Quark and Gluon Helicity at Small x

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In collaboration with:
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Based on: [2204.11898](2204.11898)

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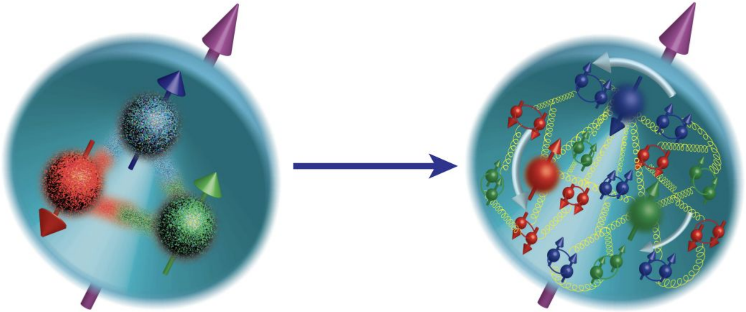
Proton Spin Puzzle

- Jaffe-Manohar sum rule: $\frac{1}{2} = S_q + S_G + L_q + L_G$
- Focus on **helicity** of quarks (S_q) and gluons (S_G)

$$S_q(Q^2) = \frac{1}{2}\int_0^1 dx\, \Delta\Sigma(x,Q^2) = \frac{1}{2}\int_0^1 dx \sum_f \left[\Delta q_f(x,Q^2) + \Delta\bar{q}_f(x,Q^2)\right]$$

$$S_G(Q^2) = \int_0^1 dx\, \Delta G(x,Q^2)$$



Accardi et al, 1212.1701

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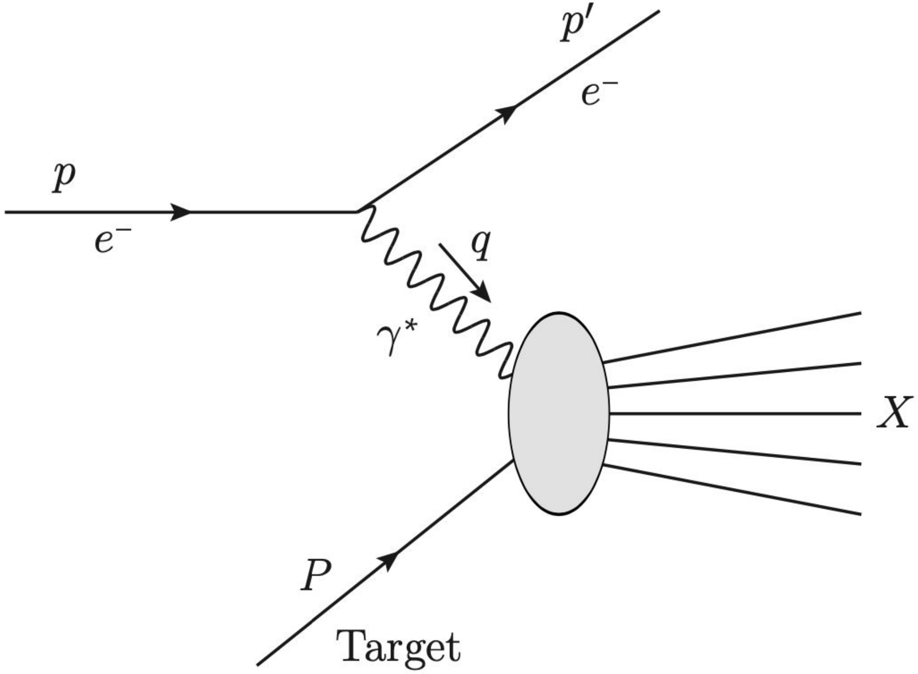
$$S_G(Q^2) = \int_0^1 dx\, \Delta G(x,Q^2)$$

- Experiments do not give helicity PDF all the way down to Bjorken x = 0.
- Determine small-x asymptotics through evolution, resumming a power of ln(1/x).

Deep-Inelastic Scattering (DIS)



- Main tool to look into the structure of protons, neutrons, etc: “target”.
- Electron - target scattering with high enough energy to break the target into pieces.
- Virtuality ~ transverse resolution:

$$Q^2 = -q^2 \sim \frac{1}{x_\perp^2}$$

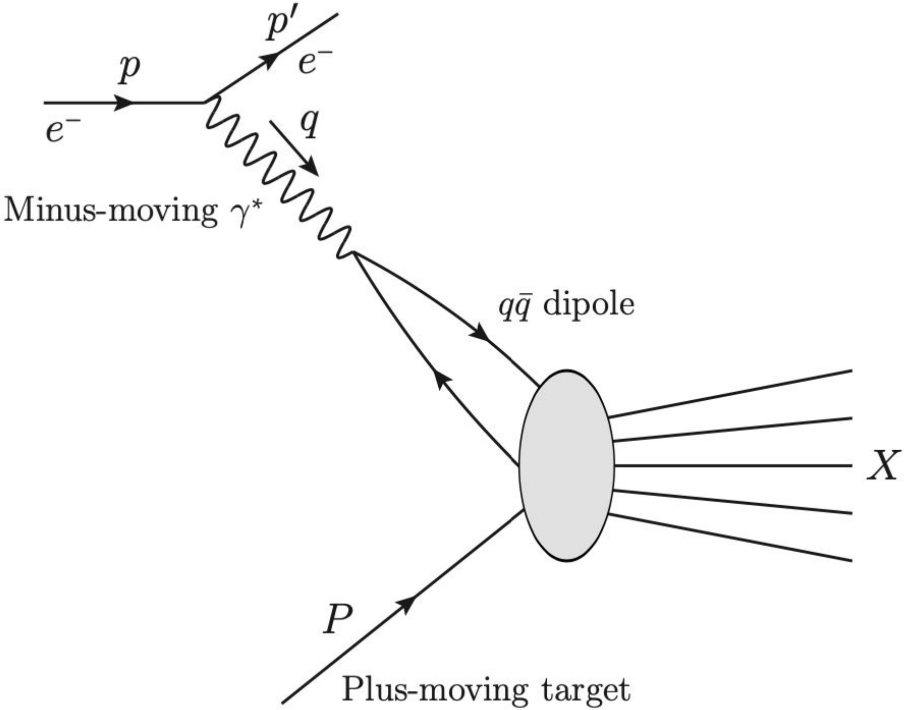
- Bjorken x:

$$x = \frac{Q^2}{2P\cdot q} \sim \frac{1}{Mx^-}$$

Small x: ɣ*+P interaction “takes forever”.

(M: target’s mass)

Deep-Inelastic Scattering (DIS)



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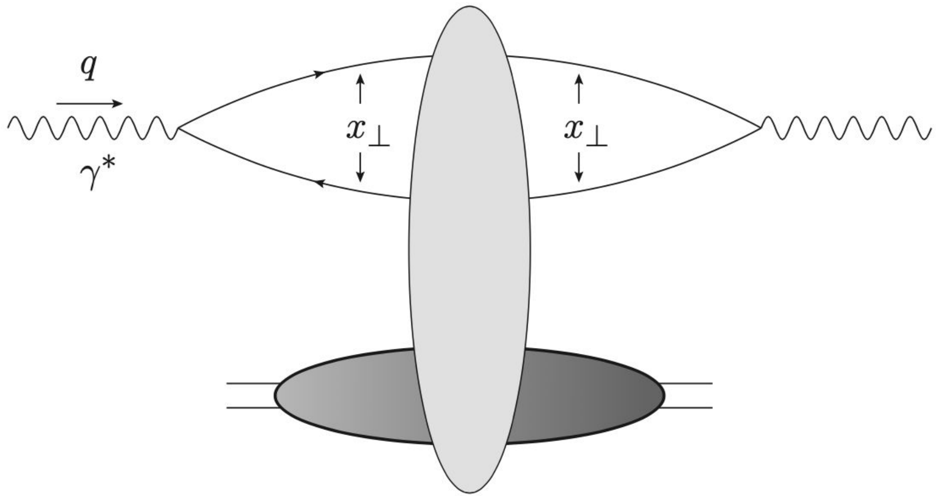

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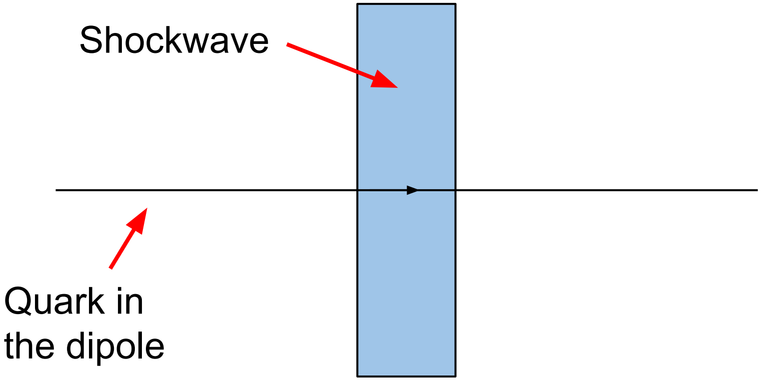
Unpolarized Dipole Amplitude

- Parton **unpolarized PDF,** Σ(x, Q^2) and G(x, Q^2), relate to **unpolarized dipole amplitude,** $S_{10}(s) = \frac{1}{N_c}\left\langle \mathrm{tr}\left[V_{\underline{1}} V_{\underline{0}}^{\dagger}\right]\right\rangle(s)$, which obeys BFKL/BK/JIMWLK evolution.
- Quark going through the shockwave at $\underline{x}_1$: unpolarized Wilson line, $V_{\underline{1}}$.
- Multiple parton exchanges at **eikonal** level (leading order in x).



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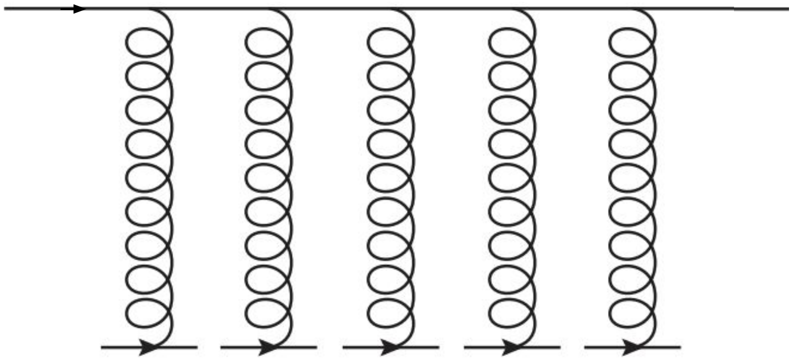


Polarized Dipole-Target Interaction

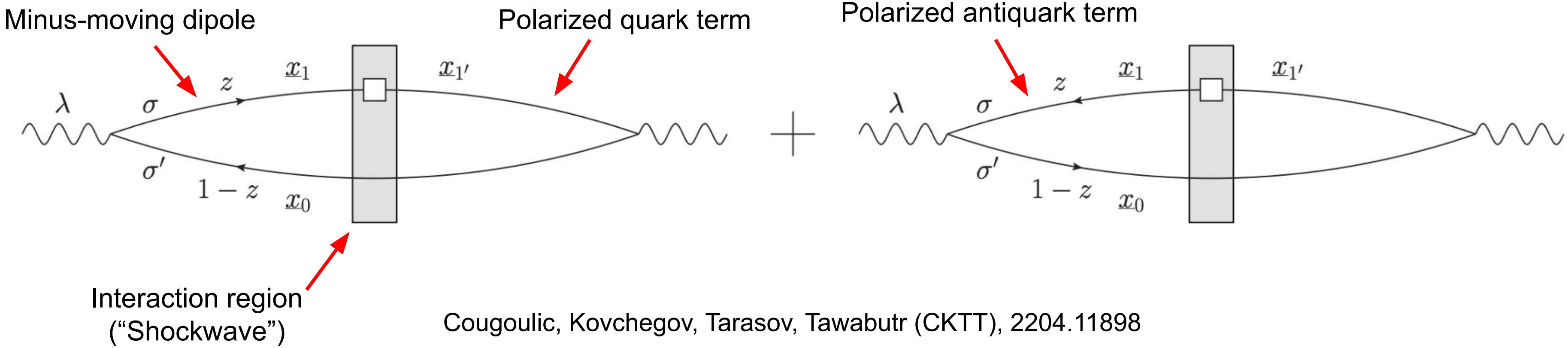
- Polarized parton PDF and structure function, $g_1(x, Q^2)$, relate to the helicity-dependent part of dipole-target scattering.

$$\hat{\sigma}_{\text{DIS}} \sim \frac{1}{x}\,\hat{\sigma}^{\text{eik}}_{\text{unpol}} + \left(\hat{\sigma}^{\text{sub-eik}}_{\text{unpol}} + \sigma S_L \cdot \hat{\sigma}^{\text{sub-eik}}_{\text{pol}}\right) + O(x)$$

σ : (anti)quark's helicity

S_L : Target's helicity

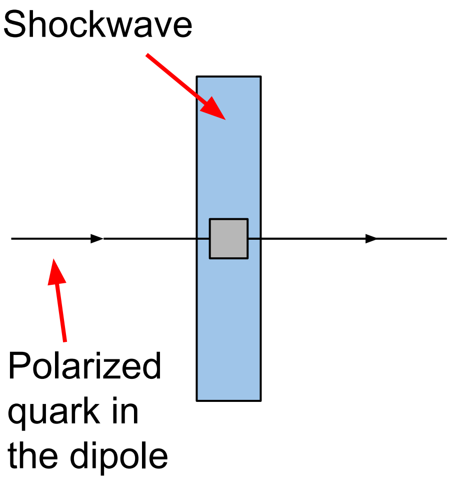
$\hat{\sigma}$: Cross section



Cougoulic, Kovchegov, Tarasov, Tawabutr (CKTT), 2204.11898

Polarized Dipole-Target Interaction

- Helicity-dependent quark line going through the shockwave corresponds to multiple eikonal parton exchanges, except for <u>**one**</u> helicity-dependent exchange, which is **sub-eikonal** (suppressed by an extra factor of x).



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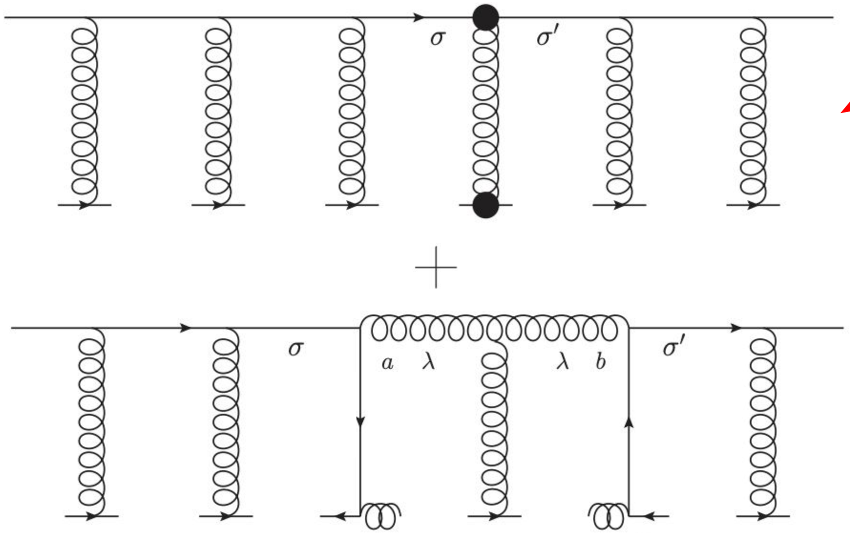


Gluon exchange term

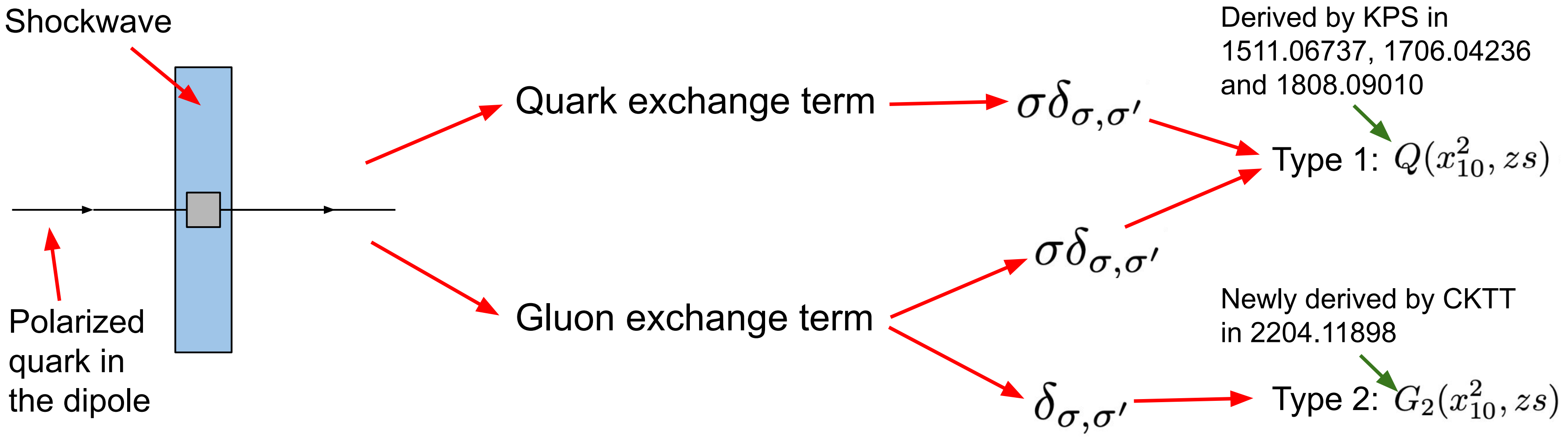
Kovchegov, Pitonyak, Sievert (KPS), 1706.04236

Quark exchange term

Kovchegov, Sievert, 1808.09010

Polarized Dipole-Target Interaction

- Each helicity-dependent interaction comes with a factor of $\sigma\delta_{\sigma,\sigma'}$ or $\delta_{\sigma,\sigma'}$.
- We group them into “polarized Wilson line” based on the spin factor.
- The trace of pol+unpol Wilson lines defines “polarized dipole amplitude.”



Relations with Helicity PDFs and g_1 Structure Function

- Through an expansion in x, **helicity PDFs,** $\Delta\Sigma(x, Q^2)$ and $\Delta G(x, Q^2)$, relate to **polarized dipole amplitudes,** $Q(x_{10}^2, zs)$ (type 1) and $G_2(x_{10}^2, zs)$ (type 2) by

$$\Delta\Sigma(x, Q^2) = -\frac{N_c N_f}{2\pi^3} \int\limits_{\Lambda^2/s}^{1} \frac{dz}{z} \int\limits_{\frac{1}{zs}}^{\min\left\{\frac{1}{zQ^2}, \frac{1}{\Lambda^2}\right\}} \frac{dx_{10}^2}{x_{10}^2} \left[Q(x_{10}^2, zs) + 2\, G_2(x_{10}^2, zs)\right]$$

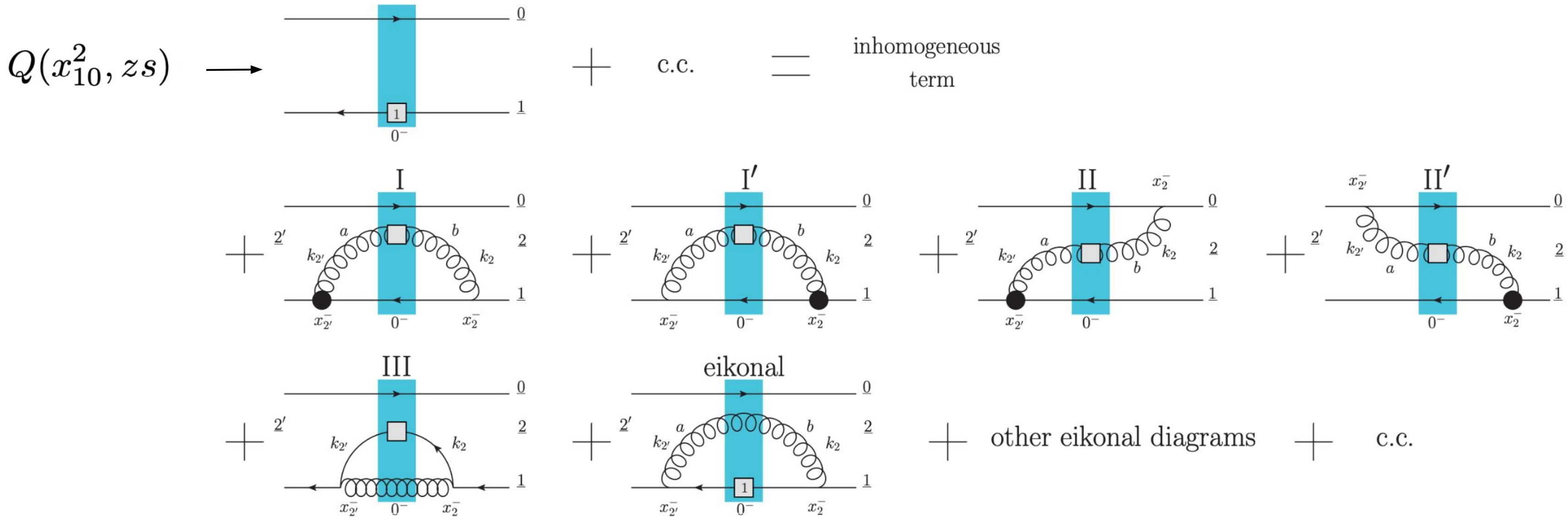
$$\Delta G(x, Q^2) = \frac{2N_c}{\alpha_s \pi^2} \left[\left(1 + x_{10}^2 \frac{\partial}{\partial x_{10}^2}\right) G_2\left(x_{10}^2, zs = \frac{Q^2}{x}\right)\right]_{x_{10}^2 = \frac{1}{Q^2}}$$

- Similarly, g_1 structure function relates to both polarized dipole amplitudes by

$$g_1(x, Q^2) = -\sum_f \frac{N_c Z_f^2}{4\pi^3} \int\limits_{\Lambda^2/s}^{1} \frac{dz}{z} \int\limits_{\frac{1}{zs}}^{\min\left\{\frac{1}{zQ^2}, \frac{1}{\Lambda^2}\right\}} \frac{dx_{10}^2}{x_{10}^2} \left[Q(x_{10}^2, zs) + 2\, G_2(x_{10}^2, zs)\right]$$

Evolution Equations

- We derive the evolution equations for quark and gluon dipoles of both types, using a technique we called **light-cone operator treatment (LCOT).**



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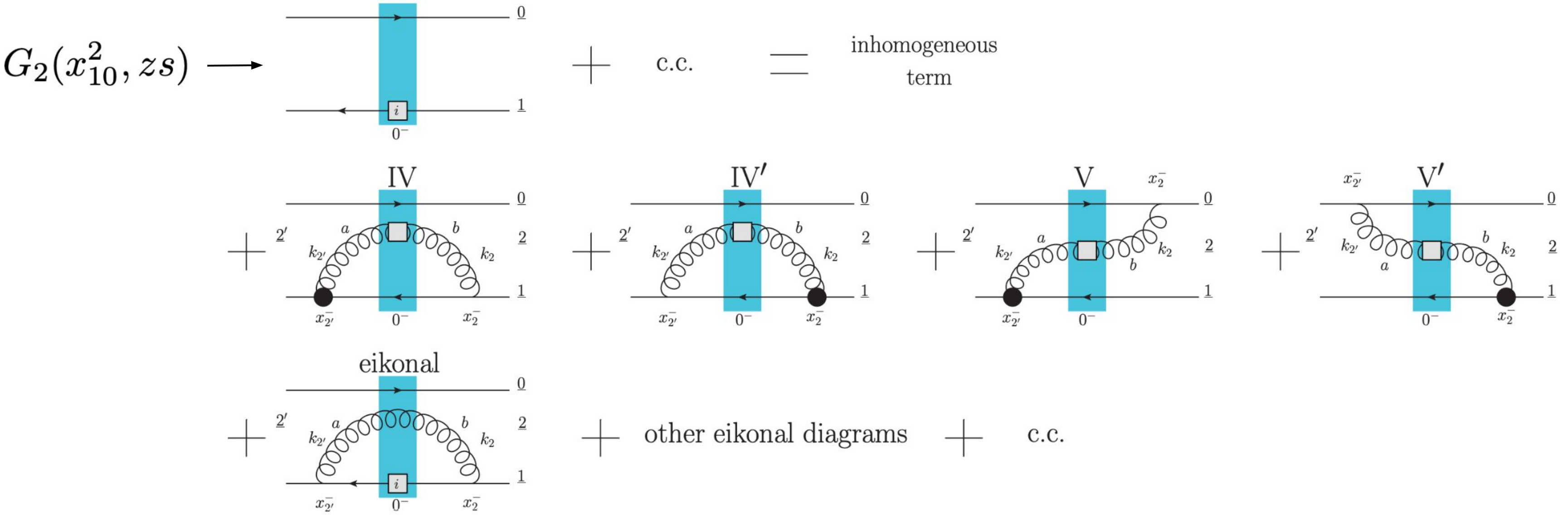
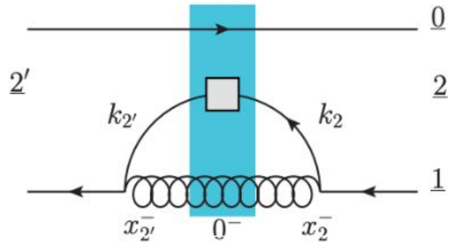


Evolution Equations

- We derive the evolution equations for quark and gluon dipoles of both types, using a technique we called **light-cone operator treatment (LCOT).**
- For all types of dipoles, the equations do not close in general.
- Similar to BK, we obtain a closed system of equations in the **large-N_c limit:**
    - $N_c \gg 1$
    - $\alpha_s N_c \ll 1$
    - Discard sub-eikonal quark-exchange terms.



Large-N_c Limit

- Define $G(x_{10}^2, zs)$ as the counterpart of $Q(x_{10}^2, zs)$, with the quark exchange term neglected.
- The equation for $G_2(x_{10}^2, zs)$ remains the same because type-2 polarized Wilson line only has gluon exchange.
- Dipole amplitudes, G and G_2, form a system of integral equations with the auxiliary **neighbor dipole amplitudes,** Γ and Γ_2.

$$\begin{pmatrix} G \\ \Gamma \\ G_2 \\ \Gamma_2 \end{pmatrix} = \begin{pmatrix} G \\ \Gamma \\ G_2 \\ \Gamma_2 \end{pmatrix}_0 + \mathcal{K} \otimes \begin{pmatrix} G \\ \Gamma \\ G_2 \\ \Gamma_2 \end{pmatrix}$$

Large-N$_c$ Limit

$$G(x_{10}^2, zs) = G^{(0)}(x_{10}^2, zs) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{sx_{10}^2}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z's}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[\Gamma(x_{10}^2, x_{21}^2, z's) + 3\, G(x_{21}^2, z's) + 2\, G_2(x_{21}^2, z's) + 2\, \Gamma_2(x_{10}^2, x_{21}^2, z's) \right]$$

Type-1 polarized dipole amplitude
(without quark exchange term)

$$\Gamma(x_{10}^2, x_{21}^2, z's) = G^{(0)}(x_{10}^2, z's) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{sx_{10}^2}}^{z'} \frac{dz''}{z''} \int\limits_{\frac{1}{z''s}}^{\min\left[x_{10}^2, x_{21}^2 \frac{z'}{z''}\right]} \frac{dx_{32}^2}{x_{32}^2} \left[\Gamma(x_{10}^2, x_{32}^2, z''s) + 3\, G(x_{32}^2, z''s) + 2\, G_2(x_{32}^2, z''s) + 2\, \Gamma_2(x_{10}^2, x_{32}^2, z''s) \right]$$

$$G_2(x_{10}^2, zs) = G_2^{(0)}(x_{10}^2, zs) + \frac{\alpha_s N_c}{\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int\limits_{\max\left[x_{10}^2, \frac{1}{z's}\right]}^{\min\left[\frac{z}{z'} x_{10}^2, \frac{1}{\Lambda^2}\right]} \frac{dx_{21}^2}{x_{21}^2} \left[G(x_{21}^2, z's) + 2\, G_2(x_{21}^2, z's) \right]$$

Type-2 polarized dipole amplitude

$$\Gamma_2(x_{10}^2, x_{21}^2, z's) = G_2^{(0)}(x_{10}^2, z's) + \frac{\alpha_s N_c}{\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z' \frac{x_{21}^2}{x_{10}^2}} \frac{dz''}{z''} \int\limits_{\max\left[x_{10}^2, \frac{1}{z''s}\right]}^{\min\left[\frac{z'}{z''} x_{21}^2, \frac{1}{\Lambda^2}\right]} \frac{dx_{32}^2}{x_{32}^2} \left[G(x_{32}^2, z''s) + 2\, G_2(x_{32}^2, z''s) \right]$$

Initial condition: deduced from experimental data

Large-N_c Results

- At high center-of-mass energy,

$$G(x_{10}^2, zs) \sim Q(x_{10}^2, zs) \sim G_2(x_{10}^2, zs) \sim (zsx_{10}^2)^{3.66\sqrt{\frac{\alpha_s N_c}{2\pi}}}$$

- With

$$\Delta\Sigma(x, Q^2) = -\frac{N_c N_f}{2\pi^3} \int\limits_{\Lambda^2/s}^{1} \frac{dz}{z} \int\limits_{\frac{1}{zs}}^{\min\left\{\frac{1}{zQ^2}, \frac{1}{\Lambda^2}\right\}} \frac{dx_{10}^2}{x_{10}^2} \left[Q(x_{10}^2, zs) + 2\,G_2(x_{10}^2, zs)\right]$$

$$\Delta G(x, Q^2) = \frac{2N_c}{\alpha_s \pi^2} \left[\left(1 + x_{10}^2 \frac{\partial}{\partial x_{10}^2}\right) G_2\left(x_{10}^2, zs = \frac{Q^2}{x}\right)\right]_{x_{10}^2 = \frac{1}{Q^2}}$$

$$g_1(x, Q^2) = -\sum_f \frac{N_c Z_f^2}{4\pi^3} \int\limits_{\Lambda^2/s}^{1} \frac{dz}{z} \int\limits_{\frac{1}{zs}}^{\min\left\{\frac{1}{zQ^2}, \frac{1}{\Lambda^2}\right\}} \frac{dx_{10}^2}{x_{10}^2} \left[Q(x_{10}^2, zs) + 2\,G_2(x_{10}^2, zs)\right]$$

Previous version:
Quarks: 2.31
Gluons: 1.88

We conclude that $\Delta\Sigma(x, Q^2) \sim \Delta G(x, Q^2) \sim g_1(x, Q^2) \sim \left(\frac{1}{x}\right)^{3.66\sqrt{\frac{\alpha_s N_c}{2\pi}}}$

- This result **agrees** with previous work [Bartels, Ermolaev, Ryskin, 9603204] that uses a different technique, in both quark and gluon helicity PDFs at large N_c .

Conclusion

- Small-x helicity evolution has been revised to include additional contribution from a term of “type 2” in helicity PDF.
- The evolution equations resum $\alpha_s \ln^2(1/x)$. They do not close in general, but form closed systems of equations at large-N_c and large-N_c&N_f limits.
- We numerically solved the equations at large N_c, obtaining

$$\Delta\Sigma(x,Q^2) \sim \Delta G(x,Q^2) \sim g_1(x,Q^2) \sim \left(\frac{1}{x}\right)^{3.66\sqrt{\frac{\alpha_s N_c}{2\pi}}}$$

in agreement with the previous results by Bartels et al.

Future Work

- Obtain the numerical solution at large N_c&N_f , c.f. [Kovchegov, Tawabutr, 2005.07285].
- Examine the quality of fits to the helicity world data, given this small-x evolution, c.f. [Adamiak et al, 2102.06159].
- Derive the single-logarithmic (SLA) corrections, resumming $\alpha_s \ln(1/x)$, which was initially studied in [Kovchegov, Tarasov, Tawabutr, 2104.11765].
- Revise the helicity JIMWLK equation, c.f. [Cougoulic, Kovchegov, 1910.04268].
- Revise the small-x OAM evolution, c.f. [Kovchegov, 1901.07453].

Bonus Slides

Crosscheck with Polarized DGLAP

- We iterate the large-N$_c$ equations to order α_s^3, starting with initial condition:

$$G^{(0)}(x_{10}^2, zs) = 0, \quad G_2^{(0)}(x_{10}^2, z's) = 1$$

- At each order in α_s, the DLA terms **agree completely** with what one would get starting from the corresponding initial condition, $\Delta G^{(0)}(x, Q^2) = \mathrm{const}$, and evolving with a large-N$_c$ polarized DGLAP evolution in the gluon sector:

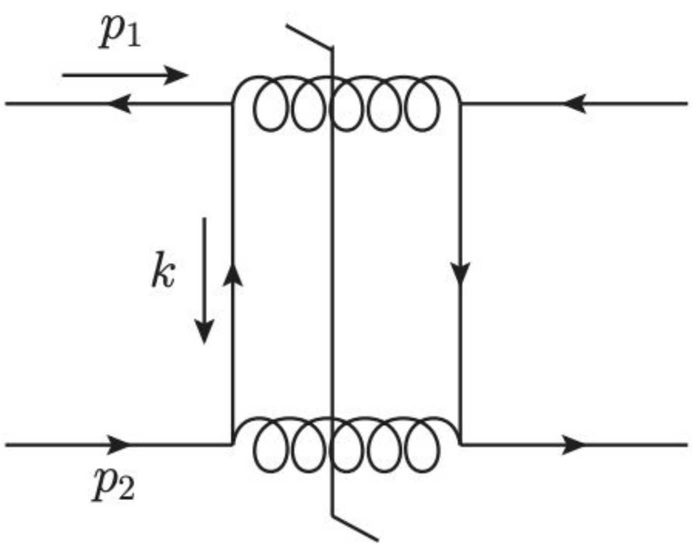
$$\frac{\partial \Delta G(x, Q^2)}{\partial \ln Q^2} = \int\limits_x^1 \frac{dz}{z} \, \Delta P_{GG}(z) \, \Delta G\left(\frac{x}{z}, Q^2\right)$$

with the small-x gluon-gluon splitting function:

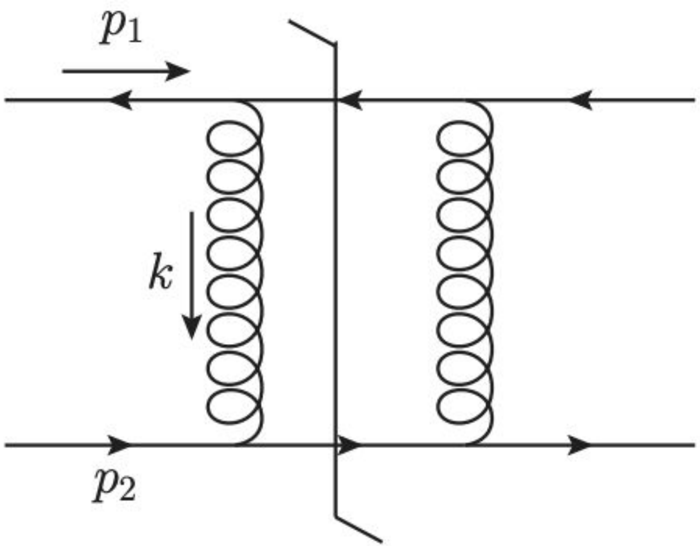
$$\Delta P_{GG}(z) = \frac{\alpha_s}{2\pi} 4N_c + \left(\frac{\alpha_s}{2\pi}\right)^2 4N_c^2 \ln^2 z + \left(\frac{\alpha_s}{2\pi}\right)^3 \frac{7}{3} N_c^3 \ln^4 z + \dots .$$

- With more iterations, this method allows for DGLAP crosscheck at all orders.

Born Level Amplitudes



Quark exchange term



Gluon exchange term

Polarized Dipole-Target Interaction

- Polarized quark line also corresponds to **polarized Wilson line.**

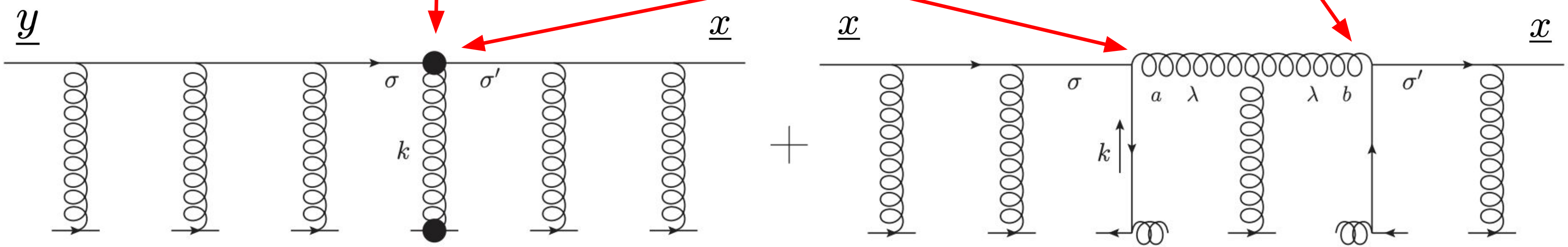
$$V_{\underline{x},\underline{y};\sigma',\sigma}\Big|_{\text{sub-eikonal}} = \sigma\delta_{\sigma,\sigma'}V_{\underline{x}}^{\text{pol}[1]}\delta^2(\underline{x}-\underline{y}) + \delta_{\sigma,\sigma'}V_{\underline{x},\underline{y}}^{\text{pol}[2]}$$

Type-1 polarized Wilson line:

$$V_{\underline{x}}^{\text{pol}[1]} = V_{\underline{x}}^{\text{G}[1]} + V_{\underline{x}}^{\text{q}[1]}$$

Type-2 polarized Wilson line:

$$V_{\underline{x},\underline{y}}^{\text{pol}[2]} = V_{\underline{x},\underline{y}}^{\text{G}[2]} + V_{\underline{x}}^{\text{q}[2]}\delta^2(\underline{x}-\underline{y})$$



Kovchegov, Sievert, 1808.09010

Type-1 Polarized Wilson Line

$$Q(x_{10}^2, zs) = \frac{zs}{2N_c} \int d^2\left(\frac{\underline{x}_0 + \underline{x}_1}{2}\right) \mathrm{Re}\left\langle \mathrm{T\,tr}\left[V_{\underline{0}} V_{\underline{1}}^{\mathrm{pol}[1]\,\dagger}\right] + \mathrm{T\,tr}\left[V_{\underline{1}}^{\mathrm{pol}[1]} V_{\underline{0}}^{\dagger}\right]\right\rangle$$

$$\Big\langle \dots \Big\rangle \equiv \frac{1}{2}\sum_{S_L} S_L \, \frac{1}{2P^+V^-} \, \langle P, S_L | \dots | P, S_L \rangle$$

$$V_{\underline{x}}^{\mathrm{pol}[1]} = V_{\underline{x}}^{\mathrm{G}[1]} + V_{\underline{x}}^{\mathrm{q}[1]}$$

Target's longitudinal spin

$$V_{\underline{x}}^{\mathrm{G}[1]} = \frac{i\,g\,P^+}{s} \int\limits_{-\infty}^{\infty} dx^- V_{\underline{x}}[\infty, x^-]\, F^{12}(x^-, \underline{x})\; V_{\underline{x}}[x^-, -\infty]$$

Kovchegov, Sievert, 1808.09010;
Chirilli, 1807.11435, 2101.12744;
Altinoluk et al, 2012.03886

$$\vec{\mu}\cdot\vec{B}$$

$$V_{\underline{x}}^{\mathrm{q}[1]} = \frac{g^2 P^+}{2\,s} \int\limits_{-\infty}^{\infty} dx_1^- \int\limits_{x_1^-}^{\infty} dx_2^- \, V_{\underline{x}}[\infty, x_2^-]\, t^b\, \psi_\beta(x_2^-, \underline{x})\, U_{\underline{x}}^{ba}[x_2^-, x_1^-]\, \left[\gamma^+\gamma^5\right]_{\alpha\beta}\, \bar{\psi}_\alpha(x_1^-, \underline{x})\, t^a\, V_{\underline{x}}[x_1^-, -\infty]$$

Axial current

Type-2 Polarized Wilson Line

$$G_2(x_{10}^2, zs) = \frac{\epsilon^{ij}(x_{10})^j_\perp}{x_{10}^2} \int d^2\left(\frac{x_0 + x_1}{2}\right) \frac{zs}{2N_c} \left\langle \mathrm{tr}\left[V_{\underline{0}}^\dagger V_{\underline{1}}^{i\,\mathrm{G}[2]} + \left(V_{\underline{1}}^{i\,\mathrm{G}[2]}\right)^\dagger V_{\underline{0}}\right]\right\rangle$$

$$\left\langle \ldots \right\rangle \equiv \frac{1}{2}\sum_{S_L} S_L \frac{1}{2P^+V^-} \langle P, S_L| \ldots |P, S_L\rangle$$

Target's longitudinal spin

Can be written in term of $V_{\underline{x},\underline{y}}^{\mathrm{G}[2]}$, which is the gluon exchange term in type-2 polarized Wilson line

$$V_{\underline{z}}^{i\,\mathrm{G}[2]} = \frac{P^+}{2s} \int_{-\infty}^{\infty} dz^- \, V_{\underline{z}}[\infty, z^-] \left[D^i(z^-, \underline{z}) - \overleftarrow{D}^i(z^-, \underline{z})\right] V_{\underline{z}}[z^-, -\infty]$$

$$V_{\underline{x},\underline{y}}^{\mathrm{G}[2]} = -\frac{i\,P^+}{s} \int_{-\infty}^{\infty} dz^- d^2z \; V_{\underline{x}}[\infty, z^-]\, \delta^2(\underline{x} - \underline{z})\, \overleftarrow{D}^i(z^-, \underline{z})\, D^i(z^-, \underline{z})\, V_{\underline{y}}[z^-, -\infty]\, \delta^2(\underline{y} - \underline{z})$$

Altinoluk et al, 2012.03886; Kovchegov, Santiago, 2108.03667; Cougoulic, Kovchegov, Tarasov, Tawabutr, 2204.11898

Numerical Computation: Setup

DLA pre-factor

- Rewrite the large-N$_c$ evolution equations in terms of

$$s_{10} = \sqrt{\frac{\alpha_s N_c}{2\pi}} \ln \frac{1}{x_{10}^2 \Lambda^2} \quad , \quad s_{21} = \sqrt{\frac{\alpha_s N_c}{2\pi}} \ln \frac{1}{x_{21}^2 \Lambda^2} \quad , \quad \eta = \sqrt{\frac{\alpha_s N_c}{2\pi}} \ln \frac{zs}{\Lambda^2} \quad , \quad \eta' = \sqrt{\frac{\alpha_s N_c}{2\pi}} \ln \frac{z's}{\Lambda^2}$$

- For example, the G$_2$-equation becomes

$$G_2(x_{10}^2, zs) = G_2^{(0)}(x_{10}^2, zs) + \frac{\alpha_s N_c}{\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int\limits_{\max\left[x_{10}^2, \frac{1}{z's}\right]}^{\min\left[\frac{z}{z'}x_{10}^2, \frac{1}{\Lambda^2}\right]} \frac{dx_{21}^2}{x_{21}^2} \left[G(x_{21}^2, z's) + 2\, G_2(x_{21}^2, z's)\right]$$

$$G_2(s_{10}, \eta) = G_2^{(0)}(s_{10}, \eta) + 2 \int\limits_0^{s_{10}} ds_{21} \int\limits_{s_{21}}^{\eta - s_{10} + s_{21}} d\eta' \left[G(s_{21}, \eta') + 2\, G_2(s_{21}, \eta')\right]$$

Numerical Computation: Setup

- Discretize the equations using left-hand Riemann sum, with step size δ.

  Define $G_{ij} = G\left(i\delta, j\delta\right)$ and $G_{2,ij} = G_2\left(i\delta, j\delta\right)$.

- For example, the G_2-equation becomes

$$G_2(s_{10},\eta) = G_2^{(0)}(s_{10},\eta) + 2\int\limits_0^{s_{10}} ds_{21} \int\limits_{s_{21}}^{\eta - s_{10} + s_{21}} d\eta' \, \left[G(s_{21},\eta') + 2\, G_2(s_{21},\eta')\right]$$

$$\Big\downarrow$$

$$G_{2,ij} = G_{2,ij}^{(0)} + 2\,\delta^2 \sum_{i'=0}^{i-1} \sum_{j'=i'}^{j-i+i'} \left[G_{i'j'} + 2\, G_{2,i'j'}\right]$$

Numerical Computation: Setup

$$G_{ij} = G_{ij}^{(0)} + \delta^2 \sum_{j'=i}^{j-1} \sum_{i'=i}^{j'} \left[\Gamma_{ii'j'} + 3\,G_{i'j'} + 2\,G_{2,i'j'} + 2\,\Gamma_{2,ii'j'}\right],$$

$$\Gamma_{ikj} = G_{ij}^{(0)} + \delta^2 \sum_{j'=i}^{j-1} \sum_{i'=\max[i,\,k+j'-j]}^{j'} \left[\Gamma_{ii'j'} + 3\,G_{i'j'} + 2\,G_{2,i'j'} + 2\,\Gamma_{2,ii'j'}\right],$$

$$G_{2,ij} = G_{2,ij}^{(0)} + 2\,\delta^2 \sum_{i'=0}^{i-1} \sum_{j'=i'}^{j-i+i'} \left[G_{i'j'} + 2\,G_{2,i'j'}\right],$$

$$\Gamma_{2,ikj} = G_{2,ij}^{(0)} + 2\,\delta^2 \sum_{i'=0}^{i-1} \sum_{j'=i'}^{j-k+i'} \left[G_{i'j'} + 2\,G_{2,i'j'}\right].$$

Numerical Computation: Setup

- Write each equation in a recursive form to save computation time.

$$G_{ij} = \begin{cases} G^{(0)}_{ij} - G^{(0)}_{i(j-1)} + G_{i(j-1)} + \delta^2 \sum_{i'=i}^{j-1} \left[\Gamma_{ii'(j-1)} + 3\, G_{i'(j-1)} + 2\, G_{2,i'(j-1)} + 2\, \Gamma_{2,ii'(j-1)}\right] & , \quad i < j \\ G^{(0)}_{ij} & , \quad i = j \end{cases}$$

$$\Gamma_{ikj} = \begin{cases} G^{(0)}_{ij} - G^{(0)}_{i(j-1)} + \Gamma_{i(k-1)(j-1)} + \delta^2 \sum_{i'=k-1}^{j-1} \left[\Gamma_{ii'(j-1)} + 3\, G_{i'(j-1)} + 2\, G_{2,i'(j-1)} + 2\, \Gamma_{2,ii'(j-1)}\right] & , \quad i < k \\ G_{ij} & , \quad i = k \end{cases}$$

$$G_{2,ij} = \begin{cases} G^{(0)}_{2,ij} - G^{(0)}_{2,i(j-1)} + G_{2,i(j-1)} + 2\, \delta^2 \sum_{i'=0}^{i-1} \left[G_{i'(i'+j-i)} + 2\, G_{2,i'(i'+j-i)}\right] & , \quad i < j \\ G^{(0)}_{2,ij} & , \quad i = j \end{cases}$$

$$\Gamma_{2,ikj} = \begin{cases} G^{(0)}_{2,ij} - G^{(0)}_{2,i(j-1)} + \Gamma_{2,i(k-1)(j-1)} & , \quad i < k \\ G_{2,ij} & , \quad i = k \end{cases}$$

Numerical Computation: Method

$$G^{(0)}(x_{10}^2, zs) = \frac{\alpha_s^2 C_F}{2N_c}\pi \left[C_F \ln\frac{zs}{\Lambda^2} - 2\ln(zsx_{10}^2)\right] \xrightarrow{\text{approxim}} G_{ij}^{(0)} = \frac{\alpha_s^2 C_F}{2N_c}\pi\sqrt{\frac{2\pi}{\alpha_s N_c}}\left[(C_F - 2)\,j + 2\,i\right]\delta$$

$$G_2^{(0)}(x_{10}^2, zs) = \frac{\alpha_s^2 C_F}{N_c}\pi \ln\frac{1}{x_{10}\Lambda} \longrightarrow G_{2,ij}^{(0)} = \frac{\alpha_s^2 C_F}{2N_c}\pi\sqrt{\frac{2\pi}{\alpha_s N_c}}\, i\delta$$

- Approximately, we take G and G_2 to be the initial condition at $i = j \Leftrightarrow x \sim 1$.
- For each trial, besides the step size, δ, we also specify the maximum rapidity, η_{max}, up to which we run the numerical computation.
- Starting from $j = 1$, compute all the amplitudes at this value of j using the results from lower j's. Repeat for increasing j until $j = j_{\max} = \frac{\eta_{\max}}{\delta}$.
- For each j, we only need the results at 0 ≤ i < j.

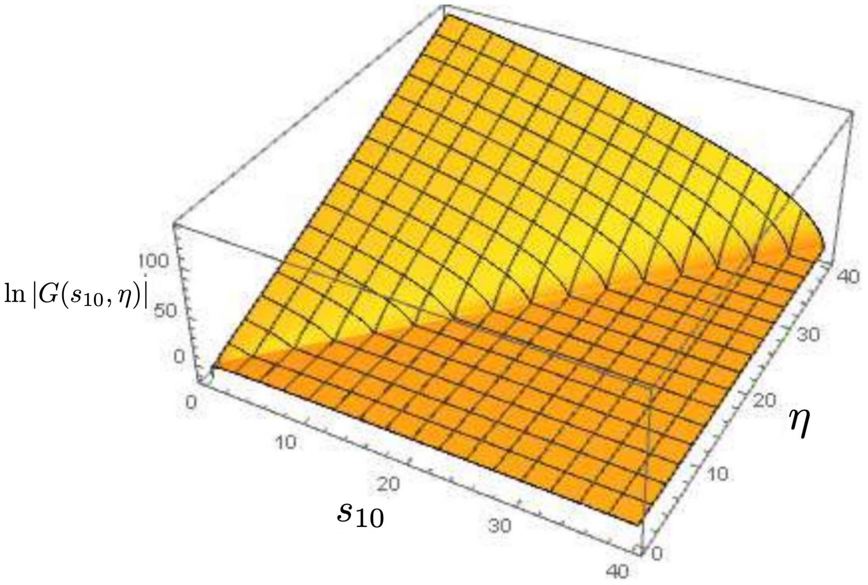
Infrared cutoff: $x_{10}^2 \ll \frac{1}{\Lambda^2}$ $\qquad x \ll 1$

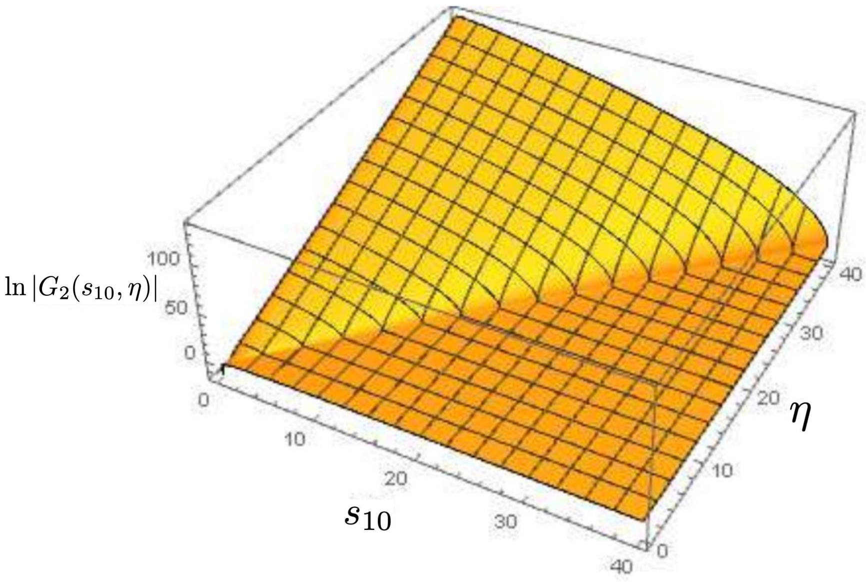
Numerical Computation: Results

$$\eta = \sqrt{\frac{\alpha_s N_c}{2\pi}} \ln \frac{zs}{\Lambda^2}$$

$$s_{10} = \sqrt{\frac{\alpha_s N_c}{2\pi}} \ln \frac{1}{x_{10}^2 \Lambda^2}$$

For example, at $\delta = 0.05$ and $\eta_{\max} = 40$, we have





Cougoulic, Kovchegov, Tarasov, Tawabutr, 2204.11898

Numerical Computation: Results

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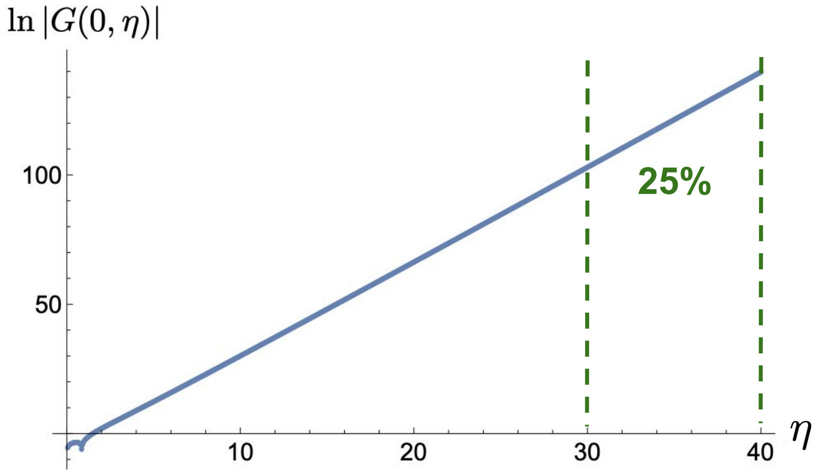

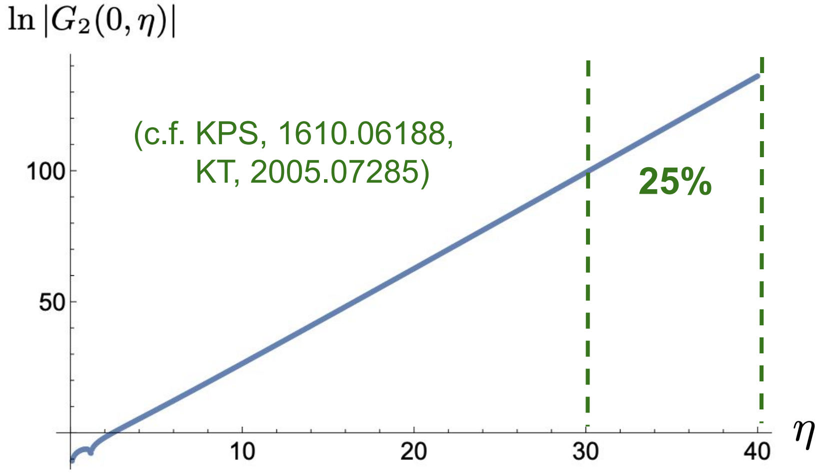

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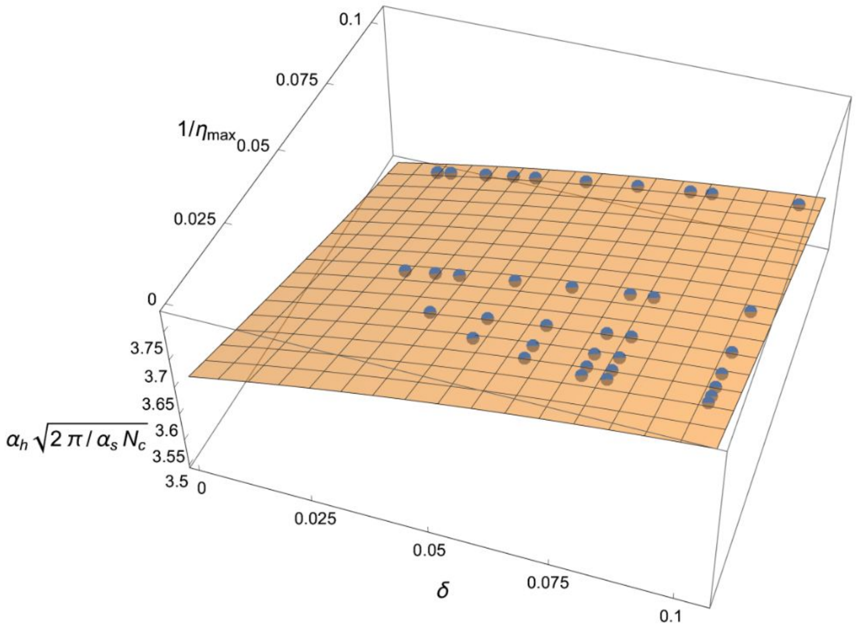
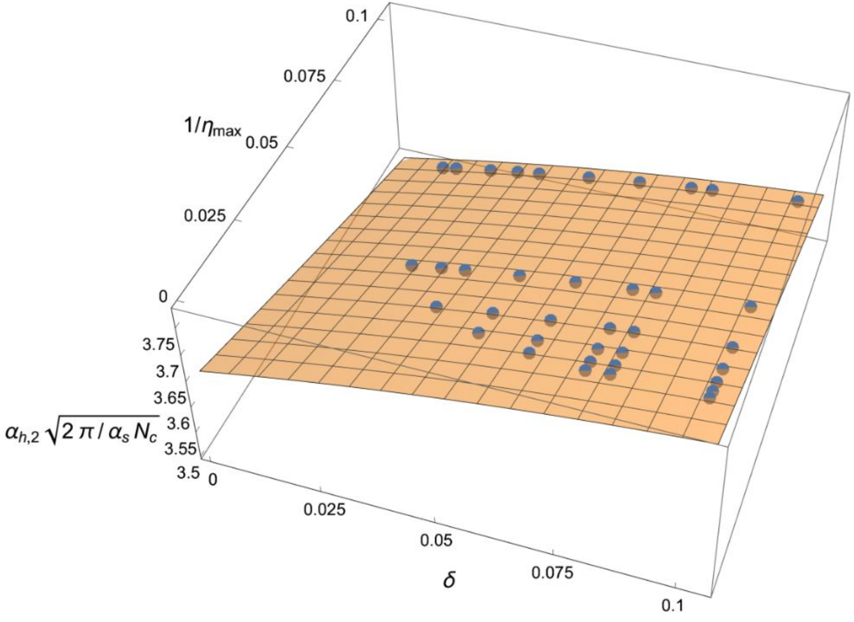
$$G(s_{10} = 0, \eta) \sim e^{\alpha_h \eta \sqrt{\frac{2\pi}{\alpha_s N_c}}}$$

$$G_2(s_{10} = 0, \eta) \sim e^{\alpha_{h,2} \eta \sqrt{\frac{2\pi}{\alpha_s N_c}}}$$

Continuum Limit

- We generally obtain different values of α_h and $\alpha_{h,2}$ with different δ and η_{max}.





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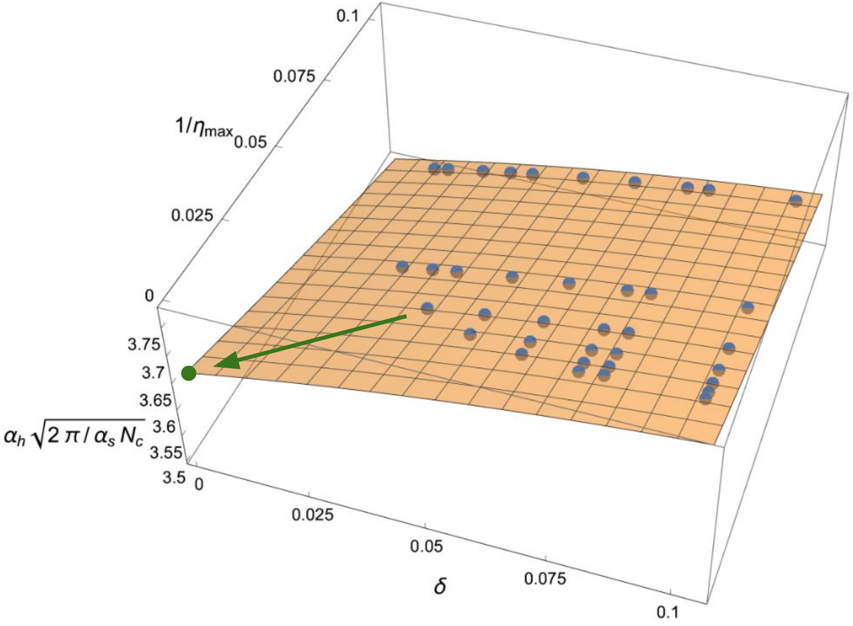

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- We fit the resulting α_h and $\alpha_{h,2}$ against δ and $1/\eta_{max}$ using polynomial regression.
- Quadratic model with interaction term fits the best based on AIC.
- The constant term gives the estimate for α_h and $\alpha_{h,2}$ at $\delta = 1/\eta_{max} = 0$, i.e. continuum limit.

(c.f. Kovchegov, Pitonyak, Sievert, 1610.06188)

Final Large-N_c Results

- At continuum limit, $\delta = 1/\eta_{max} \to 0$, the results are

$$\alpha_h = (3.661 \pm 0.006)\sqrt{\frac{\alpha_s N_c}{2\pi}}, \quad \alpha_{h,2} = (3.660 \pm 0.009)\sqrt{\frac{\alpha_s N_c}{2\pi}}$$

- The uncertainty comes from
  - Linear regression to determine the slopes of $\ln|G(0,\eta)|$ and $\ln|G_2(0,\eta)|$ for each δ and η_{max}.
  - Polynomial regression to determine α_h or $\alpha_{h,2}$ at continuum limit.
- With

$$\Delta\Sigma(x,Q^2) = -\frac{N_c N_f}{2\pi^3}\int\limits_{\Lambda^2/s}^{1}\frac{dz}{z}\int\limits_{\frac{1}{zs}}^{\min\left\{\frac{1}{zQ^2},\frac{1}{\Lambda^2}\right\}}\frac{dx_{10}^2}{x_{10}^2}\left[Q(x_{10}^2,zs) + 2\,G_2(x_{10}^2,zs)\right]$$

$$\Delta G(x,Q^2) = \frac{2N_c}{\alpha_s\pi^2}\left[\left(1 + x_{10}^2\frac{\partial}{\partial x_{10}^2}\right)G_2\left(x_{10}^2, zs = \frac{Q^2}{x}\right)\right]_{x_{10}^2 = \frac{1}{Q^2}}$$

We conclude that

$$\Delta\Sigma(x,Q^2) \sim \Delta G(x,Q^2) \sim \left(\frac{1}{x}\right)^{3.66\sqrt{\frac{\alpha_s N_c}{2\pi}}}$$

Previous version:
Quarks: 2.31
Gluons: 1.88

Kovchegov, Pitonyak, Sievert, 1610.06188, 1703.05809, 1706.04236; Chirilli, 1807.11435, 2101.12744;