Heavy quarks on the lattice at T>0

Peter Petreczky

Brookhaven National Laboratory

- Heavy quarkonium, non-relativistic potential picture and effective field theory approach (NRQCD, pNRQCD) and quarkonium melting
- NRQCD with extended meson operators and bottomonium properties at $T>0$
- Complex potential at $T>0$
- Heavy quark diffusion coefficient from lattice QCD

RBRC Workshop: Predictions for sPHENIX, *BNL, July 20-22, 2022*

Quarkonia and potential models

$m_b, m_c \gg \Lambda_{QCD}$ $\Rightarrow$ non-relativistic bound states, analogs QED positronium

1-gluon exchange, $\alpha_s \sim 0.4$

$$V(r) = -\frac{4}{3}\frac{\alpha_s}{r} + \sigma r + spin\ dep.$$

Confinement

Eichten et al, PRL 34 (75) 369, PRD 21 (80) 203

Very successful in describing charmonium and bottomonium spectrum below the the open charm and beauty threshold

Nevertheless nearly perfect agreement between the phenomenological and lattice potentials

Problems:

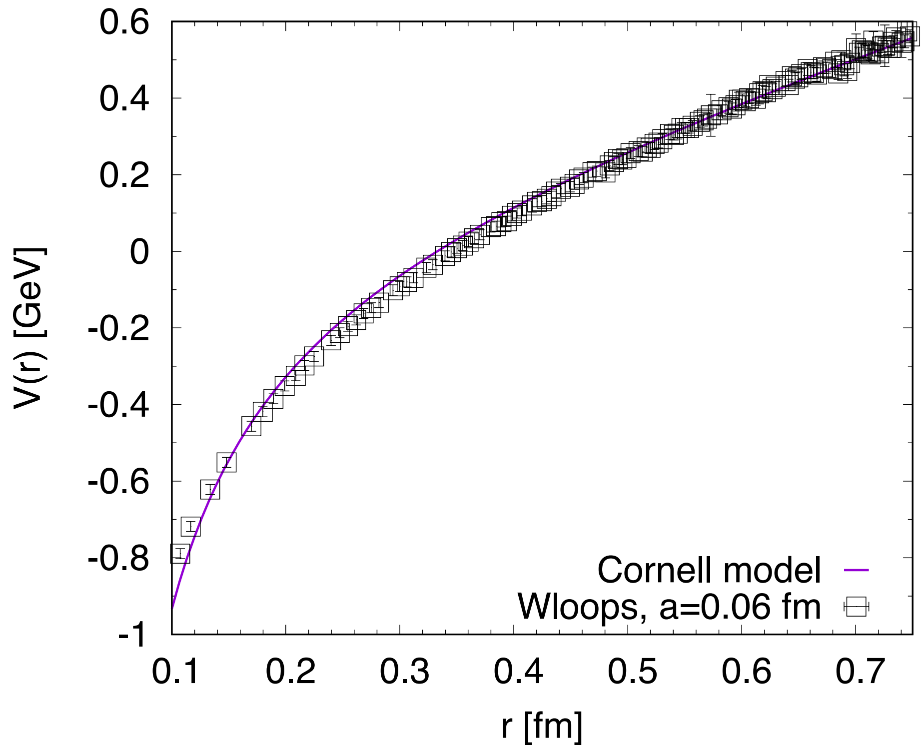
Running of α_s ?

Linear potential valid only for $r \gg 1$ fm,

$$V(r) = \sigma r - \frac{\pi}{12 r} + \ldots$$

Lüscher term

Soft gluon fields ??



Quarkonia in effective theory approach

$M \gg 1/r \sim Mv \gg Mv^2, \; M = m_{c,b}$ ➡ Effective theory (EFT) approach

Non-relativistic QCD (NRQCD) : EFT at scale $1/r$ (scale M is integrated out):

$$L_{NRQCD} = \psi^\dagger \left(iD_0 - \frac{D_i^2}{2M} \right) \psi + \chi^\dagger \left(iD_0 + \frac{D_i^2}{2M} \right) \chi + ... + \frac{1}{4} F_{\mu\nu}^2 + \bar{q}\gamma_\mu D_\mu q$$

Heavy quark fields are Pauli spinors, heavy pair creation is only present implicitly through higher dimension 4-fermion operators

Caswell, Lepage, PLB 167 (86) 437

potential NRQCD (pNRQCD): EFT at scale $E_{bin} \sim Mv^2$ (scale $1/r \sim Mv$ is integrated out):

$$L_{pNRQCD} = \int d^3\mathbf{r} \mathrm{Tr}\Big[S^\dagger \left[i\partial_0 - \left(\frac{-\nabla_r^2}{M} + V_s(r) + \ldots \right) \right] S + O^\dagger \left[iD_0 + \frac{-\nabla_r^2}{M} + V_o(r) + \ldots \right) \Big] O \Big]$$

$$+ V_A(r) \mathrm{Tr}\Big[O^\dagger \mathbf{r} g \mathbf{E} S + S^\dagger \mathbf{r} g \mathbf{E} O \Big] + V_B(r) \mathrm{Tr}\Big[O^\dagger \mathbf{r} g \mathbf{E} O + O^\dagger O \mathbf{r} g \mathbf{E} \Big] +$$

$$\mathcal{O}\left(r^2, \frac{1}{M}\right) + \frac{1}{4} F_{\mu\nu}^2 + \bar{q}\gamma_\mu D_\mu q$$

Brambilla, Pineda, Soto, Vairo, NPB 566 (00) 275

$$S = S(\mathbf{r}, \mathbf{R}, t), \; O = O(\mathbf{r}, \mathbf{R}, t), \; E = E(\mathbf{R}, t)$$

Potentials are parameters of the EFT Largangian

Tree level $\leftrightarrow$ potential model $\left(i\partial_0 + \frac{\nabla_r^2}{M} - V_s(r) \right) S(\mathbf{r}, \mathbf{R}, t) = 0$

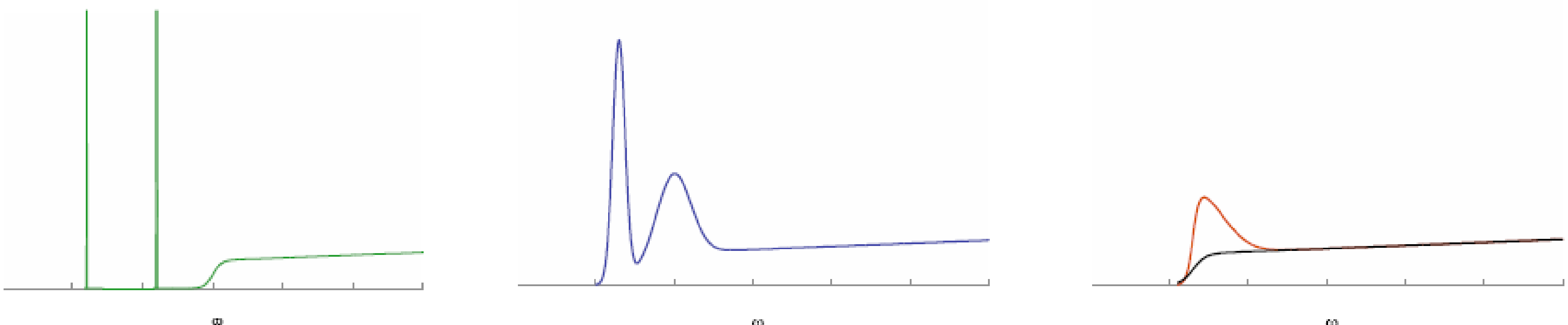
Meson correlators and spectral functions

Vacuum and in-medium properties as well as dissolution of mesons are encoded in the spectral functions:

$$\rho(\omega, p, T) = \frac{1}{2\pi} \mathrm{Im} \int_{-\infty}^{\infty} dt e^{i\omega t} \int d^3x e^{ipx} \langle [O(x,t), O(0,0)] \rangle_T, \quad O(x,t) \sim \bar{Q}(x,t) \Gamma Q(x,t)$$

Melting is seen as progressive broadening and disappearance of the bound state peaks

Modifications of quarkonium yields in heavy ion collisions   Matsui and Satz, PLB 178 (1986) 416



$$C(\tau, T) = \sum_x \langle O(x,\tau) O(0,0) \rangle_T \quad \longleftrightarrow \quad C(\tau, T) = \int_{-\infty}^{+\infty} d\omega \rho(\omega, T) e^{-\tau\omega}$$

Consider large τ behavior of $C(\tau, T=0)$:

$$C(\tau, T) \sim \sum_n |\langle 0|O|n\rangle|^2 e^{-M_n \tau} \simeq f_1 e^{-M_1 \tau} + f_2 e^{-M_2 \tau} + \ldots$$

$T > 0: \quad \tau < 1/T \Rightarrow$ reconstruct $\rho(\omega, T)$

NRQCD meson correlators

Point correlators:

Aarts et al (FASTUM) , Kim, PP, Rothkopf

$$C_p(t) = \sum_{\mathbf{x}} \langle O_p(t, \mathbf{x}) O_p(0, \mathbf{0}) \rangle,$$

$$O_p(t, \mathbf{x}) = \chi^\dagger(t, \mathbf{x}) \Gamma \psi(t, \mathbf{x}))$$

State	Irrep Λ^{PC}	Γ
η_b	A_1^{-+}	1
Υ	T_1^{--}	σ_j
h_b	T_1^{+-}	∇_j
χ_{b0}	A_1^{++}	$\boldsymbol{\sigma} \cdot \boldsymbol{\nabla}$
χ_{b1}	T_1^{++}	$(\boldsymbol{\sigma} \times \boldsymbol{\nabla})_j$
χ_{b2}	T_2^{++}	$\sigma_j \nabla_k + \sigma_k \nabla_j$

Extended correlators:

$$O_p(t, \mathbf{x}) \to O(t, \mathbf{x}) = \sum_{\mathbf{r}} \Psi(\mathbf{r}) \chi^\dagger(t, \mathbf{x}) \Gamma \psi(t, \mathbf{x} + \mathbf{r}))$$

$\Psi(\mathbf{r}) \sim e^{-|\mathbf{r}|^2/\sigma^2}$

or realistic wave-function

Optimized correlators: use several different extended meson operators with realistic wave functions and form orthogonal combinations

$$O_i \to \tilde{O}_\alpha = \Omega_{\alpha j} O_j, \ \langle \tilde{O}_\alpha(t) \tilde{O}_\beta^\dagger(0) \rangle \propto \delta_{\alpha,\beta}, \ i = 1, 2, 3, \dots$$

Mixed correlators (Bethe-Salpeter amplitudes):

Bethe-Salpeter amplitude

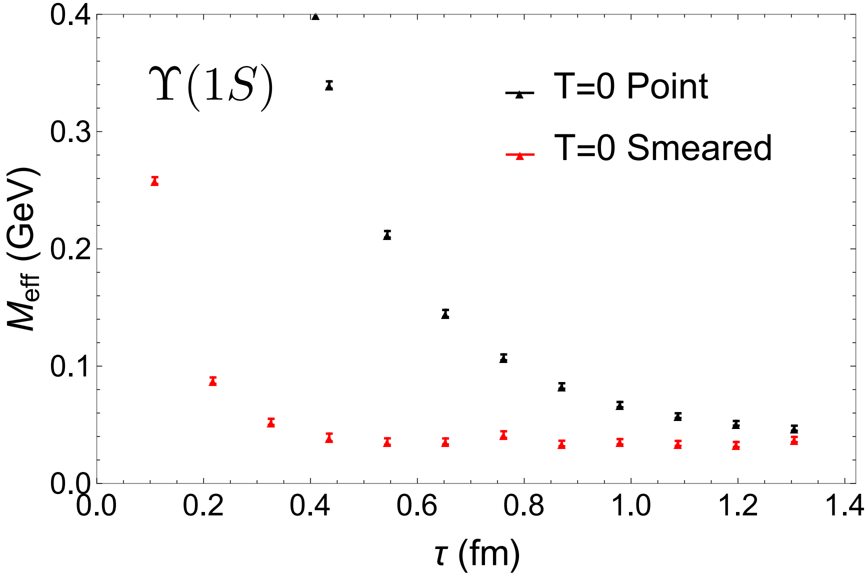
$$\tilde{C}_\alpha^r(t) = \sum_{\mathbf{x}} \langle O_{qq}^r(t, \mathbf{x}) \tilde{O}_\alpha(0, \mathbf{0}) \rangle \sim \phi_\alpha(r) e^{-E_\alpha t}, \ t \to \infty$$

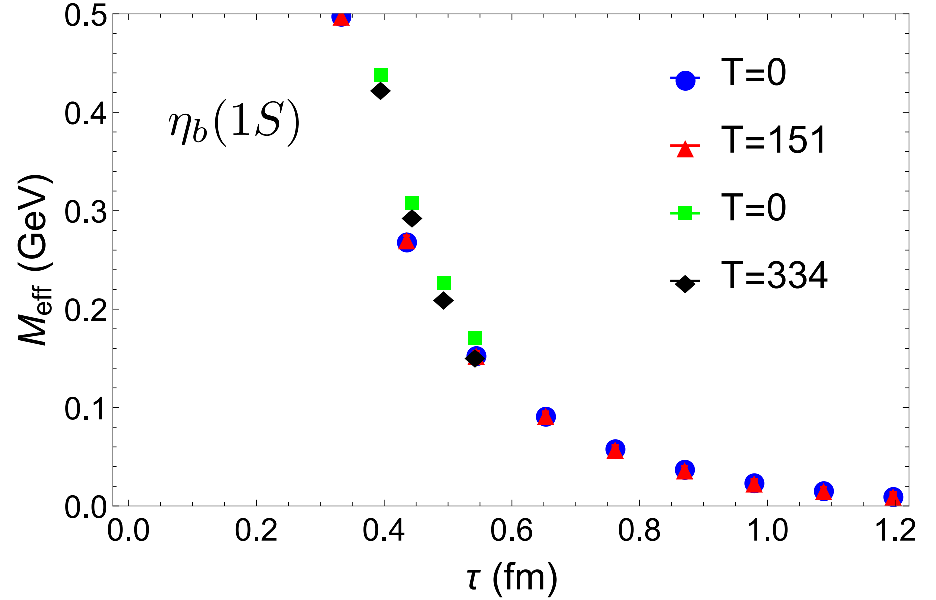
$$O_{qq}^r(t, \mathbf{x}) = \chi^\dagger(t, \mathbf{x}) \Gamma \psi(t, \mathbf{x} + \mathbf{r}))$$

Point operators vs. extended operators

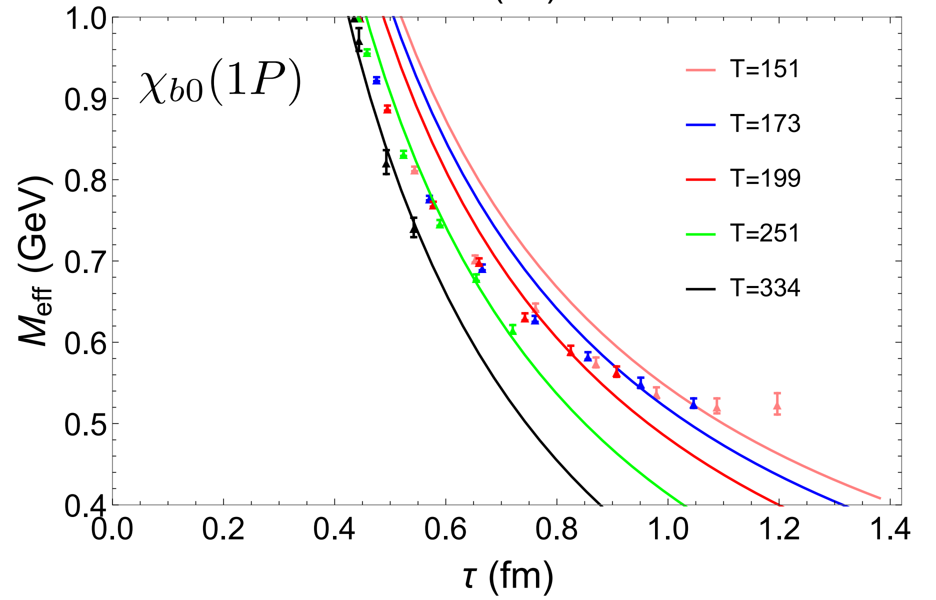
Larsen, Meinel, Mukherjee, PP, PRD100 (2019) 074506

$$M_{\rm eff}(\tau) = \frac{1}{a}\ln[C_\alpha(\tau)/C_\alpha(\tau+a)]$$





- The effective masses of point correlators do not show a plateau for $\tau < 1.2$ fm and have very small temperature dependence
- The small τ behavior of the effective masses is well described by perturbation theory for P-wave bottomonia
- The correlators of extended operators approach a plateau for $\tau < 1$ fm.

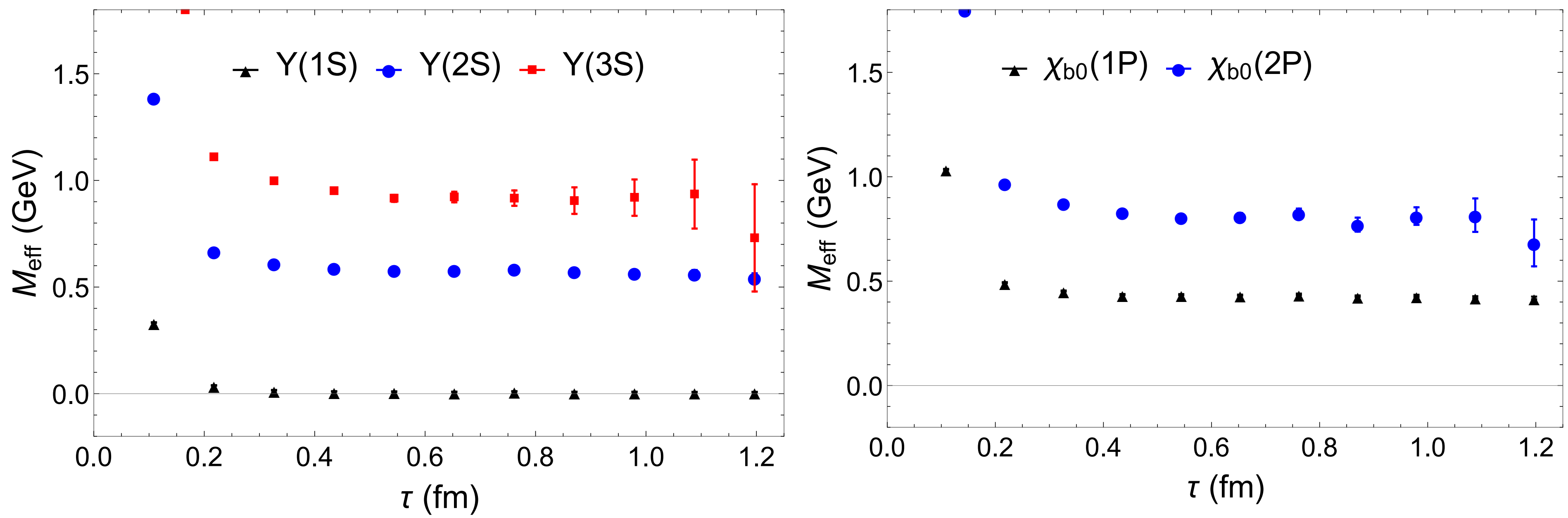


Correlators of Optimized Meson Operators at T=0

HISQ, $a = 0.109,\ 0.095,\ 0.083,\ 0.066,\ 0.060,\ 0.049$ fm

Larsen, Meinel, Mukherjee, PP, PLB 800 (20) 135119

$$M_{\rm eff}(\tau) = \frac{1}{a}\ln[C_\alpha(\tau)/C_\alpha(\tau+a)]$$



$$C_\alpha(\tau,T) = \int_{-\infty}^{\infty} d\omega \rho_\alpha(\omega,T) e^{-\omega\tau} \qquad \rho_\alpha(\omega,T) = \rho_\alpha^{\rm med}(\omega,T) + \rho_\alpha^{\rm high}(\omega)$$

$$\rho_\alpha^{\rm med}(\omega, T=0) = A_\alpha \delta(\omega - M_\alpha) \Rightarrow C_\alpha(\tau, T=0) = A_\alpha e^{-M_\alpha \tau} + C_\alpha^{\rm high}(\tau)$$

Determine A_α,M_α from single exponential fit for $\tau > 0.6$fm and then $C_\alpha^{\rm high}(\tau)$

Bottomonium Bethe-Salpeter amplitude at T=0

$$\tilde{C}^r_\alpha(\tau) = \sum_{\mathbf{x}} \langle O^r_{qq}(\tau,\mathbf{x}) \tilde{O}_\alpha(0,0)\rangle, \;\; O^r_{qq}(\tau,\mathbf{x}) = \chi^\dagger(\tau,\mathbf{x})\Gamma\psi(\tau,\mathbf{x}+r))$$

$$\tilde{C}^r_\alpha(\tau) = \sum_n \langle 0|O^r_{qq}(0)|n\rangle\langle n|\tilde{O}_\alpha(0)|0\rangle e^{-E_n\tau}|_{\tau\to\infty} \sim \langle 0|O^r_{qq}(0)|\alpha\rangle e^{-E_\alpha \tau}$$

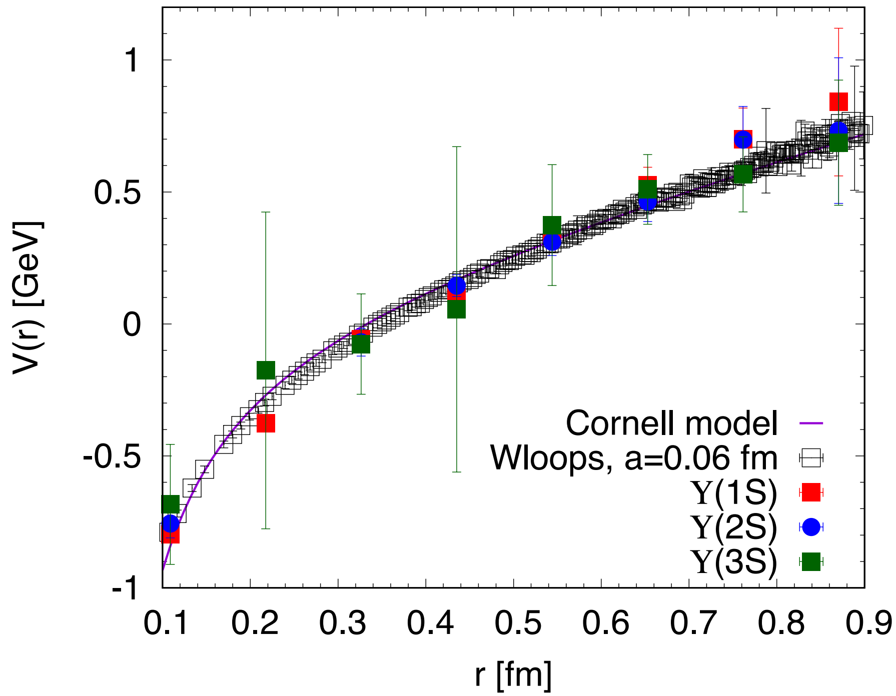
Kawanai, Sasaki, PRL 107 (11) 091601 (charmonium)

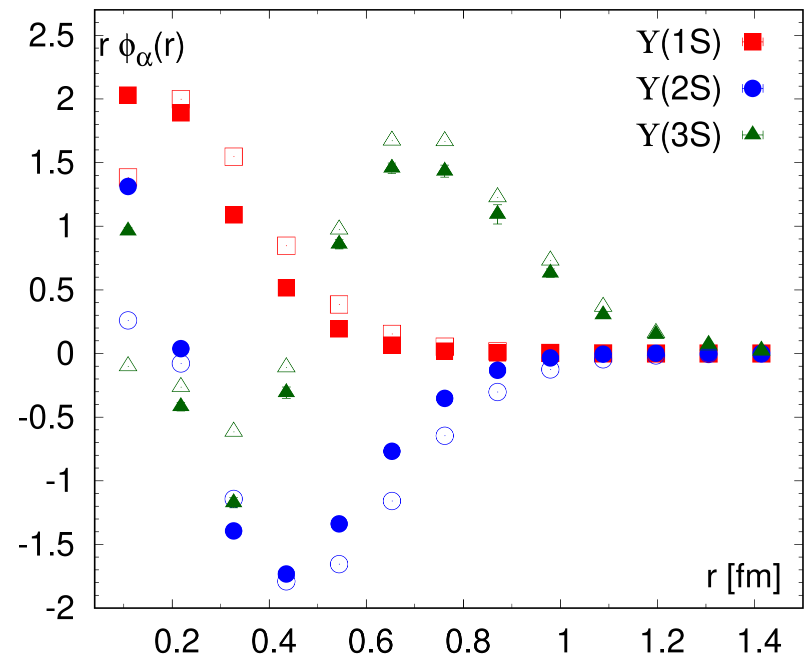
$\phi_\alpha(r)$- Bethe-Salpeter amplitude

Larsen, Meinel, Mukherjee, PP, PRD 102 (20)114508

$$\left(\frac{-\nabla^2}{m_b} + V(r)\right)\phi_\alpha = E_\alpha \phi_\alpha$$

$$m_b = 5.52 \pm 0.33 \text{ GeV}$$



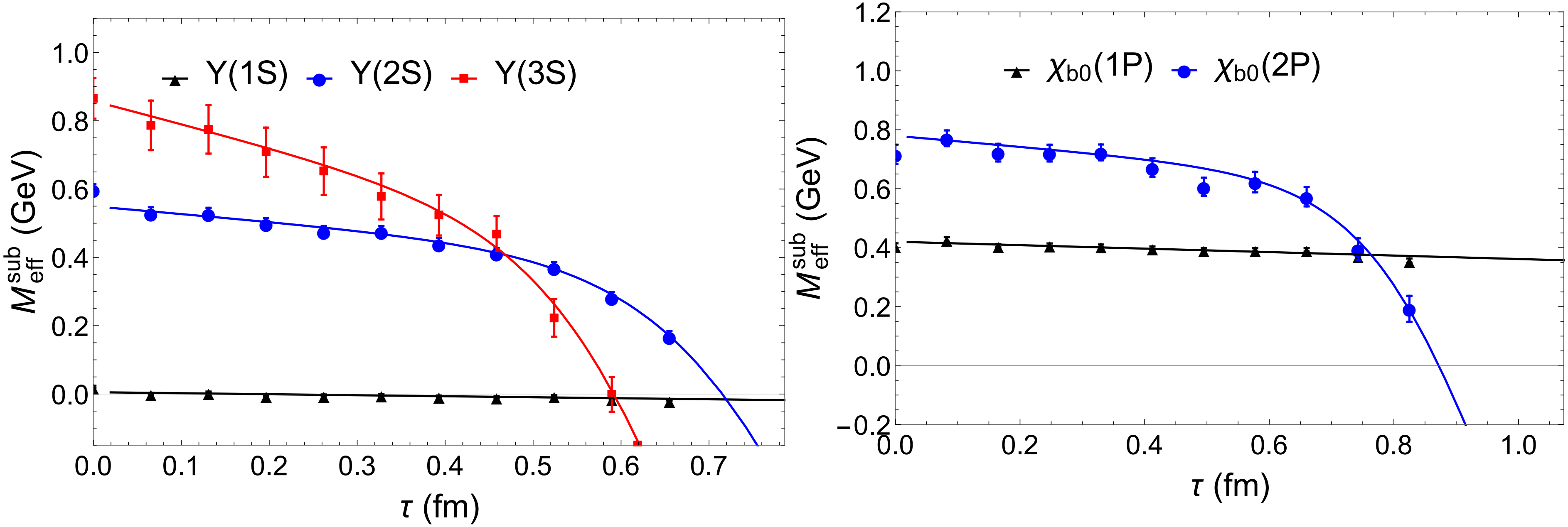


Correlators of Extended Meson Operators at T>0

HISQ, $N_\tau = 12$

Larsen, Meinel, Mukherjee, PP, PLB 800 (20) 135119

$$C_\alpha^{\rm sub}(\tau,T) = C_\alpha(\tau,T) - C_\alpha^{\rm high}(\tau) \Rightarrow aM_{\rm eff}^{\rm sub}(\tau,T) = \ln\left(C_\alpha^{\rm sub}(\tau,T)/C_\alpha^{\rm sub}(\tau+a,T)\right)$$



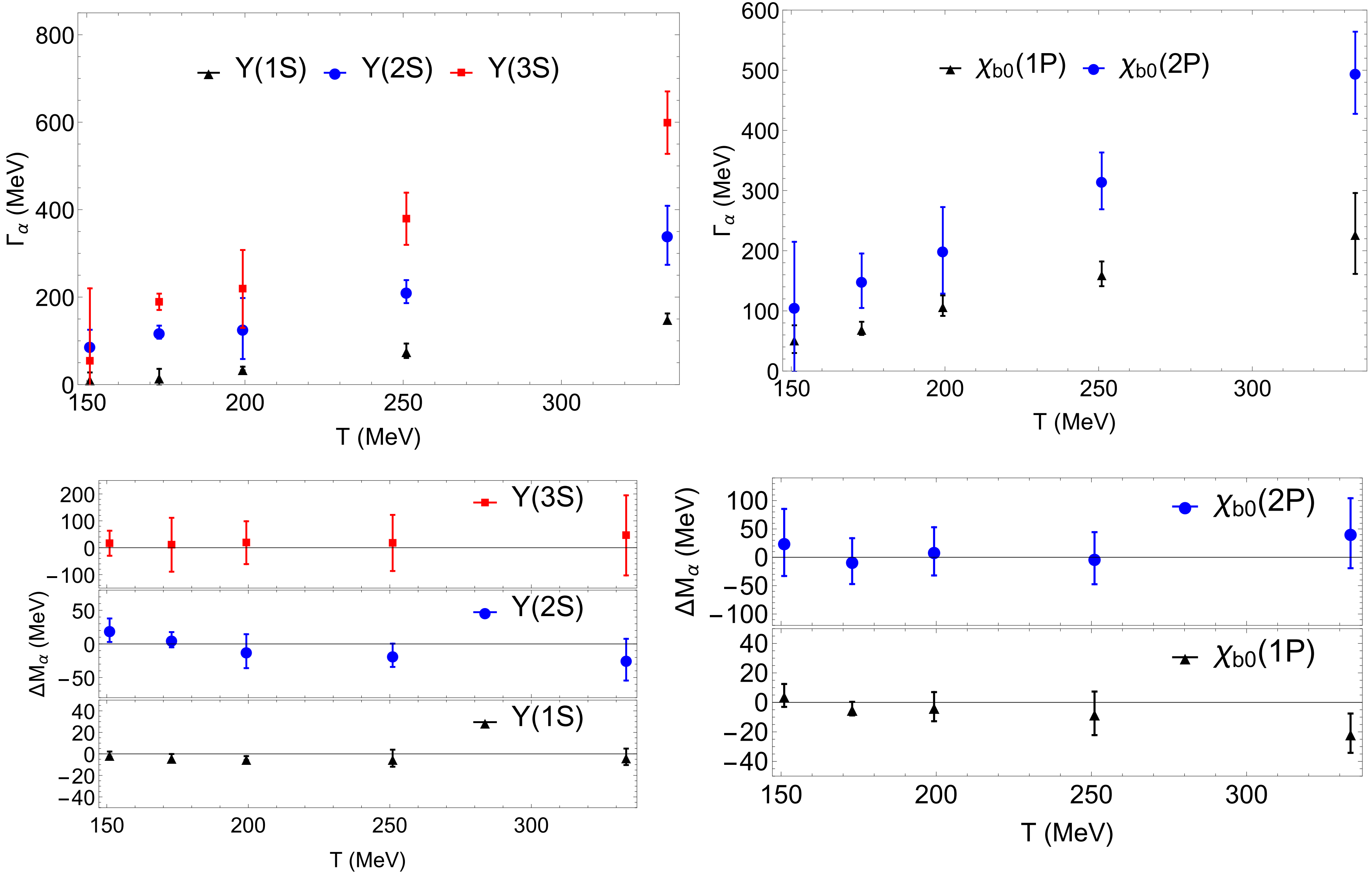
Fit $M_{\rm eff}^{\rm sub}(\tau,T)$ using a simple Ansatz:

$$\rho_\alpha^{\rm med}(\omega,T) = A_\alpha^{\rm cut}(T)\,\delta\left(\omega-\omega_\alpha^{\rm cut}(T)\right) + A_\alpha(T)\exp\left(-\frac{[\omega-M_\alpha(T)]^2}{2\Gamma_\alpha^2(T)}\right)$$

Low energy tail

$$\Rightarrow M_\alpha(T), \Gamma_\alpha(T)$$

Thermal width and mass shift of bottomonium



Quark anti-quark potential at T>0

Conjecture, Matsui and Satz, PLB 178 (86) 416 $-\frac{4}{3}\frac{\alpha_s}{r}+\sigma r \rightarrow -\frac{4}{3}\frac{\alpha_s}{r}e^{-m_D r}, T>T_c$

Extending pNRQCD to *T>0*: the potential is complex, the real part can have thermal correction but is not necessarily screened, except when $r \sim 1/m_D$

Based on weak coupling

Laine, Philipsen, Romatschke, Tassler, JHEP 03 (06) 054
Brambilla, Ghiglieri, PP, Vairo, PRD 78 (08) 014017

Calculate the potential non-perturbatively on the lattice by considering Wislon loops of size *r x* τ at *T>0*

$$W(r,\tau,T)=\int_{-\infty}^{\infty}\rho_r(\omega,T)e^{-\omega\tau}$$

If potential at $T>0$ exists the $\rho_r(\omega,T)$ should have a well define peak at $\omega \simeq \mathrm{Re}V(r,T)$, and the width of the peak is $\mathrm{Im}V(r,T)$

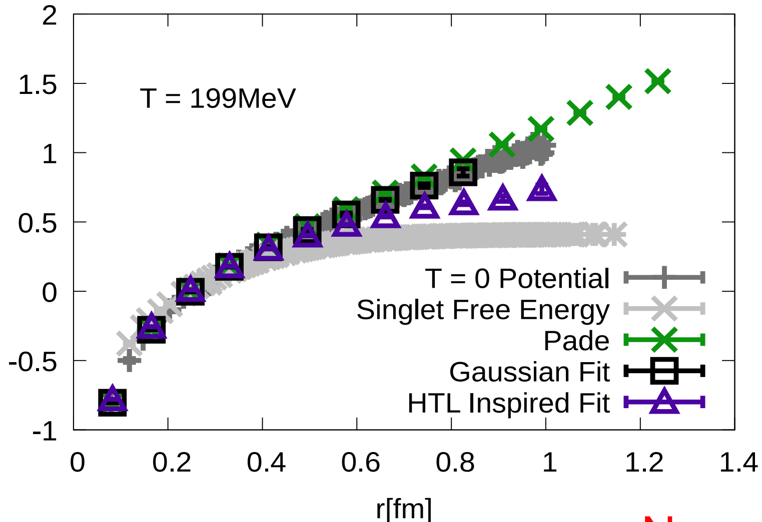
Rothkopf, Hatsuda, Sasaki, PRL 108 (2012) 162001

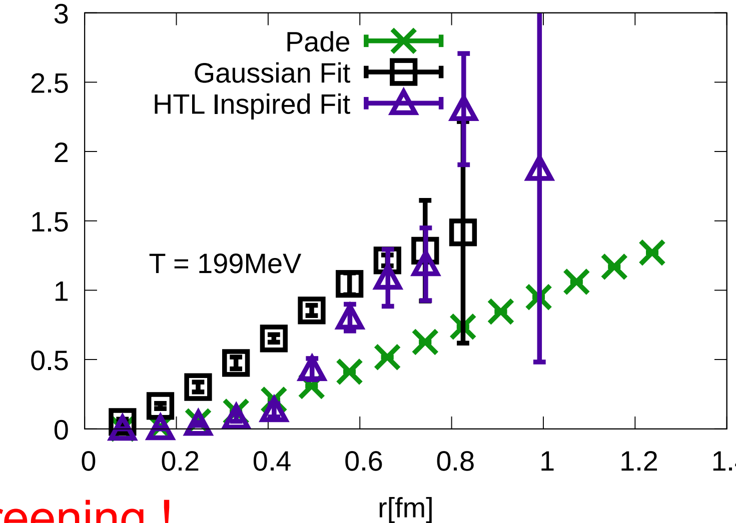
Challenge: reconstruct $\rho_r(\omega,T) \Rightarrow$ use the same approach as for reconstruction of the NRQCD bottomonium spectral functions

Quark anti-quark potential at T>0 from the lattice

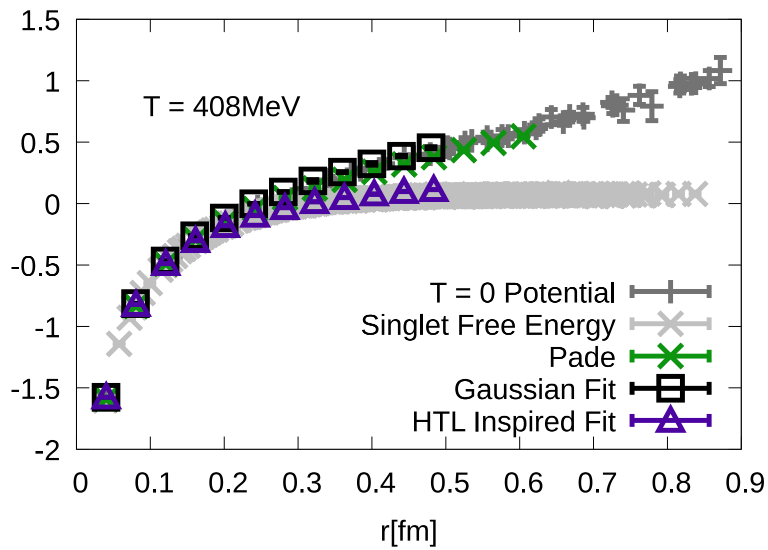
HISQ, $N_\tau = 12$

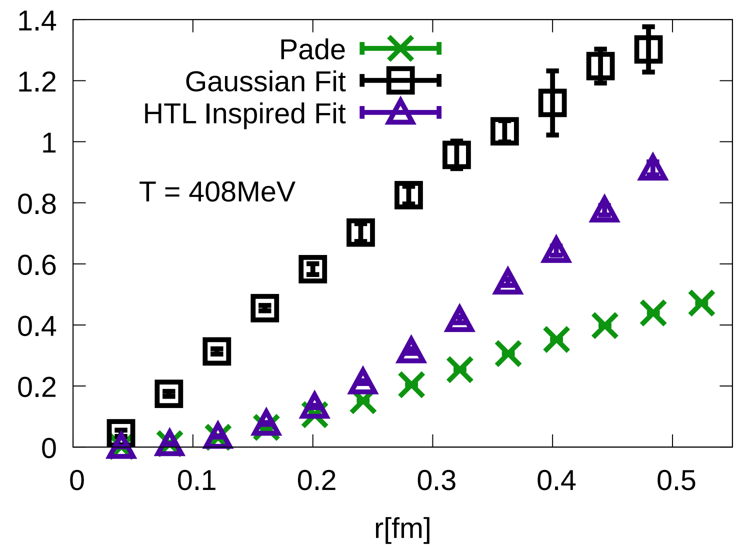
Bala et al (HotQCD), PRD 105 (2022) 054513





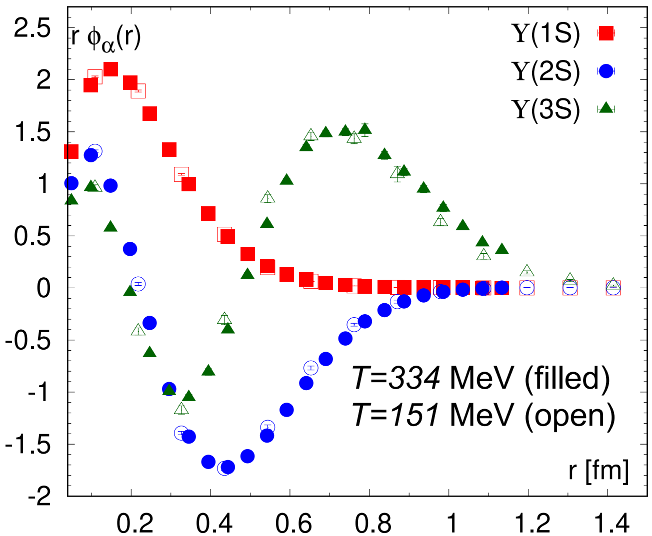
No color screening !

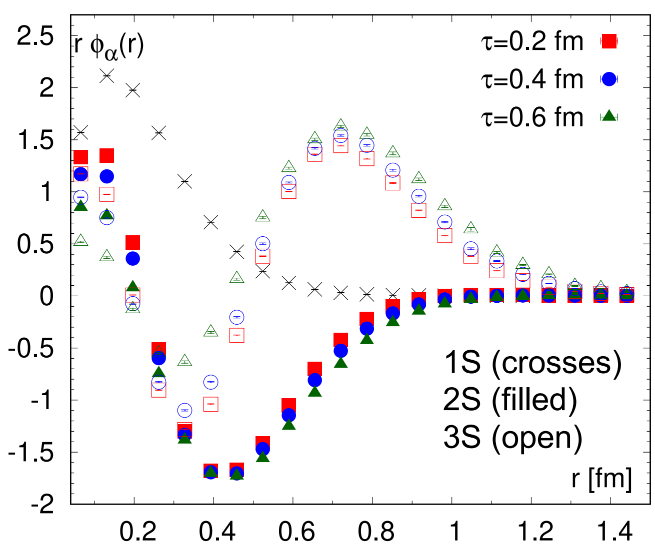




Bethe-Salpeter amplitude at *T>0* and potential model

Larsen, Meinel, Mukherjee, PP, PRD 102 (‘20) 114508

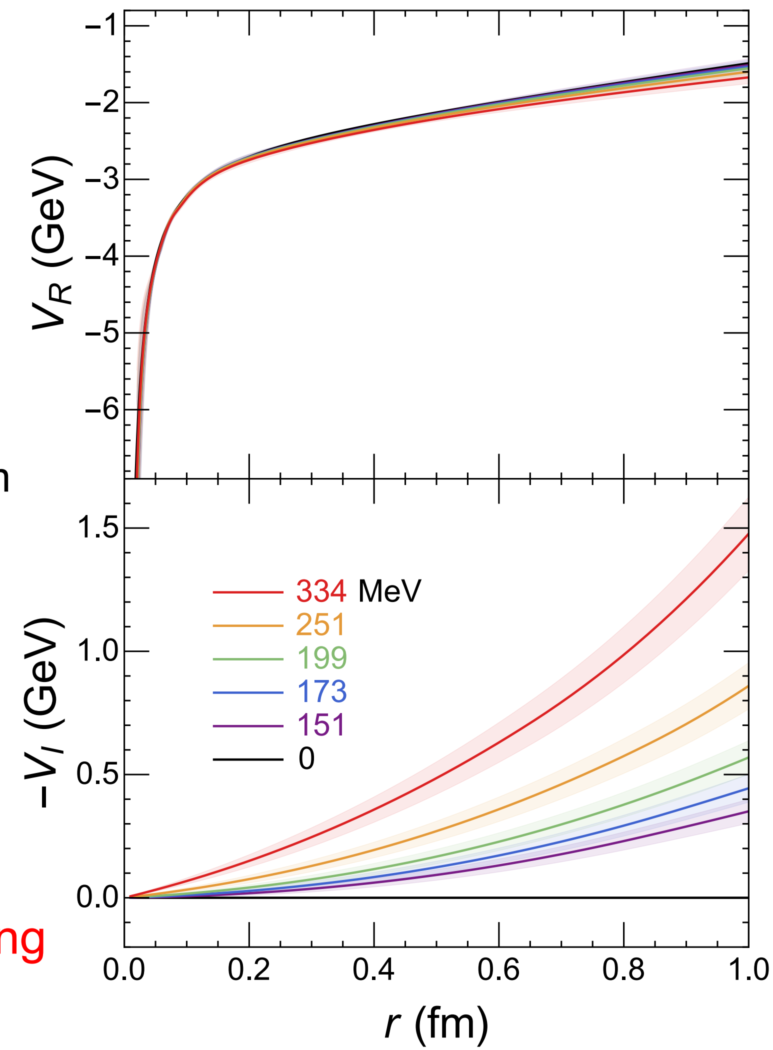




potential model
with inverse problem
+
Thermal width
from lattice
+
Machine learning

No color screening

Shi et al, PRD 105 (‘22) 014017



Current-current correlators and heavy quark diffusion

$$\partial_t p_i - \eta p_i = f_i(t),$$
$$\langle f_i(t) f_j(t') \rangle = \kappa \delta_{ij} \delta(t-t')$$

⟺

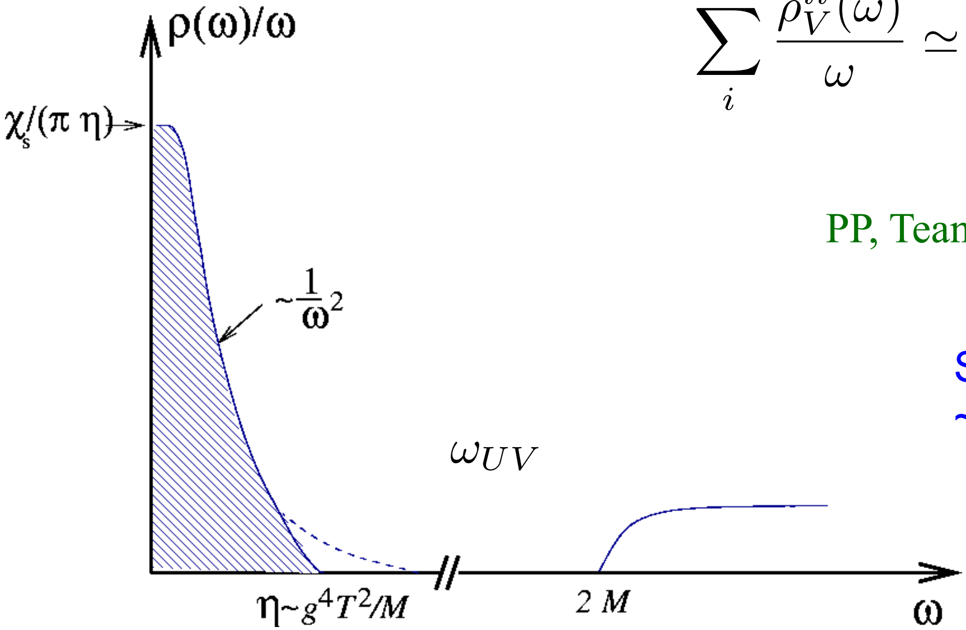
$$\rho_V^{\mu\nu}(\omega) \equiv \int_{-\infty}^{\infty} \mathrm{d}t e^{i\omega t} \int \mathrm{d}^3 x \left\langle [\hat{J}^\mu(t,\vec{x}), \hat{J}^\nu(0,\vec{0})] \right\rangle$$

Momentum diffusion coefficient $\kappa = 2MT\eta = 2T^2/D$

$$\sum_i \frac{\rho_V^{ii}(\omega)}{\omega} \simeq 3\chi_2^q \frac{T}{M} \frac{\eta}{\eta^2+\omega^2}, \quad \omega < \omega_{UV}, \quad \eta = \frac{T}{M}\frac{1}{D}$$

↑ drag constant

PP, Teaney, PRD 72 ('06) 014508



Spatial diffusion constant
~ mean free path (weak coupling) $D \sim \frac{1}{g^4 T}$

area under the peak ~ $\chi_2^q \frac{T}{M}$

heavy quark coefficient ~ width of the peak

For large quark mass the transport peak is very narrow even for strong coupling and its difficult to reconstruct it accurately from Euclidean correlator calculated on the lattice

Current-current correlators in the heavy quark limit

$$\kappa = \frac{1}{3T}\sum_{i=1}^{3}\lim_{\omega\to 0}\left[\lim_{M\to\infty}\frac{M^2}{\chi_2^q}\int_{-\infty}^{\infty}\mathrm{d}t\, e^{i\omega(t-t')}\int \mathrm{d}^3\vec{x}\left\langle \frac{1}{2}\left\{\frac{\mathrm{d}\hat{J}^i(t,\vec{x})}{\mathrm{d}t},\frac{\mathrm{d}\hat{J}^i(t',\vec{0})}{\mathrm{d}t'}\right\}\right\rangle\right]$$

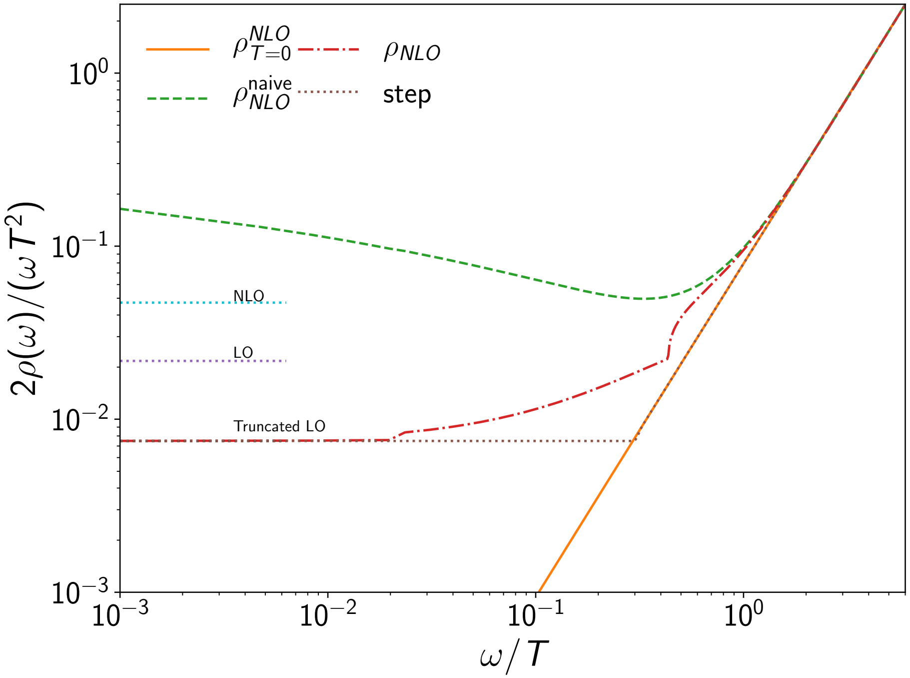
$$\frac{\mathrm{d}\hat{J}^i}{\mathrm{d}t} = \frac{1}{M}\left\{\hat{\phi}^\dagger gE^i\hat{\phi} - \hat{\theta}^\dagger gE^i\hat{\theta}\right\} + \mathcal{O}\left(\frac{1}{M^2}\right)$$

Casalderrey-Solana, Teaney, PRD 74 (2006) 085012
Caron-Huot, Laine, Moore, JHEP 0904 ('09) 053

$t \to i\tau$

$$G_E(\tau) = \frac{1}{3\chi_2^q T}\sum_i \int \mathrm{d}^3x \left\langle \left[\phi^\dagger gE_i\phi - \theta^\dagger gE_i\theta\right](\tau,\vec{x})\left[\phi^\dagger gE_i\phi - \theta^\dagger gE_i\theta\right](0,\vec{0})\right\rangle$$

Integrate out ϕ, θ



$$G_E(\tau) = -\frac{1}{3}\sum_{i=1}^{3}\frac{\left\langle \mathrm{ReTr}\left[U(\beta,\tau)\, gE_i(\tau,\vec{0})\, U(\tau,0)\, gE_i(0,\vec{0})\right]\right\rangle}{\left\langle \mathrm{ReTr}[U(\beta,0)]\right\rangle}$$

$$G_E(\tau) = \int_0^\infty \frac{\mathrm{d}\omega}{\pi}\rho_E(\omega)\frac{\cosh\left(\tau - \frac{1}{2T}\right)\omega}{\sinh\frac{\omega}{2T}}$$

Transport coefficient ~ intercept of the spectral function not its width

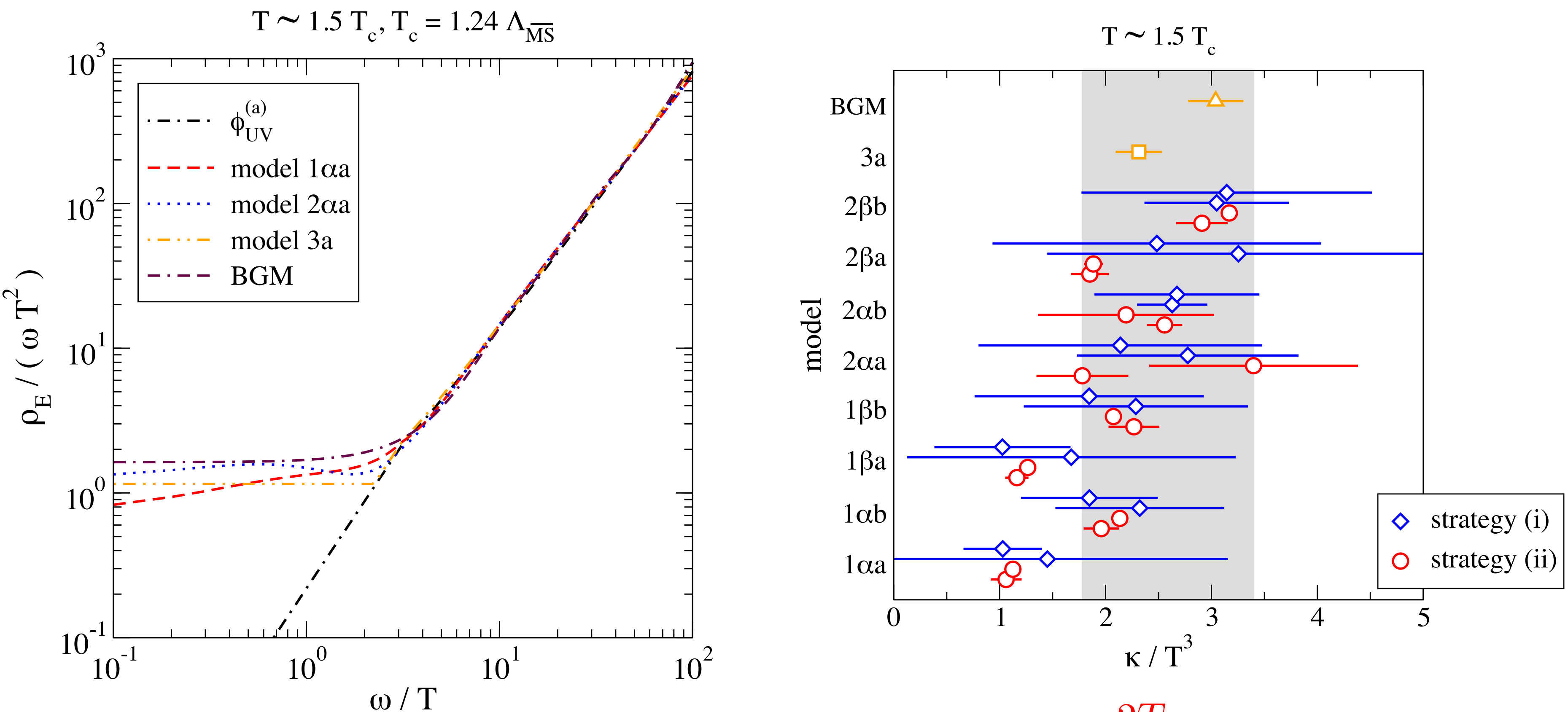
$$\kappa = \lim_{\omega\to 0}\frac{2T}{\omega}\rho_E(\omega)$$

Extracting the spectral function and the diffusion constant

Major challenge noise reduction → multilevel algorithm → only quenched QCD

Fit the lattice using a forms of the spectral function constrained by low and high energy asymptotic behavior + corrections

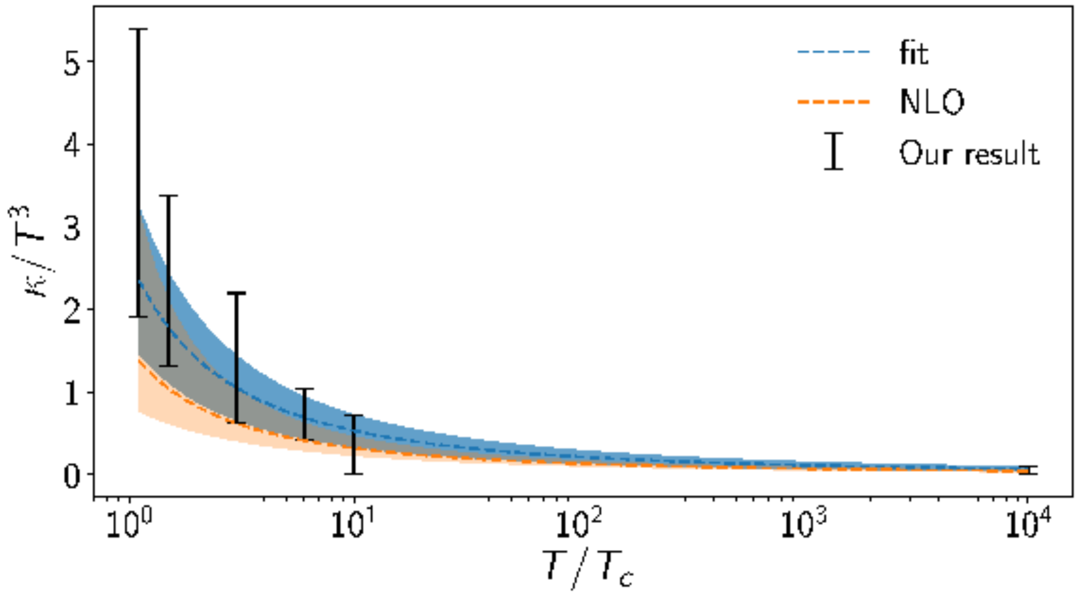
$$\rho^{low}(\omega) = \frac{\kappa\omega}{2T} \qquad \rho^{high}(\omega) = \frac{g^2(\mu_\omega)C_F}{6\pi}\omega^3, \mu_\omega = max(\omega, \pi T)$$



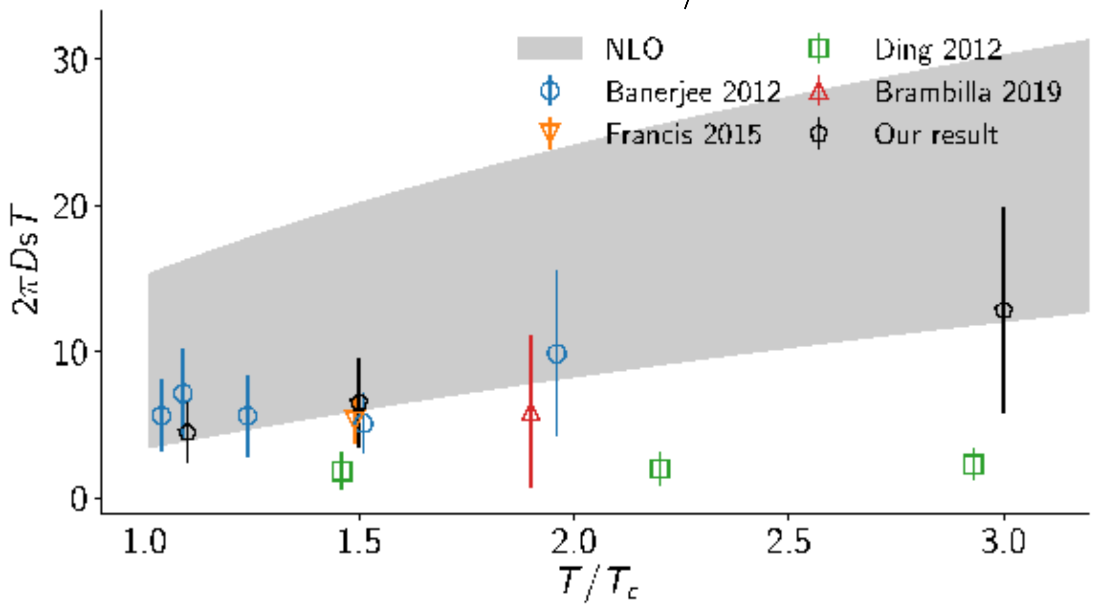
Francis, Kaczmarek, Laine, et al, PRD 92 ('15) 116003

$$\kappa = \lim_{\omega\to 0} \frac{2T}{\omega}\rho(\omega) = (1.8 - 3.4)T^3$$

Diffusion constant as function of the temperature



$$\kappa = 2T^2/D$$



The calculations have been performed recently at different temperature using similar approach

Brambilla, Leino, PP, Vairo,
PRD 102 ('20), arXiv:2206.02861

The new results also agree with estimates by Banerjee et al,
PRD 85 ('12) 0145010; arXiv:2206.15471
Altenkort et al, PRD 103 ('21) 014511

New development: noise reduction through gradient flow

The estimate from current-current correlator is too low
Ding et al, PRD 86 ('12) 014509
is too low

The width transport peak is difficult to estimate ?

1/M effects in the heavy quark diffusion

$$M\mathbf{v} \to \mathbf{p} = \gamma M \mathbf{v} \qquad M \to M_{kin}(T)$$

$$\langle f_i(t) f_j(t) \rangle = \langle E_i(t) E_j(t') \rangle + \frac{1}{3} \langle \mathbf{v}^2 \rangle \langle \delta_{ij} B_k(t) B_k(t') - B_i(t') B_j(t) \rangle$$

$$\lim_{t\to\infty} \langle \mathbf{p}^2(t) \rangle = \frac{3\kappa}{2\eta} \qquad \langle \gamma \mathbf{v}^2 \rangle = \frac{3T}{M_{kin}} \qquad \eta \simeq \frac{\kappa}{2 M_{kin} T} \left(1 - \frac{5T}{2M_{kin}}\right)$$

Bouttefeux, Laine, JHEP 12 (2020) 150

$$G_B(\tau) = -\frac{1}{3} \sum_{i=1}^{3} \frac{\left\langle \mathrm{ReTr}\left[U(\beta,\tau)\, gB_i(\tau,\vec{0})\, U(\tau,0)\, gB_i(0,\vec{0})\right] \right\rangle}{\left\langle \mathrm{ReTr}[U(\beta,0)] \right\rangle}$$

$$G_B(\tau) = \int_0^\infty \frac{\mathrm{d}\omega}{\pi} \rho_B(\omega) \frac{\cosh\left(\tau - \frac{1}{2T}\right)\omega}{\sinh \frac{\omega}{2T}}$$

$$\kappa = \kappa_E + \frac{2}{3} \langle \mathbf{v}^2 \rangle \kappa_B$$

$$\kappa_E = \lim_{\omega\to 0} \frac{2T}{\omega} \rho_E(\omega) \qquad \kappa_B = \lim_{\omega\to 0} \frac{2T}{\omega} \rho_B(\omega)$$

sensitive to the non-perturbative "magnetic mass" at LO

$$1.5T_c : \kappa_B = (1.23 - 2.54)T^3,$$

Brambilla et al, arXiv:2206.02861

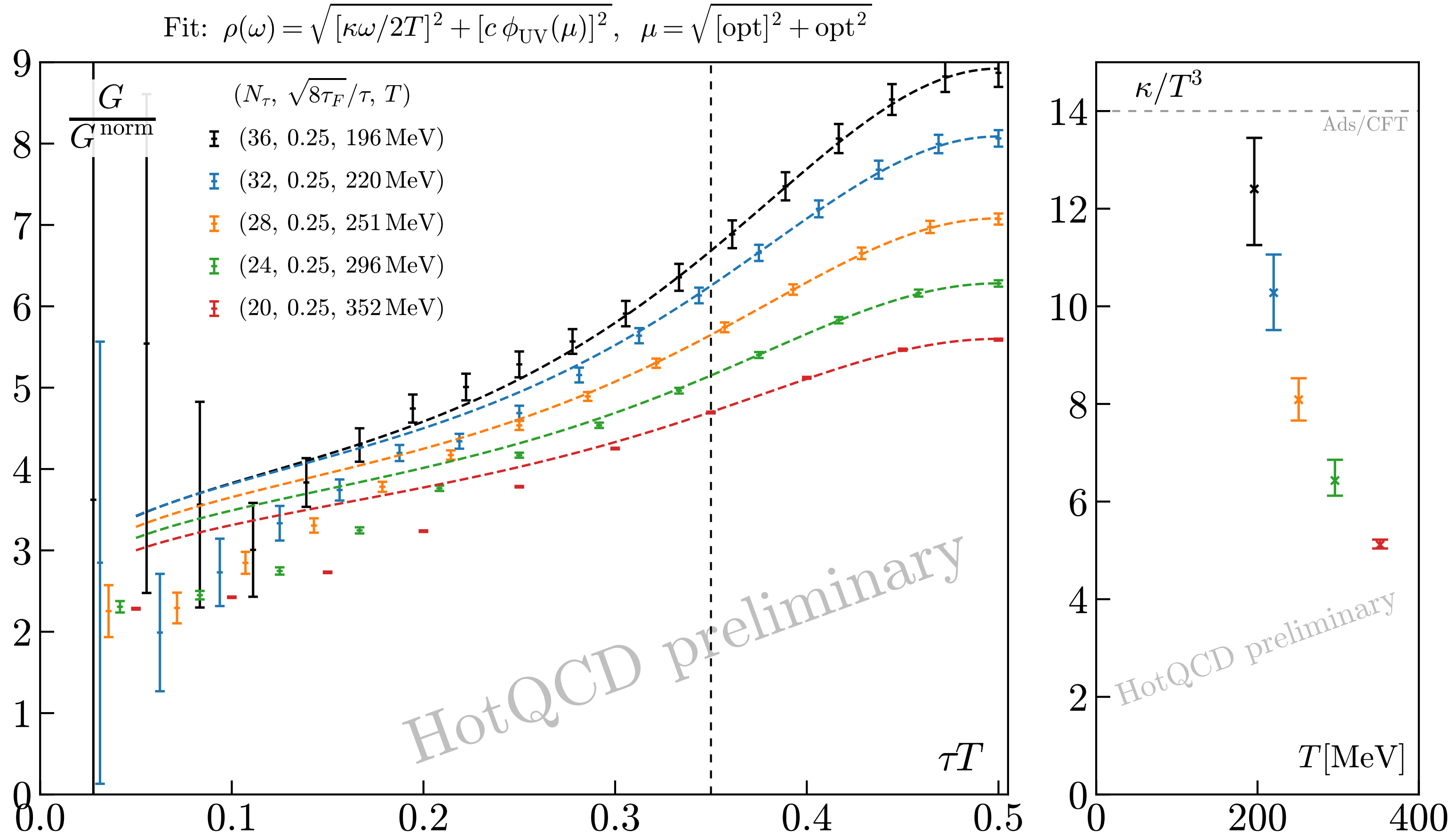
$$\kappa_B = (1.0 - 2.1)T^3$$

Banerjee et al, arXiv:2204.14075

10-20% correction for bottom quark, ~30% correction for charm quark

Heavy quark diffusion constant in 2+1 f QCD

2+1 flavor QCD with $m_\pi = 300$ MeV, $96^3 \times N_\tau$ lattice; Gradient flow for noise reduction



κ/T^3 is significantly larger in 2+1 flavor QCD than in quenched QCD, close to the AdS/CFT limit

Summary

- The thermal width of $\Upsilon(1S)$, $\Upsilon(2S)$, $\Upsilon(3S)$, $\chi_{b1}(1P)$ and $\chi_{b2}(2P)$ states was calculated in lattice NRQCD and found that the value of the thermal width follows the hierarchy of the bottomium sizes

- No significant thermal modification of bottomonium masses have been found in contrast with the expectations based on potential models with screened potential

- Lattice calculations confirm the existence of the imaginary part of the potential; There is no evidence for the screening of the real part of the potential $\Rightarrow$ Matsui and Satz picture is not correct, quarkonium melting is not related to color screening

- The heavy quark diffusion coefficient has been estimated in quenched lattice QCD and different lattice results seem to agree well, and the uncertainty in κ mostly comes from the systematic uncertainties in the reconstruction of the spectral function; The quark mass suppressed effects in the heavy quark diffusion coefficient have been estimated in quenched QCD to be around $10-20\%$ for bottom quarks and around 30% for charm quarks

- First full QCD calculation of the heavy quark diffusion coefficient become available now and indicate that κ/T^3 is larger than un quenched QCD and close to the Ads/CFT bound

Back-up: NRQCD on the Lattice

Advantages: No large cutoff effects $\sim aM_b$, large τ range for $T > 0$

Inverse lattice spacing provides a natural UV cutoff for NRQCD, provided $a^{-1} \leq 2M_Q$ (lattices cannot be too fine)

Quark propagators are obtained as initial value problem:

$$G_\psi(\mathbf{x},t) = \langle \psi(\mathbf{x},t)\, \psi^\dagger(\mathbf{0},0)\rangle \qquad G_\chi(\mathbf{x},t) = -G_\psi^\dagger(\mathbf{x},t)$$

$$G_\psi(t) = K(t) G_\psi(t-1),$$

$$K(t) = \left(1 - \frac{a\delta H|_t}{2}\right)\left(1 - \frac{aH_0|_t}{2n}\right)^n U_4^\dagger(t) \times \left(1 - \frac{aH_0|_{t-1}}{2n}\right)^n \left(1 - \frac{a\delta H|_{t-1}}{2}\right),$$

$$t = \tau/a,\;\; H_0 = \frac{-\Delta^{(2)}}{2M_b},\quad \delta H \sim v^4,\;\; v^6(\text{ spin} - \text{dep.})$$

@ Tree level

Meinel, PRD 82 (2010) 114502

masses are only defined up to *a*-dependent shift: $M_{\Upsilon(1S)} = E_{\Upsilon(1S)} + C_{\text{shift}}(a)$

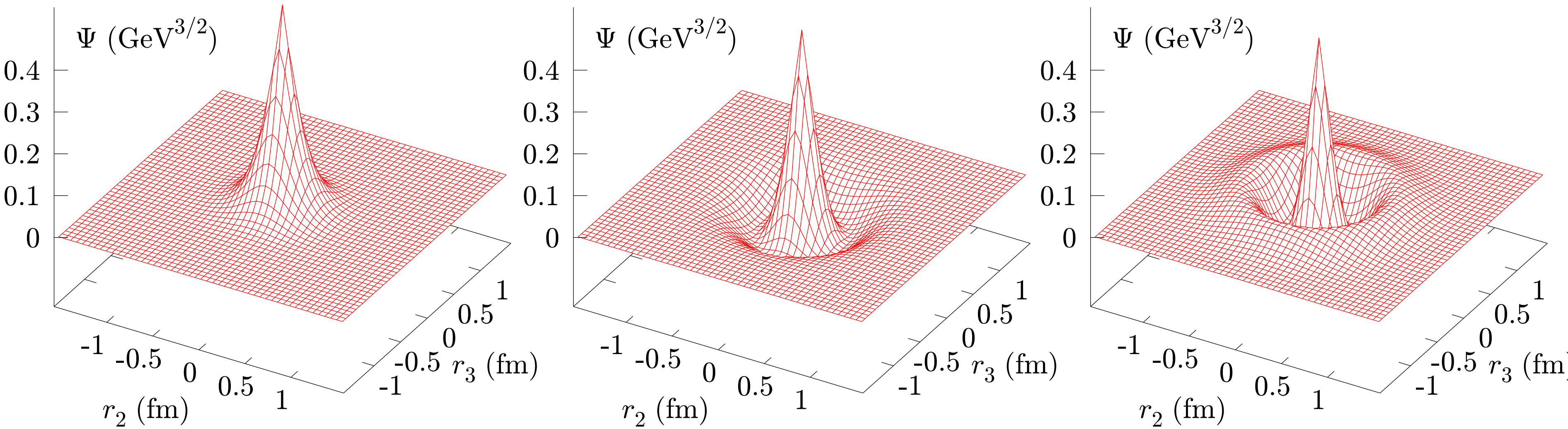
Use kinetic mass instead: $E_{\Upsilon(1S)}(p) = E_{\Upsilon(1S)} + C_{\text{shift}}(a) + \dfrac{p^2}{2M^{kin}_{\Upsilon(1S)}}$

Tune M_b such that $M^{kin}_{\Upsilon(1S)} = M^{PDG}_{\Upsilon(1S)}$

Back-up: Optimized Meson Operators

$O_i(\mathbf{x},t) = \sum_{\mathbf{r}} \Psi_i(\mathbf{r})\chi^\dagger(\mathbf{x}+\mathbf{r},t)\Gamma\psi(\mathbf{x},t)$ $\Psi_i(\mathbf{r})$ from potential model with Cornell potential

Meinel, PRD 82 (2010) 114502



Good overlap with bottomonium states but

$C_{ij}(t) = \langle O_i(t) O_j^\dagger(0)\rangle \neq 0$ for $i \neq j$

$O_i \to \tilde{O}_\alpha = \Omega_{\alpha j} O_j$ such that

$\langle \tilde{O}_\alpha(t)\tilde{O}^\dagger_\beta(0)\rangle \propto \delta_{\alpha,\beta}$

$\Omega_{\alpha j}$ can be obtained as

$G_{ij}(t)\Omega_{\alpha j} = \lambda_\alpha(t,t_0) G_{ij}(t_0)\Omega_{\alpha j}$.

Back-up: Lattice results on bottomonium spectrum

$$\Delta M = M - \overline{M}(1S), \quad \overline{M}(1S) = (M_{\eta_b(1S)} + 3M_{\Upsilon(1S)})/4$$

state	ΔM [MeV]	$\Delta M(PDG)$ [MeV]
$\Upsilon(3S)$	906.0(25.0)(5.2)	910.3(0.7)
$h_b(2P)$	804.4(35.8)(4.7)	814.9(1.3)
$\chi_{b2}(2P)$	809.2(36.2)(4.7)	823.8(0.9)
$\chi_{b1}(2P)$	802.2(34.9)(4.7)	810.6(0.7)
$\chi_{b0}(2P)$	786.8(32.7)(4.6)	787.6(0.8)
$\Upsilon(2S)$	582.7(9.8)(3.4)	578.4(0.6)
$h_b(1P)$	454.5(4.7)(2.6)	454.4(0.9)
$\chi_{b2}(1P)$	463.3(4.8)(2.7)	467.3(0.6)
$\chi_{b1}(1P)$	448.9(4.6)(2.6)	447.9(0.6)
$\chi_{b0}(1P)$	421.3(4.7)(2.4)	414.5(0.7)
hyperfine(3S)	13.4(6.2)(0.1)	NA
hyperfine(2S)	24.1(1.0)(0.1)	24.5(4.5)

Larsen, Meinel, Mukherjee, PP,
PLB 800 (20) 135119

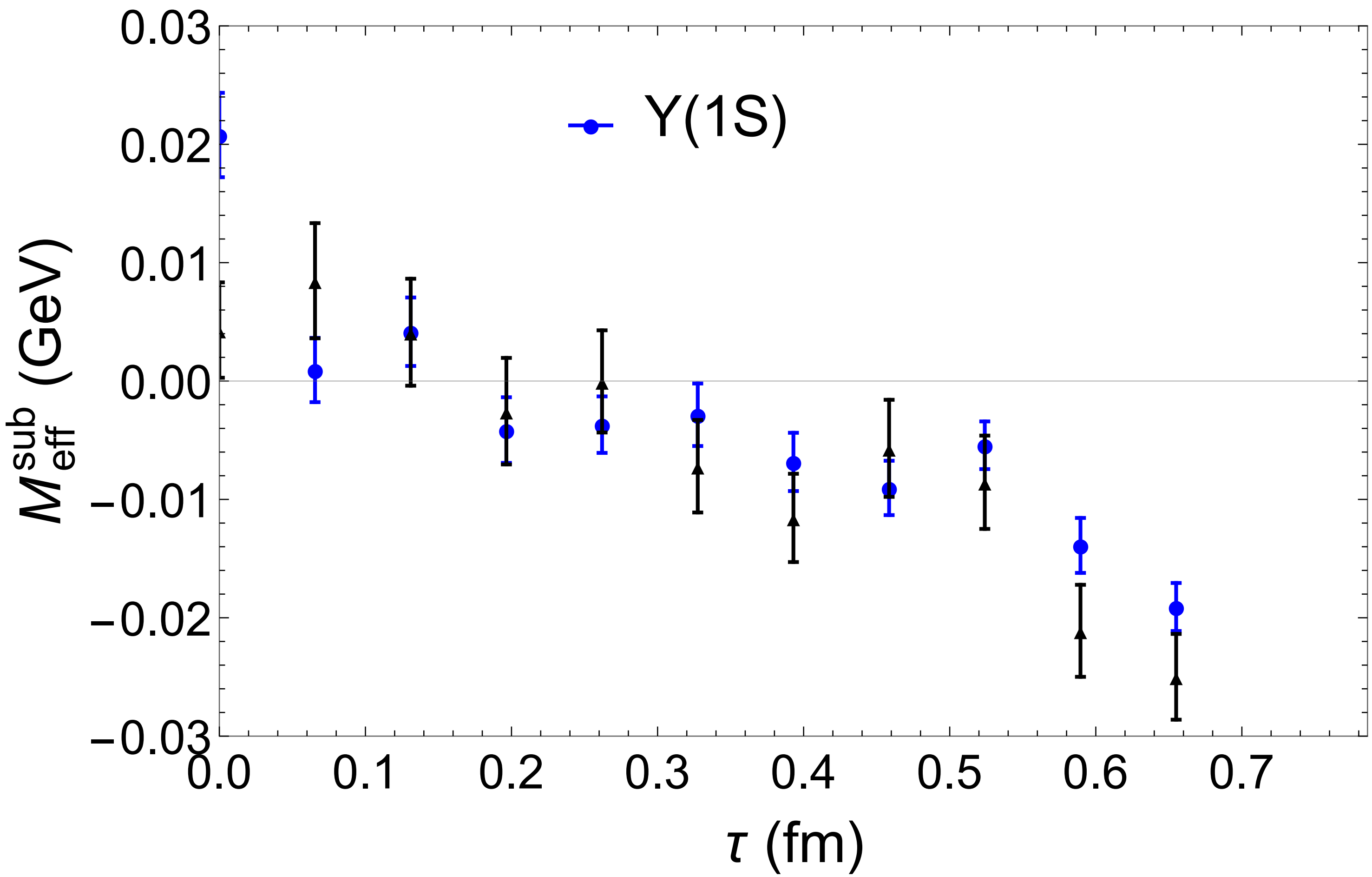
Prediction for $\eta_b(3S)$!

$1S$ hyperfine splitting, $M_{\Upsilon(1S)} - M_{\eta_b(1S)}$ is not reproduced within this approach because it is very sensitive to short distance physics $\Rightarrow$ need radiative correction in the NRQCD Lagrangian Dowdal et al (HPQCD), PRD 85 (12) 054509; PRD 89 (14) 031502(R)

or relativistic approach Hatton et al, PRD 103 (21) 054512

For charmonium it is better to use relativistic lattice formulation, Burch et al, PRD 81 (2010) 034508

Back-up:Comparison of different Meson Operators

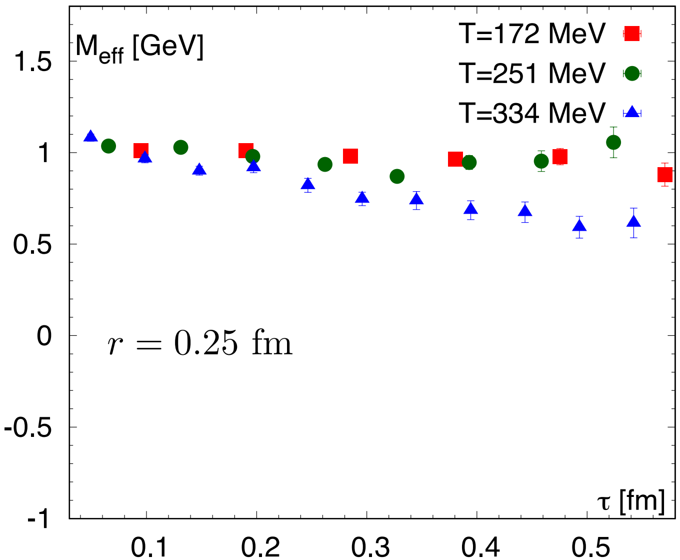


Blue circles: optimized operators

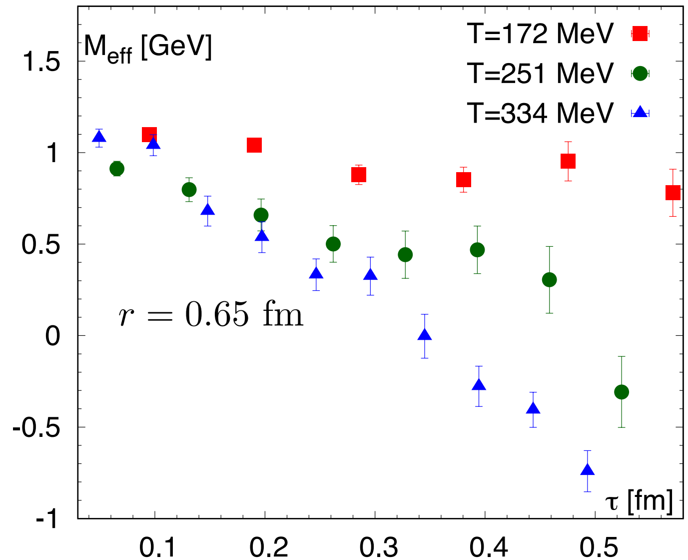
Black triangles: extended operators with Gaussian smearing

Back-up: Bottomonium Bethe-Salpeter amplitude at T>0

$\Upsilon(3S)$

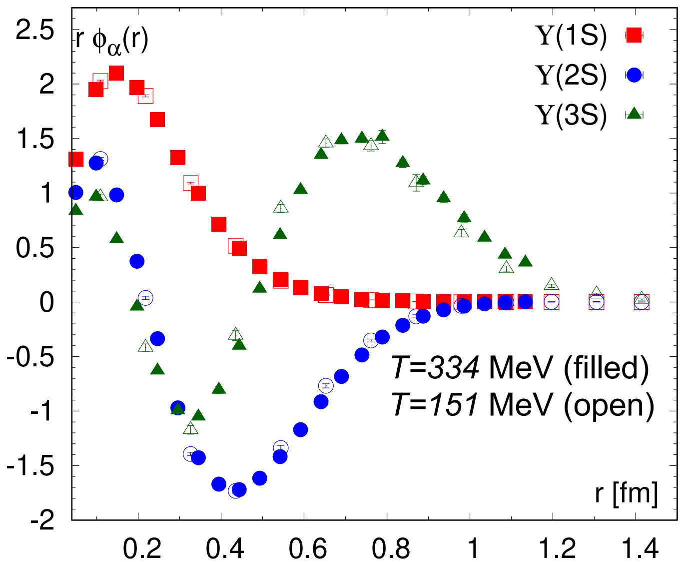


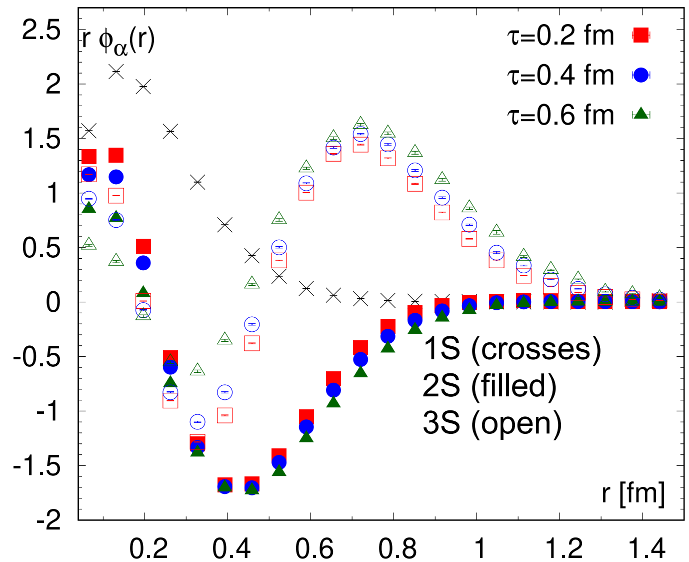
$\Upsilon(3S)$



M_{eff} shows similar thermal effects similar to one obtained from optimized correlators;

Thermal effects are larger for larger r





Thermal effects can be seen but ϕ_α is similar to the $T = 0$ result at qualitative level