and 3+1 dimensions.

We will now consider point-like defects/impurities in a system at a second-order phase transition.



$$H = J_0 \vec{T} \cdot \vec{S} + H_{bulk}$$

Impurities are localized in space and they are lines in space-time. Therefore, in QFT, we think about impurities as line operators/line defects in space-time.



The subject of line defects has been historically extremely productive. The Kondo line defect in 2d has led to the **renormalization group** [Wilson...], to substantial progress on **integrability** [Andrei, Tsvelick-Wiegmann...], and of course to the development of **conformal symmetry** at the end points of the RG flow.

We will touch briefly upon several subjects:

- RG flows on line defects and a new theorem.
- Magnetic field defects as an application.

Consider a straight line in a d-dimensional CFT. It can be conformal or non-conformal. A conformal line preserves

$$SL(2,\mathbb{R}) \times SO(d-1)$$

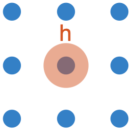
(we assume the line has no transverse spin). A non-conformal line preserves

$$\mathbb{R} \times SO(d-1)\ .$$

It describes a point-like impurity in space at zero temperature, with a critical bulk. At long distances, the impurity becomes critical.

A central question is if the impurity becomes trivial (=screened) at long distances.

We will consider below a simple example of an impurity: a localized magnetic field in a quantum anti-ferromagnet.

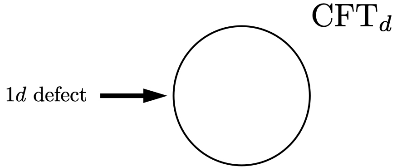


As we go to longer distances, what are the effects of such a magnetic field?

An interesting quantity that we can define is "defect entropy." We make the line into a Euclidean circle of radius R and compute the expectation value of the circle:

$$s = \left(1 - R\frac{\partial}{\partial R}\right)\log\langle L\rangle\ .$$

The differential operator $\left(1 - R\frac{\partial}{\partial R}\right)$ ensures that the answer is independent of the "cutoff" i.e. details on the lattice scale.



- At the fixed point of the impurity renormalization group the value of s is R-independent.
- For trivial impurities (=screened impurities) we have $s = 0$.
- It is hard to imagine how s is going to be measured in quantum phase transitions, but it is theoretically a very useful device.
- The main claim to fame of this defect entropy is that

$$R\frac{\partial s}{\partial R} \leq 0$$

- Therefore also $s_{ir} < s_{uv}$.

$$s(M_0 R) \to \begin{cases} s_{uv} \text{ as } R \to 0 \\ s_{ir} \text{ as } R \to \infty \end{cases}$$

The renormalization group flow is implemented by changing the radius of the circle of the defect worldline.

The trivial line defect is just the unit line operator. It is completely transparent. It has $s = 0$.

Corollary: If the line is trivial in the short distance limit then $s_{ir} < 0$ and hence any relevant perturbation cannot lead to a screened impurity.

The localized magnetic field in a quantum anti-ferromagnet is an example of a defect with $s_{uv} = 0$. More generally: integrate a bulk operator with $\Delta < 1$ on the line:

$$S = S_{bulk} + M_0^{1-\Delta_O} \int dt O(t)$$

An external magnetic field corresponds to $O = \phi_1$ in the $O(N)$ Wilson-Fisher fixed point and indeed we know $\Delta(\phi_1) < 1$.

The Pinning Field in $O(N)$ Models

Consider the $O(N)$ model in $2 \leq d \leq 4$ with an external localized magnetic field:

$$S = S_{O(N)} + h \int dt \phi_1(t)$$

where $S_{O(N)}$ stands for the critical bulk $O(N)$ model in d space-time dimensions and ϕ_1 is the first component of $\vec{\phi}$.

This is a relevant perturbation in $2 \leq d \leq 4$. This must flow to a nontrivial infrared conformal defect in any $2 \leq d < 4$. Hence, the external magnetic field cannot be "screened" and cannot disappear.

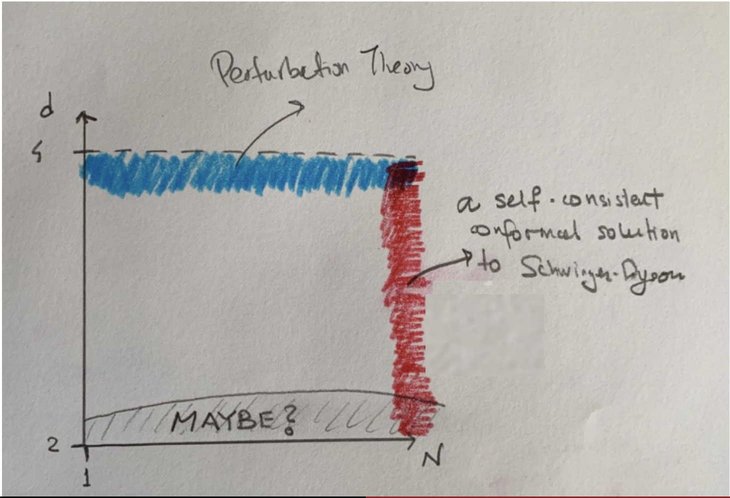
The Pinning Field in $O(N)$ Models

This is physically realizable as a localized magnetic field at zero temperature at a bulk quantum critical point and it can be tested in quantum critical points and also in Monte Carlo [....Assaad, Herbut; Parisen Toldin, Assaad, Wessel....].

This infrared fixed point will have no nontrivial relevant operators whatsoever.

The Pinning Field in $O(N)$ Models

In principle, understanding the infrared is a strongly coupled problem.



The Pinning Field in $O(N)$ Models

Here is a sample of results at large N and $d = 3$ in the deep infrared:

$$s = -0.1536N + \mathcal{O}(N^0)$$

$$\Delta(\hat{\phi}_1) = 1.542... + \mathcal{O}(N^{-1})$$

Note: s is arbitrarily negative at large N.

The Pinning Field in $O(N)$ Models

Combining all the data we amassed suggests that in $d = 3$ one should expect $\Delta(\hat{\phi}_1) \sim 1.5$ with rather weak N dependence. This is the first nontrivial $O(N-1)$ singlet operator. It is roughly consistent with Monte Carlo simulations and this along with several other predictions should be testable.

Spin Impurities

Another important line defect especially for the $O(3)$ magnet comes about as follows: We begin with QM with a spin s representation of $SO(3)$, so just a QM system with Hilbert space of dimension $2s+1$. We then couple the $SO(3)$ generators S^a to the interacting bulk:

$$S = S_{O(3)} - \gamma \int dt S^a(t)\phi^a(t) \ .$$

This is the line operator

$$Tr_s P e^{\gamma \int dt S^a \phi^a} \ .$$

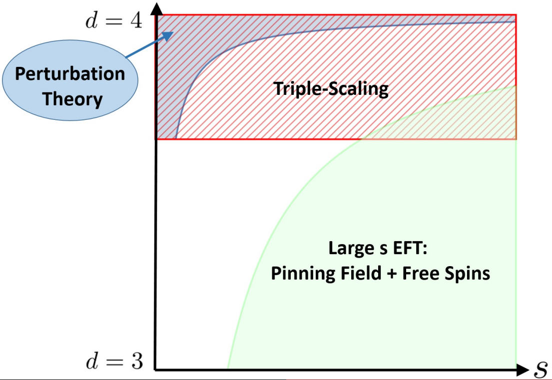
It is similar to Wilson lines but it is just a line defect in a magnet. It is essentially a 2+1 dimensional generalization of the Kondo problem.

Spin Impurities

Physically this is realizable by putting an external atom of spin s in a quantum anti-ferromagnet at the critical point. While there is a lot to say about this problem here I will mention one general result.

Spin Impurities

At $s \to \infty$ the spin impurity breaks up into two almost-decoupled DCFTs, one being the pinning field DCFT we studied above and the other being just the theory of a free spin s. There is a systematic $1/s$ expansion. This statement leads to many predictions that can be checked in the future.



Spin Impurities

Some very special case of this was verified in QMC in a paper that appeared today by Manuel Weber and Matthias Vojta.

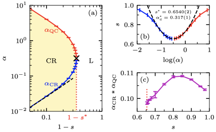


FIG. 5. Fixed-point duality of the SU(2)-symmetric spin-boson model. (a) Location of the two intermediate-coupling fixed points CR and QC, as determined from crossing points of $T^s \chi_x$ [33], as a function of the bath exponent s. The black dashed line indicates the prediction (4) of the perturbative RG for α_{CR}. (b) Close to s^*, the fixed-point collision is well approximated by $s = s^* + \frac{B_0}{A_1} \ln^2(\alpha/\alpha_{\mathrm{d}}^*)$ from which we extract $s^* = 0.6540(2)$, $\alpha_{\mathrm{d}}^* = 0.317(1)$, and $A_1/B_0 = 17.7(2)$. (c) Product of the two fixed-point couplings as a function of s: This is approximately constant despite the couplings varying over several orders of magnitude.

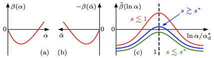


FIG. 6. (a) Weak-coupling and (b) its dual strong-coupling beta function, as in Eqs. (3) and (8). (c) Reflection symmetry of $\tilde{\beta}(\ln \alpha)$ around α_{d}^*. The zeros of $\beta(\alpha)$ disappear for $s < s^*$.

between the two fixed-point theories which we discuss in the following.

First, the properties of QC near $s = 1$ can be deduced from the dual of the weak-coupling expansion (3); we note that a suitable field theory for that is not known. Introducing $\bar{\alpha} \equiv {\alpha_{\mathrm{d}}^*}^2/\alpha$ we have by duality

$$\beta(\bar{\alpha}) = (1 - s)\bar{\alpha} - 4\bar{\alpha}^2 + 8\bar{\alpha}^3 . \qquad (8)$$

where the sign change compared to (3) arises from $d \ln \alpha / d \ln \mu = -d \ln(1/\alpha)/d \ln \mu$. The QC fixed point is then located at

Wilson Lines

I will close with one brief comment about Wilson lines. Let us for instance consider Wilson lines in some conformal gauge theory with gauge group $SU(2)$:

$$Tr_s P e^{i\int dt S^a A^a}$$

Alternatively we can seek a conformal defect labeled by s with a Coulomb field

$$A_0 \sim \frac{g_{YM}^2 s}{r}$$

However for large s there will be pair creation due to this huge electric field a-la Schwinger.
So what is the space of Wilson line defects seems like an open question.

Thank you for your attention!