Machine Learning for Collinear QCD

Lecture 1

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The three lectures

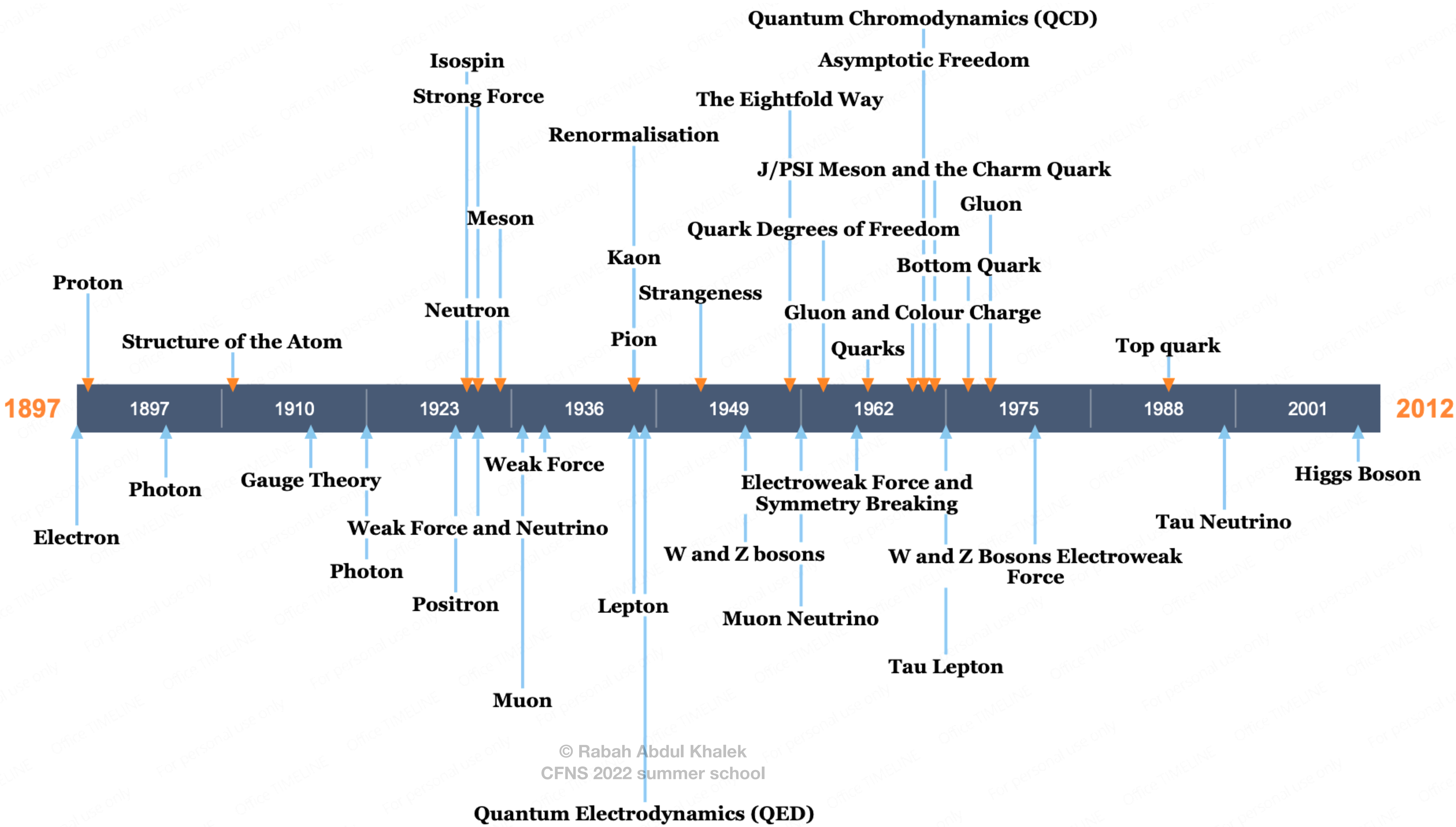
1 Recap on collinear QCD
Proton substructure

2 ML and statistics + tutorial 1
Artificial Neural Networks

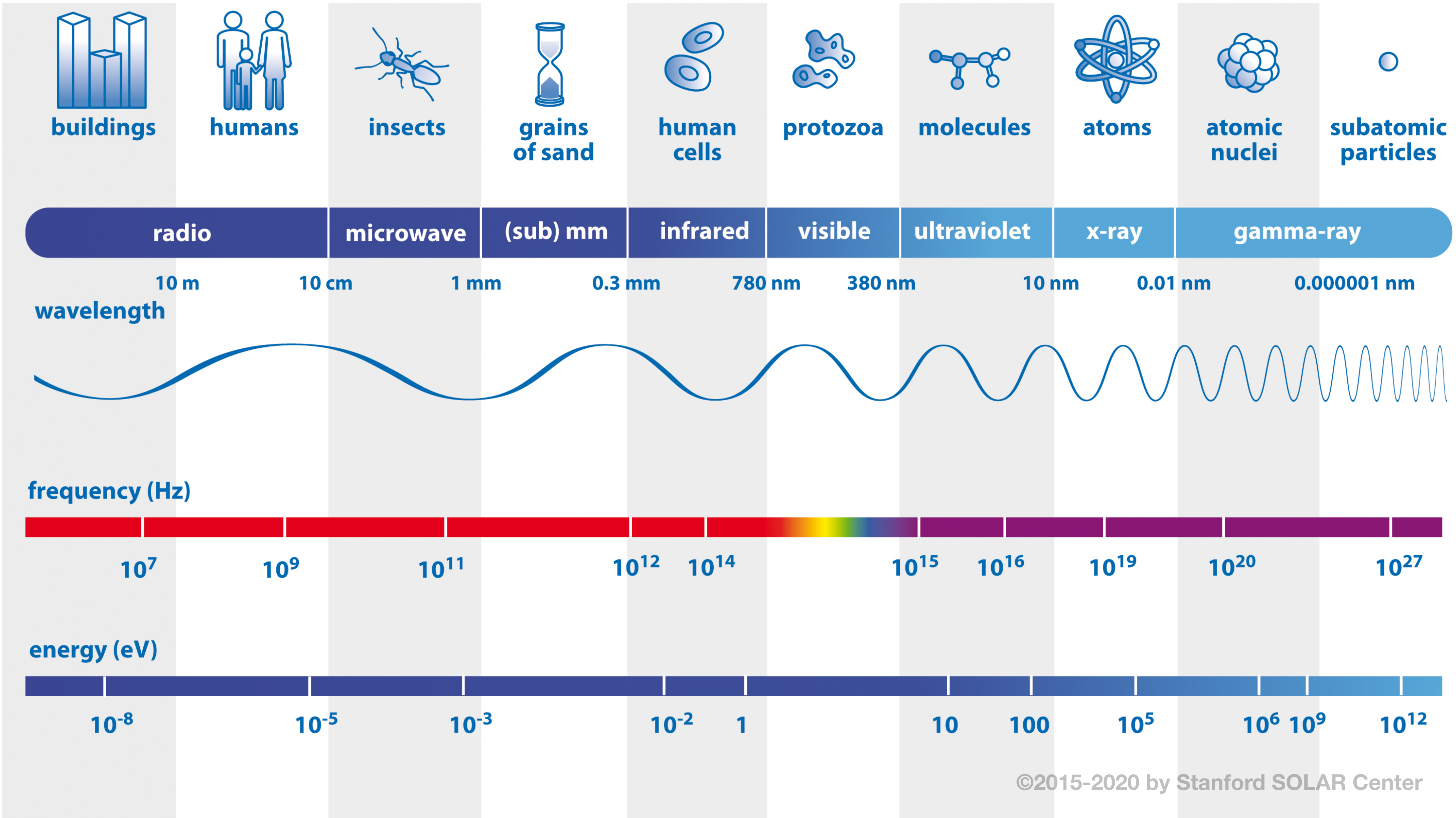
3 Mixing of lectures 1 and 2 + tutorial 2
Extracting collinear QCD objects using ML

This lecture's outline

QCD (non-exhaustive) timeline



Probing Hadrons



$$\text{Radius}(^{208}\text{Pb}) \simeq r_0 A^{1/3} \simeq 7.4 \times 10^{-15}\ \text{m} \ll \text{Visible limit} \simeq 3.1 \times 10^{-11}\ \text{m}$$

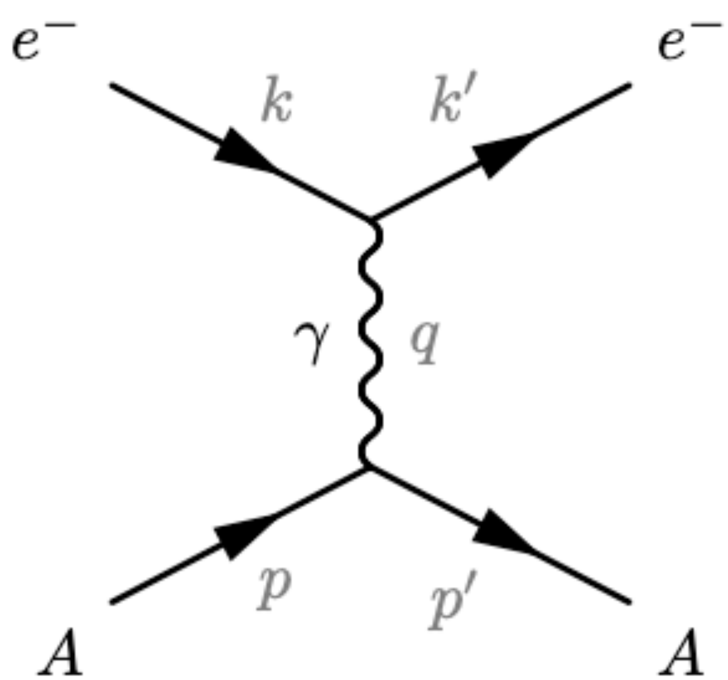
with $r_0 \sim 1.25 \times 10^{-15}$ m the average radius of a nucleon and A the nuclei atomic mass.

Elastic scattering

Elastic scattering is the dominant lepton-hadron scattering process if a lepton has an energy E much lower than the mass m_A of a hadron A ($E \ll m_A$).

At the lowest order in perturbation theory:

$i\mathcal{M} =$

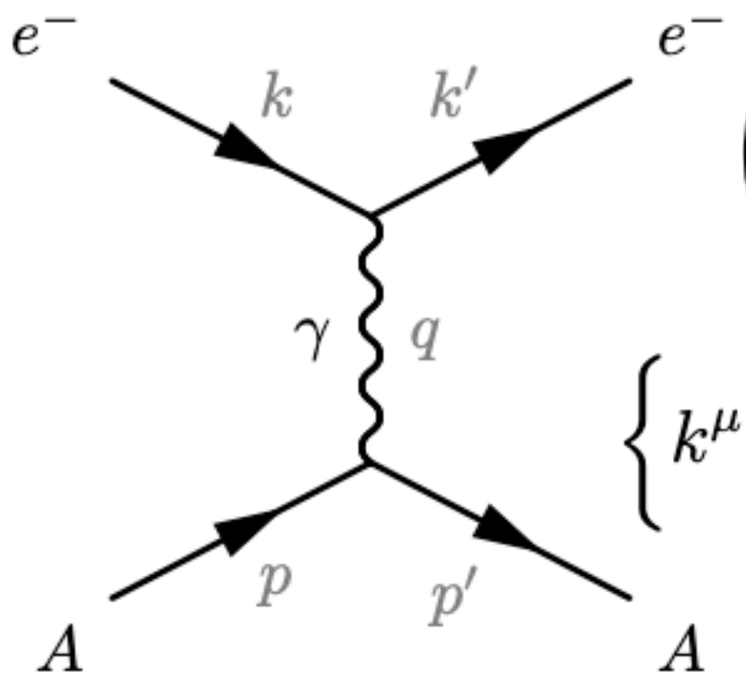


$$= (-ie)\bar{u}(k')\gamma^\mu u(k)\frac{-ig_{\mu\nu}}{q^2}(-ie)\bar{u}(p')\gamma^\nu u(p)$$

$$|\tilde{\mathcal{M}}| = \frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{2e^4}{t^2}\left[u^2 + s^2 + 4t(m_e^2 + m_A^2) - 2(m_e^2 + m_A^2)^2\right]$$

$$s = (k+p)^2, \qquad t = (k-k')^2, \qquad u = (k-p')^2$$

Elastic scattering



$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{1}{64\pi^2 E_{cm}^2}\frac{|\boldsymbol{k'}|}{|\boldsymbol{k}|}|\tilde{\mathcal{M}}|^2 \quad \text{with: } d\Omega = \sin\theta d\theta d\phi$$

$$\left\{ k^\mu = (E, \boldsymbol{k}), \quad p^\mu = (m_A, 0), \quad k'^\mu = (E, \boldsymbol{k'}), \quad p'^\mu = (m_A, 0) \right\}$$

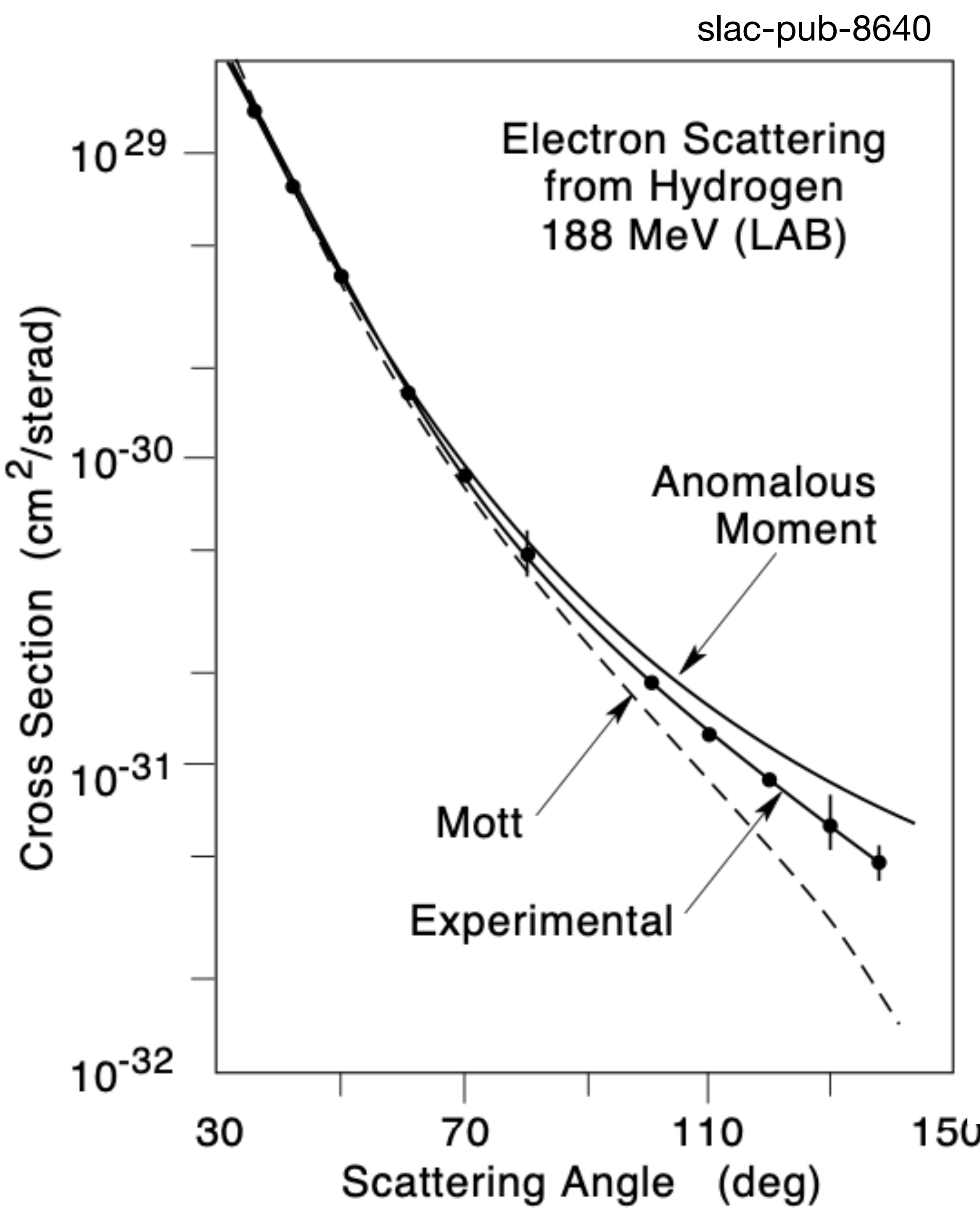
$$k = |\boldsymbol{k}| = |\boldsymbol{k'}|, \qquad \boldsymbol{k}\cdot\boldsymbol{k'} = k^2\cos\theta, \qquad q^2 = 0 \text{ (on-shell)}, \qquad \Theta = \frac{\alpha^2}{(1-\cos\theta)^2}$$

Mott expression includes relativistic corrections to Rutherford's. Both of which describe the electron **elastic scattering** off a **point-like** hadron A at **low-energies**.

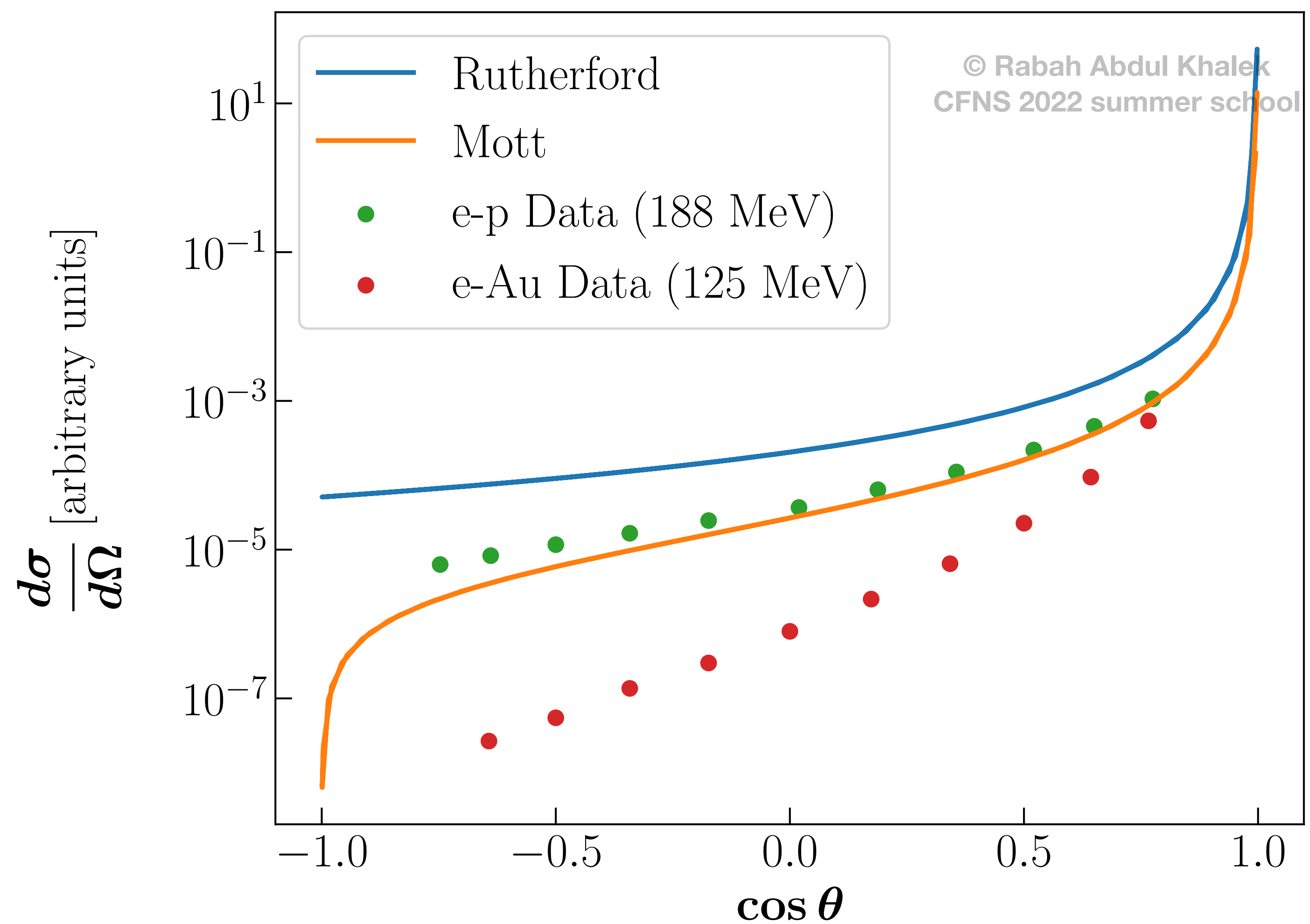
$$(k \ll E \sim m_e \ll m_A) \xrightarrow[m_A\to\infty]{} \left(\frac{d\sigma}{d\Omega}\right)^{\text{Rutherford}}_{\text{CM}} = \Theta \cdot \frac{m_e^2}{k^4}$$

$$(E \ll m_A) \xrightarrow[m_A\to\infty]{} \left(\frac{d\sigma}{d\Omega}\right)^{\text{Mott}}_{\text{CM}} = \Theta \cdot \frac{E^2}{k^4}\left(1 - \frac{k^2}{E^2}\sin^2\frac{\theta}{2}\right)$$

Elastic scattering



The early data hinted that the "proton was not a point"



Elastic scattering

$$(k \ll E \sim m_e \ll m_A) \underset{m_A\to\infty}{\longrightarrow} \quad \left(\frac{d\sigma}{d\Omega}\right)^{\text{Rutherford}}_{\text{CM}} = \Theta \cdot \frac{m_e^2}{k^4}$$

$$(E \ll m_A) \underset{m_A\to\infty}{\longrightarrow} \quad \left(\frac{d\sigma}{d\Omega}\right)^{\text{Mott}}_{\text{CM}} = \Theta \cdot \frac{E^2}{k^4}\left(1 - \frac{k^2}{E^2}\sin^2\frac{\theta}{2}\right)$$

Let us now consider the high energy regime ($E \gg m_e$) where the nucleon in the final state cannot be assumed to be at rest, i.e it has to recoil.

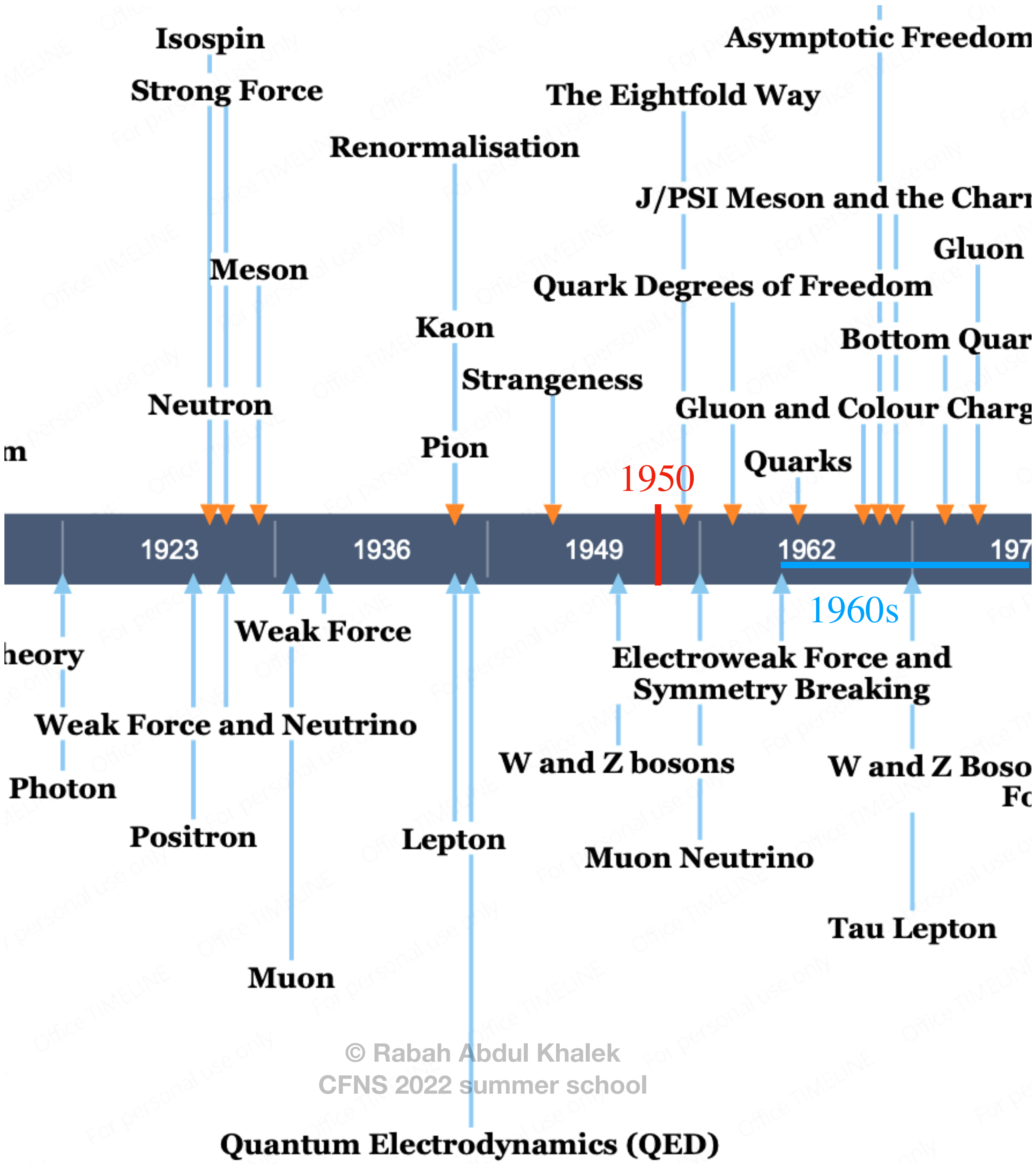
$$\left\{k^\mu = (E, \boldsymbol{k}), \quad p^\mu = (m_A, \mathbf{0}), \quad k'^\mu = (E', \boldsymbol{k'}), \quad p'^\mu = k^\mu - k'^\mu + p^\mu\right\}$$

$$q^2 = -2k^\mu k'_\mu = -4E'E\sin^2\frac{\theta}{2} = -2m_A(E - E'), \qquad \Theta = \frac{\alpha^2}{(1-\cos\theta)^2}$$

$$(E \gg m_e) \longrightarrow \quad \left(\frac{d\sigma}{d\Omega}\right)^{\text{High energy}}_{\text{lab}} = \Theta \cdot \frac{1}{E^2}\frac{E'}{E}\left(\cos^2\frac{\theta}{2} + \frac{E - E'}{m_A}\sin^2\frac{\theta}{2}\right)$$

$$= \Theta \cdot \frac{1}{E^2}\frac{E'}{E}\left(\cos^2\frac{\theta}{2} - \frac{q^2}{2m_A^2}\sin^2\frac{\theta}{2}\right)$$

Hints of proton compositeness



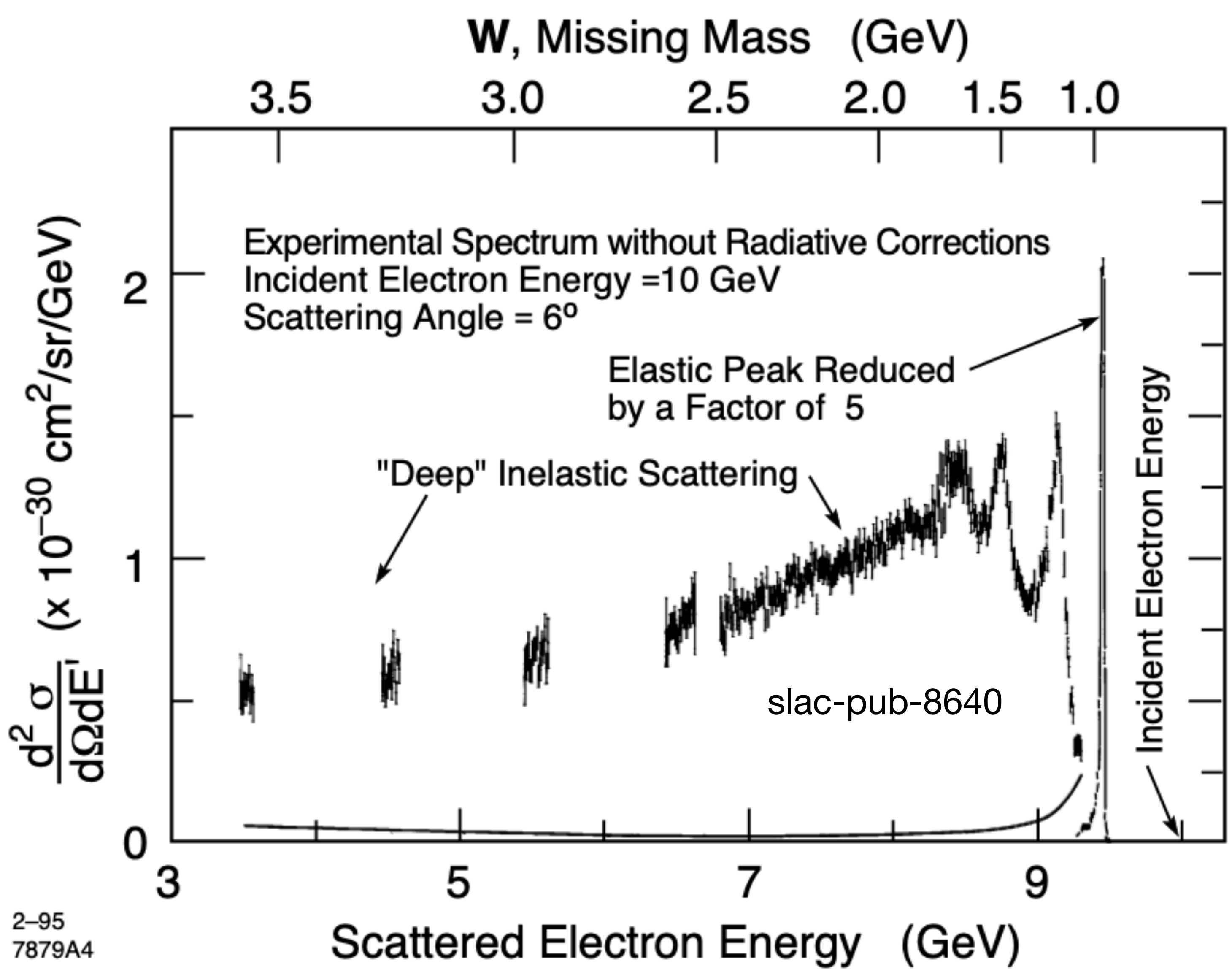
In the 1950s the proton and neutron were considered to be the final elementary constituents of matter.

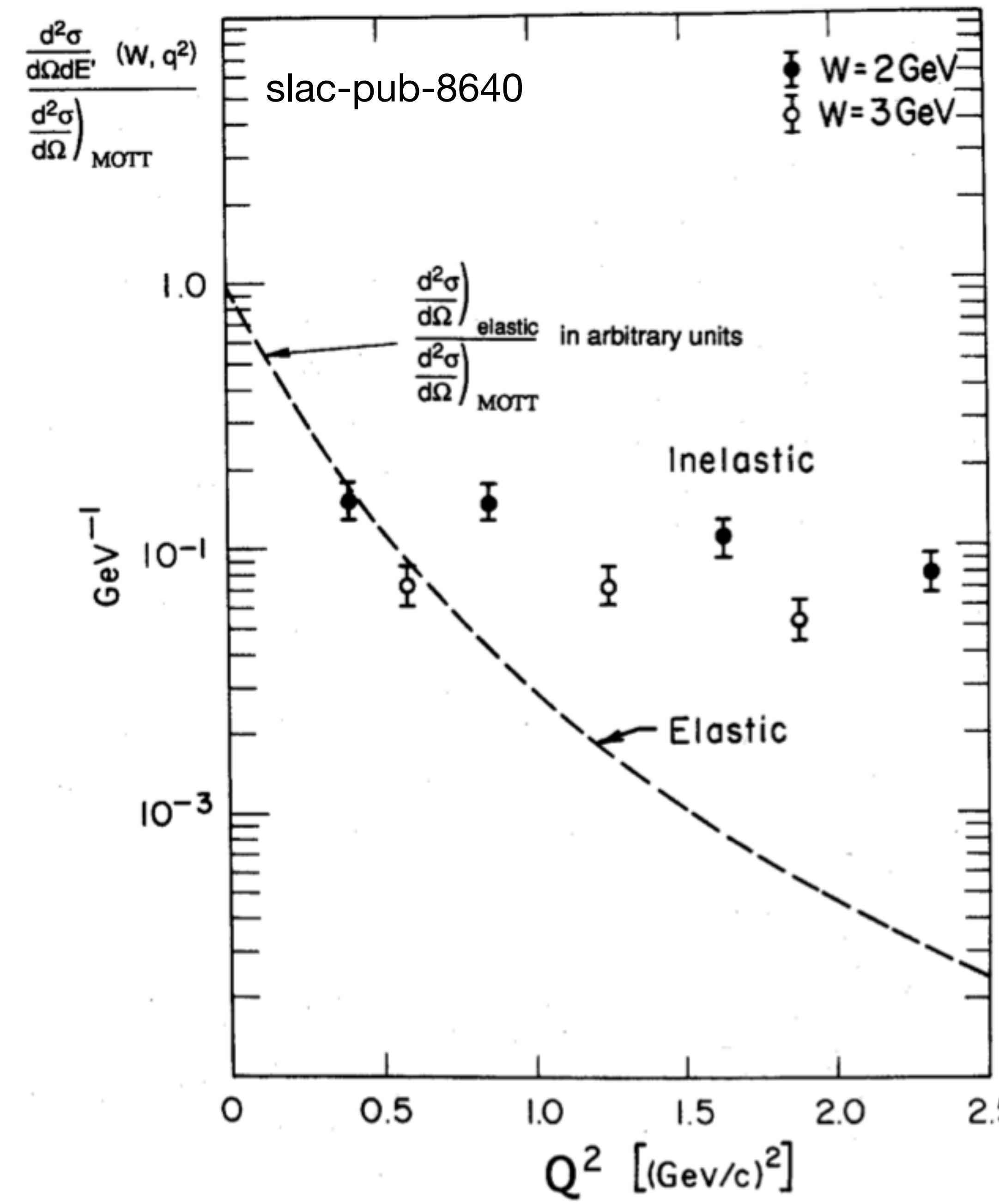
While all of the above **point-like** expressions were able to describe the scattering at low energies ($\sim$ 5 MeV in Rutherford's gold foil experiment and early Hofstadter experiments ~188 MeV), they fail at higher energies.

This was first examined in the 1960s by a series of experiments initiated at Stanford by Hofstadter, focused on **inelastic** proton-nucleon scattering.

Hints of proton compositeness

Early electron scattering cross-sections from the first inelastic scattering experiments performed at SLAC by the SLAC-MIT collaboration.



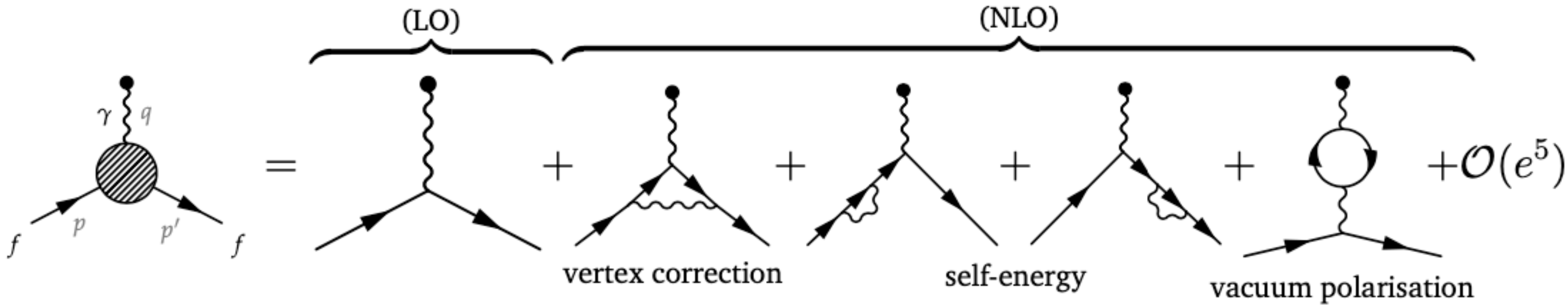


Early results on electron proton scattering, showing the striking difference in Q^2 dependence between elastic and inelastic cross-sections.

Beyond leading-order

Let us now consider the correction of the photon-nucleon vertex in elastic scattering. Starting from the NLO calculations and with minimal assumptions, we will see how inelastic scattering seems almost describable purely by QED.

The photon-fermion vertex correction reads:



$$= (-ie)\bar{u}(p')\Gamma^\mu u(p)$$

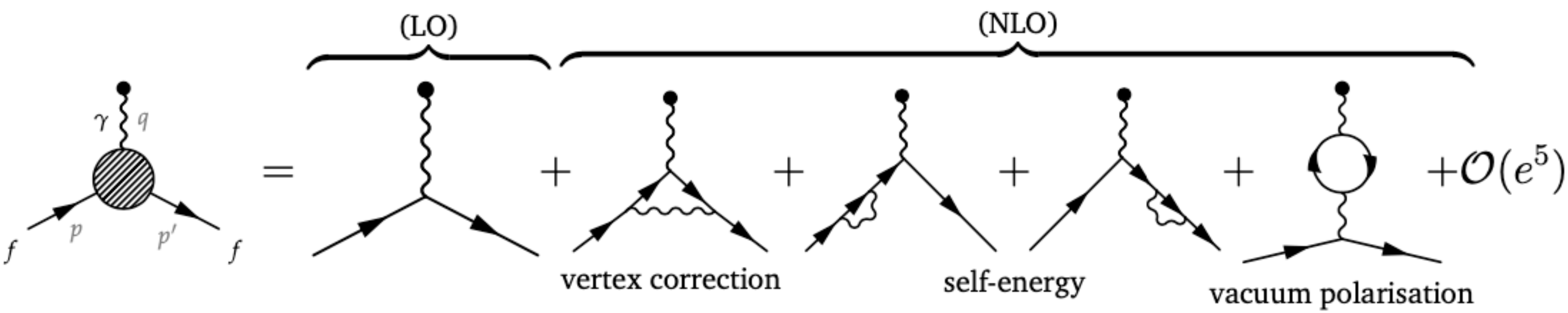
Using Lorentz invariance and the Ward identity ($q_\mu \Gamma^\mu = 0$), we can generally write:

$$\bar{u}(p')\Gamma^\mu u(p) = \bar{u}(p')\left[\gamma^\mu \mathcal{F}_1^f(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m_f}\mathcal{F}_2^f(q^2)\right]u(p)$$

F_1^f and F_2^f are the only unknown functions of q^2 allowed by relativistic invariance called **form factors**, where at the lowest-order $F_1^f = 1$ and $F_2^f = 0$

QED accuracy test

At NLO, the self-energy and vacuum polarisation diagrams involve a correction to the fermion and photon propagators respectively. These diagrams will result in divergent terms proportional to γ^μ, therefore contributing only to F_1^f .



However, the vertex correction is convergent at all orders and contributes solely to F_2^f which in the limit $q^2 \to 0$ provides the most accurate test of QED.

$$\mathcal{F}_2^f(q^2 \to 0) = |\boldsymbol{\mu_f}| - 1 = g\frac{e}{2m_f}|\boldsymbol{S}_f| - 1 \overset{\text{spin}-1/2}{=} \frac{\alpha}{2\pi}\frac{m_e}{m_f}$$

μ_f and S_f are the spin magnetic and angular moment of the fermion f respectively and g is their proportionality constant called g-factor. The prediction yielding the anomalous magnetic moment of an electron and equivalently the fine-structure constant **agrees with the measurements within ten parts in a billion (10^{-8})**.

Rosenbluth formula

However, the measurement of the nucleons anomalous magnetic moment in 1933 and 1939 turned out to be significantly different from that of point-particles, like electrons. A further strong indication to the nucleons compositeness.

Now we can rederive:

$$\boxed{(E \gg m_e) \longrightarrow \quad \left(\frac{d\sigma}{d\Omega}\right)_{\text{lab}}^{\text{High energy}} = \Theta \cdot \frac{1}{E^2}\frac{E'}{E}\left(\cos^2\frac{\theta}{2} - \frac{q^2}{2m_A^2}\sin^2\frac{\theta}{2}\right)}$$

with the vertex correction.

The resulting expression is called the *Rosenbluth* formula and reads:

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{\text{lab}} = \Theta \cdot \frac{1}{E^2}\frac{E'}{E}\left[\left((\mathcal{F}_1^p)^2 - \frac{q^2}{4m_A^2}(\mathcal{F}_2^p)^2\right)\cos^2\frac{\theta}{2} - \frac{q^2}{2m_A^2}(\mathcal{F}_1^p + \mathcal{F}_2^p)^2\sin^2\frac{\theta}{2}\right]}$$

While this expression agrees well with the measured cross section of lepton-lepton scattering (f = e, μ, τ), it deviates for the lepton-nucleons at high energies.

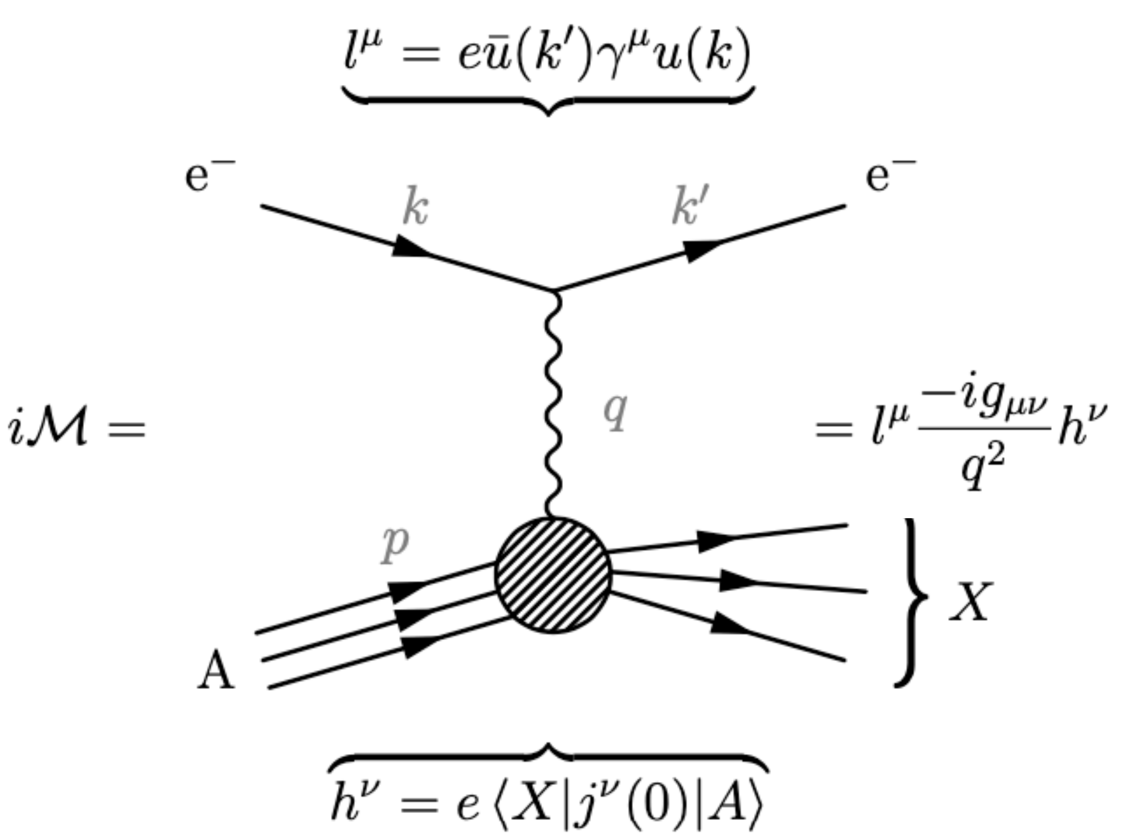
Deep-inelastic scattering (DIS)

The dominant scattering process off hadrons at high energies, the deep-inelastic one.

DIS was first measured in 1969 by the Stanford Linear Accelerator (SLAC) experiments covering electron energies ranging from 7 to 17 GeV.

At such high center-of-mass energies (Deep), a hadron A breaks apart (Inelastic) and the dominant scattering process becomes the so-called deep-inelastic scattering.

That directly implies a different vertex correction derivation where instead we have to consider the photon-hadron-X vertex, with X is whatever the hadron can break apart into as in:



$$L_{\mu\nu} = \frac{1}{2e^2}\sum_{e^\pm \text{spins}} l_\mu l_\nu^* \equiv \frac{1}{2}\sum_{e^\pm \text{spins}} \bar{u}(k')\gamma^\mu u(k)\bar{u}(k)\gamma^\nu u(k')$$
$$= 2(k'^\mu k^\nu + k'^\nu k^\mu - k\cdot k' g^{\mu\nu})$$

$$W^{\mu\nu} = \int d\Pi_X (2\pi)^4 \delta^{(4)}(q + p - p_X)\frac{1}{e^2}\sum_{X,\text{spins}} h^\mu h^{*\nu}$$
$$= W_1\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) + W_2\left(p^\mu - \frac{p\cdot q}{q^2}q^\mu\right)\left(p^\nu - \frac{p\cdot q}{q^2}q^\nu\right)$$

$$\downarrow$$

$$\partial_\mu j^\mu = 0 \implies \begin{cases} q_\mu W^{\mu\nu} & = 0 \\ q_\nu W^{\mu\nu} & = 0 \end{cases} \qquad d\Pi_X = \prod_{\text{final states } i} \frac{d^3 p_{X,i}}{(2\pi)^3}\frac{1}{2E_{X,i}}$$

Deep-inelastic scattering (DIS)

Let us use a set of Lorentz-invariant variables that are particularly suited for the description of the *Bjorken scaling* feature (see next slides). A feature that was predicted by the parton model but in fact violated in perturbative QCD. These variables read:

$\nu = \dfrac{p \cdot q}{m} = E - E'$	Lepton energy loss
$Q^2 = -q^2 > 0$	Energy scale squared
$x = \dfrac{Q^2}{2p \cdot q} = \dfrac{Q^2}{2m\nu}$	Bjorken-x
$y = \dfrac{q \cdot p}{k \cdot p} = \dfrac{\nu}{E}$	Inelasticity
$W^2 = (p+q)^2 = m_p^2 + Q^2 \dfrac{1-x}{x}$	Mass squared of the final state X
$s = (k+p)^2 = \dfrac{Q^2}{xy} + m_p^2$	Center-of-mass energy squared

Deep-inelastic scattering (DIS)

Taking into account the newly introduced Lorentz-invariant kinematic variables, we can finally write the double differential cross section as:

$$\boxed{\begin{aligned}\left(\frac{d\sigma}{d\Omega dE'}\right)_{lab} &= \frac{\alpha^2}{4\pi m_p q^4}\frac{E'}{E}L_{\mu\nu}W^{\mu\nu} \\ &= \frac{\alpha^2}{4\pi E^2(1-\cos\theta)^2}\left(\frac{m_p}{2}W_2(x,Q^2)\cos^2\frac{\theta}{2}+\frac{1}{m_p}W_1(x,Q^2)\sin^2\frac{\theta}{2}\right)\end{aligned}}$$

It has the same form as the Rosenbluth formula but with the quantities W_1 and W_2 encoding the hadronic structure instead of elastic form factors F_1^f and F_2^f that are purely electromagnetic.

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)^{Rosenbluth}_{\text{lab}} = \Theta\cdot\frac{1}{E^2}\frac{E'}{E}\left[\left((\mathcal{F}_1^p)^2-\frac{q^2}{4m_A^2}(\mathcal{F}_2^p)^2\right)\cos^2\frac{\theta}{2}-\frac{q^2}{2m_A^2}\left(\mathcal{F}_1^p+\mathcal{F}_2^p\right)^2\sin^2\frac{\theta}{2}\right]}$$

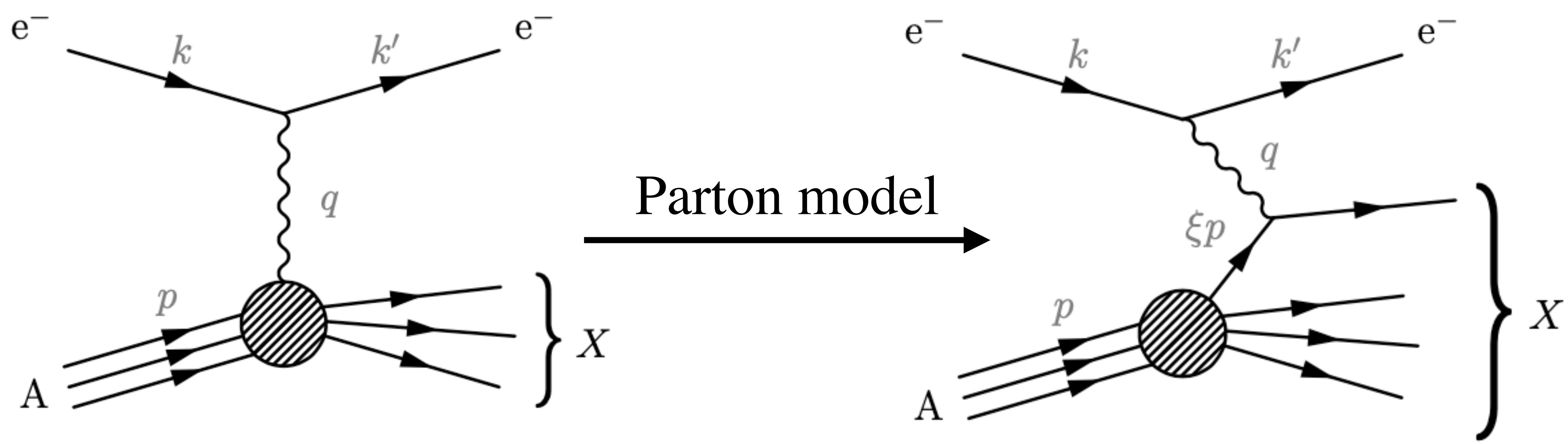
However, both of them are completely determined by measuring the energy and angular dependence of the outgoing electron. A feature that makes the DIS process one of the most experimentally accessible processes.

Parton model

At very high energy, the agreement with the elastic point-like scattering indicated the presence of point-like constituents within the hadron.

Based on this observation, the parton model was developed by Feynman, assuming that some particles (possibly charged) called *partons* within the hadron are very weakly interacting at high energy scales Q^2.

Within this context, the electron scatters elastically off partons of mass m_q inside a hadron A which will allow us to constrain the quantities W_1 and W_2:



$p_q = \xi p$ is the parton's fraction of the hadron's momentum p. It directly implies that:

$$m_q^2 \overset{\text{parton model}}{=} (\xi p + q)^2 = m_q^2 + 2\xi p \cdot q - Q^2 \implies \frac{Q^2}{2\xi p \cdot q} = 1 \qquad x \overset{\text{def}}{=} \frac{Q^2}{2p \cdot q} \overset{\text{parton model}}{=} \xi$$

Parton model

In the parton model, we assume that the DIS off hadron cross section of $e^- A \to e^- X$ is given by the off parton one $e^- q \to e^- X$. Therefore, the task of determining the unknown quantities W_1 and W_2 reduces to determining $f_i(\xi)d\xi$ the probability of the photon hitting parton type i that has a fraction ξ of the hadron's momentum p. We call $f_i(\xi)$ **the bare parton distribution functions (PDFs)**.

$$\sigma_{(e^- A \to e^- X)} \overset{\text{parton model}}{=} \sum_i \int_0^1 d\xi \, f_i(\xi) \; \hat{\sigma}_{(e^- q \to e^- X)}$$

where the hat refers to the partonic cross section of $e^- q \to e^- X$

$$\left(\frac{d\sigma}{d\Omega dE'}\right) \overset{\text{parton model}}{=} \sum_i f_i(x) \frac{\alpha^2 e_{q_i}^2}{E^2(1-\cos\theta)^2} \left(\frac{2m_p}{Q^2} x^2 \cos^2\frac{\theta}{2} + \frac{1}{m_p}\sin^2\frac{\theta}{2}\right)$$

$$\Longrightarrow \begin{cases} W_1(x,Q) & = 2\pi \sum_i e_{q_i}^2 f_i(x) \\ W_2(x,Q) & = 8\pi \frac{x^2}{Q^2} \sum_i e_{q_i}^2 f_i(x) \end{cases} \overset{\text{or}}{\longleftrightarrow} \begin{cases} F_1(x) & = \frac{1}{4\pi} W_1(x) \\ F_2(x) & = \frac{Q^2}{8\pi x} W_2(x) \end{cases}$$

Parton model

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where F_1 and F_2 are the dimensionless form of W_1 and W_2. The fact that F_1 and F_2 are independent of the energy scale Q is called Bjorken scaling that was the first success of the parton model as it was confirmed by the SLAC data… up to a violation caused by QCD radiative corrections as we will see next.

Another success of the parton model is the *Callan-Gross* relation:

$$W_1(x,Q^2) = \frac{Q^2}{4x^2} W_2(x,Q^2), \qquad Q \gg m_p$$

This relation originates from the fact that $x = 1 \implies \dfrac{Q^2}{2m_q^2} = \dfrac{Q^2}{2x^2 m_p^2}$ and the LO Rosenbluth is retrieved which describes scattering of a spin-1/2 particle. Therefore this relation, verified by the same data, indicated that partons are actually spin-1/2.

Overview

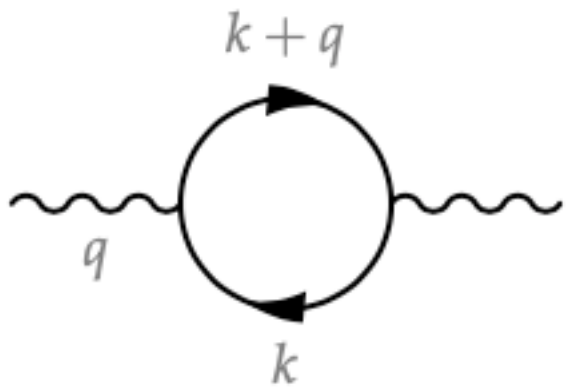
1. **Beyond QED structure of the proton:** Assuming the nucleon as a point-like particle is a good approximation only at low center-of-mass energies. While the QED photon-fermion vertex corrections describes well the leptons, they deviate for the nucleon that has an anomalous magnetic moment significantly different from that of a point-like particle.

2. **Weakly interacting partons at large-energies:** The parton model and Bjorken scaling (although approximate) were very successful in describing the first measured DIS data, which suggested that a proton is made of almost free partons of spin-1/2, also reinforced by the Callan-Gross relation when compared to the data.

3. **Asymptotic freedom and non-abelian theories**: QCD seemed the best candidate to explain, on one hand, the success of the parton model at short-distances (asymptotic freedom) and, on the other hand, the strong binding at long-distances together with the fact that quarks are not observed as isolated particles. It was shown that *non-abelian* (or non-commutative) gauge theories are the only asymptotically free field theories in four dimensions. Such theories can be constructed as generalisations of QED and this was the final piece that lead to QCD as the theory of quarks bound together by interacting vector bosons called gluons.

QCD through the coupling constant

In QED, the vacuum polarisation, is of particular importance as it will result in the renormalisation of the electric charge, thus the coupling constant as follows:



$$\Longrightarrow \quad \mu_R^2 \frac{d}{d\mu_R^2}(\alpha_{\text{eff}}) \equiv \frac{d}{d\log\mu_R^2}(\alpha_{\text{eff}}) \equiv \beta(\alpha_{\text{eff}}) \overset{\text{(NLO)}}{=} \frac{\alpha_{\text{eff}}^2}{3\pi}$$

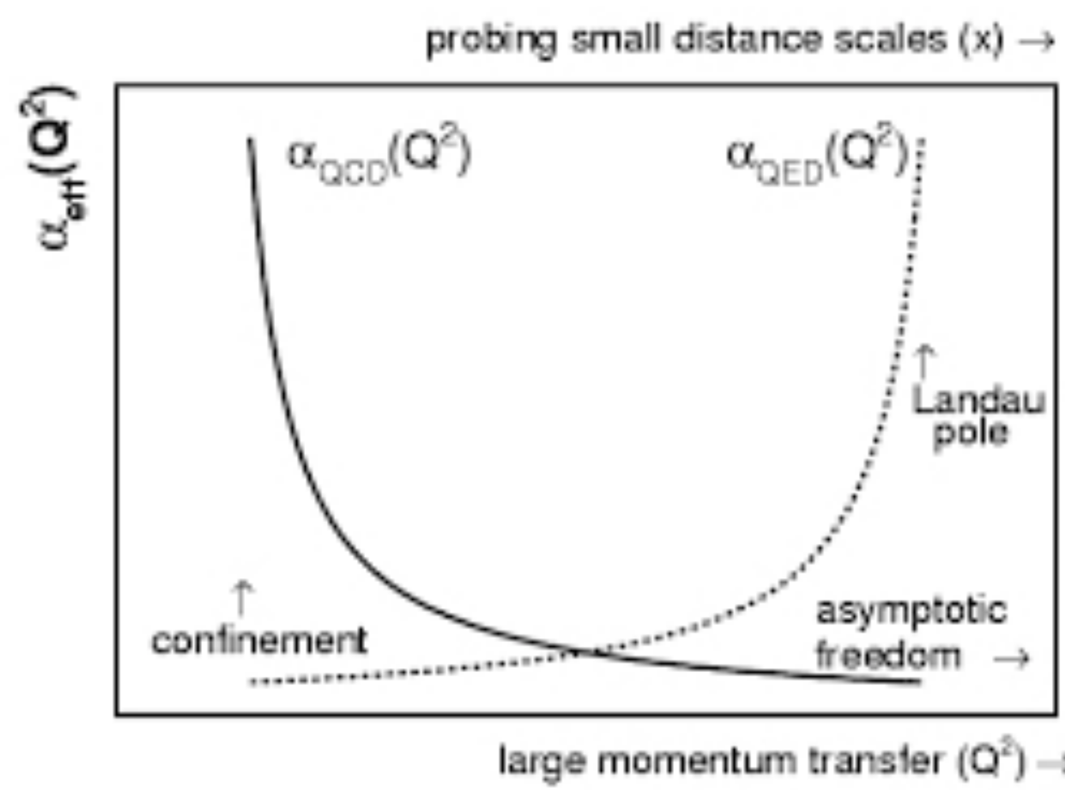
$$\boxed{\alpha_{\text{eff}}(-q^2) \overset{\mathcal{O}(\alpha_R)}{=} \frac{\alpha_R(\mu_R^2)}{1 - \frac{\alpha_R(\mu_R^2)}{3\pi}\log\left(\frac{-q^2}{\mu_R^2}\right)}}$$

In QCD, the corrections take a different form:

$$\mu^2 \frac{d}{d\mu^2}\alpha_s = \beta(\alpha_s) = -(b_0\alpha_s^2 + b_1\alpha_s^3 + b_2\alpha_s^4 + \cdots)$$

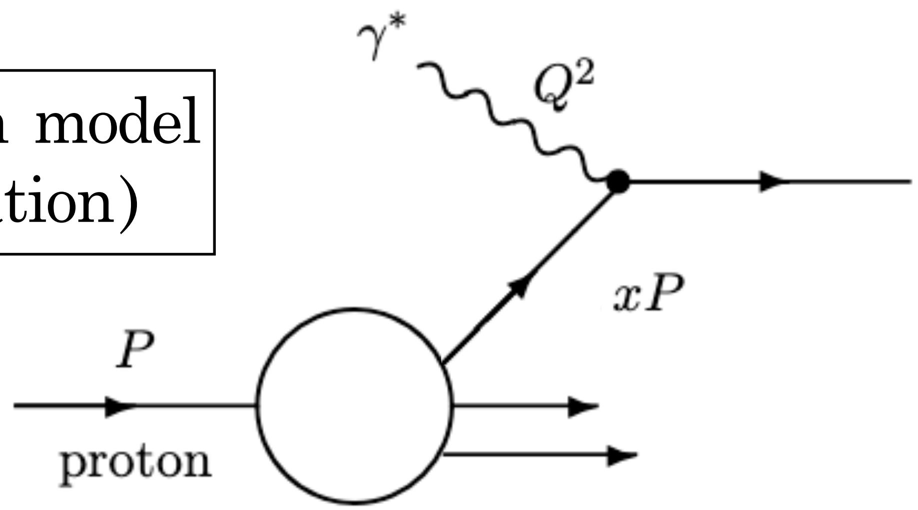
$$b_0 = (\overbrace{11C_A}^{\text{gluon loops}} - \overbrace{4n_f T_R}^{\text{quark loops}})/(12\pi)$$

$$\boxed{\alpha_s(Q^2) = \frac{\alpha_s(M_Z^2)}{1 + b_0\alpha_s(M_Z^2)\log\frac{Q^2}{M_Z^2} + \mathcal{O}(\alpha_s^2)}}$$



Gluon Radiation

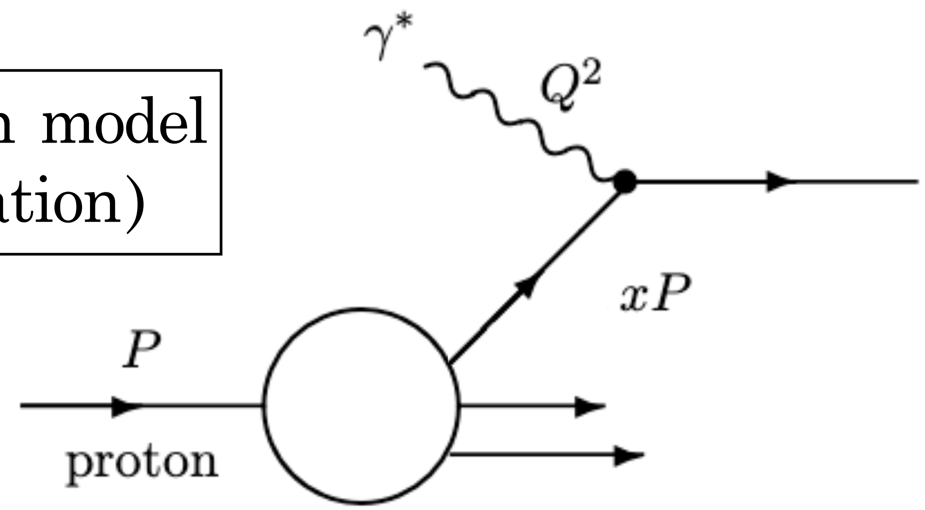
$\gamma^* p$ in the parton model
(no gluon radiation)



$$F_2^{\mathrm{ep}}(x) = \sum_i e_i^2 x f_i(x)$$

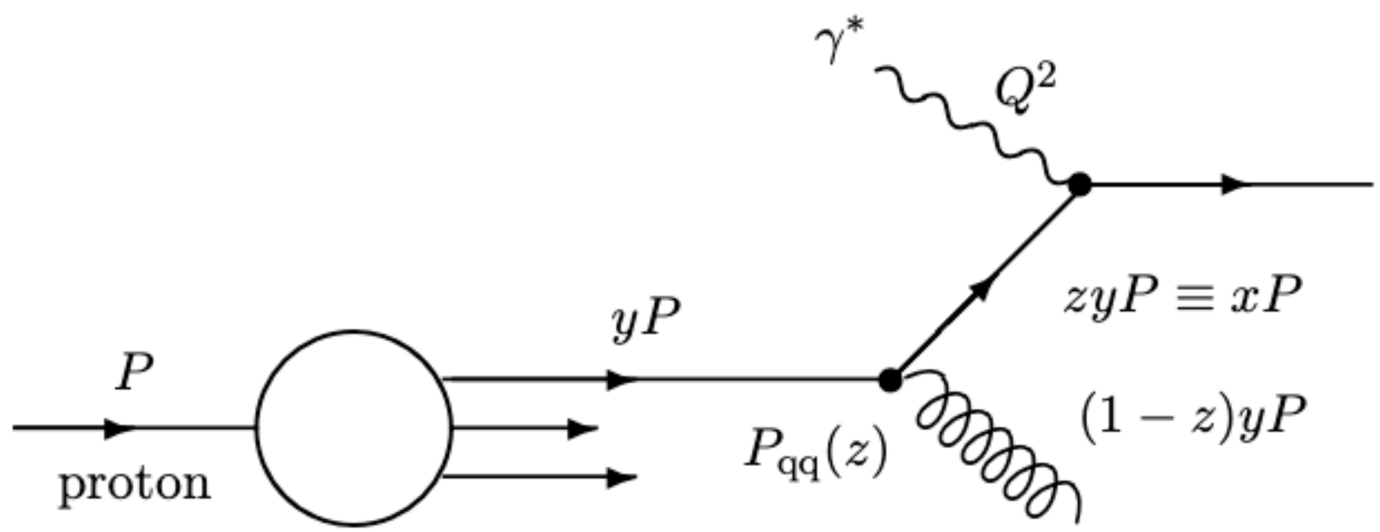
Gluon Radiation

$\gamma^* p$ in the parton model (no gluon radiation)



$$F_2^{\rm ep}(x) = \sum_i e_i^2 x f_i(x)$$

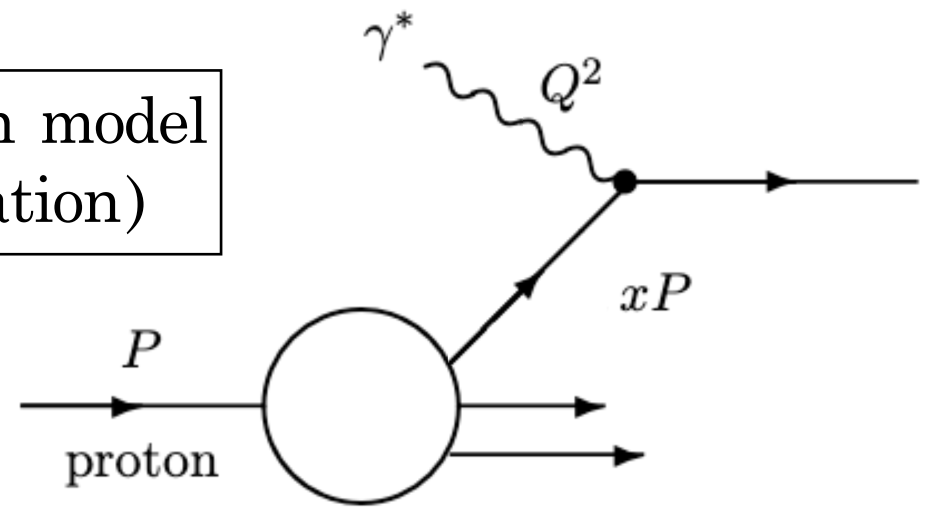
$\gamma^* p$ with gluon radiation



$$\frac{F_2(x,Q^2)}{x} = \sum_i e_i^2 \int_x^1 \frac{{\rm d}y}{y} f_i(y) \left[\delta\left(1-\frac{x}{y}\right) + \frac{\alpha_{\rm s}}{2\pi} P_{\rm qq}\left(\frac{x}{y}\right) \ln\frac{Q^2}{m^2} \right]$$

Gluon Radiation

$\gamma^* p$ in the parton model (no gluon radiation)



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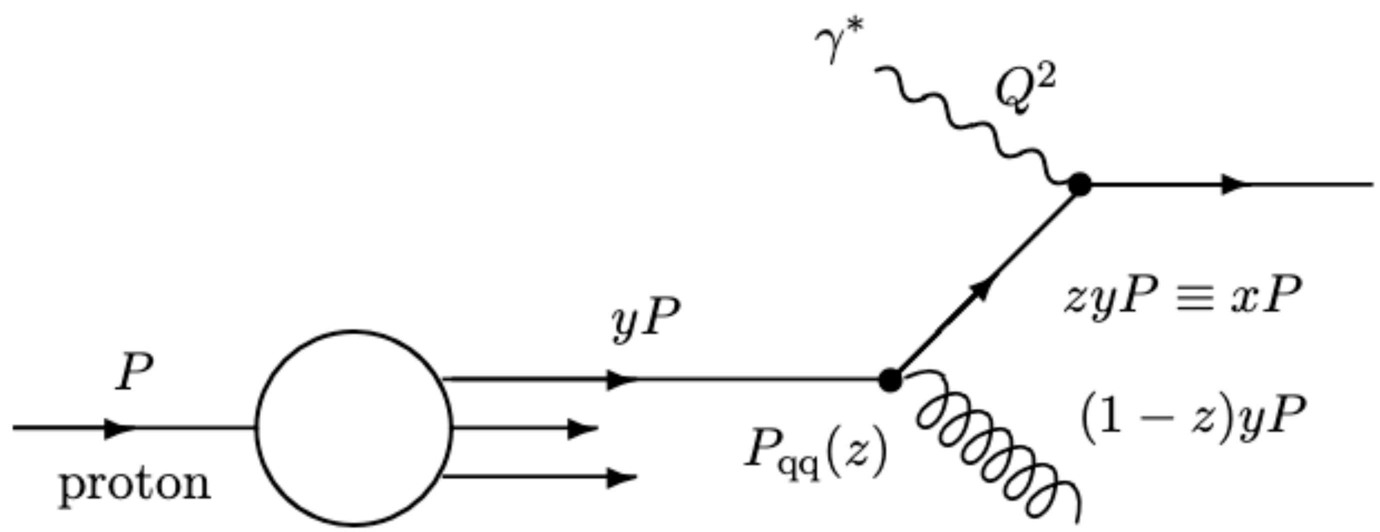
$\gamma^* p$ with gluon radiation



m^2 is a cut-off regularising the divergence arising when the gluon becomes collinear with the quark

$$\frac{F_2(x,Q^2)}{x} = \sum_i e_i^2 \int_x^1 \frac{\mathrm{d}y}{y} f_i(y) \left[\delta\left(1 - \frac{x}{y}\right) + \frac{\alpha_{\mathrm{s}}}{2\pi} P_{\mathrm{qq}}\left(\frac{x}{y}\right) \ln \frac{Q^2}{m^2} \right]$$

The probability that a quark emits a gluon with the daughter quark retaining a fraction z of its momentum

$$P_{\mathrm{qq}}(z) = \frac{4}{3}\left(\frac{1+z^2}{1-z}\right)$$

Renormalised PDFs

$$\boxed{\frac{F_2(x,Q^2)}{x} = e^2 \int_x^1 \frac{\mathrm{d}y}{y} f(y) \left[\delta\left(1-\frac{x}{y}\right) + \overbrace{\frac{\alpha_s}{2\pi} P_{qq}\left(\frac{x}{y}\right)}^{I_{qq}} \ln\frac{Q^2}{m^2} \right]}$$

Diverges when ($m^2 \to 0$) $\implies$ Introduction of a factorisation scale μ to separate the singular from calculable factor

$$\frac{F_2(x,Q^2)}{x} = e^2 \underbrace{\left[\underbrace{f(x) + I_{qq}(x) \ln\frac{\mu^2}{m^2}}_{f(x,\mu^2)\,\to\,\text{renormalised distribution}} + I_{qq}(x) \ln\frac{Q^2}{\mu^2} \right]}_{f(x,Q^2)}$$

$$f(x,Q^2) = f(x,\mu^2) + \frac{\alpha_s}{2\pi} \int_x^1 \frac{\mathrm{d}y}{y} f(y,\mu^2) P_{qq}\left(\frac{x}{y}\right) \ln\frac{Q^2}{\mu^2} + \mathrm{O}(\alpha_s^2)$$

DGLAP evolution

F_2 should not depend on the choice of factorisation scale μ
$\Longrightarrow$ Renormalisation group equation (RGE):

$$\frac{1}{e^2 x}\frac{\partial F_2(x,Q^2)}{\partial \ln \mu^2} = \frac{\partial f(x,\mu^2)}{\partial \ln \mu^2} + \frac{\alpha_{\mathrm{s}}}{2\pi}\int_x^1 \frac{\mathrm{d}y}{y}\left[\frac{\partial f(y,\mu^2)}{\partial \ln \mu^2}\ln\left(\frac{Q^2}{\mu^2}\right) - f(y,\mu^2)\right] P_{\mathrm{qq}}\left(\frac{x}{y}\right) = 0$$

$\Longrightarrow$ Dokshitzer, Gribov, Lipatov, Altarelli and Parisi (DGLAP)
Evolution equation:

$$\boxed{\frac{\partial f(x,\mu^2)}{\partial \ln \mu^2} = \frac{\alpha_{\mathrm{s}}}{2\pi}\int_x^1 \frac{\mathrm{d}y}{y} f(y,\mu^2) P_{\mathrm{qq}}\left(\frac{x}{y}\right) + \mathrm{O}(\alpha_{\mathrm{s}}^2)}$$

DIS at NLO

Now we can generally write, the mixing of PDFs flavours under DGLAP when considering real gluon emission:

$$\mu_F \frac{d}{d\mu_F} \begin{pmatrix} f_i(x,\mu_F) \\ f_g(x,\mu_F) \end{pmatrix} = \sum_j \frac{\alpha_s}{\pi} \int_x^1 \frac{d\xi}{\xi} \begin{pmatrix} P_{q_i q_j} & P_{q_i g} \\ P_{g q_j} & P_{gg} \end{pmatrix} \begin{pmatrix} f_j(\xi,\mu_F) \\ f_g(\xi,\mu_F) \end{pmatrix}$$

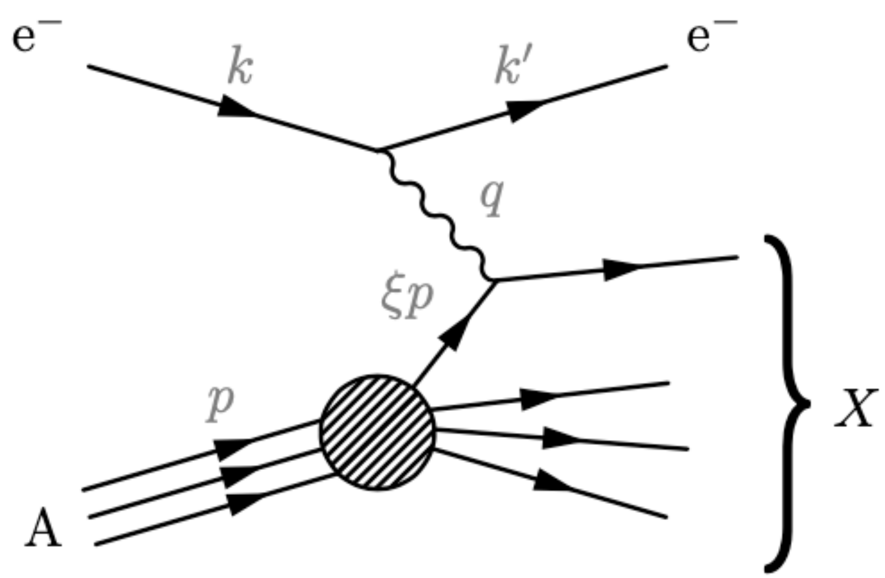
And the general form of the factorised structure functions expression in the limit $Q \gg m$, neglecting contributions which are suppressed by powers of $1/Q^2$ these include Higher twist effects (HT) and target mass corrections (TMC)

$$F_1(x,Q) = \frac{1}{4\pi} W_1(x) = \sum_i \int_x^1 \frac{d\xi}{\xi} \underbrace{f_i(\xi,\mu_F)}_{\text{non-perturbative}} \underbrace{H_{1i}\left(\frac{x}{\xi}, \frac{Q}{\mu_F}, \alpha_s(\mu_R)\right)}_{\text{perturbative}} + \mathrm{HT}$$

$$F_2(x,Q) = \frac{Q^2}{8\pi x} W_2(x,Q) = x \sum_i \int_x^1 \frac{d\xi}{\xi} \underbrace{f_i(\xi,\mu_F)}_{\text{non-perturbative}} \frac{\xi}{x} \underbrace{H_{2i}\left(\frac{x}{\xi}, \frac{Q}{\mu_F}, \alpha_s(\mu_R)\right)}_{\text{perturbative}} + \mathrm{HT}$$

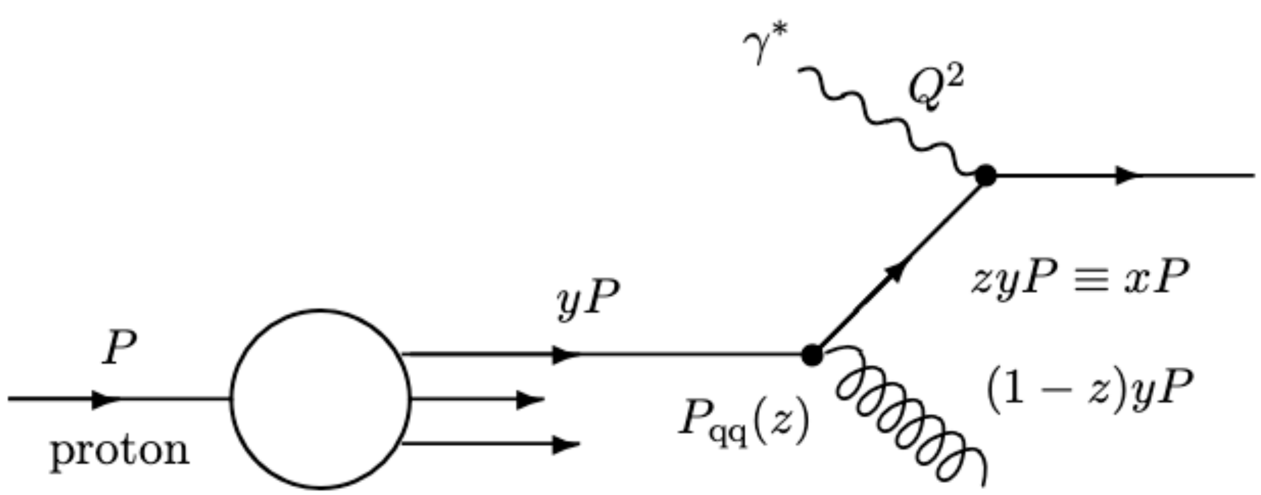
Generically we can write the compact form: $F(x,Q) = \sum_i f_i \otimes H_i^{\mathrm{DIS}}$

Take-away points



Parton model in QED
introduced bare PDFs

$$F_2^{\mathrm{ep}}(x) = \sum_i e_i^2 x f_i(x)$$

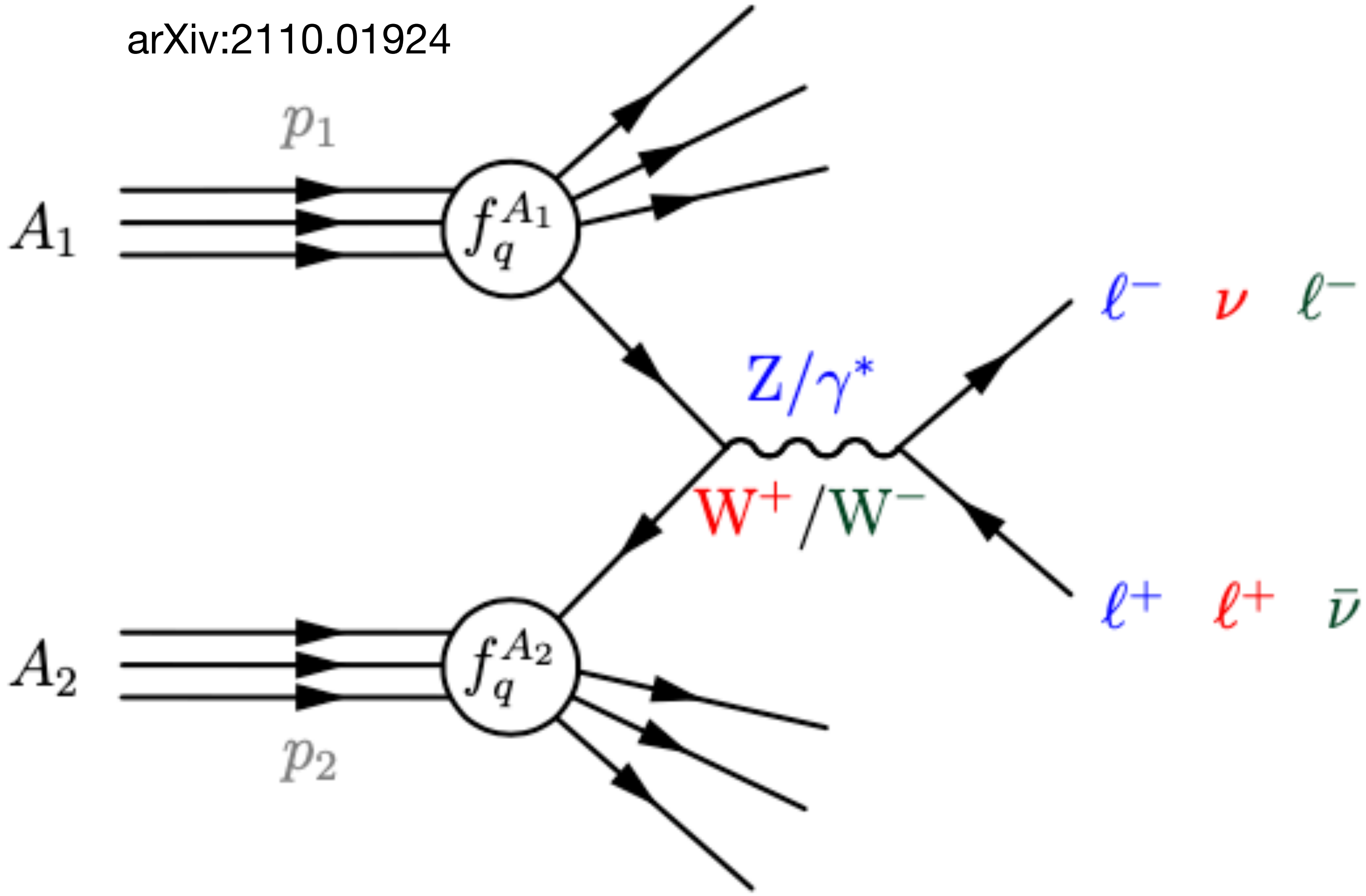


Parton model + gluon radiation in QCD
introduced renormalised PDFs
that needs to be extracted from data

$$F_2(x,Q) = \frac{Q^2}{8\pi x} W_2(x,Q) = x \sum_i \int_x^1 \frac{d\xi}{\xi} \underbrace{f_i(\xi,\mu_F)}_{\text{non-perturbative}} \underbrace{\frac{\xi}{x} H_{2i}\left(\frac{x}{\xi}, \frac{Q}{\mu_F}, \alpha_s(\mu_R)\right)}_{\text{perturbative}}$$

Other processes: Drell-Yan

arXiv:2110.01924



DY LO Kinematics

$$x_{1,2} = \frac{M}{\sqrt{s}} e^{\pm y}$$

$$M^2 = x_1 x_2 s$$

$$s = (p_1 + p_2)^2$$

$$y_{\ell\ell} = \frac{1}{2} \ln \frac{E_{\ell\ell} + p_{\ell\ell,z}}{E_{\ell\ell} - p_{\ell\ell,z}}$$

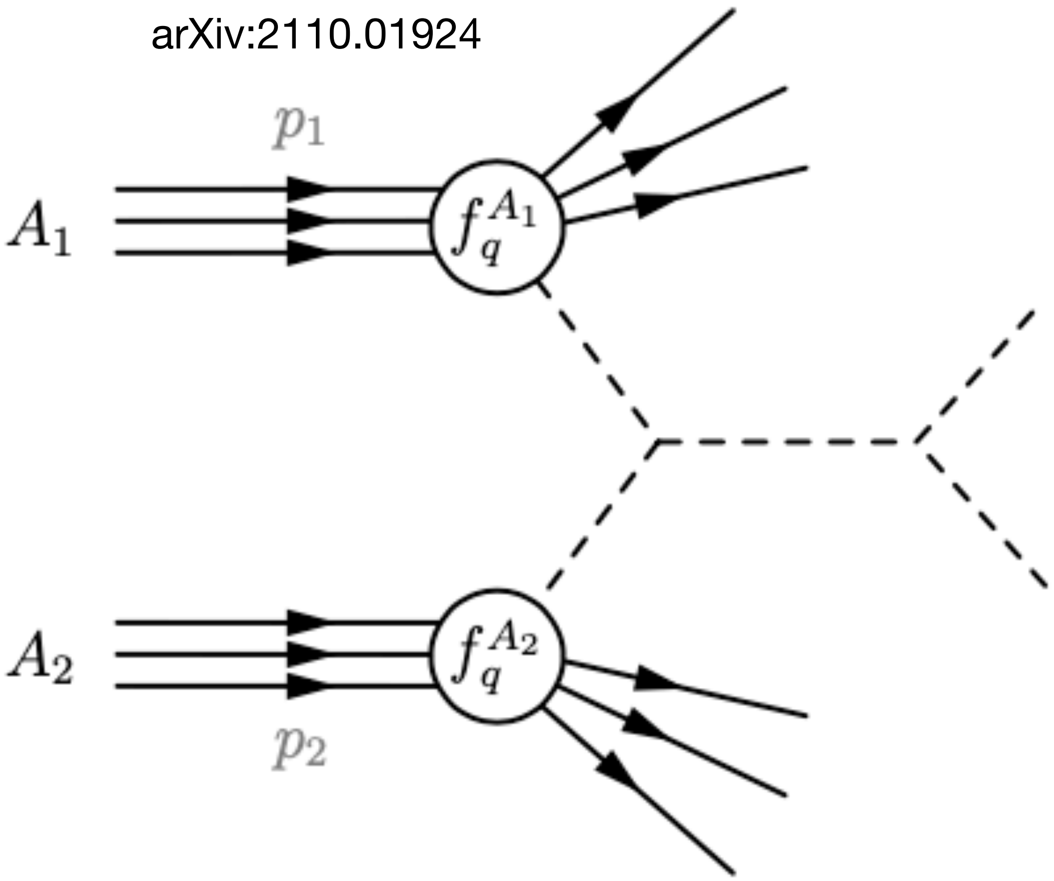
$$\frac{d^2\sigma^a}{dM^2 dy} = \sum_n \alpha_s^n(\mu_R^2) \sum_{ij} \int_{x_1}^1 d\xi_1 \int_{x_2}^1 d\xi_2$$

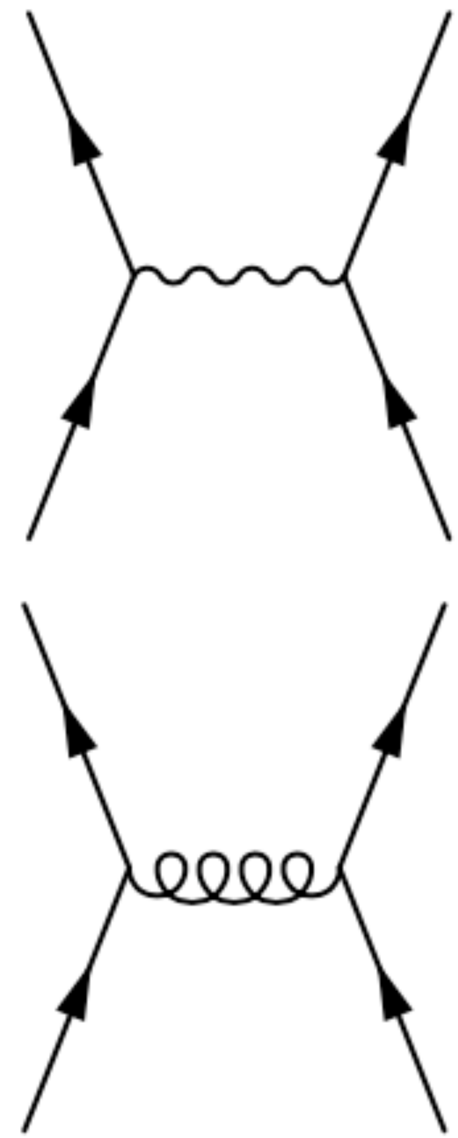
$$f_i^{A_1}(\xi_1, \mu_F^2) f_j^{A_2}(\xi_2, \mu_F^2) \times \frac{d^2\hat{\sigma}^a_{ij\to W/Z+X}}{dM^2 dy}\left(\frac{x_1}{\xi_1}, \frac{x_2}{\xi_2}, Q, \mu_R^2, \mu_F^2\right)$$

$$= \sum_{ij} \underbrace{f_i^{A_1} \otimes f_j^{A_2}}_{\text{non-perturbative}} \otimes \underbrace{\frac{d^2\hat{\sigma}^a_{ij\to W/Z+X}}{dM^2 dy}}_{\text{perturbative}}$$

Other processes: Jet-Production

arXiv:2110.01924





Jet production LO Kinematics

$$x_{1,2} = \frac{p_T e^{\pm y}}{\sqrt{s}}$$

$$p_T^2 = x_1 x_2 s$$

$$s = (p_1 + p_2)^2$$

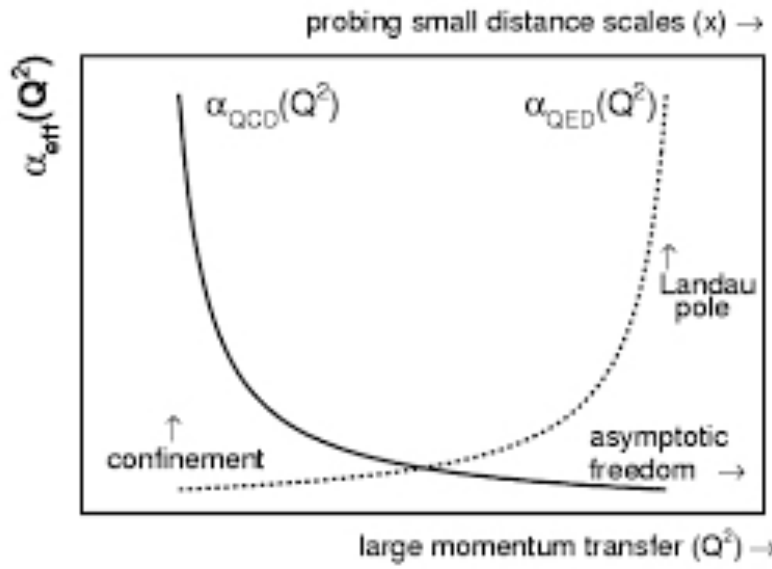
$$y = \frac{1}{2} \ln \frac{E + p_z}{E - p_z}$$

$$\frac{d^2\sigma^a}{dp_T^2 dy} = \sum_{ij} \underbrace{f_i \otimes f_j}_{\text{non-perturbative}} \otimes \underbrace{\frac{d^2\hat{\sigma}^a_{ij\to n \text{ jets}+X}}{dp_T^2 dy}}_{\text{perturbative}}$$

Collinear Fragmentation

Due to the nature of the strong force and the increasing strong coupling at low- energies, quarks and gluons never appear as asymptotic states.

It is rather energetically more favourable for partons to *fragment* into *jets* of quark-antiquark pairs until they end up *hadronising*.



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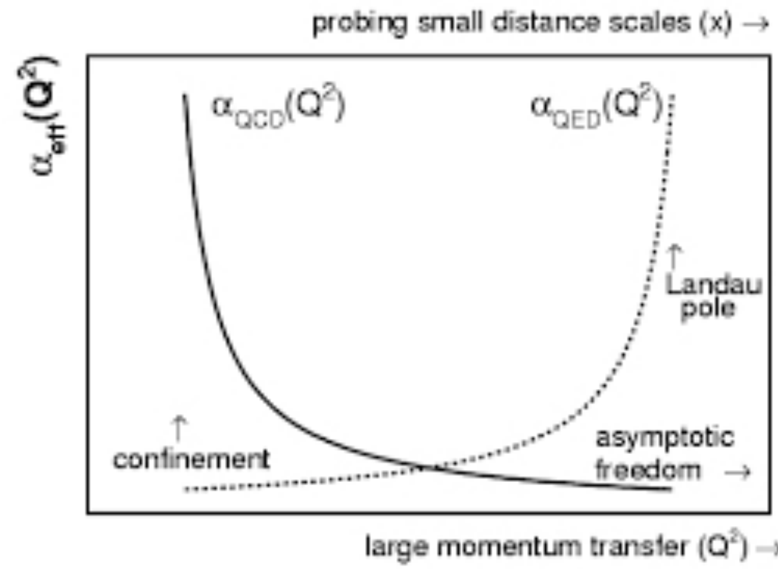
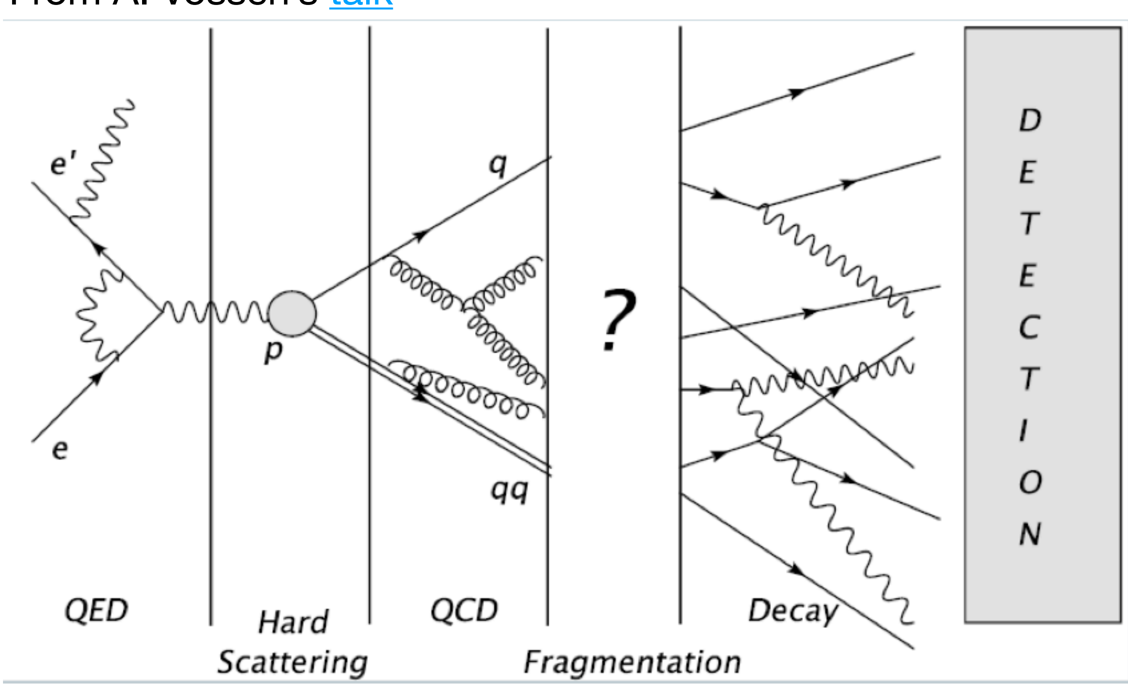


From A. Vossen's talk

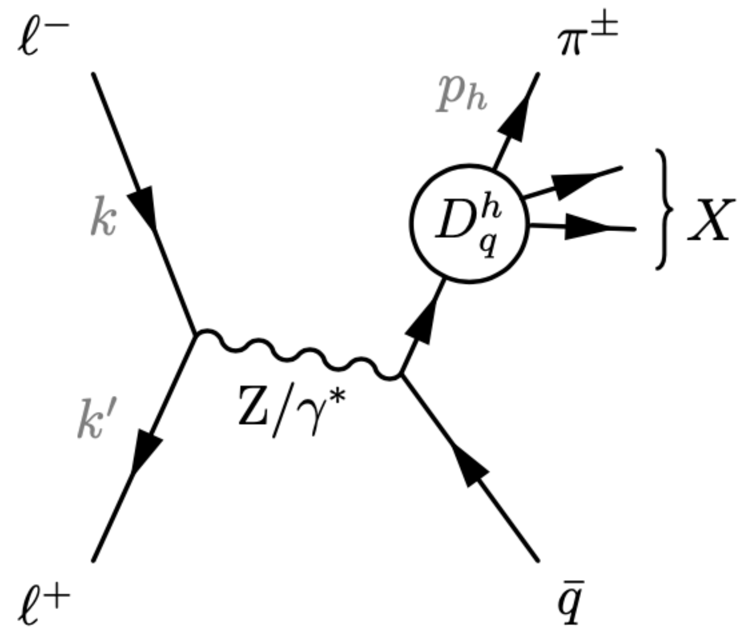


In inclusive deep-inelastic scattering, we only need to measure the incoming and outgoing lepton momenta to calculate the cross section.

However, whenever specific particles are identified in the final state, parton **Fragmentation Functions (FFs)** appear as non-perturbative ingredient.

Single-inclusive annihilation

$$e^+(k_1) + e^-(k_2) \to \pi^\pm(p_h) + X\,.$$



SIA LO Kinematics

$$Q^2 = q^2$$

$$z = \frac{2p_h \cdot q}{Q^2}$$

$$s = Q^2$$

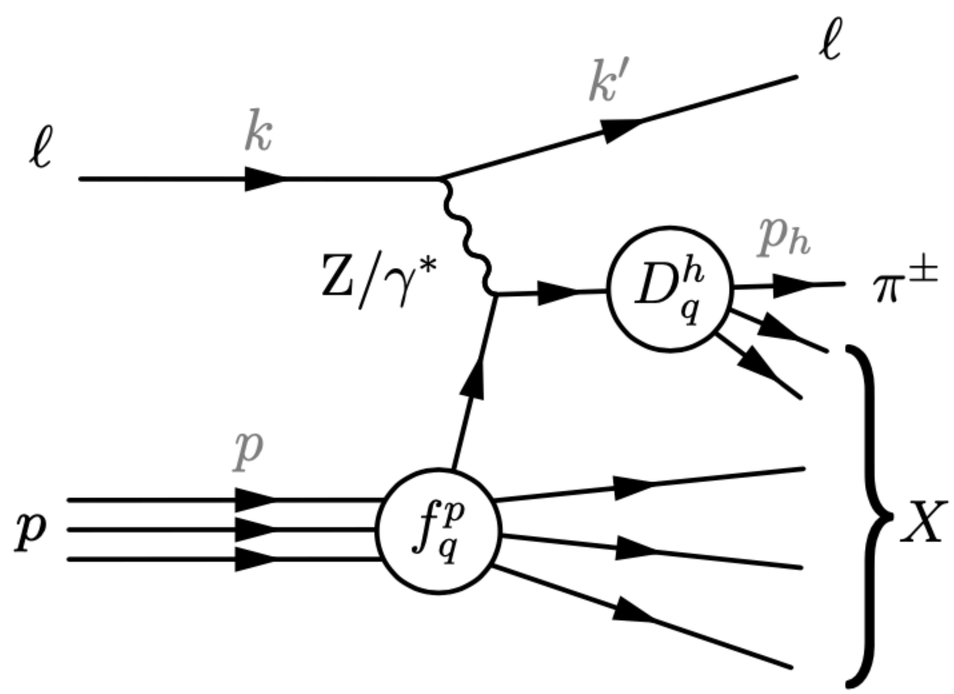
- **Cross section:** $$\frac{d\sigma^h}{dz}(z,Q) = \frac{4\pi\alpha(Q)}{Q^2} F_2^h(z,Q)$$

- **Structure functions:** $$F_2^{\mathrm{NC},\pi^\pm} = \frac{1}{n_f}\left(\sum_j^{n_f} \hat{e}_{q_j}^2\right) F_{2,\mathrm{S}}^{\mathrm{NC},\pi^\pm} + F_{2,\mathrm{NS}}^{\mathrm{NC},\pi^\pm}$$

$$F_{2,\mathrm{S}}^{\mathrm{SIA},\pi^\pm} = C_{2,\mathrm{S}}^{\mathrm{SIA}} \otimes D_{\underline{\Sigma}}^h + C_{2,g}^{\mathrm{SIA}} \otimes D_{\underline{g}}^h, \qquad F_{2,\mathrm{NS}}^{\mathrm{SIA},\pi^\pm} = C_{2,\mathrm{NS}}^{\mathrm{SIA}} \otimes \sum_j^{n_f} \hat{e}_{q_j} D_{j,\mathrm{NS}}^h$$

Semi-inclusive DIS

$$\ell(k) + N(p) \to \ell(k') + \pi^{\pm}(p_h) + X\,.$$



SIDIS LO Kinematics

$$Q^2 = -q^2$$
$$x = \frac{Q^2}{2p\cdot q}$$
$$z = \frac{p\cdot p_h}{p\cdot q}$$
$$y = \frac{Q^2}{xs}$$
$$s = (k^2 + p^2)$$

- **Cross section ($Q \ll M_Z$):** $\dfrac{d^3\sigma}{dxdQdz} = \dfrac{4\pi\alpha^2(Q)}{xQ^3}\left[\left(1+(1-y)^2\right)F_2(x,z,Q^2) - y^2 F_L(x,z,Q^2)\right]$

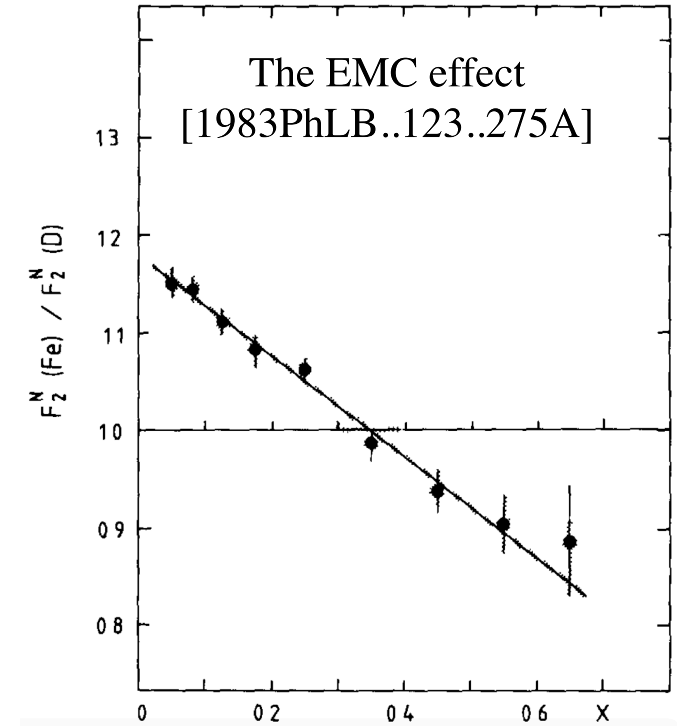
$$\begin{aligned} F_i(x,z,Q) &= x\sum_{q\bar{q}} e_q^2\Big\{\left[C_{i,qq}(x,z,Q)\otimes \underline{f_q(x,Q)} + C_{i,qg}(x,z,Q)\otimes \underline{f_g(x,Q)}\right]\otimes D_q^{\pi^{\pm}}(z,Q) \\ &+ \left[C_{i,gq}(x,z,Q)\otimes \underline{f_q(x,Q)}\right]\otimes D_g^{\pi^{\pm}}(z,Q)\Big\}\,, \qquad i = 2, L\,. \end{aligned}$$

$$C(x,z)\otimes f(x)\otimes D(z) = \int_x^1 \frac{dx'}{x'}\int_z^1 \frac{dz'}{z'} C(x',z') f\left(\frac{x}{x'}\right) D\left(\frac{z}{z'}\right)$$

Nuclear modification

Partons in nuclei have significantly different momentum distributions from those measured in free nucleons such as a proton. This was first discovered by the European Muon Collaboration (EMC).

Even at higher energies where the nucleon is expected to be composed of weakly interacting quarks and gluons, effects of the nucleon being in the nucleus were still playing a role.



This was at the least surprising as MeV-scale nuclear binding effects were expected to be negligible compared to the typical momentum transfers ($Q \geq 1$ GeV) in hard-scattering reactions.

$$F_2(x, Q^2, A) = \sum_i^{n_f} \sum_j^{n_f} \underline{C_i(x, Q^2)} \otimes \underline{\Gamma_{ij}(x, Q_0^2, Q^2)} \otimes \underline{q_j(x, Q_0^2, A)}$$

a) the fundamental interactions are the same but PDFs are different.
b) The fundamental interactions are different in the medium but PDFs are the same.
c) Both a) and b).
d) The factorisation picture is no longer valid.

Nuclear modification

Starting from the assumption that the nucleons are the only degrees of freedom of the nuclear wave function, we can write:

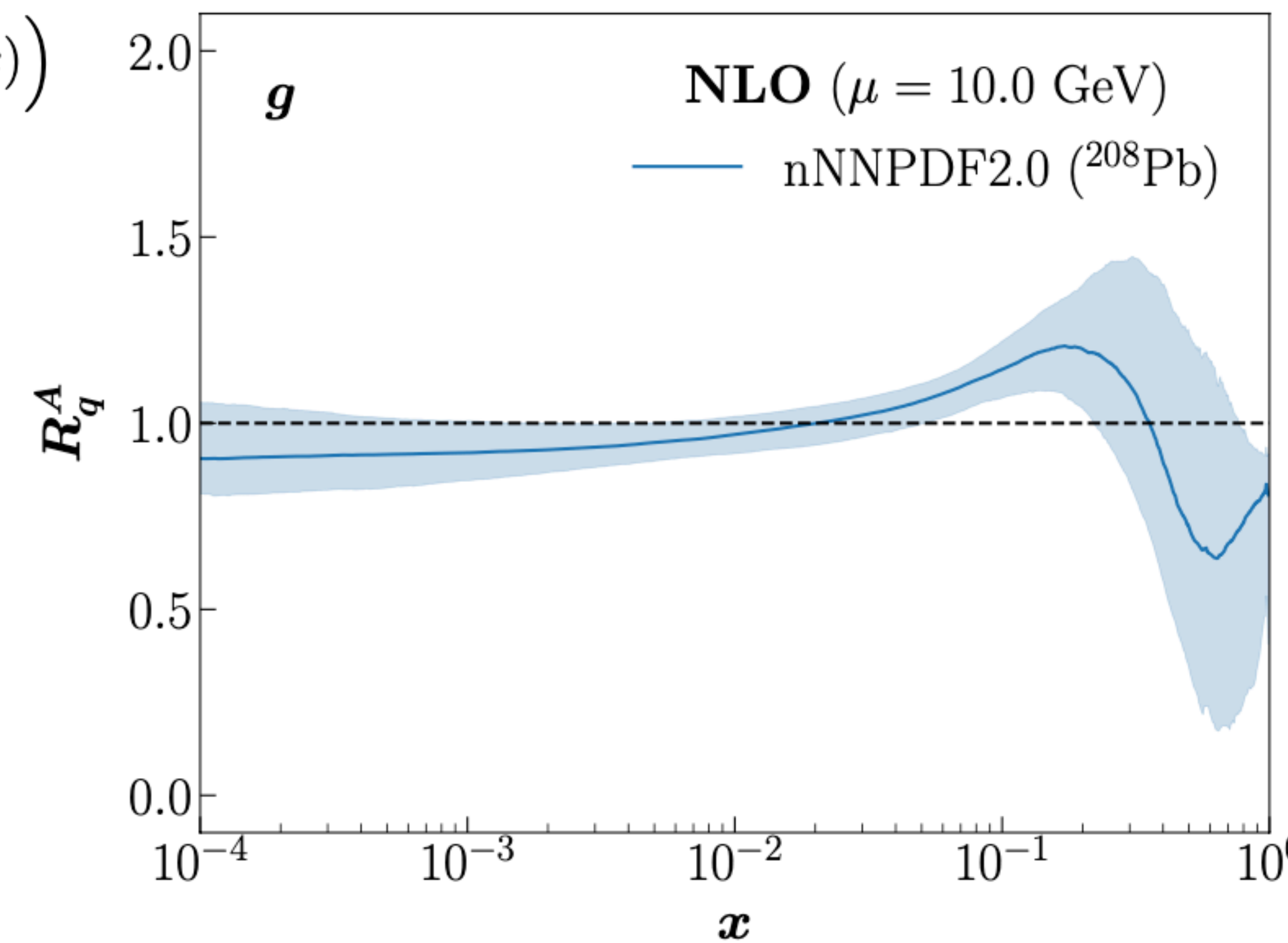
$$p_q = x_A p_A = x_A (A p_N) = (A x_A) p_N \implies x \equiv x^{N/A} = A x_A$$

And thus, we can define the PDF of a nucleon (N) bound to a nucleus (A) as:

$$f_i^{(N/A)}(x) = \frac{1}{A}\left(Z f_i^{(p/A)}(x) + (A-Z) f_i^{(n/A)}(x)\right)$$

Taking the ratio of nuclear PDFs over the free proton PDFs:

$$R_i^A = \frac{\left(Z f_i^{(p/A)}(x) + (A-Z) f_i^{(n/A)}(x)\right)}{\left(Z f_i^{(p)}(x) + (A-Z) f_i^{(n)}(x)\right)}$$



Physical Properties

Charge conjugation (C) and SU(2) isospin symmetry (I) constraints on FFs and (n)PDFs

	FFs	PDFs	nPDFs
$\mathcal{C}$	$D^{h^+}_{q(\bar{q})} = D^{h^-}_{\bar{q}(q)}$	N/R	N/R
$\mathcal{I}$	$D^{\pi^+}_{u(d)} = D^{\pi^-}_{d(u)}$ $D^{\pi^+}_{\bar{u}(\bar{d})} = D^{\pi^-}_{\bar{d}(\bar{u})}$	$f^{(p)}_{u(d)} = f^{(n)}_{d(u)}$ $f^{(p)}_{\bar{u}(\bar{d})} = f^{(n)}_{\bar{d}(\bar{u})}$	$f^{(N/A)}_{u(d)} = 2f^{(D)}_{d(u)} - f^{(N/A)}_{d(u)}$ $f^{(N/A)}_{\bar{u}(\bar{d})} = 2f^{(D)}_{\bar{d}(\bar{u})} - f^{(N/A)}_{\bar{d}(\bar{u})}$
$\mathcal{C}+\mathcal{I}$	$D^{\pi^+}_{u(d)} = D^{\pi^-}_{d(u)} = D^{\pi^+}_{\bar{d}(\bar{u})} = D^{\pi^-}_{\bar{u}(\bar{d})}$	N/R	N/R

Sum rules for FFs and (n)PDFs

	FFs	PDFs	nPDFs
Momentum $(i = q, \bar{q}, g)$	$\sum_h \int_0^1 z D^h_i dz = 1$	$\sum_i \int_0^1 x f^{(p)}_i dx = 1$	$\sum_i \int_0^1 x f^{(p/A)}_i dx = 1$
Number $(j = s, c, b, t)$	N/A	$\int_0^1 f^{-,(p)}_d dx = 1$ $\int_0^1 f^{-,(p)}_u dx = 2$ $\int_0^1 f^{-,(p)}_j dx = 0$	$\int_0^1 f^{-,(p/A)}_d dx = 1$ $\int_0^1 f^{-,(p/A)}_u dx = 2$ $\int_0^1 f^{-,(p/A)}_j dx = 0$