Calculation of Hadronic observables with Canonical approach in finite density lattice QCD

Contents

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*for **Zn Collaboration***

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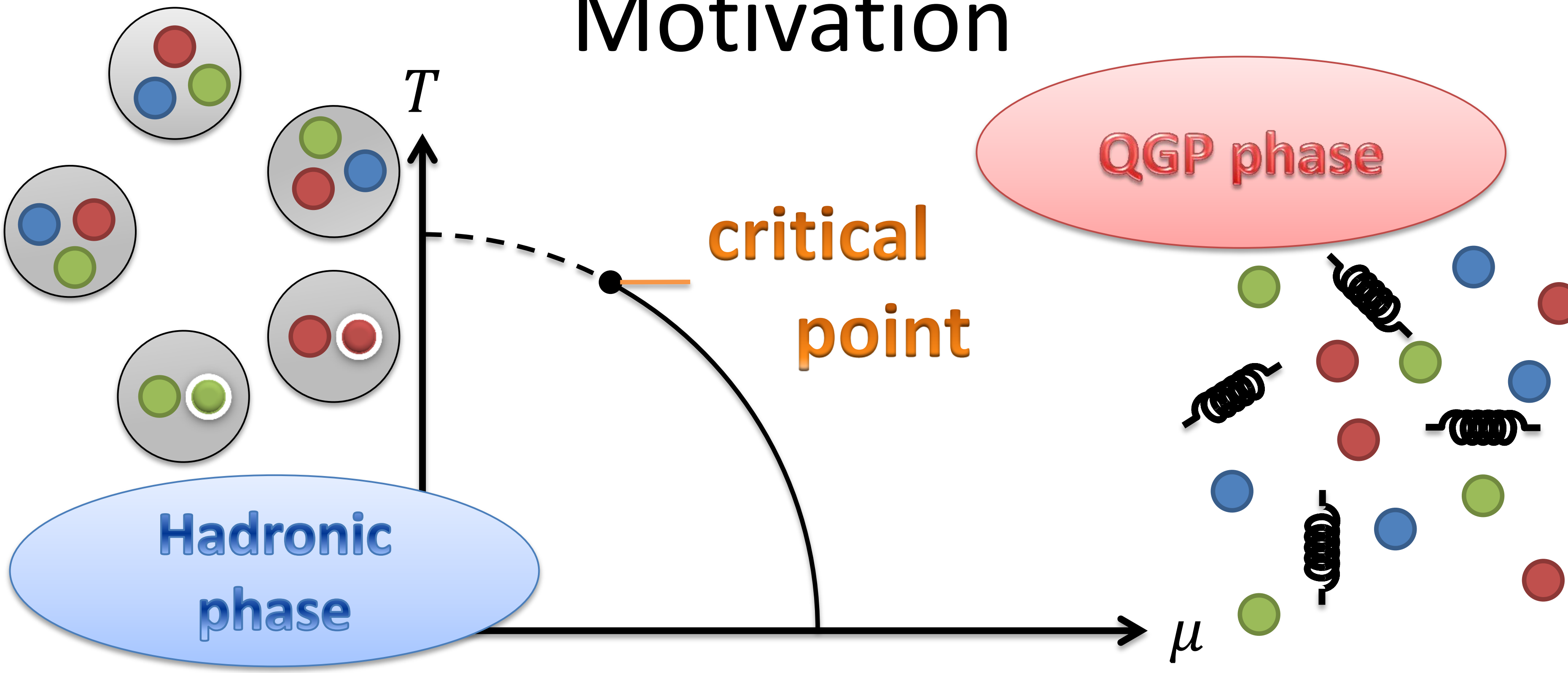
Motivation



✓ **confinement-deconfinement phase transition**

Experiment

RHIC, LHC

FAIR, NICA, J-PARK

Simulation

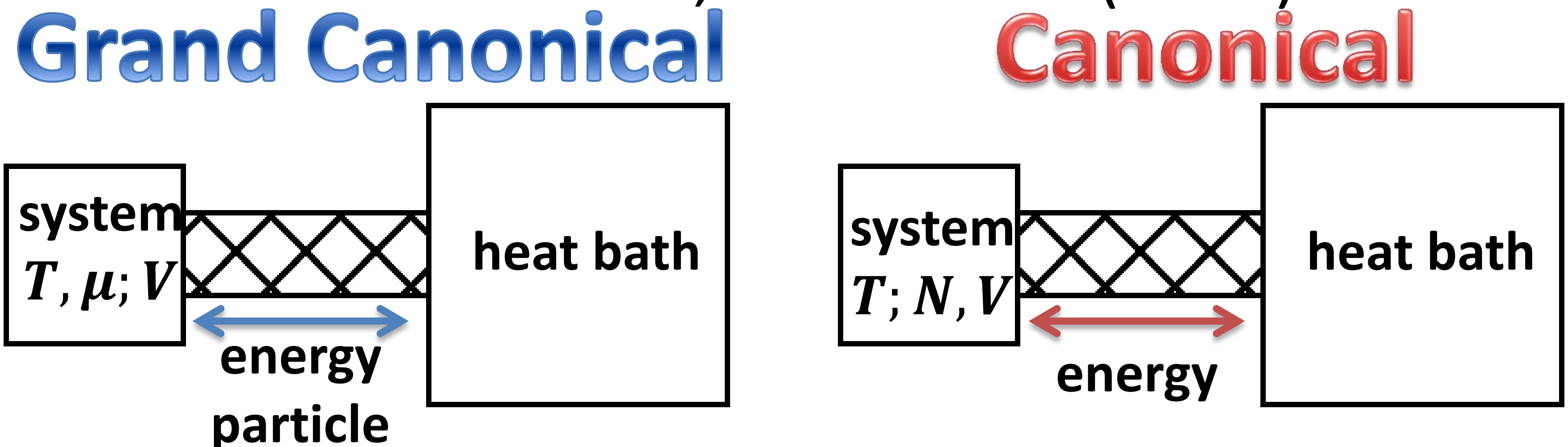
Reweighting, Complex Langevin

Lefschetz thimble

Canonical approach

Canonical approach

A.Hasenfratz , D.Toussaint (1992)

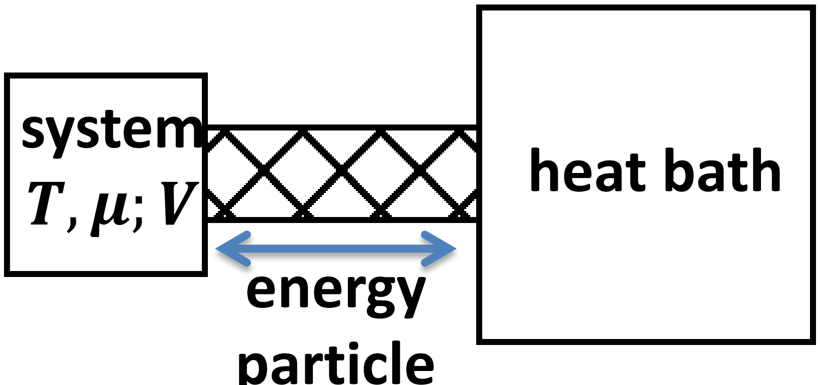


✓ **Both ensembles describe same thermodynamics**

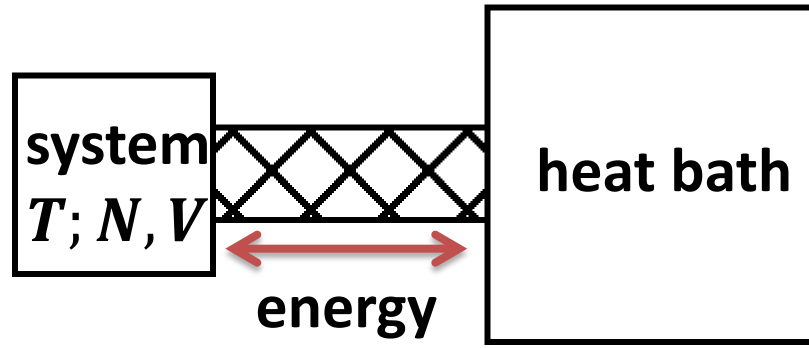
Canonical approach

A.Hasenfratz , D.Toussaint (1992)

Grand Canonical



Canonical



✓ **Both ensembles describe same thermodynamics**

$$\boldsymbol{Z_{G.C.}(T,\mu;V) := \sum_{i,N} \langle E_i, N | e^{-(\hat{H}-\mu\hat{N})/T} | E_i, N \rangle}$$

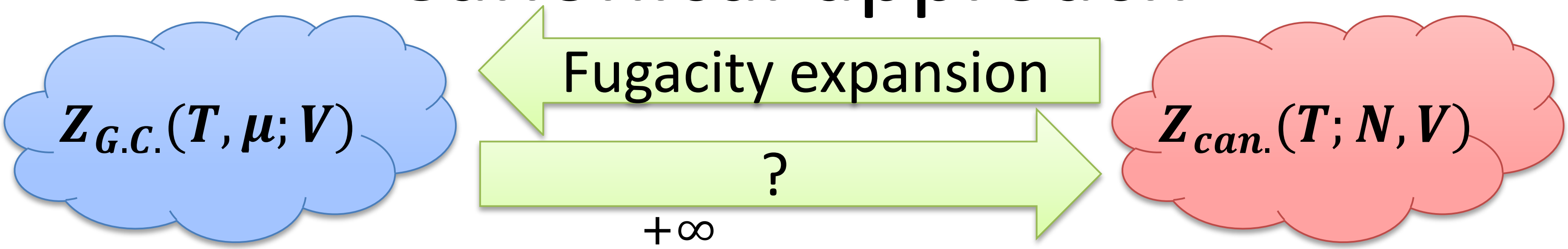
$$\boldsymbol{Z_{can.}(T;N,V) := \sum_{i} \langle E_i, N | e^{-\hat{H}/T} | E_i, N \rangle}$$

✓ **Relation between partition functions ?**

- **Fugacity expansion**

$$Z_{G.C.}(T,\mu;V) = \sum_N Z_{can.}(T;N,V)\xi^N \,, \xi = e^{\frac{\mu}{T}}$$

Canonical approach



$$Z_{G.C.}(T,\mu;V) = \sum_{N=-\infty}^{+\infty} Z_{can.}(T;N,V)\xi^N \,, \xi = e^{\frac{\mu}{T}}$$

✓ **Regard as Laurent series with respect to fugacity**

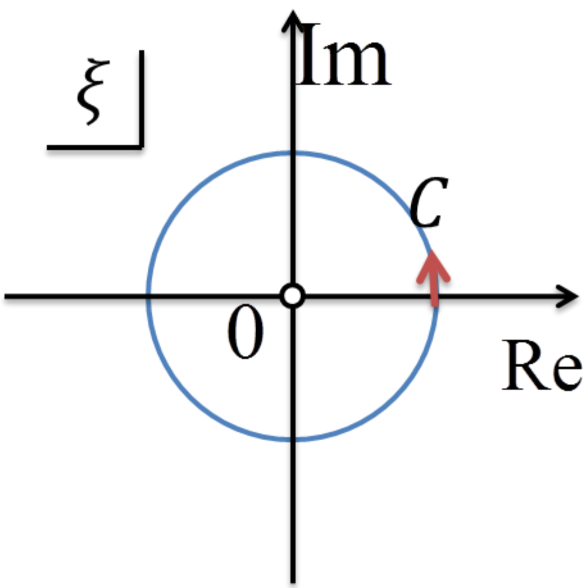
Canonical approach

$Z_{G.C.}(T,\mu;V)$ ← Fugacity expansion — ? → $Z_{can.}(T;N,V)$

$$Z_{G.C.}(T,\mu;V)=\sum_{N=-\infty}^{+\infty} Z_{can.}(T;N,V)\xi^{N}\,,\xi=e^{\frac{\mu}{T}}$$

✓ **Regard as Laurent series with respect to fugacity**

$$Z_{can.}(T;N,V)=\frac{1}{2\pi i}\oint_{C} d\xi\,\xi^{-(N+1)}Z_{G.C.}(T;\xi,V)$$



$$C:\xi=e^{i\frac{\mu}{T}}\;,\frac{\mu}{T}\in[-\pi,\pi)$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi} d\frac{\mu}{T}\,e^{-i\frac{\mu}{T}N}Z_{G.C.}(T;i\mu,V)$$

pure imaginary

✓ $\boldsymbol{Z_{can.}}$ **is given by**

Fourier transformation

Winding number expansion

Li, X. Meng, A. Alexandru,K. F. Liu (2008)

$$Z_{can.}(T;N,V) = \left\langle \frac{1}{2\pi}\int_{-\pi}^{\pi} d\frac{\mu}{T}\, e^{-i\frac{\mu}{T}N}\, \frac{Det\{D(i\mu)\}}{Det\{D(0)\}} \right\rangle_g$$

- ✓ **discrete Fourier transf. is expensive!**
- ✓ **low cost calculation of $\mathbf{Det}(\boldsymbol{D}(\boldsymbol{i\mu}))$**

$$Det\{D(\mu)\} = Det\{1-\kappa Q(\mu)\} = e^{Tr\{\log(1-\kappa Q)\}}$$

κ : hopping parameter, Q : hopping term

➢ **Winding number expansion method**

$$\boxed{Tr\{\log(1-\kappa Q)\} = \sum_k W_k e^{\frac{\mu}{T}k}}$$

- ✓ $\{\boldsymbol{W_k}\}$ **does not depend on $\boldsymbol{\mu}$**

Construction of $\{W_k\}$(our method)

✓ **Wilson fermion**

$\left(Q_i^{(\pm)} : \pm i - \text{directed hopping}\right)$

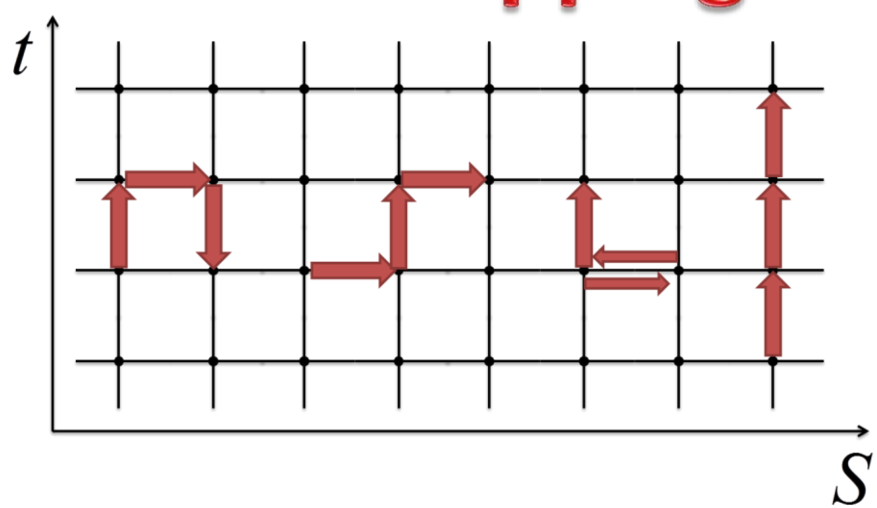
$$Q = \sum_{i=1}^{3}\left(Q_i^{(+)} + Q_i^{(-)}\right) + e^{\mu}Q_4^{(+)} + e^{-\mu}Q_4^{(-)}$$

$$Tr\{\log(1-\kappa Q)\} = -\sum \frac{\kappa^n}{n} Tr\{Q^n\} = -\sum \frac{\kappa^n}{n} Tr\left\{X_m^{(n)}\right\} e^{m\mu}$$

$X_m^{(n)}$

number of hopping

hopping for time direction



ex. $n = 3$

part of $X_0^{(3)}, X_1^{(3)}, X_3^{(3)}$

Construction of $\left\{X_m^{(n)}\right\}$

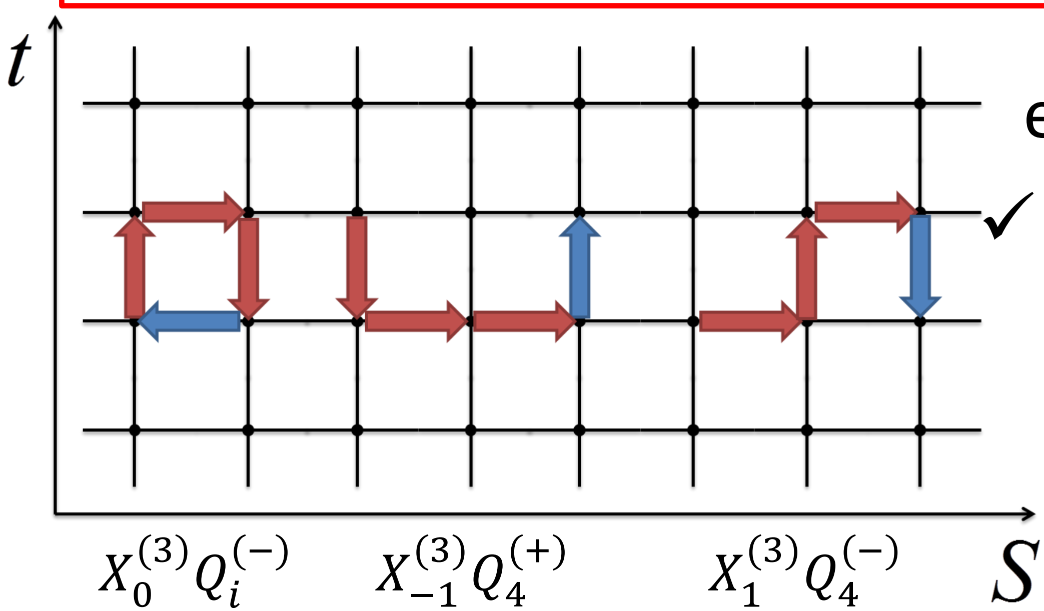
$X_m^{(n)}$

number of hopping → (n)

hopping for time direction → m

✓ **recurrent formula $n \rightarrow n+1$**

$$X_m^{(n+1)} = \sum_{i=1}^{3}\left(Q_i{}^{(+)} + Q_i{}^{(-)}\right)X_m^{(n)} + Q_4{}^{(+)}X_{m-1}^{(n)} + Q_4{}^{(-)}X_{m+1}^{(n)}$$



ex. $X_m^{(3)} \rightarrow X_0^{(4)}$

✓ **$m = N_t k$ can contribute**

$$W_k = -\sum_n \frac{\kappa^n}{n} Tr\left\{X_{N_t k}^{(n)}\right\}$$

Construction of $\left\{X_m^{(n)}\right\}$

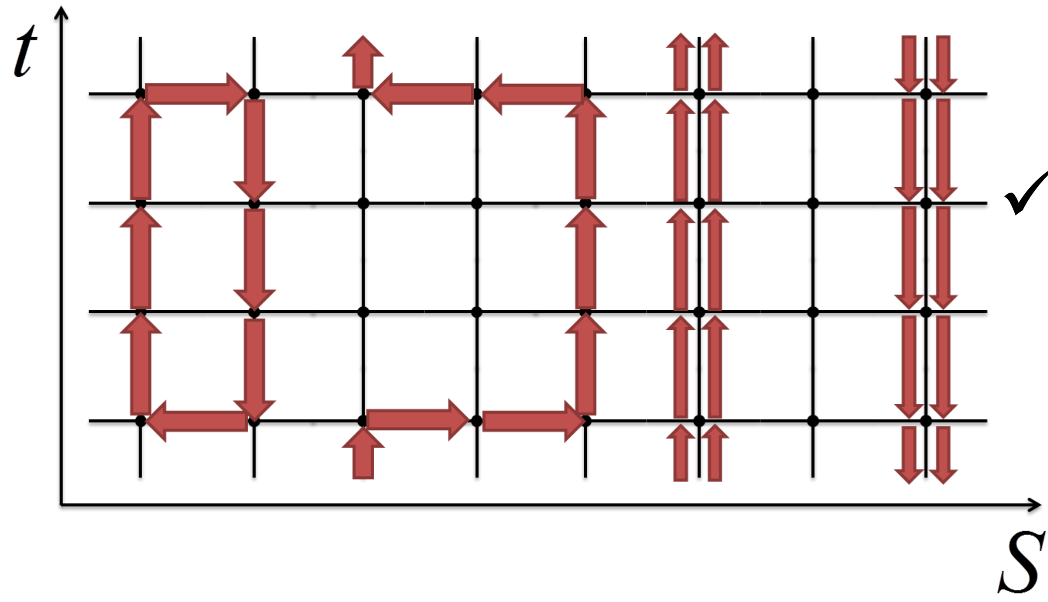
$X_m^{(n)}$

number of hopping

hopping for time direction

✓ **recurrent formula $\boldsymbol{n} \to \boldsymbol{n}+\mathbf{1}$**

$$\boxed{X_m^{(n+1)} = \sum_{i=1}^{3} \left(Q_i{}^{(+)} + Q_i{}^{(-)}\right) X_m^{(n)} + Q_4{}^{(+)} X_{m-1}^{(n)} + Q_4{}^{(-)} X_{m+1}^{(n)}}$$



ex. $X_0^{(8)}, X_4^{(8)}, X_8^{(8)}, X_{-8}^{(8)}$

✓ **$\boldsymbol{m = N_t k}$ can contribute**

$$\boxed{W_k = -\sum_n \frac{\kappa^n}{n} Tr\left\{X_{N_t k}^{(n)}\right\}}$$

k ; winding number

Method of calculation –summary-

$$Z_{can.}(T;N,V) = \left\langle \frac{1}{2\pi} \int_{-\pi}^{\pi} d\frac{\mu}{T}\, e^{-i\frac{\mu}{T}N}\, \frac{Det\{D(i\mu)\}}{Det\{D(0)\}} \right\rangle_g$$

$$Det\{D(i\mu)\} = \exp\left\{\sum_k W_k e^{i\frac{\mu}{T}k}\right\}$$

1. generate gauge conf. at $\mu = 0$
2. calculate $\{W_k\}$ from $\{X_m^{(n)}\}$
3. perform the Fourier transformation
   - **obtain $Z_{can}(N)$** (arXiv:1504.06351 R. Fukuda, A. Nakamura, S. Oka)
4. reconstruct Hadronic observable **at real μ**

$$Z_{G.C.}(\mu) = \sum_N Z_{can.}(N) e^{\frac{\mu}{T}N},\ \langle \hat{N}^k \rangle = \frac{\sum_N N^k Z_{can.}(N) e^{\frac{\mu}{T}N}}{\sum_N Z_{can.}(N) e^{\frac{\mu}{T}N}}$$

$$\langle \bar{\psi}\psi \rangle(\mu) = \sum_N O(N) e^{\frac{\mu}{T}N}$$

(arXiv:1504.04471 A. Nakamura, S. Oka, Y. Taniguchi)

Numerical setup

Action :

Iwasaki gauge action

2-flavor Wilson clover action

Lattice size : $8^3 \times 4$, $12^3 \times 4$

✓configulation : $100 - 200$

✓ β : confined phase to deconfined phase

✓ $T_C = 222.5(11) MeV$

✓ κ : small

✓ HPE : up to 480th order

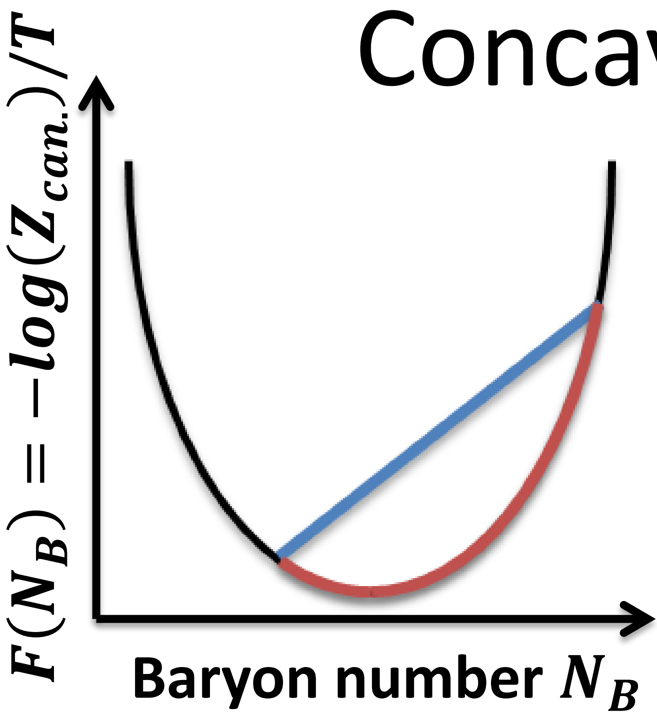
(winding number $\in [-120,120]$)

✓ Discrete Fourier transf. : 512

parameters

β	κ	Csw	T/T_C	m_π/m_ρ
0.9	0.137	1.1	0.644	0.8978(55)
1.1	0.133	1.1	0.673	0.9038(56)
1.3	0.133	1.1	0.706	0.8770(52)
1.5	0.131	1.1	0.813	0.8486(58)
1.6	0.130	1.1		
1.7	0.129	1.1	1.00	0.770(13)
1.8	0.126	1.1		
1.9	0.125	1.1	1.68	0.714(15)
2.1	0.122	1.1	3.45	0.836(47)

Concavity of Free energy



✓ required from thermodynamics

$$\lambda\, F(N_1) + (1-\lambda)\, F(N_2) > F(\lambda\, N_1 + (1-\lambda) N_2)$$

✓ Special case :

$$\lambda = 1/2, N_1 = N, N_2 = N + 2$$

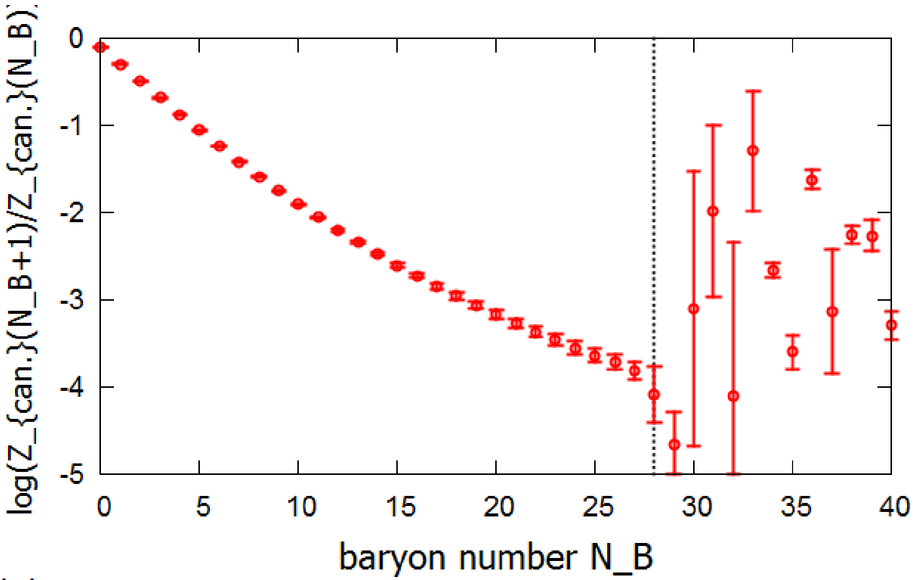
$$\log Z_{can}(N+1) > \log \sqrt{Z_{can}(N) Z_{can}(N+2)}$$

$$\Updownarrow$$

$$\frac{Z_{can}(N+1)}{Z_{can}(N)} > \frac{Z_{can}(N+2)}{Z_{can}(N+1)}$$

✓ monotonic decrease of $Z_{can}(N+1)/Z_{can}(N)$

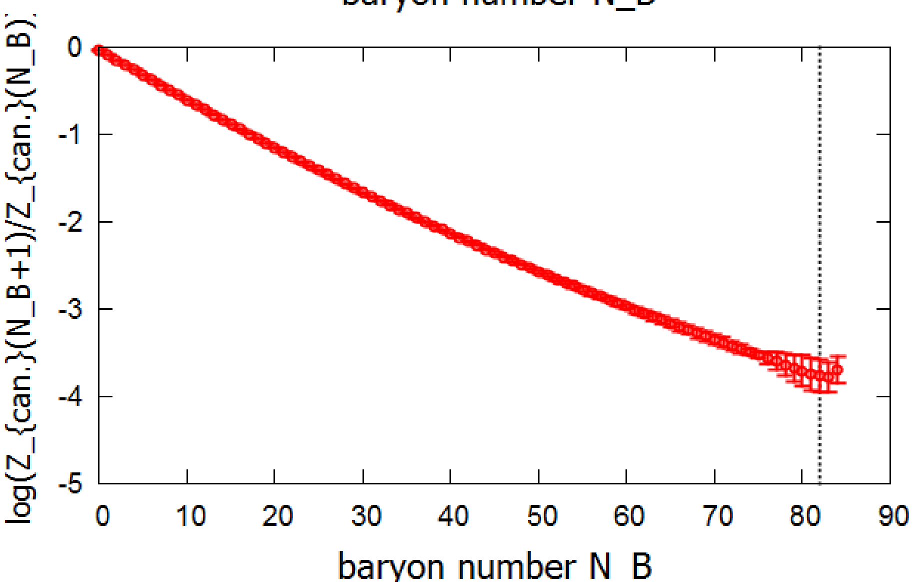
Application range of winding number expansion method



- $\log\left(\frac{Z_{can}(N+1)}{Z_{can}(N)}\right)$

✓ $ex.\ \beta = 1.9\,, 8^3 \times 4\ lattice$

✓ set $Z_{can}(N > N_{cut} = 28) = 0$
(truncate the fugacity expansion)



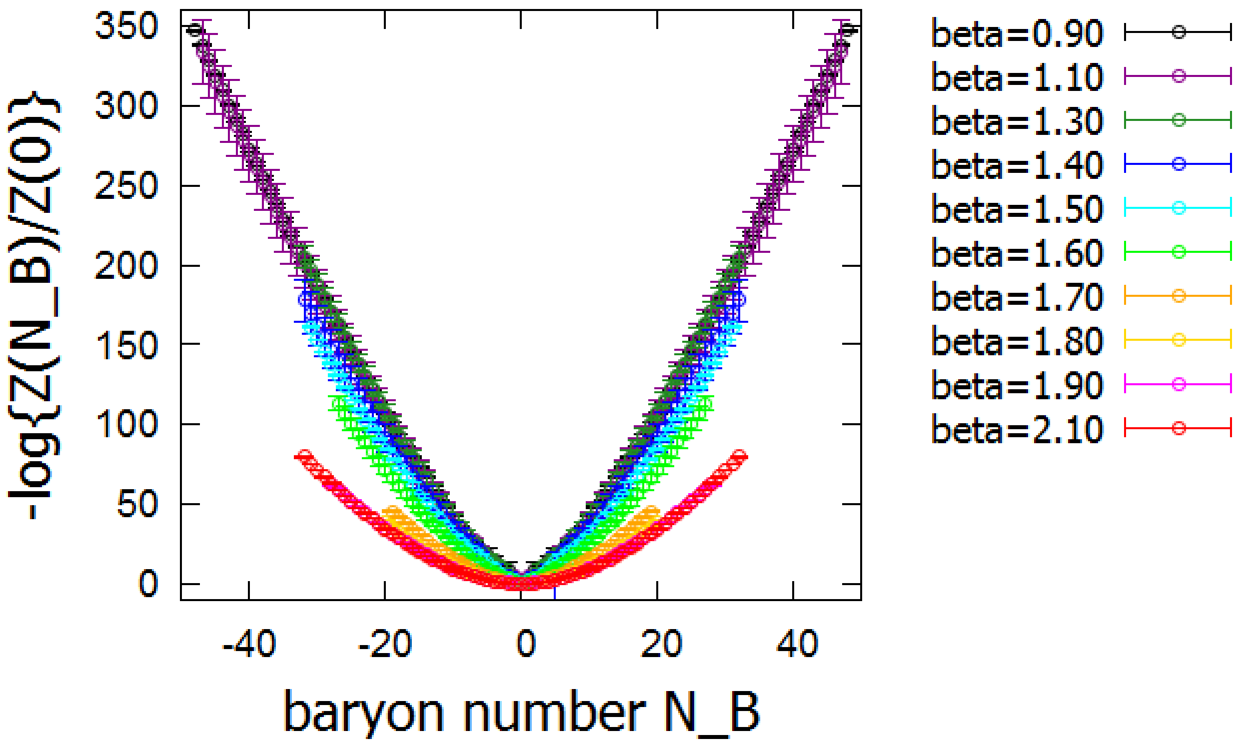
✓ $ex.\ \beta = 1.9\,, 12^3 \times 4\ lattice$

✓ $N_{cut} = 82$

✓ density (and β) decide N_{cut}

$$\frac{28}{8^3} \sim \frac{82}{12^3} \sim 0.05$$

Free energy



β	T/T_C
0.9	0.644
1.1	0.673
1.3	0.706
1.5	0.813
1.6	
1.7	1.00
1.8	
1.9	1.68
2.1	3.45

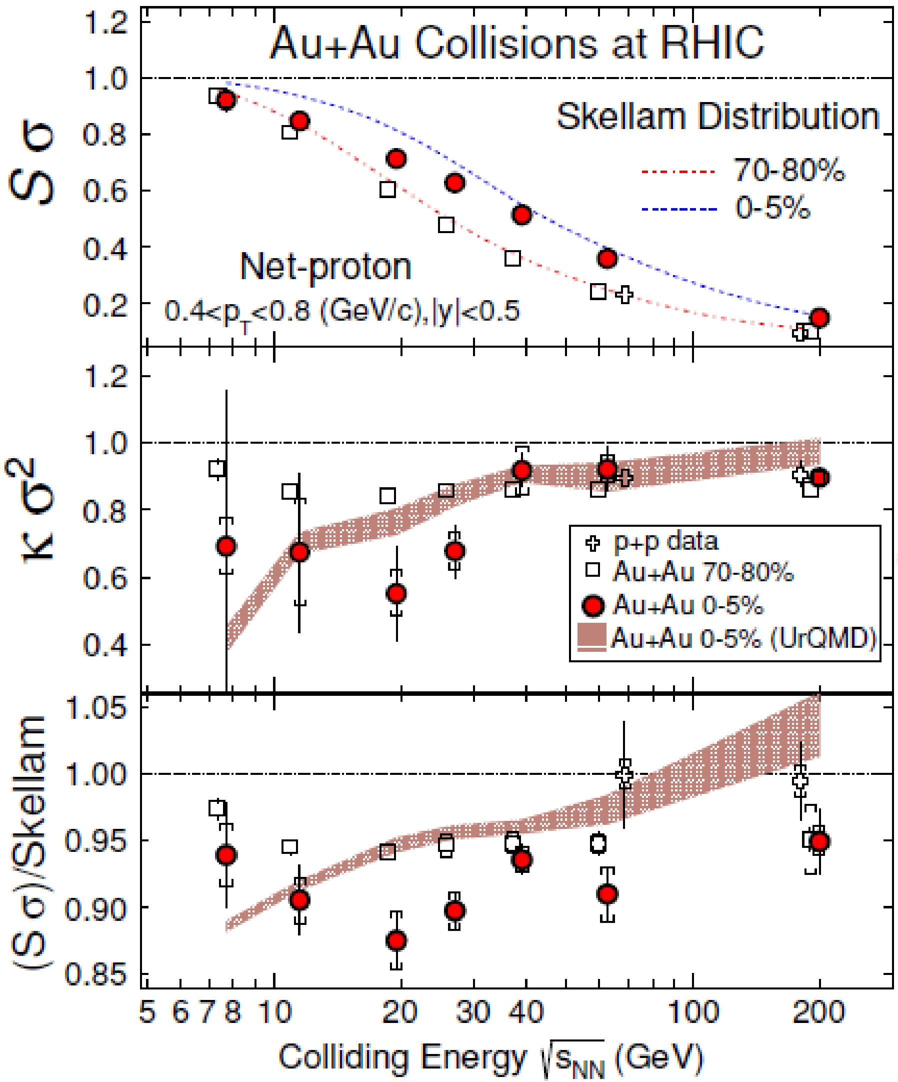
✓High temp. $(T > T_C)$: flat

✓Low temp. $(T < T_C)$: pointed

➢ **evidence of phase transition?**

Quark number cumulant

$$\langle \hat{N}^k \rangle_C := \frac{\partial^k}{\partial(\mu/T)^k} \log Z_{G.C.}$$



STAR Collaboration
Phys. Rev. Lett. 112 (2014) 032302

Hadron Resonance Gas

$$\frac{1}{3}\frac{\langle \hat{N}^3 \rangle_C}{\langle \hat{N}^2 \rangle_C} = \tanh\left(3\frac{\mu}{T}\right)$$

(dotted line)

$$\frac{1}{9}\frac{\langle \hat{N}^4 \rangle_C}{\langle \hat{N}^2 \rangle_C} = 1$$

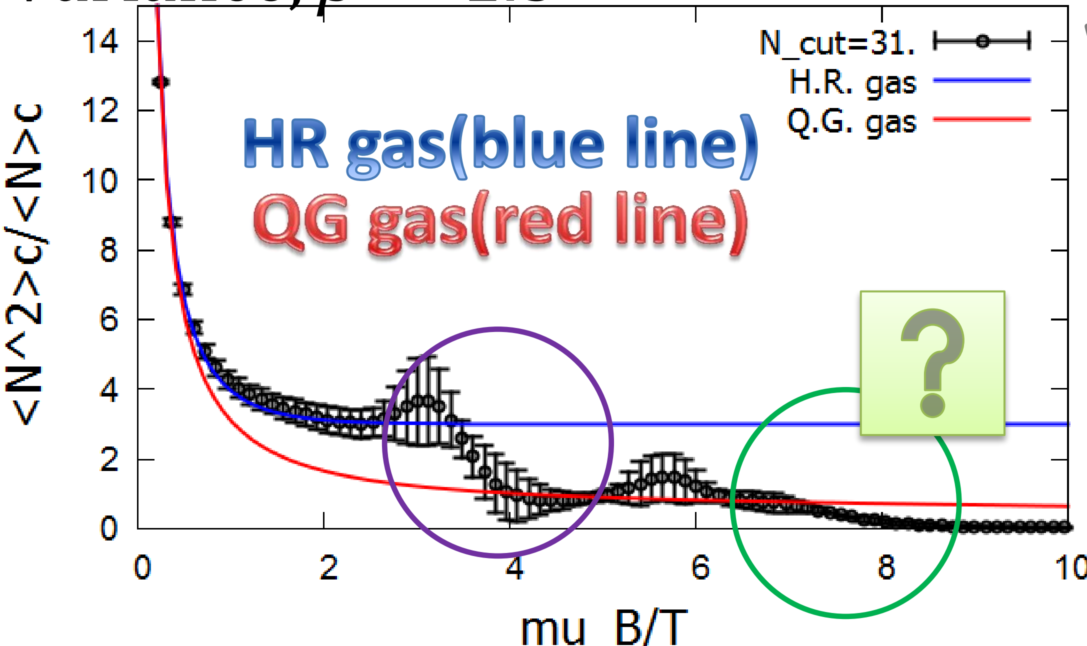
$$\frac{1}{9}\frac{\langle \hat{N}^3 \rangle_C}{\langle \hat{N}^1 \rangle_C} = 1$$

✓ deviation from H.R. Gas

➡ something interesting?

Variance $\langle \hat{N}^2 \rangle_c / \langle \hat{N}^1 \rangle_c$

Variance, $\beta = 1.5$



We truncate the fugacity expansion

$$Z_{G.C.}(\mu_B/T) \sim \sum_{n=-N_{cut}}^{N_{cut}} Z_{can.}(n) e^{n\frac{\mu_q}{T}}$$

$$Z_{can.}(n) = 0 \ , |n| > N_{cut}$$

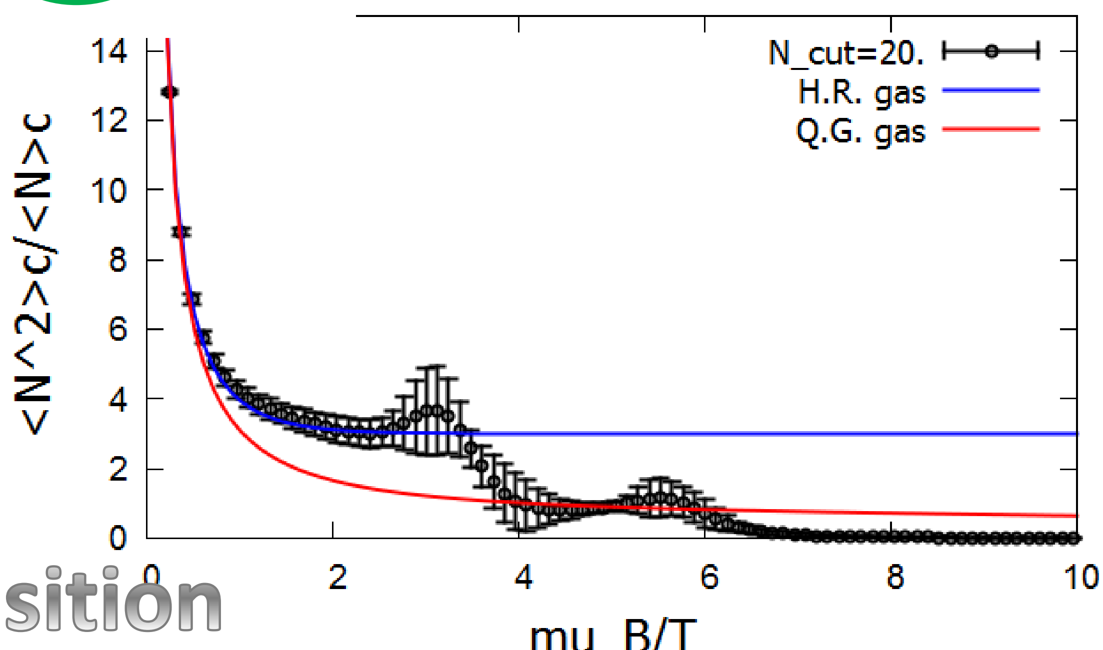
deviate from HR gas

approach to QG gas

deviate from QG gas

approach to 0

N_{cut} creates artificial phase transition



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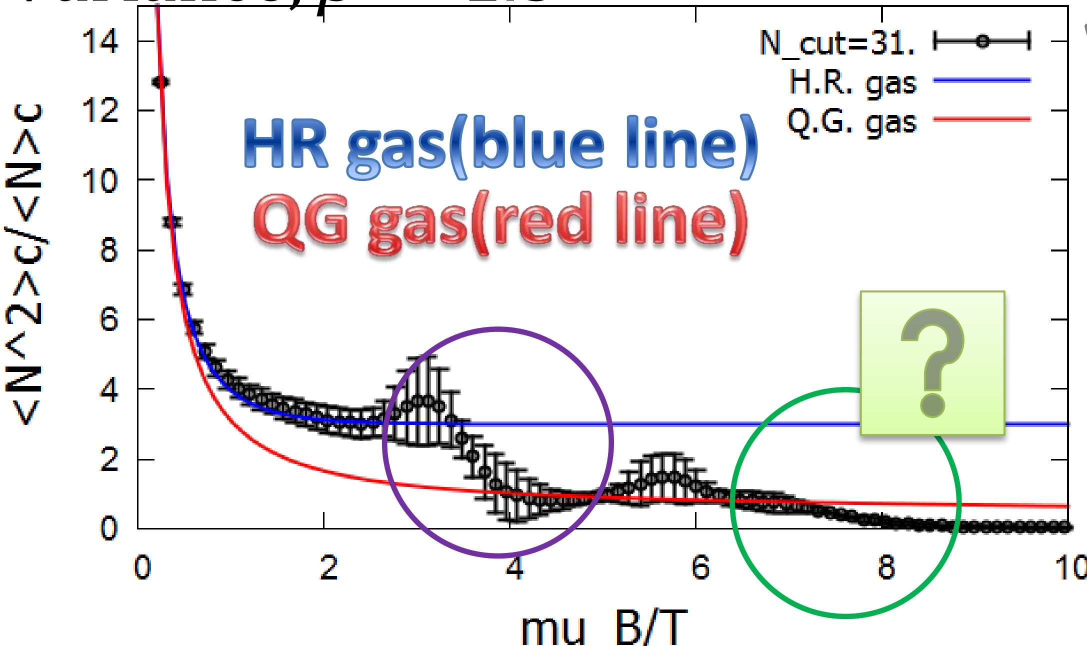

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deviate from HR gas

approach to QG gas

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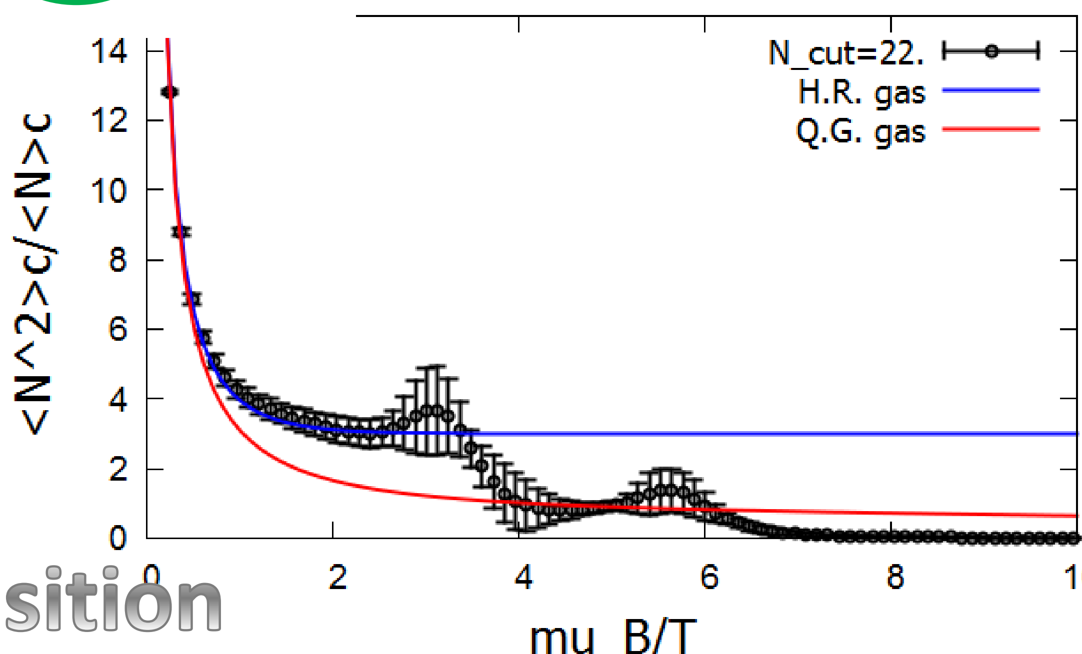

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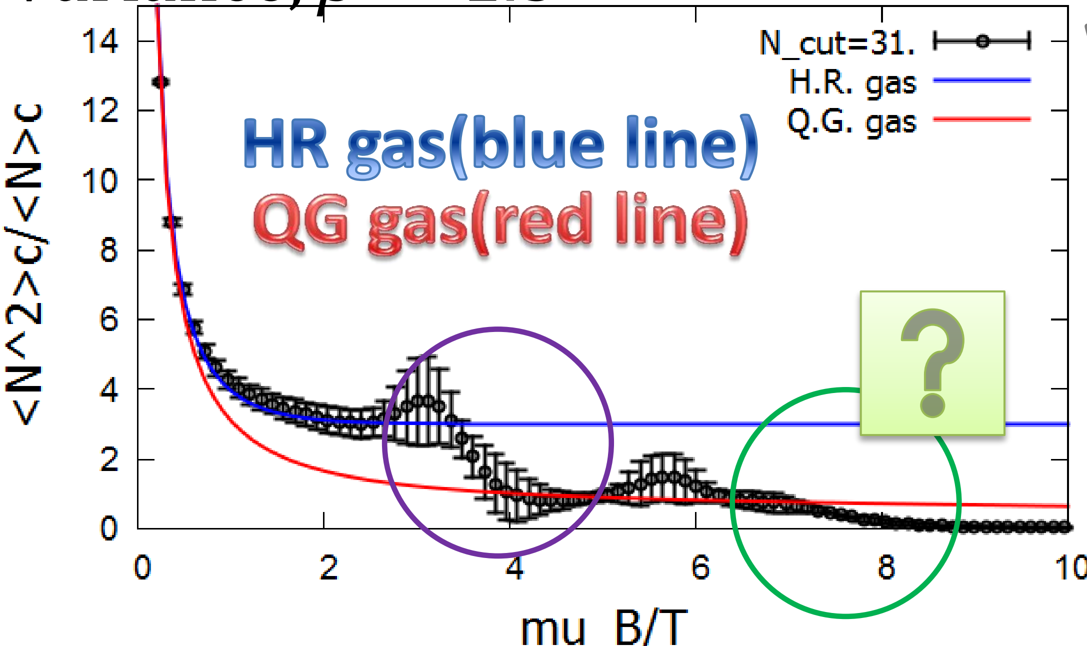

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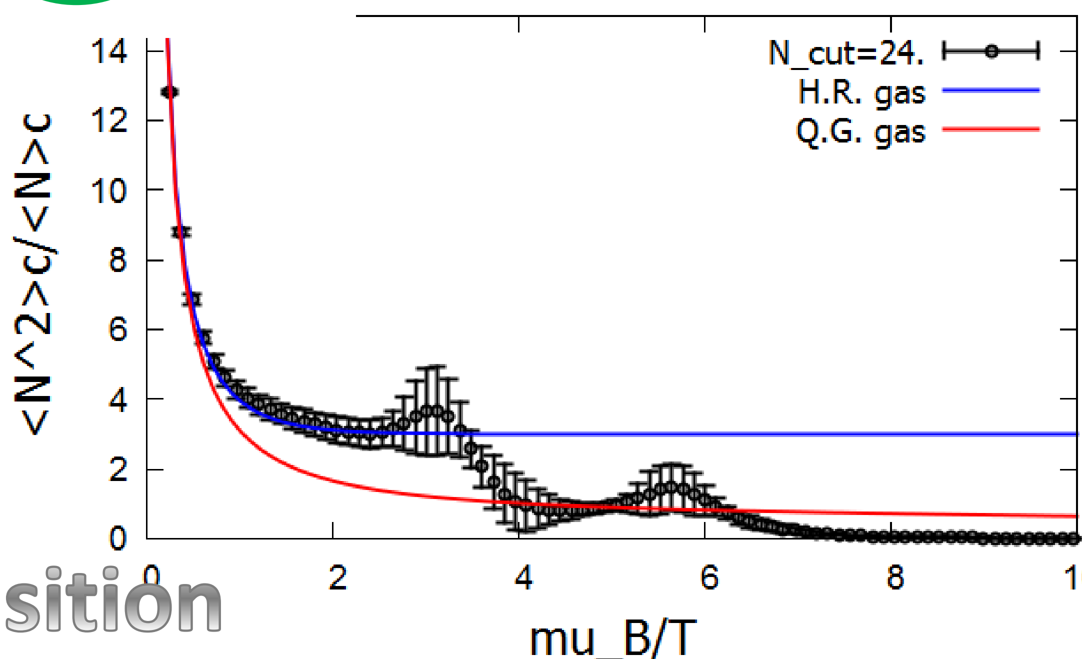

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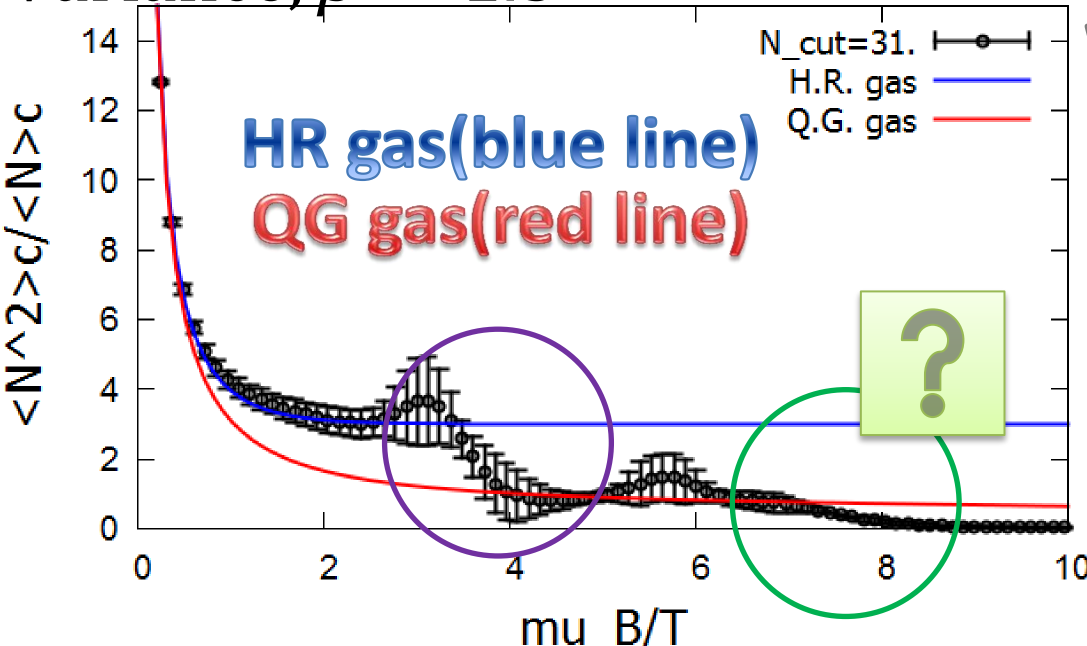
Variance, $\beta = 1.5$



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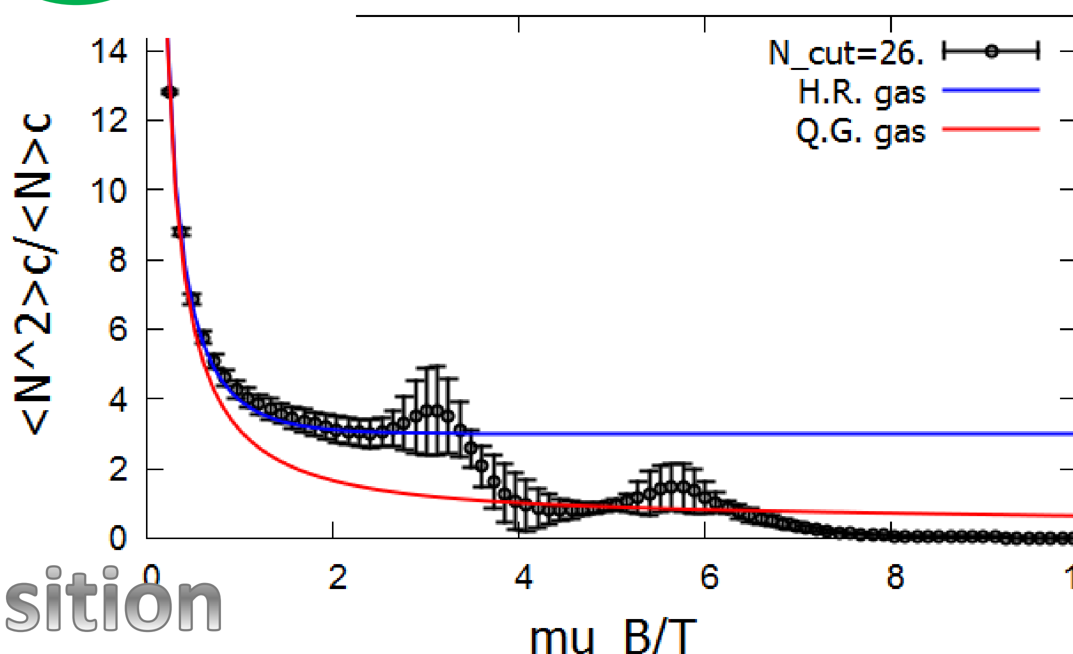

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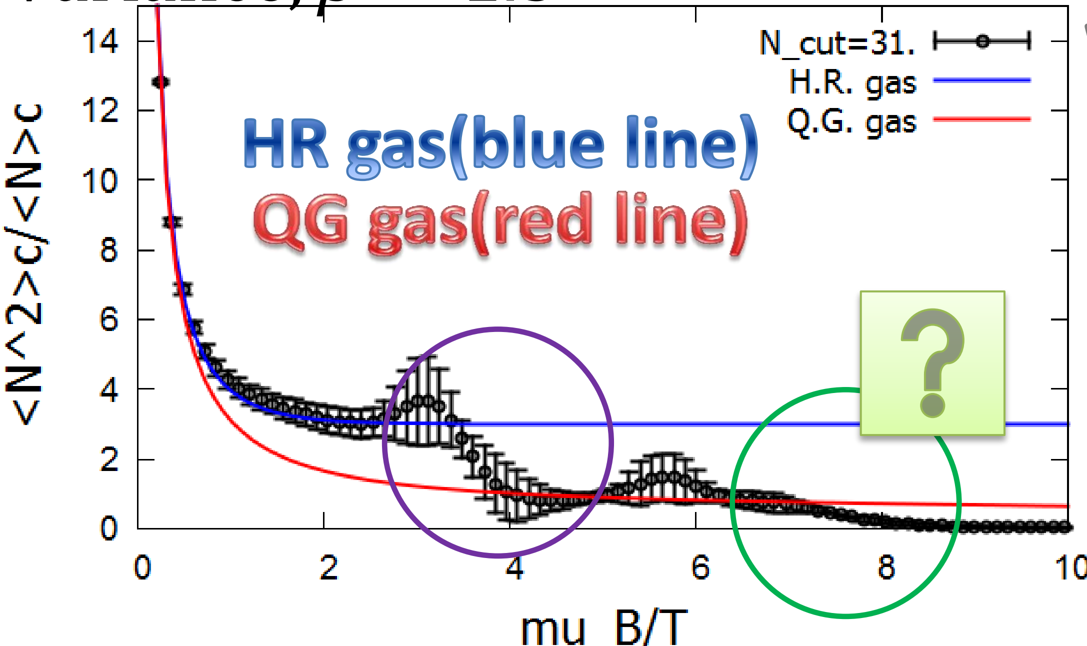

HR gas(blue line)

QG gas(red line)

We truncate the fugacity expansion

$$Z_{G.C.}(\mu_B/T) \sim \sum_{n=-N_{cut}}^{N_{cut}} Z_{can.}(n) e^{n\frac{\mu_q}{T}}$$

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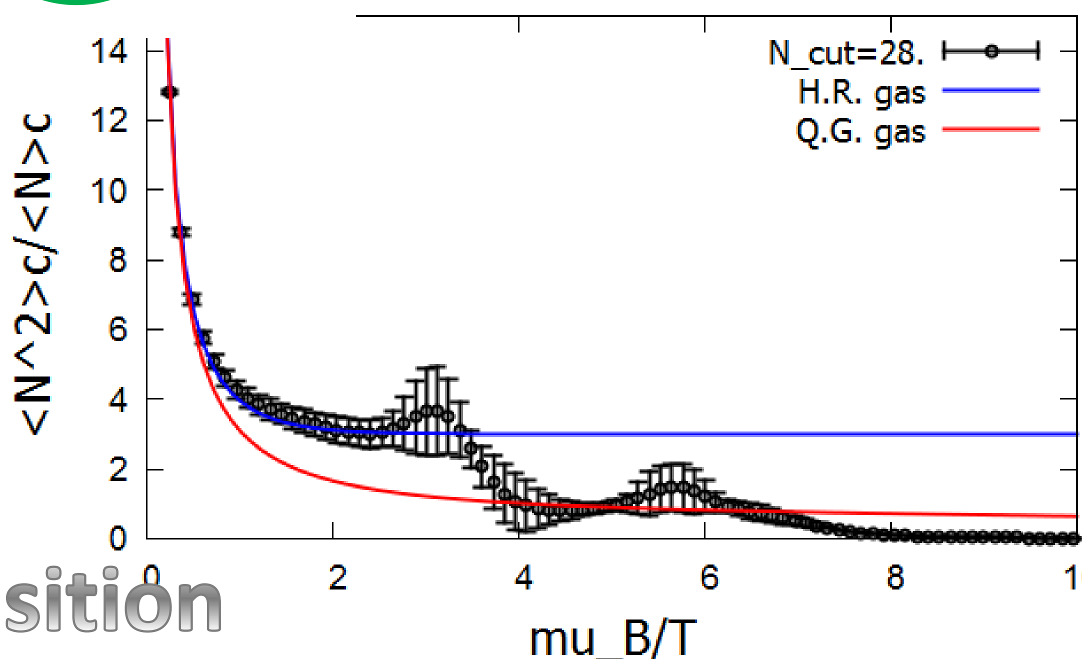

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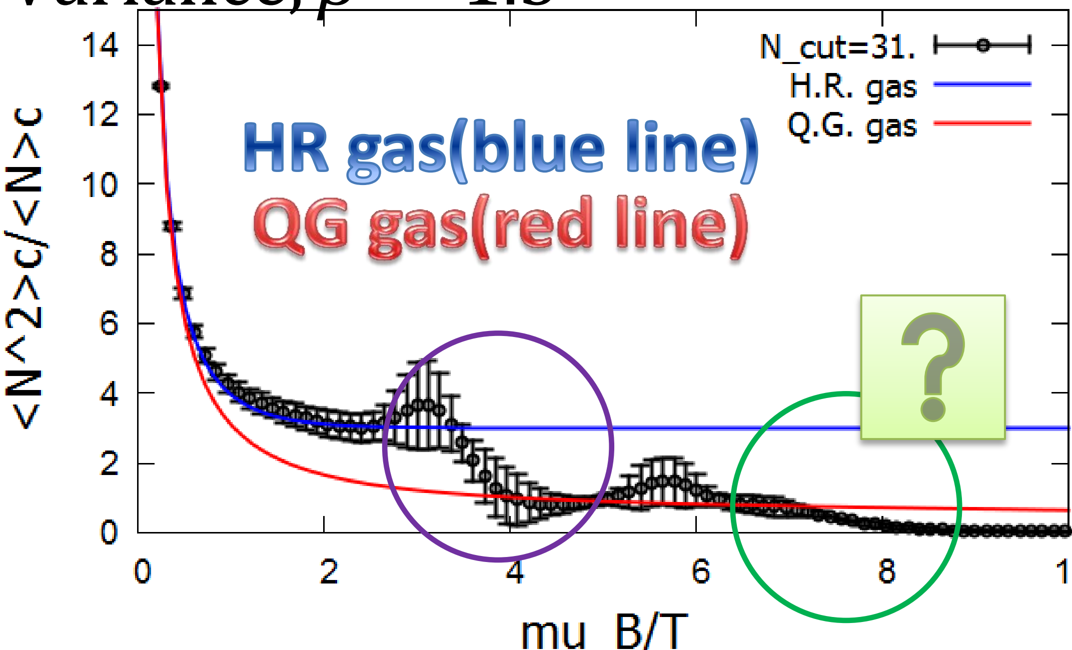

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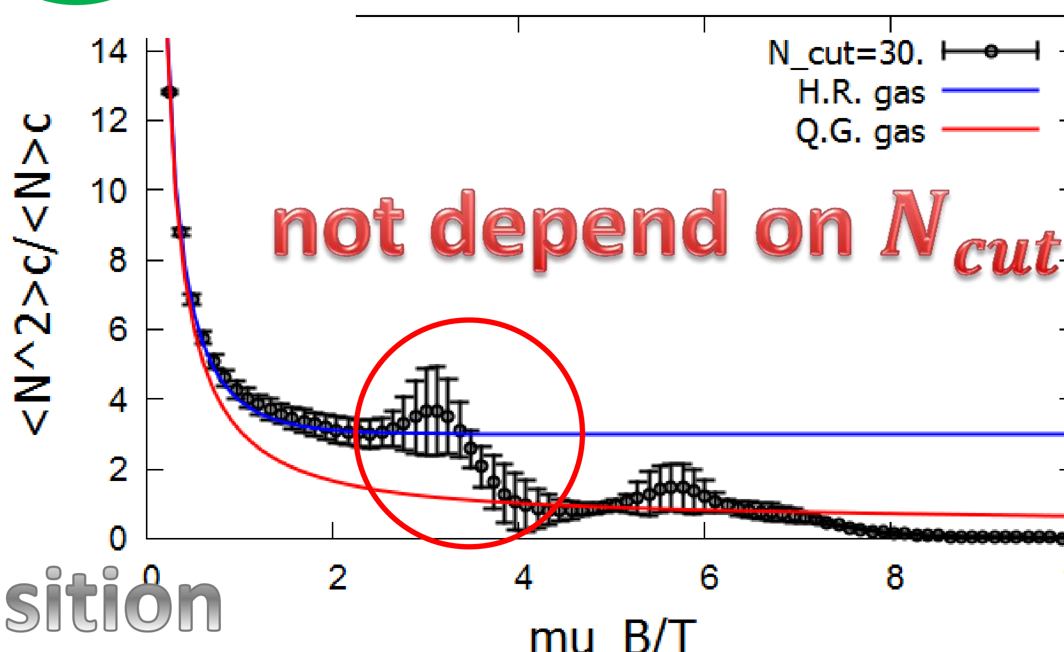
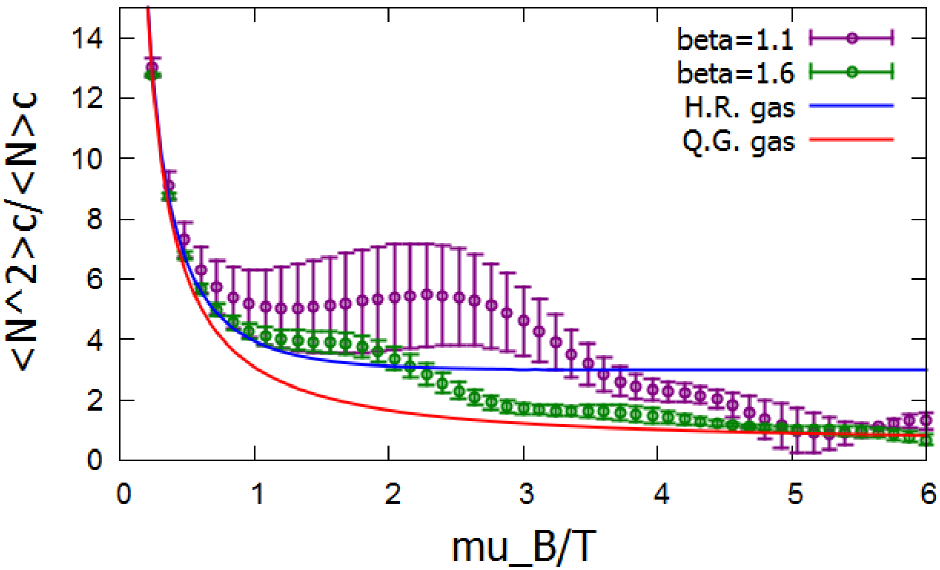
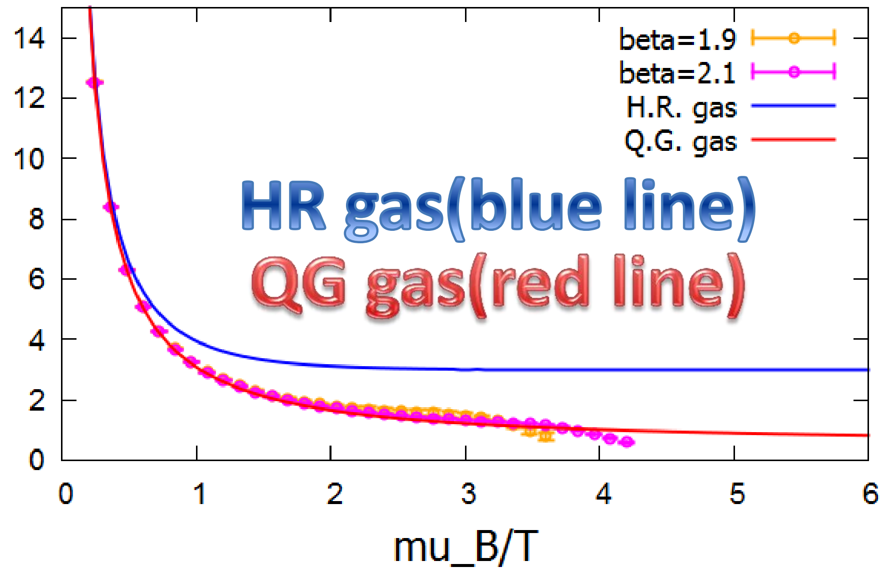


Variance $\langle \hat{N}^2 \rangle_c / \langle \hat{N}^1 \rangle_c (\mu_B/T)$





✓ **Low temp**

HR gas → QG gas transition

✓ **High temp**

approach to Q.G. gas

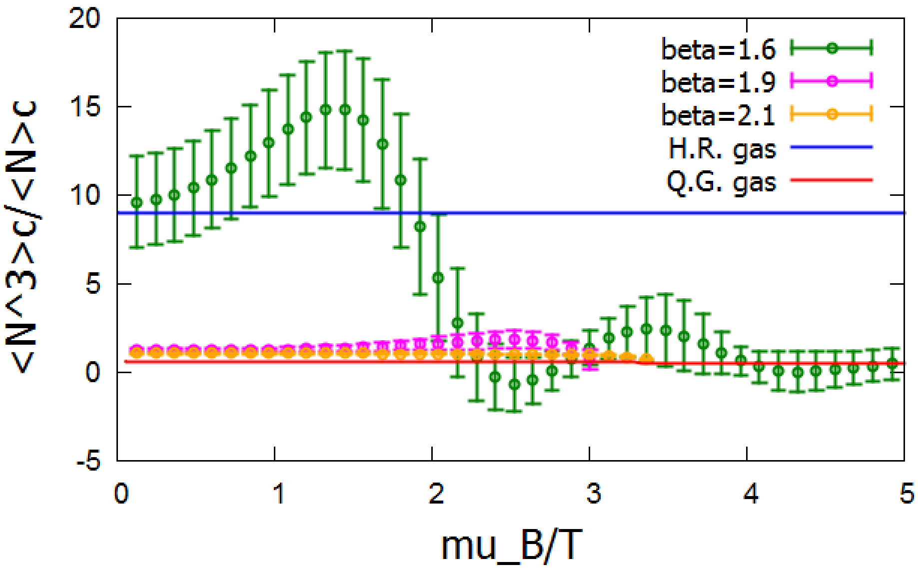
expected from phase diagram

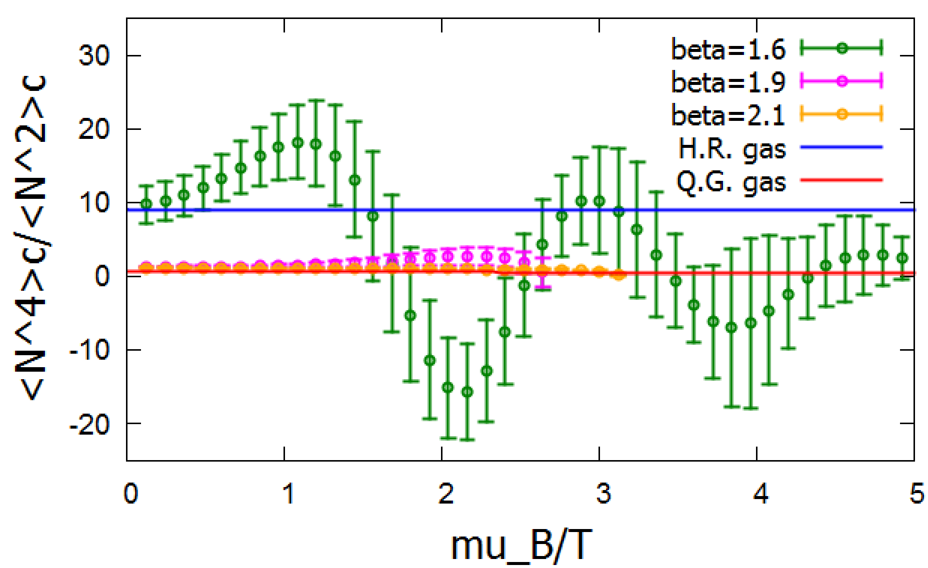
✓ **sign problem remains**

➢ **large error bars**

✓ **sign problem is controlled**

Skewness $\frac{\langle \hat{N}^3 \rangle_c}{\langle \hat{N}^1 \rangle_c}$, Kurtosis $\frac{\langle \hat{N}^4 \rangle_c}{\langle \hat{N}^2 \rangle_c}$





✓ **Low temp**

H.R. gas → Q.G. gas transition

✓ **High temp**

approach to Q.G. gas

◆Kurtosis $(\beta = 1.6)$

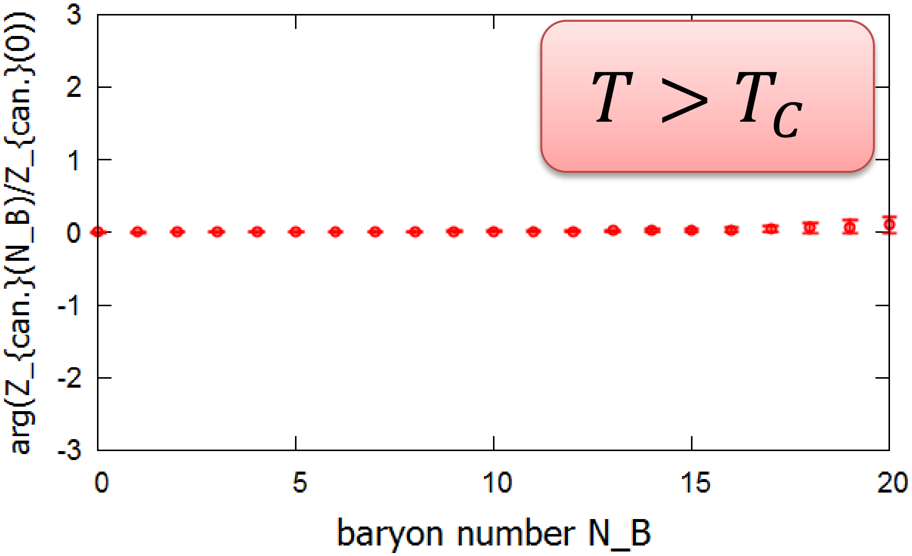
oscillate around $\mu_B/T \sim 2$

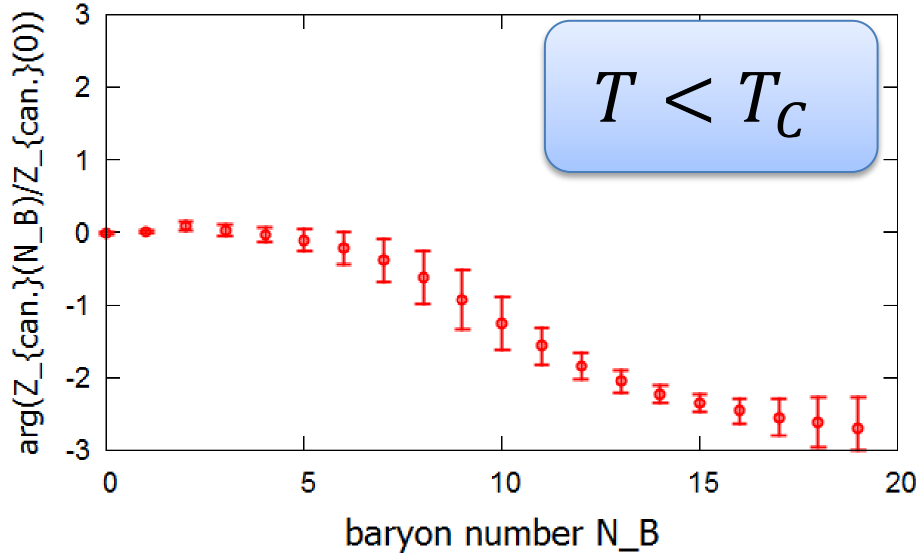
between HR gas region and QG gas region

indicate phase transition?

Dark side of Canonical approach

✓ $\boldsymbol{arg(Z_{can.}(N_B)) = 0}$?



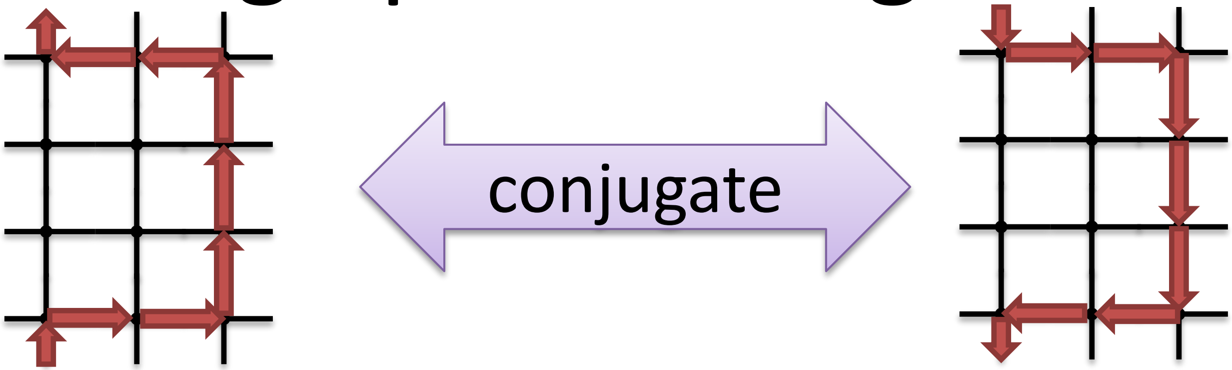


$T > T_C$: consistent $\quad$ $T < T_C$: sign problem!

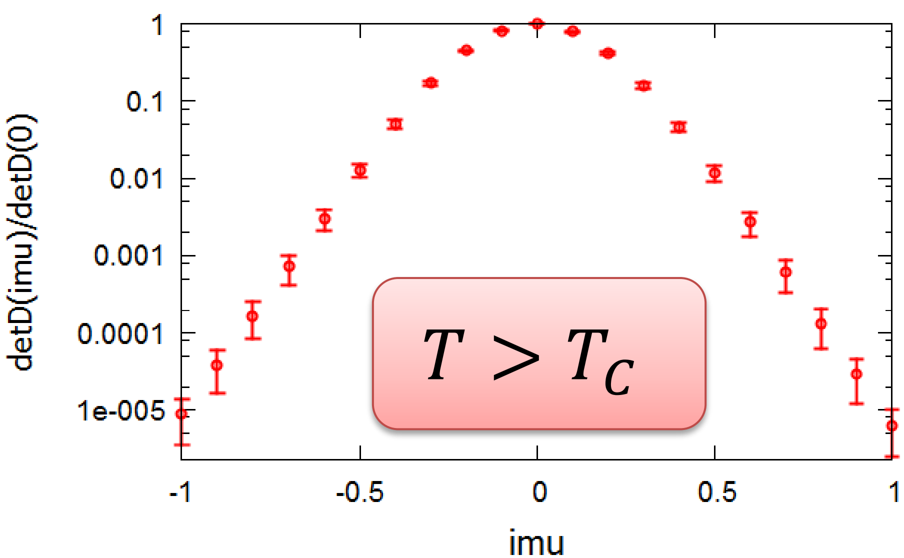
$$Z_{can.}(T;N,V) = \left\langle \boxed{\frac{1}{2\pi}\int_{-\pi}^{\pi} d\frac{\mu}{T}\, e^{-i\frac{\mu}{T}N}\, \frac{Det\{D(i\mu)\}}{Det\{D(0)\}}} \right\rangle_g$$

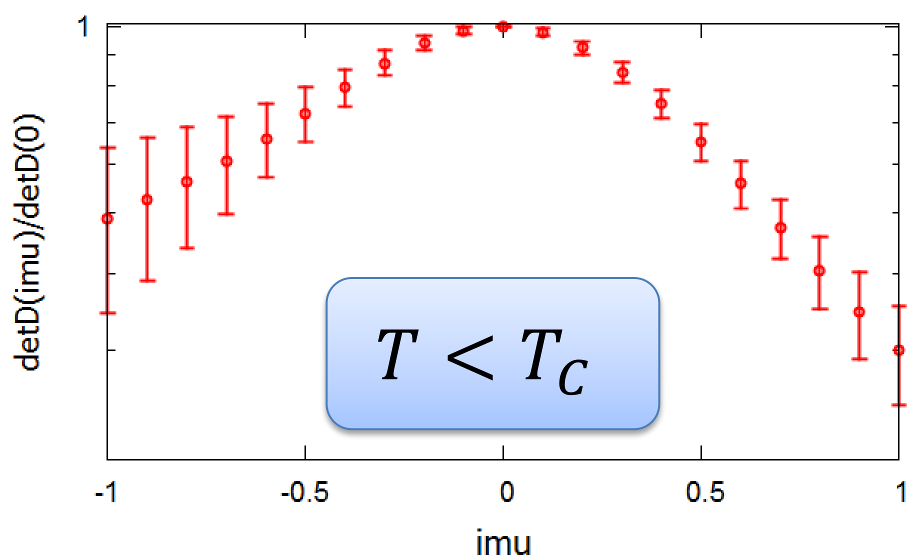
✓ $\boxed{}$ has nonzero phase ! $\quad$ ➢ Sign problem!

Sign problem again !

$W_k \ni$  $\in W_{-k}$

- ✓ $\det D(i\mu) = \exp\left\{\sum_k W_k e^{i\frac{\mu}{T}k}\right\}$ is real
- ✓ γ_5-hermiticity $\det D(i\mu) = \det D(-i\mu)$ is broken





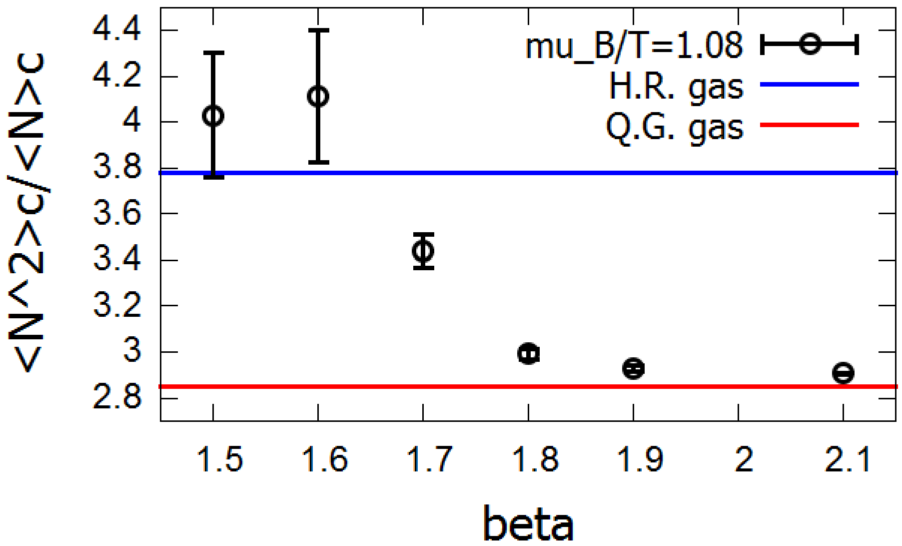
- ➢ another form of sign problem? overlap problem?

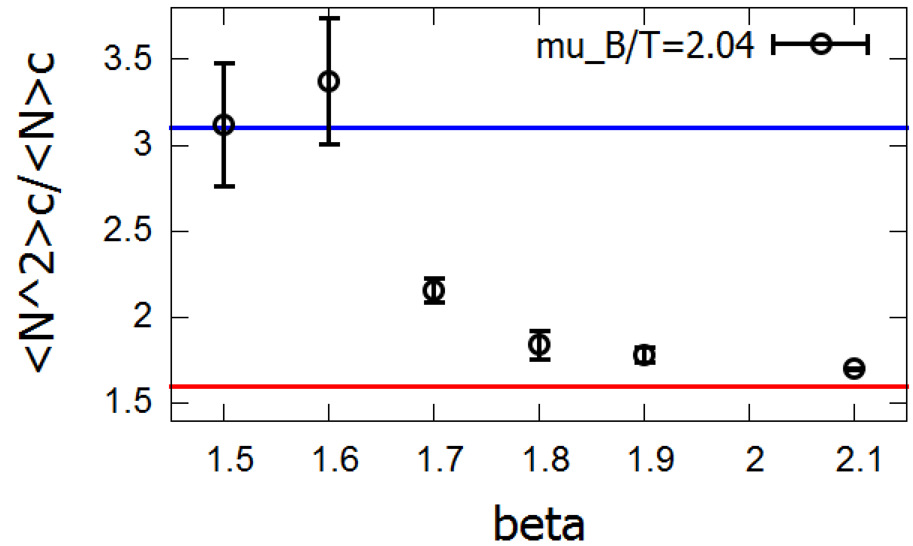
Conclusion

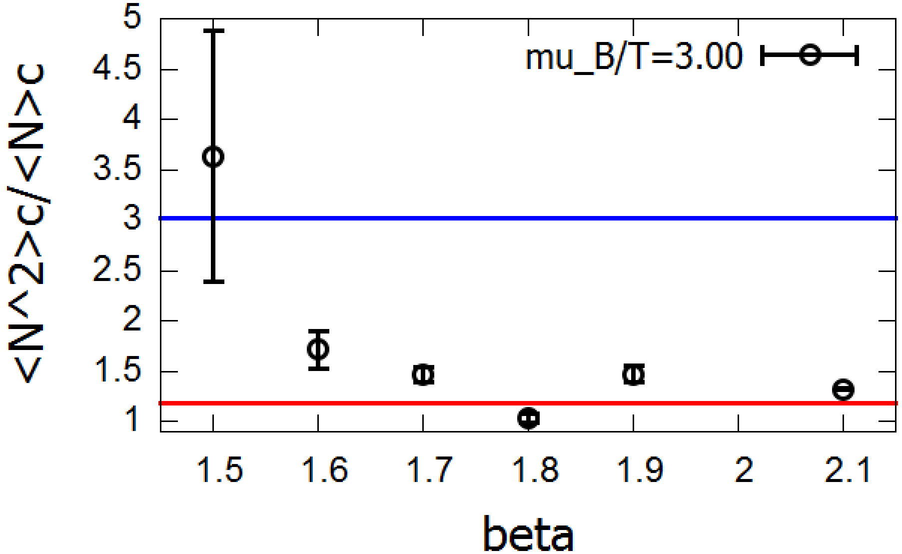
- Canonical approach + winding number expansion
  - truncate the fugacity expansion at some N_{cut}
  - beware to the artificial phase transition
- Canonical approach work well at high temp.
- Sign problem remains at low temp.
  - γ_5 hermiticity is broken
  - large error bars
- Evidence of phase transition
  - shape of the Free energy
  - transition from HR gas to QG gas
  - behavior of Kurtosis

back up

Variance $\langle \hat{N}^2 \rangle_c / \langle \hat{N}^1 \rangle_c (\beta)$







✓ **Low temp :**

Hadron resonance gas

(blue solid line)

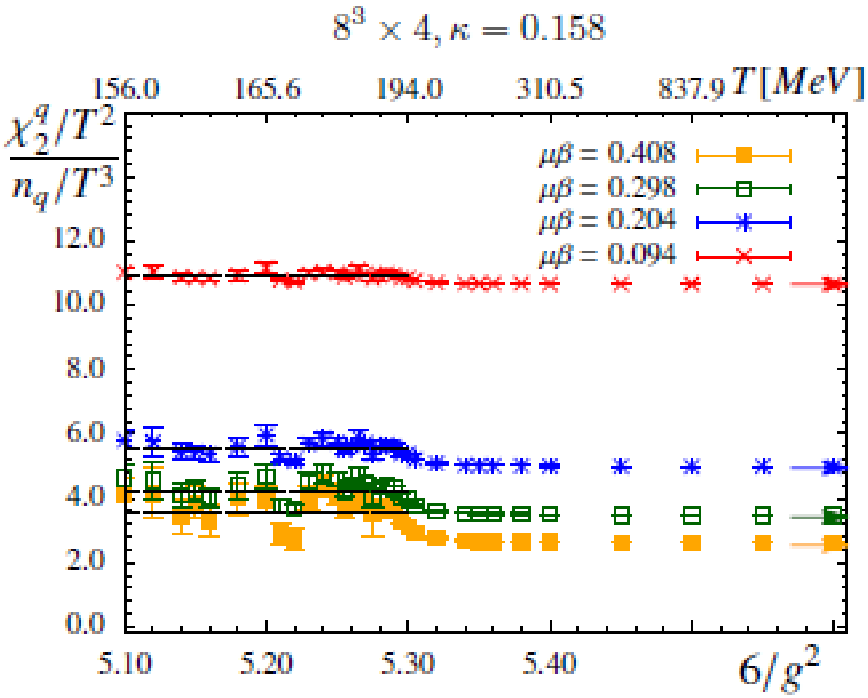
transition

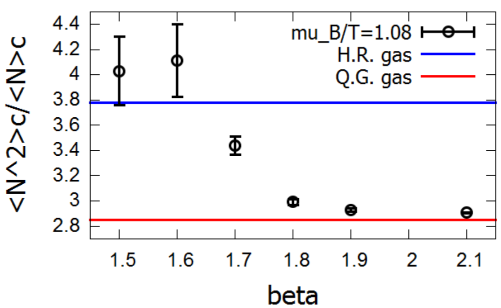
✓ **High temp :**

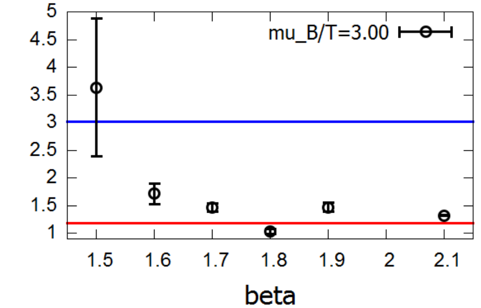
Quark gluon gas

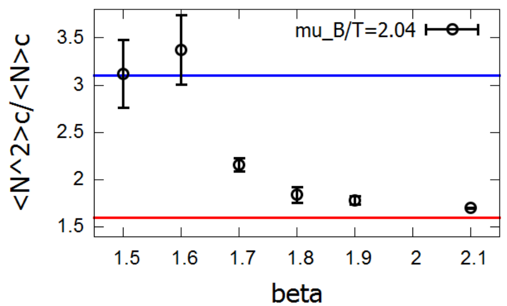
(red solid line)

Comparison with previous work









Christof Gattringer and Hans-Peter Schadler

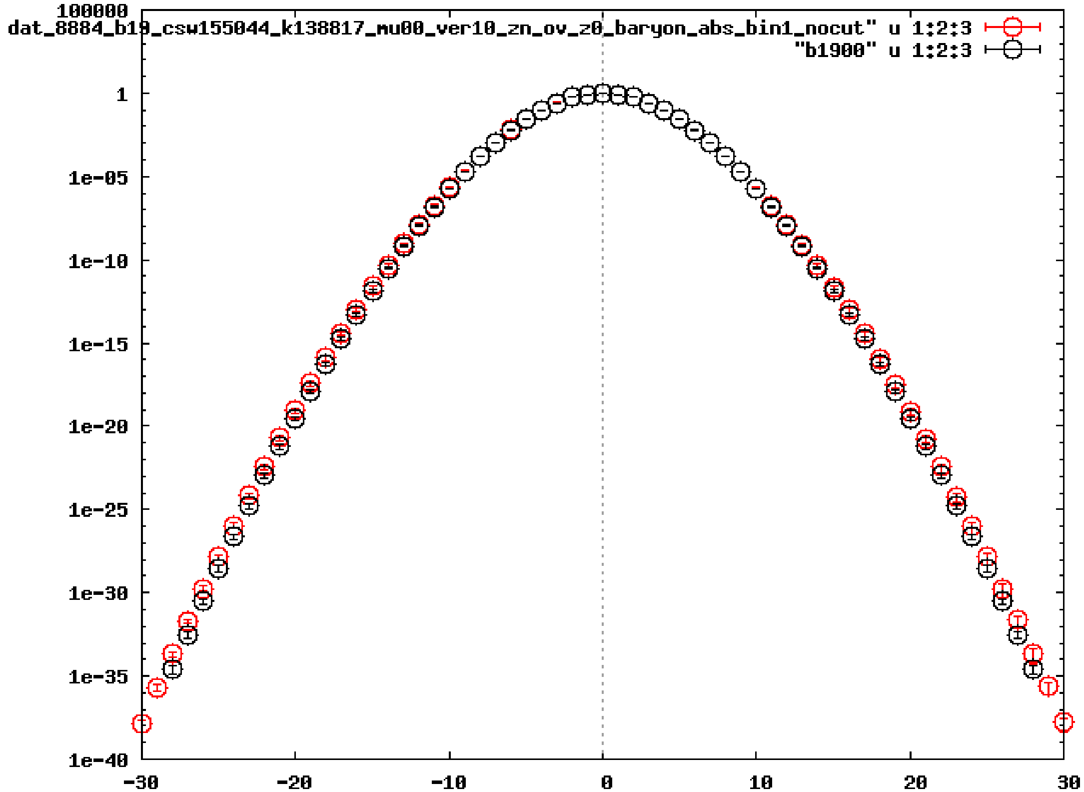
Phys. Rev. D 91, 074511

This work

consistent behavior

✓ application range of HR gas becomes small as μ/T increases

✓ measure the H.R. gas / Q.G. gas transition

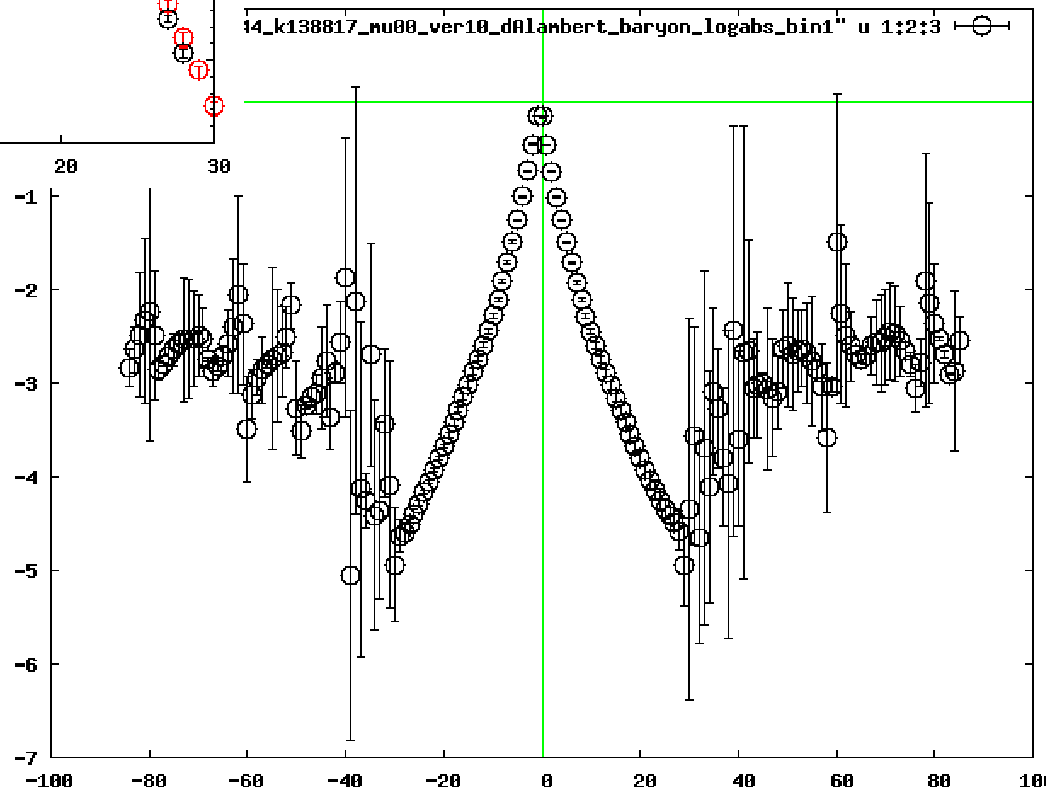


$Z(N)/Z(0)$

○ Can.

○ Can. + HPE

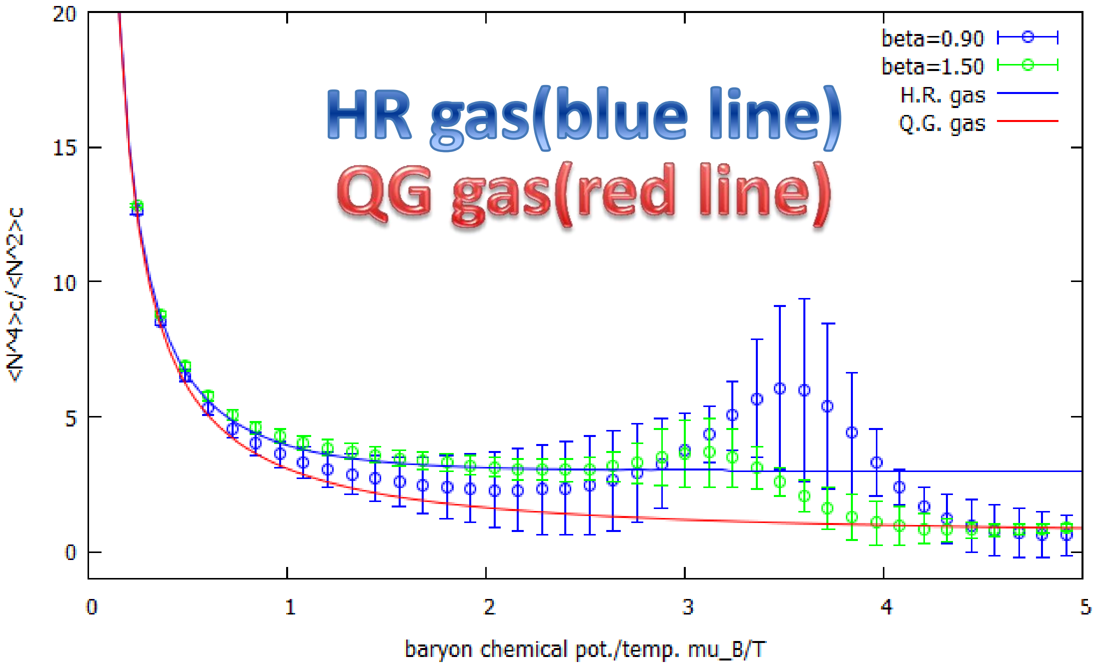
consistent n < 28

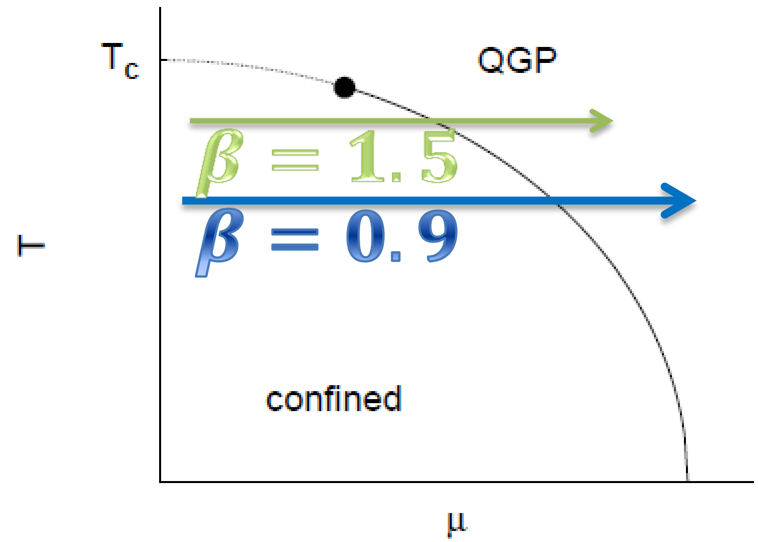


$Z(N+1)/Z(N)$

$N_{cut} = 28$

$\langle \hat{N}^2 \rangle_C / \langle \hat{N}^1 \rangle_C (\mu_B/T)$ at low temperature





consistent

small $\mu_B/T \sim$ H.R. gas

$\mu_B/T \sim 4$ for $\beta = 0.9$

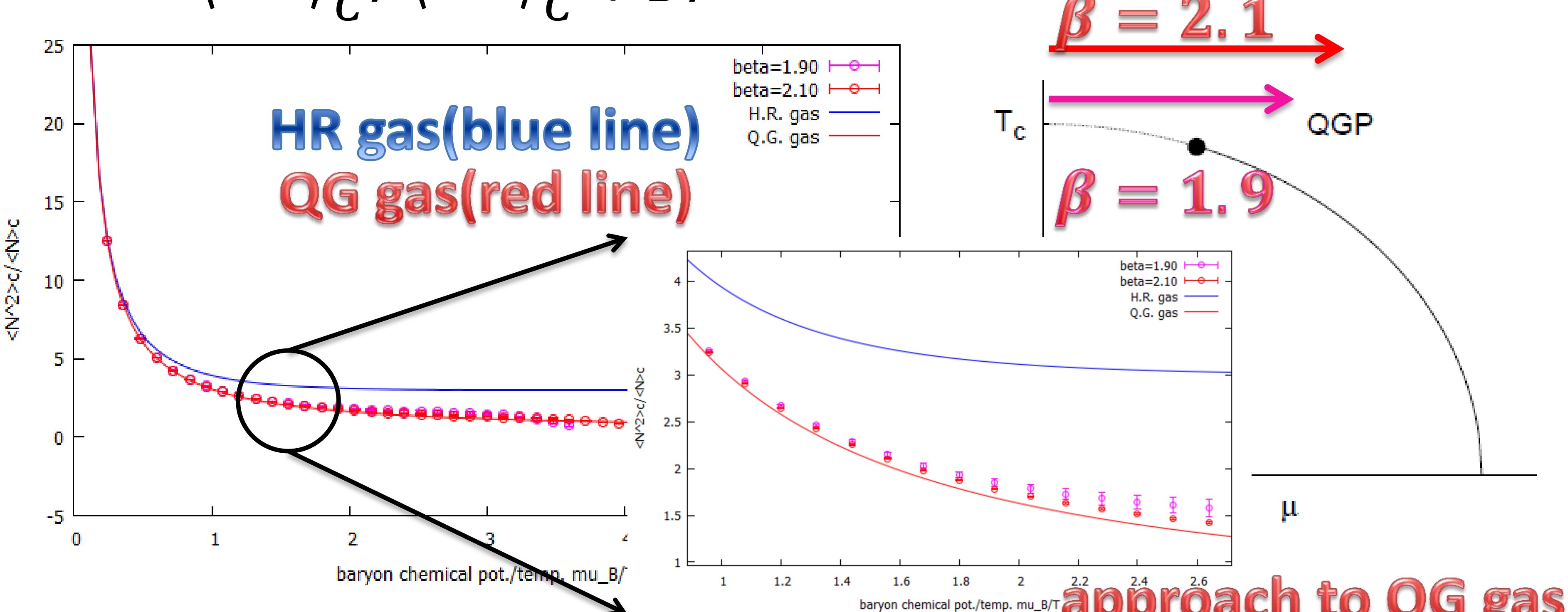
$\mu_B/T \sim 3.5$ for $\beta = 1.5$

large $\mu_B/T \sim$ Q.G. gas

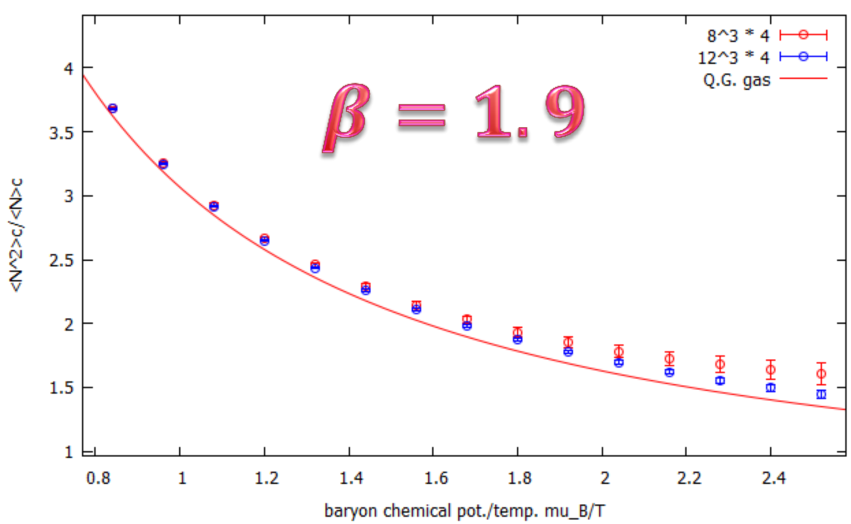
Application range of HR gas becomes small as β increases.

circumstantial evidence of phase transition

$\langle \hat{N}^2 \rangle_C / \langle \hat{N}^1 \rangle_C (\mu_B/T)$ at high temperature



approach to QG gas
as β increases



volume dependence

$8^3 \times 4 \Rightarrow 12^3 \times 4$

approach to QG gas
as volume increases

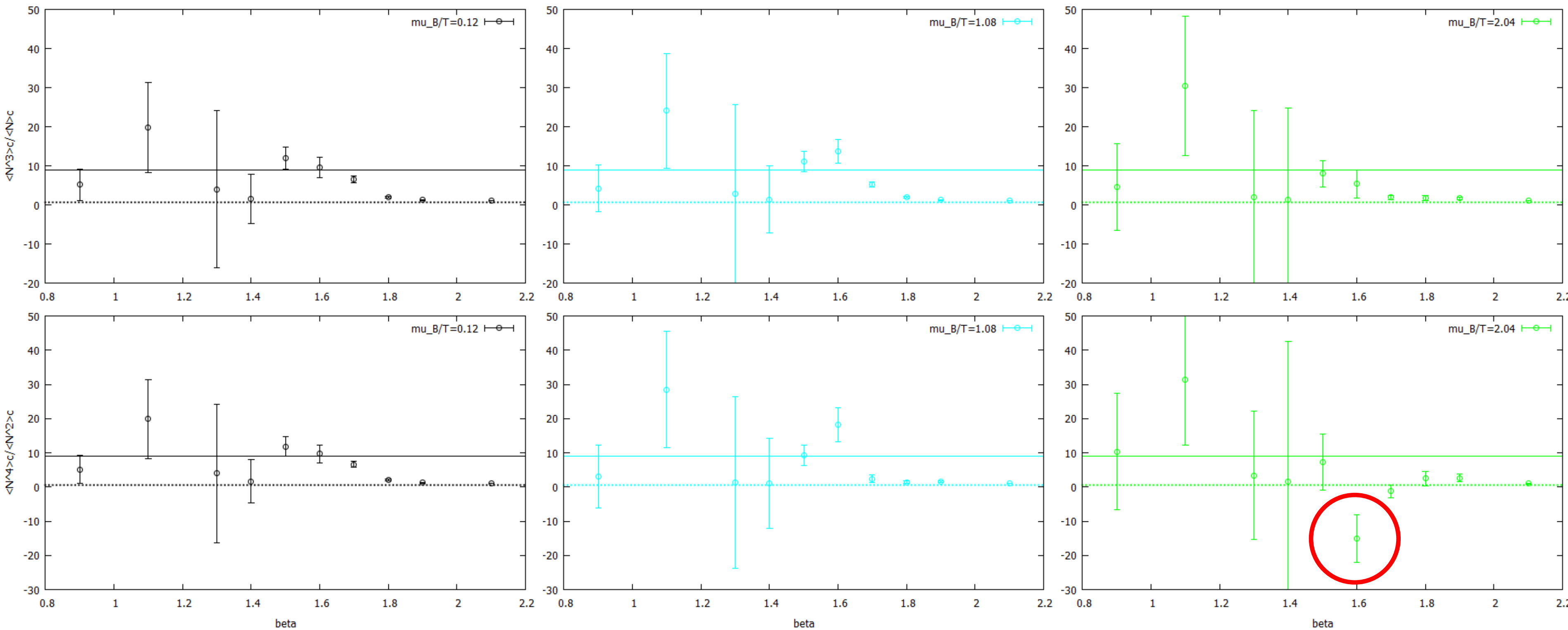
interaction remains

Skewness $\frac{\langle \hat{N}^3 \rangle_c}{\langle \hat{N}^1 \rangle_c}$, Kurtosis $\frac{\langle \hat{N}^4 \rangle_c}{\langle \hat{N}^2 \rangle_c}$ $(vs. \beta)$

solid line : H.R. gas

mu_B/T=0.12
mu_B/T=1.08
mu_B/T=2.04

dotted line : Q.G. gas



Skewness (fig. above)

$\mu_B/T = 0.12\,, 1.08\,, 2.04 \quad \beta \lesssim 1.6$

consistent with H.R. gas

as same as $\langle \hat{N}^2 \rangle_c / \langle \hat{N} \rangle_c$

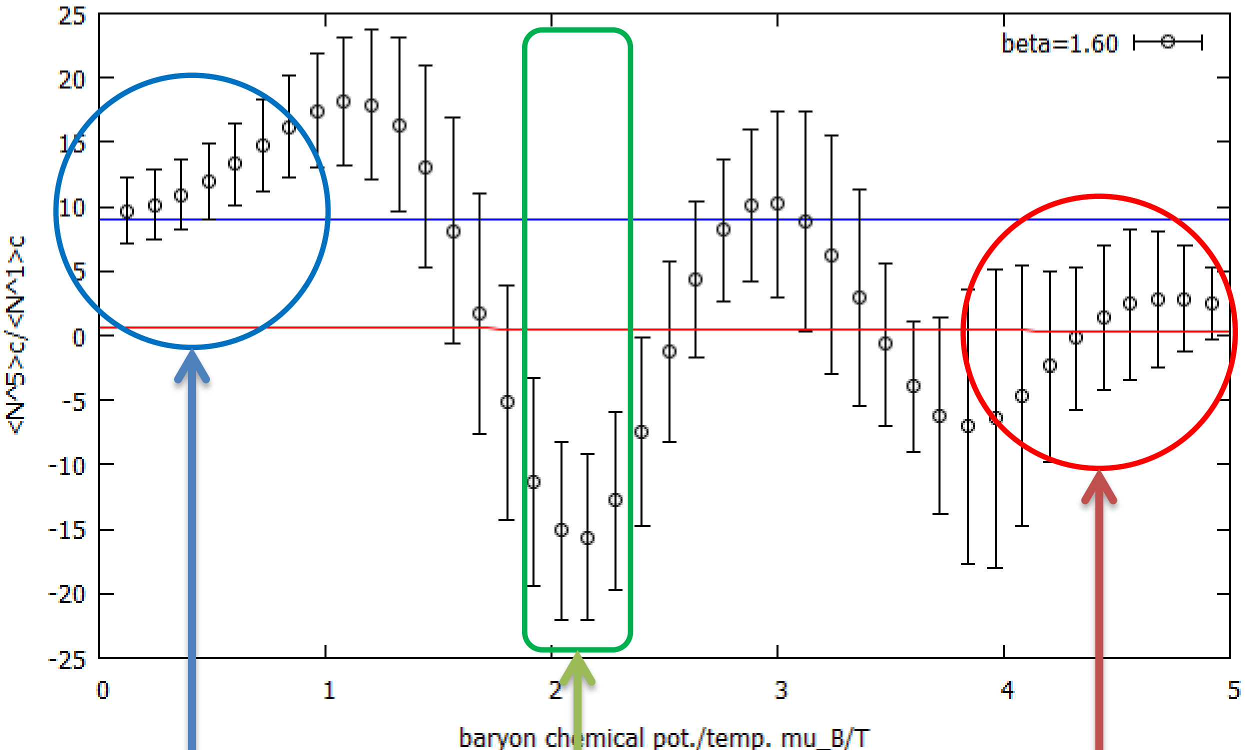
Kurtosis (fig. bottom)

$\mu_B/T = 2,04\,, \beta = 1.6$

deviate from H.R. gas

negative value

Kurtosis $(vs.\mu_B/T)$,$\beta = 1.6$



consistent with H.R. gas

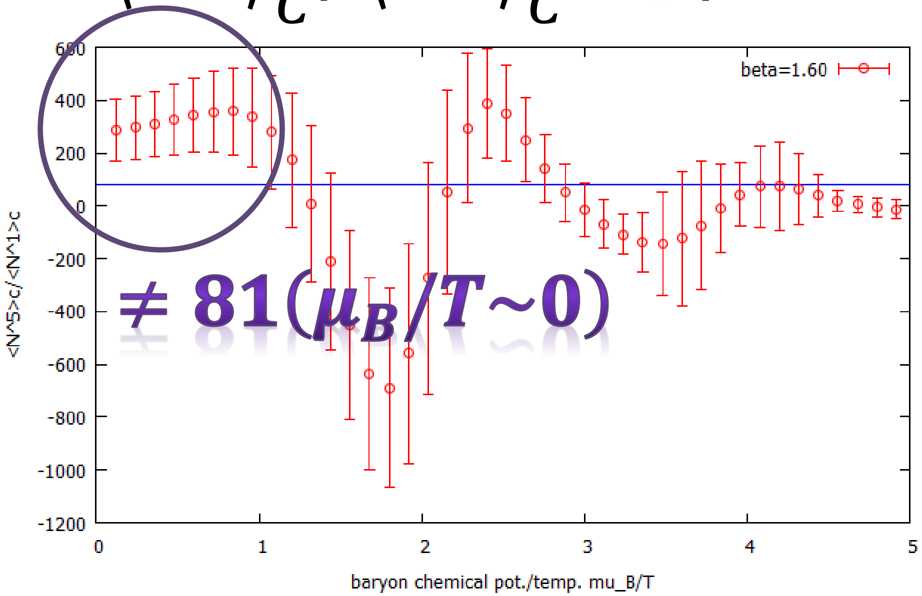
approach to Q.G. gas

singularity

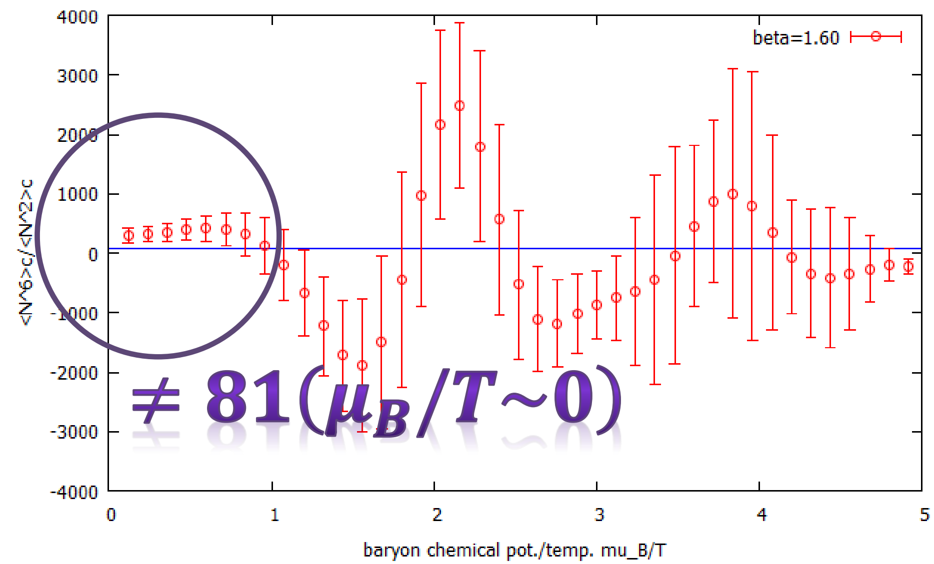
circumstantial evidence of phase transition

more high-order cumulant ($\beta = 1.6$)

$\langle \hat{N}^5 \rangle_C / \langle \hat{N}^1 \rangle_C (\mu_B/T)$



$\langle \hat{N}^6 \rangle_C / \langle \hat{N}^2 \rangle_C (\mu_B/T)$



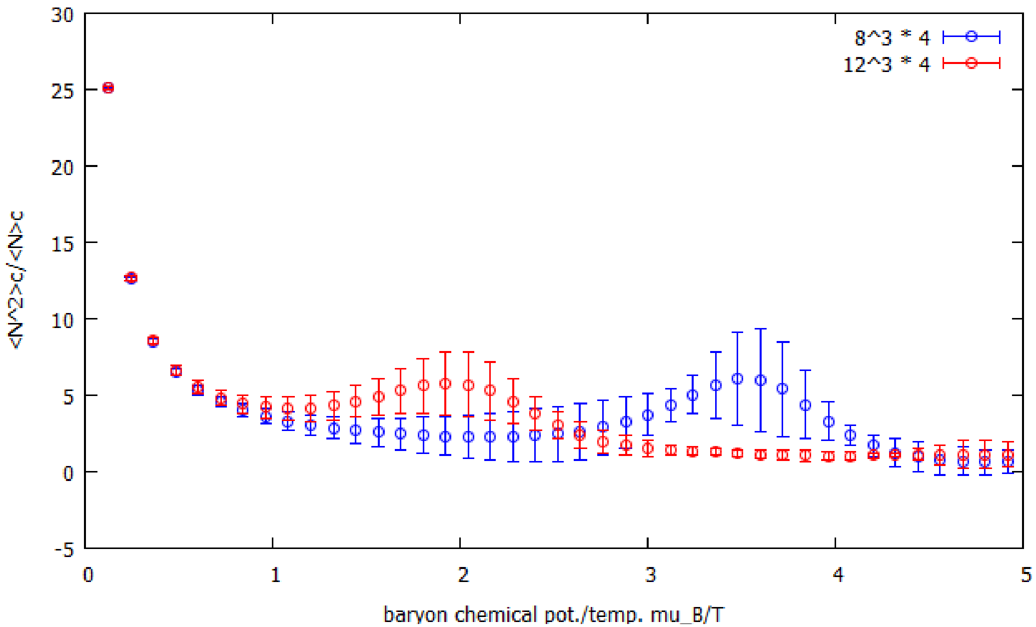
H.R. gas
(blue line)

$$\frac{\langle \hat{N}^5 \rangle_C}{\langle \hat{N} \rangle_C} = \frac{\langle \hat{N}^6 \rangle_C}{\langle \hat{N}^2 \rangle_C} = 81$$

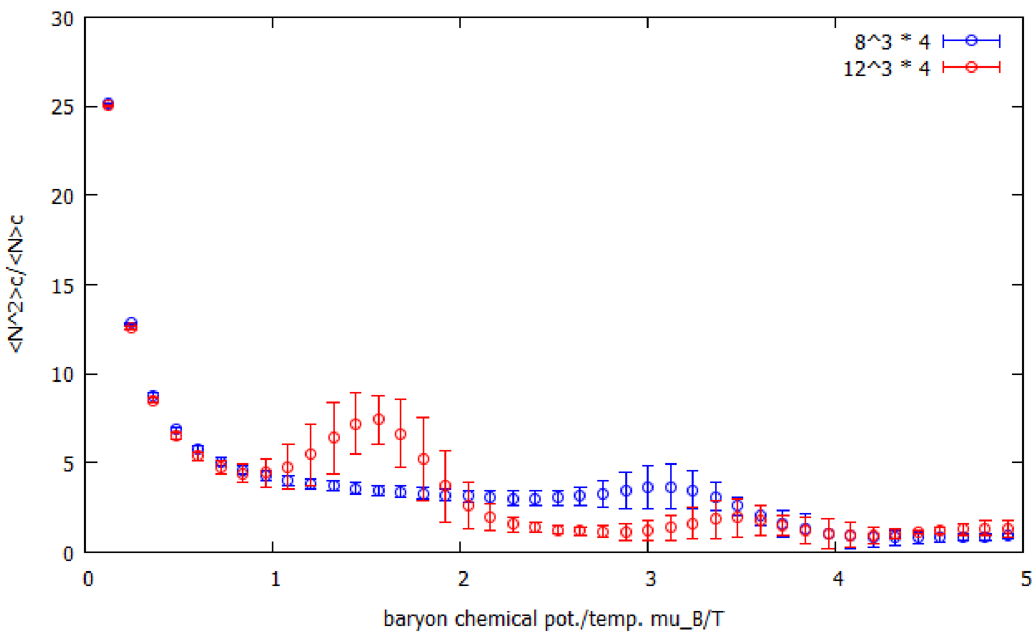
2nd ,3rd ,4th : consistent with H.R. gas

5th ,6th : inconsistent with H.R. gas
(suggestion)

Volume dependence $8^3 \times 4 \; vs. \; 12^3 \times 4$



$$\langle \hat{N}^2 \rangle_C / \langle \hat{N}^1 \rangle_C (\mu_B/T)$$
$$\beta = 0.9$$



$$\langle \hat{N}^2 \rangle_C / \langle \hat{N}^1 \rangle_C (\mu_B/T)$$
$$\beta = 1.5$$